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ORIGINAL ARTICLE

Instructional materials as a site to study teachers' planning and learning

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Abstract
While reports of teachers' use of curriculum materials are common, that of teachers as designers of their own materials are far less so. We argue that these (rare) instructional materials, defined as materials that are classroom-ready and that carry the teachers' actual instructional goals, are 'objects' that are suitable as records of teachers' planning and learning when developed alongside professional development. We provide supporting evidence of this claim and unpack the complexities of interacting instructional goals through a case study of a teacher who (re-)designed her own instructional materials as she participated in professional development. From the findings of the case, we reflect on the educational and methodological implications of pursing this research approach.

Keywords Instructional materials · Professional development · Proportionality · Intercontextual boundary object

Introduction

The recent shift in research and thinking about teacher professional development indicates a re-focus in the weight one should place if the aim is for changes in the classroom towards greater alignment to that envisioned by reform initiatives. In particular, there is recognition that pedagogical decisions in the classrooms are made primarily by teachers; thus, the key is in the quality of teachers' decision-making which is in turn influenced by their resources, orientations, and goals (or ROG, Schoenfeld, 2010). The main lever from the policy perspective to affect teachers' ROG is professional development (PD). In other words, policy makers cannot 'get to' classrooms to bring about *direct* change at scale; such efforts are mediated *always* through teachers.

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This means that PD which is effective at persuading teachers to imbibe and implement certain educational reform ideas lies at the heart of this enterprise.

But this persuasion project is by no means straightforward. The work of teaching is a complex enterprise that involves teachers trying to fulfil multiple goals of teaching often all at once (Lampert, 2001). This perspective of teachers having to balance these many and often competing goals (Leong & Chick, 2007) challenges the simplistic view of teacher PD as uni-dimensionally couched as about 'content top-up' or about 'method of teaching'. The reality is that teachers' 'craft knowledge' (Malara & Zan, 2002) is located practically on a day-to-day basis in the *complex interactions* of numerous overlapping knowledge domains to cope with actual problems of teaching. In other words, instead of studying PD contexts and teachers' instructional contexts separately—and usually, the latter as an 'after-effect' of the former, there is a need to study how PD works its way *within* the context of teachers' consideration of complex interplay of multiple goals for actual instruction.

One way to capture this complexity of interactions in teachers' instructional intents is in the instructional materials that teachers themselves design for actual use in in-class teaching (Leong et al., 2019a, b). Different from curriculum materials designed by others and adapted by teachers, instructional materials that are designed by teachers are 'concretisations' (Leong et al., 2016, 2019c) of their lesson conceptualizations, and the former is the classroom-ready version of the latter. And, as teachers draft and re-draft instructional materials alongside participation in PD, the changes in the materials in response to influences from PD provide a suitable site to examine how PD works its way into the instructional goal-mix of the teachers.

The study reported in this paper indeed locates one such context: teachers' design of their own instructional materials with a view of incorporating relevant ideas from PD. In particular, we report a case of how a teacher projected her instructional intents (in the form of design principles), including adjusting them to bring in goals acquired from PD, in her design of a set of instructional materials for her students. Two questions guide our study: (1) What were the design principles adopted by the teacher and how were they applied when crafting the set of instructional materials? (2) How did PD influence the mix of these design principles as revealed in the changes in the drafts of the instructional materials? We argue that research of this nature would inform us viscerally, from a teacher's perspective, how teachers may (not) appropriate contents of PD in a way that coheres with their actual goals of teaching mathematics in the classrooms. There are then implications in education research and methodological developments.

Closing the PD-classroom gap: intercontextual boundary objects

From the perspective of PD as a means to realize reform-oriented educational goals by teachers in the classroom, it is important that teachers take something of value from PD sessions that will consequently influence their classroom teaching. In our experience as PD providers (subsequently known as teacher educators or TEs), however, we often find that while teachers may appear to participate productively and even avow learning during these PD sessions, there are often little teaching changes

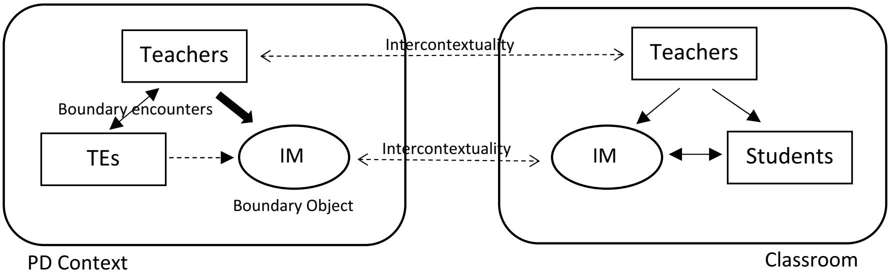


Fig. 1 Teacher-designed instructional materials as intercontextual boundary object

corresponding to the avowed learning as observed in classroom enactments. This common phenomenon (e.g., Hill, 2009; Wallace, 2009) is known as the 'PD-Classroom gap'—presented visually as the space between the 'PD setting' and 'Classroom' as shown in Fig. 1, modified from Leong et al. (2019c).

There are numerous ways, as found in the literature, to conceptualize the gap. One is that of 'boundary' which is rooted in a metaphor of natural (say, geographical) boundaries between peoples of different cultural heritage and practices. These boundaries serve as contact lines for them to exchange culture and practices. In viewing PD as a boundary encounter between TEs and teachers (presented as a bidirectional arrow in Fig. 1), there is a recognition that a boundary (Wenger, 2000) exists between these two parties because they ordinarily operate in different domains. PD settings thus provide a common platform where knowledge exchange, transfer, or creation can occur, which in theory can then lead to boundary crossings (Akkerman & Bakker, 2011) of this knowledge benefitting both parties. While the potential for fruitful engagements within boundary encounters is evident, creating the conditions for successful boundary crossings is not straightforward. Indeed, focusing on one side of the exchange, where teachers do not benefit from these boundary crossings in practical ways, they are unlikely to effect productive changes to their classroom practices, which accounts for the PD-classroom gap.

Closer to the theme of this paper and hence of interest to us is the construct of 'Boundary Objects'. Star and Griesemer (1989), from whom the term was originally attributed, described boundary objects as follows:

> objects which are both plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites … They may be abstract or concrete … The creation and management of boundary objects is key in developing and maintaining coherence across intersecting social worlds. (p. 393)

We think this notion of boundary objects as 'representations of knowledge' (Sztajn et al., 2014) that can potentially be shared across TEs and teachers a promising way to close the PD-classroom gap, especially when these objects are not limited in use within PD settings but can in some ways 'carry over' as holders of this shared knowledge to the classroom. On the left of Fig. 1, the boundary object is seen as acted upon jointly, and hence shared, by both TEs and teachers. As will be explained later, the differentiation of a perforated arrow and a bold thick arrow from TEs and teachers respectively is to highlight the work of design in this project as primarily that of teachers, with TEs' use of the boundary object as a means to focus the teachers' attention for the purpose of PD.

This notion of shared objects that can be easily transferred from the PD setting to the classroom is reinforced by the notion of *intercontextuality*. According to Engle (2006), 'Intercontextuality occurs when two or more contexts become linked with one another. When this occurs between learning and transfer contexts, the content established during learning is considered relevant to the transfer context' (p. 456). Thus, to increase the likelihood of transfer, one way is to intentionally build in these links that strengthen intercontextuality, as shown in the arrow connecting 'Boundary Object' across PD setting and classroom in Fig. 1.

There are, thus, two aspects to 'objects': First, objects that bring TEs and teachers together into a productive boundary encounter; second, objects that strengthen intercontextual links between teachers' experiences in the PD setting and teachers' actual instructional work in the classroom. These aspects are conceptually different but can be combined: intercontextual boundary objects. The intentional use of such objects hold promise in closing the PD-Classroom gap.

Teacher-designed instructional materials as intercontextual boundary objects

There is a long tradition of designing instructional materials for teachers' use in classroom instruction. More recently, as reflected in the ICMI Study 22 topic on task design (Watson & Ohtani, 2015), there is a revived interest in the affordances of quality task design as a key lever in making a difference to students' mathematics learning in the classroom. But an area of inquiry that is relatively scarce is as follows: teachers themselves as designers of instructional materials. By 'instructional materials' (IM), we mean more specifically materials that are prepared by teachers into a 'form that is considered [by the teachers as] … classroom-ready and that carries the teachers' actual instructional goals' (Leong et al., 2019a, p. 50). By definition, we make a distinction between IM and other materials that the teachers, in the process of design, may refer to. We term the latter 'reference materials'. Indeed, teachers do draw heavily upon tasks presented in reference materials, but previous research on teachers' design reveal that the transformation from reference materials to IM presents this summary picture: teachers do not merely lift task items from reference materials and re-arrange them into a set of IM; rather, the selection, modification, and inclusion of novel ideas reflect a complex interplay of teachers' own instructional goals that were customized for their students (e.g., Cheng et al., 2021; Leong et al., 2021a, b).

Furthermore, when we consider the process of IM-design alongside that of ongoing PD, we have an even more complex mix. Teachers imbue their own instructional goals into the design of IM while negotiating with new ideas that they interact with TEs in the PD setting, but this at the same time also renders teacher-designed IM to be suited as intercontextual boundary objects (Fig. 1) for these reasons: (1)

unlike materials that are designed by others and used by teachers, these materials are designed and owned by the teachers in the sense that they are vested in it—not just for the purpose of learning in PD contexts but for ownership of use in their actual classroom instruction. There is a natural strengthening of intercontextual links between the PD setting and the classroom for the teachers (hence, the intercontextuality arrow connecting 'Teachers' in Fig. 1); (2) the IM is a boundary object in the sense that it concretizes the knowledge interaction of TEs and teachers, not merely in abstract forms, but in a form that teachers would directly use in their classroom instruction (as shown in IM's role on the right side of Fig. 1. Details to explain the interactional arrows on this right side of the figure will not be given here as the focus of this study is not on this aspect of the overall conception. Readers may refer to Leong et al. (2019c) for details). The IM is an 'object' in a most tangible sense of the term: it is visible, manipulable (in that revisions can be easily made), shareable across the TE-teacher boundary, and moveable across the PD-classroom divide.

Interestingly, these layers of complexity in teachers' design of IM fits well methodologically for the purpose of this research: (a) it is consistent with our commitment, as indicated in the Research Question 1, to capture the complex intersection among design principles; and (b) insofar as the IM is an objectification of these complexities, it is a suitable primary artefact to locate teachers' adjustments when negotiating their existing design principles with inputs from TEs in PD settings—in line with Research Question 2.

Method

We adopted a case study approach for this project as it is particularly suited to bring out, in visceral and nuanced ways (Yin, 2018), the interaction of goals and shifts in the mix of design principles adopted by teachers, as required in the research aims. Also, as will be explicated in the later sections, the bringing together of various data sources around the primary source of the IM to build a coherent picture of how the design was conceived is indeed a case of a teacher who designs her own IM with a view of incorporating relevant ideas from PD. This approach was also undertaken by similar studies of IM-design (e.g., Leong et al., 2019a, 2021a, b).

As mentioned earlier, our conception of the relation between PD setting and planning for classroom instruction is not one where the latter is an after-effect of the former; rather, we view it as one where PD works back and forth with teachers' design of drafts of IM. This bidirectionality was rendered practicable due to the common work of examining the IM to ready it for classroom instructional work. TEs and teachers discuss ideas in these boundary encounters with a view of how these ideas can be worked into the IM (the intercontextual boundary object) during PD. We claim that this is novel methodology when it comes to the study of the evolvement of teacher-designed IM as (not after) the teacher participates in PD. The point of departure lies in viewing teacher-designed IM not merely as something teachers do to prepare to teach, but also as an object that captures the teachers' changing goals, including changes in response to PD.

In this case study, we followed closely the process of Teacher Meng Tin's (pseudonym) design of IM for her Year 7 mathematics class. While she had been teaching mathematics at the secondary levels for more than 15 years, her deployment was mainly at the Year 9/10 levels. The last time she was assigned to teach this strand of Year 7 mathematics was eight years ago and she did not recall prior design of IM for this level. She was thus personally motivated to develop a new set of IM for this class for the topic under consideration: Ratio and Rates. A further motivation was provided as she was a participant in an ongoing PD on Proportionality as a Mathematical Big Idea; she was keen to incorporate suitable ideas into her IM-design. Big Idea talk was recent among Singapore mathematics teachers; it was kickstarted when the Ministry of Education included "Mathematical Big Ideas" in the latest syllabus revision (Ministry of Education, 2020).

Meng Tin drew from the boundary encounters with the TEs to design a draft of the IM; the draft, as a boundary object, was then brought back as an object of further examination by TEs and teachers during PD. This cycle was repeated five more times before it was finalized by Meng Tin for her classroom instruction. [Meng Tin did not draft the whole set of IM at one go but gradually and chronologically through the duration of the six cycles. So, in subsequent cycles, there were discussions both of previously drafted materials as well as new materials on later sub-topics]. The team members in these PD sessions were mathematics teachers appointed by the Mathematics Department of the school and the authors of this paper (as TEs). The composition of the mathematics teachers in the team were: Meng Tin; a beginning teacher who covered the same mathematics topics as Meng Tin in another Year 7 class; an experienced teacher who did not teach Year 7 mathematics during the period of the project but had taught it before; the Subject Head of mathematics who oversaw Year7-8 mathematics programme in the school.

Prior to the start of these cycles we had two 1-h PD sessions where the TEs illustrated how proportionality is indeed a big idea, in that it is utilized (explicitly and implicitly) in many content areas, in Singapore school mathematics and also specified examples within the Ratio and Rate topic where proportionality can be usefully foregrounded to the students. The examples were not restricted to the syllabus content of Year 7 mathematics. The subsequent PD sessions were essentially discussions on the drafts of IM designed by Meng Tin. The comments by TEs in these sessions were confined to how proportionality can be more explicitly and usefully incorporated into the IM. The TEs also responded to questions on content and pedagogy raised by teachers. Decisions on actual changes (or non-change) to the IM drafts were left to Meng Tin.

Data collection

The primary data for this study were the drafts of IM done by Meng Tin. She designed it for the teaching of the entire topic of Ratio and Rate which was scheduled by the school for a total duration of around five teaching hours across six lessons. The final set of IM was 36 A4-sized pages in length. The IM included spaces for students to do working for each task.

Because our study examined the design principles, including that which were influenced by the PD sessions, adopted by Meng Tin in her IM-design, we also drew data from two other sources: video records of the PD sessions and interviews with her.

The interviews were conducted over a teleconferencing platform and recorded. These interviews, varying in duration from 20 – 45 min, were conducted each time Meng Tin informed us through email that she had made a new draft of the IM. The purpose of the interviews was to find out the reasons behind her selection and sequencing of the items in the IM, and changes made to subsequent drafts. Where relevant and raised by Meng Tin, the questions were connected to how ideas from the PD sessions influenced her decisions. The goal was to inquire as fine-grained as achievable during the interview for goal-interaction details in Meng Tin's IM-design. As such, a semi-structured interview format was adopted to allow space for Meng Tin to describe her intentions in the context of suitable prompts for details from the interviewer. The actual process of preparing for the interviews in order to achieve this effect will be described in the next section of this paper.

The chronology of these data sources is presented here for ease of the readers' reference:

- First PD session to introduce proportionality as a big idea in various content areas in school mathematics
- Second PD session to specify ways to foreground proportionality as a big idea in the topic
- 1st draft of Meng Tin's IM
- Interview on 1st draft
- Third PD session with TEs on 1st draft
- Subsequently, another 5 more cycles of draft, interview, and PD Session
- All the PD sessions and interviews were video-recorded

Analysis

We proceeded with the analysis in three phases. Phase 1 was based on the first draft of the IM. We would argue that this draft reflected more heavily Meng Tin's own instructional goals for the class while the influences from the PD sessions (only two of them up to that point and not directly about items on IM) were minimal. Thus, findings to address research question 1 were drawn from phase 1. Phase 2 was based on the last draft of the IM, and by contrast, it would reflect the strongest influence from the PD sessions. Phase 3 involved an examination of the shifts in Meng Tin's design principles as highlighted in the findings of phase 1 and phase 2, and the focus of analysis was on the intervening drafts. Thus, to address research question 2, we drew findings from phases 2 and 3.

For phase 1, we parsed the IM draft into units of analysis. We consider a unit as a section in the IM where there was a conspicuous common goal that tied the tasks in the section together; the separation between units was marked by a change in goal that can be easily explicated. To illustrate this process of parsing, the first page

SN	Ratio 1	Ratio 2	Justifications
1	21 : 63	1 : 3	Yes/ No
2	24 : 56	3 : 7	Yes/ No
3	$2\frac{3}{5} : 1\frac{4}{9}$	9 : 5	Yes/ No
4	0.36 : 1.2	10 : 3	Yes/ No

Fig. 2 First draft of unit 1 of the IM

of the IM started with "Example 1" which consisted of a table of 4 tasks (Fig. 2). All these tasks shared an identifiable common structure that can be summarized as 'check if ratios are equivalent'. This example was thus considered a unit of analysis. 'Example 2' in the next page consisted of another table with 5 tasks which shared a common feature of 'reduce given ratio to its simplest form' and so, it was considered a separate unit of analysis.

We started the analysis of each unit with initial codes drawn from earlier related research work in this area (Cheng et al., 2021; Leong et al., 2021a, b; Toh et al., 2021). We applied these codes because there was evidence to suggest that some of these principles of design are pervasive among Singapore mathematics teachers (Leong & Kaur, 2021; Leong et al., 2019b). The codes that were relevant for this study, and the brief explanations of them, are given in Table 1. The details of these codes and how they interact will be described in 'Fndings from phase 1: first draft of the IM' of this paper. Here, we illustrate the code of 'gradation' for the first unit of the IM (Fig. 2). We noticed that the numbers chosen for the first item was such that it is easier to see 63 as 3 times that of 21 so that the ratio

Table 1 Codes of Meng Tin's principles of IM-design

Codes of Meng Tin's principles of IM-design	Brief explanation
Gradation	Increase the difficulty level of items progressively
Activeness	Make provision for students to actively engage with tasks
Implicit-explicit	Introduce an idea implicitly first before making it explicit later in IM
Think-slow	Address students' tendency to follow method mindlessly
Challenge	Require students to use higher-order thinking
Test preparation	Prepare students for tests/exams
Connection	Help students see mathematical content-to-be-learnt as conceptually connected

reduces to 1:3. The subsequent items involved more 'difficult' numbers such as numbers that are not simple multiples of the other (24:56) or are non-integers (in the 3rd and 4th items), including a non-example of equivalent ratios (item 4). We reckoned that there was an intended 'gradation' in the degree of difficulty for students to ease them into the work of completing the tasks.

The codes we assigned after we studied the IM units were confirmed, rejected, or adjusted when we examined the interview immediately following this draft. The interviewer started by requesting that Meng Tin talked through the sections of the IM to explicate the reasons for designing each section and task in the way she did. There was no suggestion by the interviewer with respect to specific codes, such as 'were you doing gradation here?' as that would run the risk of post hoc justification. Instead, where Meng Tin made unsolicited references to design principles such as that of 'gradation', the interviewer probed for further explanation and connections with other considerations that she mentioned earlier, thus seeking to increase the likelihood of uncovering interactions among her other goals of instruction. With respect to the specific illustration of 'gradation' for the section as shown in Fig. 2, the confirmatory interview transcripts were as follows:

> 1.27 So, the numbers used in this Example 1 [the first two items] are not so difficult for students to simplify
> 6.25 … [N]ot just whole numbers. Want to include fractions and decimals …

When the analysis of each of these units of the IM was put together, we could see a pattern of repeated codes applied throughout the design of the whole IM. But there were also differences across some units based on a number of factors which Meng Tin brought into play, and especially explicated during the interviews. To probe into these differences, we referred to the transcripts of the PD sessions where relevant, such as instances when Meng Tin herself, during the interview, made references to contents of the PD sessions. The aim of this first phase of analysis was to find out the main principles that most influenced Meng Tin's design considerations, as well as the ways in which she coped with the interaction and competition of the instructional goals underlying these principles as she sought to concretize them in the IM. This addressed the first research question.

Phase 2 analysis was conducted in a very similar way as that of Phase 1 but on the *final* draft of the IM. There was no need, however, of re-parsing of units as Meng Tin did not make changes to the main sections of the IM throughout the drafts. Neither was there a need to re-analyse those portions that were unchanged from the first draft. The focus in this phase was on the changes made. As an illustration, the second item in Fig. 2 was changed: the order of the ratios was inverted (that is, ratio 1 became 3:7 and ratio 2 became 24:56). We went through the same process of conjecturing the reasons, and verifying/amending them from the interview data.

In phase 3, we focused on the changes as evidenced from the analysis of phase 2. To account for these differences, we looked for the relevant transcripts of the PD sessions (and when needed, viewed the videos for visual information during interactions) and sought confirmatory evidence from the interviews with Meng Tin to strengthen the links. Being aware that using ideas from PD for the design of IM was not a straightforward matter, we took into consideration the confluence of factors that shaped the changes that Meng Tin decided to make.

In the next sections, we present the findings. The order of analyses follows the order of the phases. Analysis of phase 1 is presented first. We present the findings of phase 2 and phase 3 together as the connections between changes and the reasons for the changes can be seen clearer this way without duplication of the same data support. Due to limitations in space, the detailed analyses will be presented only for the first 6 pages of the IM, with references to other parts of the IM where relevant. Also, elaborations of how design principles were derived from the analysis will only be provided once for each principle.

Findings from phase 1: first draft of the IM

Unit 1: check if ratios are equivalent

The first draft of this unit of IM is extracted as Fig. 2. As briefly illustrated in the earlier 'Analysis' of this paper, Meng Tin consciously built in a gradation principle in the selection and sequencing of the items in this unit. There were at least two other principles: 'activeness' and 'implicit-explicit'.

On activeness, we notice the column labeled 'Justifications'. In the context of the tasks, we think it has less to do with formal mathematical justification; rather, we surmise it meant that students were encouraged to show through working steps how one ratio is (or is not) equivalent to the other, instead of waiting for the teacher to provide the answers. That this was indeed the intent of Meng Tin was evidenced by: 'I also want them to justify their workings' (1st interview, 1.38, emphases added) and 'I want them to do their own working and to justify' (1st interview, 21.07, emphases added). By activeness, we mean a goal of wanting students, while interacting with the IM task, to exercise activeness in learning and thinking instead of merely following the teacher.

On implicit-explicit, it was tied to 'without the use of calculator' as the prefatory instruction of the unit (Fig. 2). Based on the IM unit alone, we initially thought this had to do with the intention of getting students to work by-hand instead of relying too much on the outputs of the calculator. But Meng Tin built in deeper roots behind this instruction: 'I purposely don't want them to use calculator, because I want them to divide or times by a common constant to bring out the idea of proportionality' (1st interview, 21.00). In other words, if students were pressed to work by-hand, they would have to show the working of multiplying the same constant to both parts of one ratio to obtain the given equivalent ratio. This, to Meng Tin, was *implicitly* the idea of proportionality, which would otherwise be bypassed if students just used the calculator for simplifying ratios. It is implicit at this point of the IM because she did not intend to mention 'proportionality' but merely plant the underlying idea, only to make it more *explicit* later: 'After Example 1 and Example 2 then I will discuss the idea of their workings. This is the part that I will bring in the term "proportion"' (1st Interview, 21.37). (For more on implicit-explicit as a design strategy of IM of another Singapore mathematics teacher, the reader may refer to Leong et al., (2019a, b, c).)

Unit 2: express ratio in its 'simplest form'

Instead of comparing the equivalence of two given ratios (unit 1), the task in this unit was for students to express a given ratio as one where both the quantities are co-prime integers (expressed to students simply as 'simplest form'). It is clear that the underlying working method was meant to be similar to that of the previous unit and can be seen as an extension of it: here, the student expresses each given ratio into a *series* of equivalent ratios (instead of just one given equivalent ratio) to obtain the simplest form. The layout of this unit was almost identical to that in unit 1. There were 5 tasks presented in table form instead of 4. The given ratios were 144: 132, $1\frac{1}{2} : 4\frac{1}{2}$, $0.48 : 2\frac{2}{15}$, 850 g: 3.4 kg, 1.4: 7: 6.3. Instead of 'Justifications', the column was labeled 'Working'. The page ended with a text box that read '"Idea" behind the workings'.

The principle of gradation remained evident in this unit. Like in the previous unit, the numbers chosen were gradated. The activeness principle continued with Meng Tin requiring students to do the working steps, cued by the 'Working' column, without prescribing a fixed way. The implicit-explicit principle was similar to that in unit 1: proportionality was utilized implicitly in working through each of these tasks but would only be made explicit in the discussion under '"Idea" behind the workings' in the textbox at the end of the section.

There were two other principles that became more evident in this unit: 'think slow' and 'challenge'. The item on '850 g:3.4 kg' was meant to break the 'coasting along' mentality of students as if every task was similar to the previous one. Here, if students 'think fast',[1] they were likely to disregard the different units and treat it as 850:3.4. Meng Tin said, '[This item is] not new to students but I don't want to just

[1] The reader may realize that we are using language that approximates that of "thinking fast and slow" by Kahneman (2011). In a separate interview with Meng Tin, she acknowledged that she was indeed influenced by Kahneman's different systems of thinking and she was seeking opportunities to train students to switch to a "think slow" mode at suitable junctures.

keep working with numbers. … I want them to have a change of momentum. I don't want them to keep working [in the same mode] … [they get to see that] suddenly, there are units, [and pause to ask] what should I do?' (1st interview, 7.55). Overlapping with this 'think slow' principle was the 'challenge' intention in this item and the next: 'Then I introduce 3 ratios to break the momentum of it ….They are quite high ability. So [the earlier items] might be too easy for them' (1st Interview, 8.42). It seemed clear that Meng Tin intended to introduce tasks—especially at the end of a set of similar tasks—that would be seen as challenging in a way that was suited to the students in her class.

At the end of these two sets of tasks (example 1 and example 2), Meng Tin's intention was that students, on their own accord, would be able to reflect and tease out commonalities in the working steps for her to explicate it as proportionality: 'I want, at the end of the exercises, [for students] to realize that it is the same thing, no matter what form it takes, [about proportionality]' (1st interview, 8.53).

Unit 3: word problems involving 2-quantity ratios

There were two problems presented in this unit; the first was for 'teacher demonstration' and the second was for 'do it yourself':

Problem 1: The ratio of the number of fiction books to the number of non-fiction books in a library is 5:2. If there are 1421 fiction and non-fiction books altogether, how many more fiction books than non-fiction books are there in the library?

Problem 2: Cheryl and Shufen each have a sum of money. The ratio of the amount of money Cheryl has to that of Shufen is 3:5. After Shufen gives $150 to Cheryl, the ratio of the amount of money Cheryl has to that of Shufen becomes 7:9. Find the sum of money Cheryl had initially.

This unit was rather different from the previous two units. The task types were different. These were problems lifted directly from the school-prescribed textbook which was the main reference material. We think that they were selected on the basis that they were deemed typical test-type questions, and that Meng Tin took it as her social responsibility as a teacher, in a context where students' test performance holds high consequences (Leong & Kaur, 2021), to prepare her students for such test situations. The underlying principle behind the selection of these tasks was mainly 'test preparation'.

None of the principles discussed earlier was directly applied here, except arguably gradation—the 'teacher demonstration' first followed by 'do it yourself' by students was possibly a way to ease the students into this type of tasks. And, the activeness principle appeared contradicted. Instead of letting students have a go at the problems using the method that most appealed to them, she wanted students to follow her working steps: 'You know for Secondary One students [if] you don't show them the *proper way of presenting* the working first ….[K]nowing the students, they tend to be all over the place in their presentation. … I want them to see the required presentation of working instead of having them undo their working [later]' (1st interview, 28.05, emphases added). Meng Tin had a view that students who were in the first year of secondary schooling were not used to presenting their word problem working in a 'proper way'. She judged that instead of 'undoing' their way of presentation later, it was important for this type of tasks where working 'properly' was critical that she prescribed a presentation style for students to follow.

During the interview, Meng Tin admitted frankly that for the problems in this unit, 'Problem sums like this I don't know how to bring proportion in …' (27.03). But in the next moment, she said, 'So, even if they find one unit right, even that is proportion [right]? … I want to use this idea of proportion to show the working'. These apparently contradictory statements by Meng Tin can be explained: At this point in time, she did not feel she had good a grasp of how proportionality, in all its facets, can be applied to these problems. But she was aware that the 'unitary method' worked. To illustrate the unitary method in the case of the first problem, 7 units represent 1421 books, so 1 unit represents 1421/7 books, thus 3 units represent $1421/7 \times 3$ books. In summary, Meng Tin knew that the unitary method, which implicitly utilizes proportional reasoning, worked for her and the students here but she was not clear how the proportionality talk developed in the previous sections of the IM *connected* smoothly to the task presentation in unit 3. She experienced difficulty in fulfilling her 'connection' principle.

This connection difficulty became more pronounced in problem 2. Meng Tin had thought to solve it this way: Express the initial amount of Cheryl and Shufen (in $) as $3\times$and $5\times$respectively. After the transfer, their amounts were $3x+150$ and $5x-150$ respectively. These quantities are also in the ratio 7:9. So $\frac{3x+150}{7}=\frac{5x-150}{9}$. The rest is about solving the equation and obtaining the value of $3x$. The proportionality step is found in the equation but she struggled to explicate the connection; moreover, the solution became to her too much about algebraic manipulation than it is about proportional reasoning.

Unit 4: ratios involving three quantities

The contents for this unit is extracted as Fig. 3. All the principles utilized in the first two units were conspicuous again in this unit.

The critical layout difference in this unit as against the first two units lied in the use of a table here for each of the tasks. The reasons for this insertion at this stage of the IM were given by Meng Tin during the interview:

> 17.35. I put the table in because [the first author] actually presented the table form in solving some of the questions [during the second 1-hr PD Session]. I actually think it is quite useful for the lower progress students. For the high progress students, they will have no difficulty solving using their own methods; but even for them, [the table] adds to their repertoire of strategies. For the lower progress students, this way of presenting can help them with the rate questions later on, esp the ones demonstrated by [the first author] on foreign exchange. This part of the worksheet I am targeting more of the low progress students.

To provide the context of what the first author presented during the second 1-h PD session which directly influenced her use of the table for 3-quantity ratios, a brief

SN	Question
1	Find the ratio of \$7.60 to 84 cents to \$6
2	If $x : y = 5 : 6$ and $y : z = 4: 9$, find $x : y : z$ *Workings*:
3	If $x : y = 3 : 4$ and $y : z = 5 : 8$, find $x : y : z$ *Workings*:
4	If $p : q = \frac{3}{4} : 2$ and $p : r = \frac{1}{3} : \frac{1}{2}$, find $p : q : r$. *Workings*:

\$7.60					
84 cents					
\$6					

x					
y					
z					

x					
y					
z					

p					
q					
r					

Fig. 3 First draft of unit 4 of the IM

description of a relevant portion of the PD Session is described here. The 'foreign exchange' question that Meng Tin referred to is as follows: 'One day, the exchange rate between the euro (€) and the Singapore dollar (S\$) was €1 = S\$1.5997. On the same day, the exchange rate between the New Zealand dollar (NZ\$) and euro was NZ\$1 = €0.5801. (i) Li Ting changed €460 into Singapore dollars. Calculate how many Singapore dollars she received, correct to the nearest dollar. (ii) Raju changed S\$800 into euros before converting the full amount into New Zealand dollars.

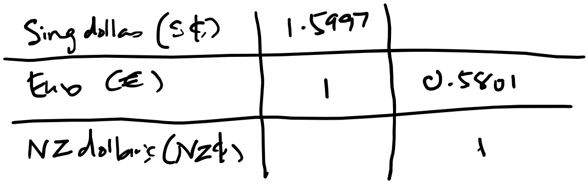


Fig. 4 The representation on the board by the first author: the givens of the foreign exchange item

Calculate how many New Zealand dollars she received, correct to the nearest dollar'. During the PD session, the first author sought to demonstrate how proportionality (defined as the relationship between two quantities such that one instantiation of these quantities is produced by multiplying the same constant to another instantiation) can be made more visible and explicit in the context of this foreign exchange question. He first transferred the given conditions of the problem onto a table as shown in Fig. 4. At this point in the PD Session, Meng Tin uttered, 'Oh! That's very nice …. Oh yes, I never saw it this way'.

The proportionality relationship of multiplying by constants was then shown to be captured and objectified in the use of arrows across columns (instantiations), as presented in Fig. 5. Much later, in the final PD Session, Meng Tin revealed that she was drawn to this representation as it helped reduce the level of difficulty of such problems for herself. In terms of design principle, this can be considered an elaboration of 'challenge'—where Meng Tin was cognizant that these 3-quantity ratios presented a challenge to some students and she wanted to make the learning of how to solve these challenging tasks easier for them. She was at this point more drawn to the ease of visualizing than for its potential to foreground proportionality, although she was aware of the latter: 'I only start bringing in [the table] when it is 3-quantity. so I use this table using [the first author's] example. I thought 2-quantity is relatively simple' (third PD after 1st draft, 26.32). This view of the role of the table explains its introduction at this juncture of the IM, and not earlier.

Summary response to research question 1

Some of the principles adopted by Meng Tin in the design of the IM, such as gradation, activeness, implicit-explicit, think slow, challenge, test preparation, and connection (Table 1), are common with those used by other Singapore Secondary

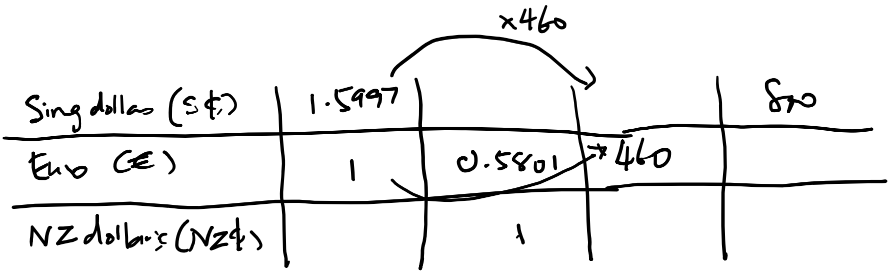


Fig. 5 The representation on the board by the first author: the arrows to highlight proportionality

SN	Ratio 1	Ratio 2	Justifications (Show your workings)
1	21 : 63	1 : 3	Yes/ No 21 63

Fig. 6 First item in unit 1 of the final draft

Mathematics teachers in their design of IM (Leong et al., 2019a, b, 2021a, b). But in these previous research findings, the focus of each report was on one or two of these elements at play; in the case of Meng Tin, we see how these principles came together in her design. The interactions were complementary in some situations where the set of tasks were more of the practice-variety, as shown in unit 1, unit 2, and unit 4, but were more conflictual in another situation, such as when detailed steps were needed to be followed by the students to fulfil test requirements, as was the case of tasks in unit 3, and with the connection difficulty. Insofar as these principles are in common with many Singapore mathematics teachers, we may say that the some-complementary-some-conflictual way of coping with multiple goals of teaching is a usual experience for Meng Tin, as it is for other mathematics teachers (Lampert, 2001; Leong & Chick, 2007).

But the additional commitment to foregrounding proportionality as a big idea appeared to pose further challenges to Meng Tin in her IM-design. Up to this point, that is, the first draft, Meng Tin avowed desire to incorporate proportionality in her IM (as evidenced by the numerous proportionality-talk in the quoted transcripts), but she admitted to struggling to pull all units of the IM together around the ambit of proportionality as a big idea in an explicit way. In unit 3, for example, she resorted to her more comfortable unitary method and hoped that this was also 'considered' proportionality. Also, even when she took an implement from PD (that is, the table) for her IM-design, she was more motivated by its potential as a help for students confronting a challenging problem than for its potential in using the big idea of proportionality. Overall, Meng Tin's first draft was largely driven by her existing goals and principles of design, while making room, where it fitted with these existing goals (such as 'challenge'), for proportionality talk as an add-on non-essential. That she was self-aware of her own struggles opened a way for learning through more boundary encounters (in the subsequent PD sessions).

Findings from phase 2 and phase 3: subsequent drafts of IM

Figure 6 shows the changes in relation to the first item for unit 1 (see Fig. 2 for comparison). The main difference was in the introduction of the table under the 'Justifications' column. The rest of the items in unit 1 were presented in an identical layout as the first item. In this version, there was an explicit requirement for students

a. Representation on the board by the first author as multiplying by same number to obtain other instantiations and as same rate

b. Representation on the board by the first author as producing instantiations of fractions

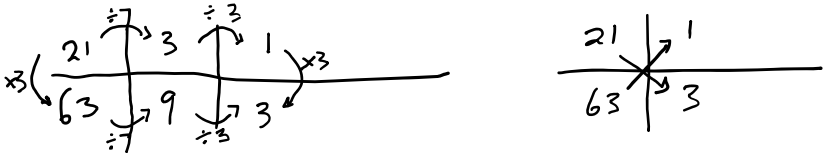


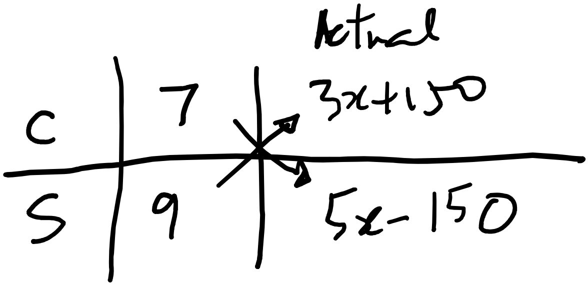
Fig. 7 **a** Representation on the board by the first author as multiplying by same number to obtain other instantiations and as same rate. **b** Representation on the board by the first author as producing instantiations of fractions

to use the table format to 'show [the] workings' to justify if the given ratios were equivalent.

The first push for this change was, surprising, from other teachers in the team who participated in the PD Session. After reviewing the first draft, a teacher commented, 'How do we bring in the idea of proportionality louder … I am wondering if we can insert [proportionality] into the … worksheet so they can [use it to] do working to justify. Get them to keep thinking about it. Right now, the word "ratio" is louder than the word "proportion" …' (third PD session, 25.11, emphases added). Another teacher followed up this comment by adding, 'I don't know whether we need to keep repeatedly mentioning the word proportionality. If so, this worksheet is not enough' (emphases added).

The first author, in his capacity as TE, seized upon that opportunity raised by the teachers to highlight that proportionality can indeed be suitably foregrounded using the existing tasks by demonstrating concretely the various ways to harness proportionality in the first task (as shown in Fig. 6). Briefly, he took on the personas of different imaginary students who would work out the answer to the task differently: one would divide 3 on both 21 and 63 to obtain 7 and 21 in the next column, then divide by 7 to obtain 1 and 3 in the next column (that is, viewing proportionality as multiplying/dividing by same constant to one instantiation to obtain another instantiation of proportional quantities); another would notice that 63 is 3 times that of 21, so if he puts '1' in the next column beside 21, the number below should also be 3 times that of 1 to yield 3 (that is, viewing proportionality as 'same rate' in all instantiations of proportional quantities), yet another would place 1 and 3 in the next column beside 21 and 63. To test that they are equivalent ratios, one can 'cross-multiply' 1 to 63 and 3 to 21 to check they result in equal numbers (that is, viewing proportionality as producing instantiations of fractions, and thus all the relevant operations of fractions apply). He presented all these using the table (as shown in Fig. 7a, b) as a convenient way to 'hold' all these talks together in a coherent visual structure: 'Once you put things in a table, you can see proportionality through different lenses' (third PD session, 10.21). He also presented these different ways of viewing proportionality as different 'affordances' which might be 'useful' for the students when they work on a range of such tasks. (He then went on to convince the teachers, again

Fig. 8 Representation on the board by the first author to show how proportionality as yielding fractions can be used for problem 2



within the structure of the table, but using *x* and *y* in the first column, and *kx* and *ky* as representing all other instantiations that these conceptions of proportionality are in general equivalent.) He reminded the teachers that the goal of 'making proportionality a big idea' was so that students will not merely see ratio, rate, speed, and percentages (and others) as separate topics to be studied with separate methods to be acquired and also to see them as applications of a common idea of proportionality undergirding them all.

Since Meng Tin expressed difficulty with connecting proportionality to problem 2 (see the earlier reported findings in the first phase of unit 3), the first author went on to show (Fig. 8) that by taking the conception of proportionality as yielding fractions, as discussed earlier (and visually expressed in Fig. 7b), the connection to this problem can now be easily made using 'cross-multiplying' as an application of proportionality.

Throughout the presentation by the first author, Meng Tin nodded intermittently but there were occasions where she displayed visibly the intensity of furrowed brows. She did not commit to any change at this PD session but appeared to be dealing with some internal tension as she took in these new perspectives.

The table form for working was also carried over to unit 2 tasks. An excerpt of the first task in unit 2 is shown in Fig. 9.

When asked in the interview following the second draft for the reason for this change, she mentioned:

> 3.15I change … the table … [The first author] posed me this question … "Is there an *anchoring* that I can put it in my worksheet so that I can bring out the idea of proportion more explicitly?". I thought I should put the table in it. … So I think table is an *anchoring* point in *my whole worksheet*. [Emphases added]

SN	Ratio	Workings				
1	144 : 132	144				
		132				
		Answer: ___________				

Fig. 9 First item in unit 2 of the final draft

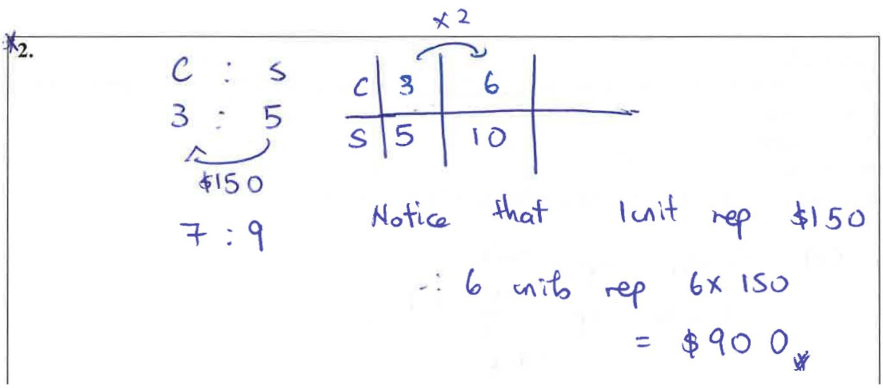


Fig. 10 Meng Tin's working for problem 2

Contrary to Meng Tin's claim, the first author did not literally pose such a question to her nor was the word 'anchoring' used. Obviously, this was Meng Tin's own interpretation of what transpired in the previous PD session. This was, to her, the impact of the boundary encounter: she felt she needed to respond by having a recognizable 'anchoring' (presumably to her and to her students) of proportionality consistently throughout her IM. And she became convinced that the table form was such a suitable 'anchoring'.

That the table form as anchoring was also continued into unit 2 provided evidence that Meng Tin was at this juncture more committed to making proportionality a big idea throughout her IM than merely leaving it at the background of her design (as was the case in the first draft).

The commitment to using the table as an 'anchor' of proportionality became more obvious in unit 3, despite this: '[N]otice when it comes to the word problems I removed [the table in the worksheet]. Actually, I am a teacher who is not very comfortable with students following my method. … I believe there is more than one method to solve the problem. So long as students can understand I am fine with the method, so long as the concept and process is correct' (post-2nd draft interview, 3.26). This looks like a conflict against her 'activeness' principle. Nevertheless, Fig. 10 shows her working for problem 2 which utilized the table representation.

In other words, even though she did not include the table form in the printed IM, she intended to show it, as an anchoring for proportionality, in her working for the problem during class instruction. Moreover, her working as shown here was a decisive move away from the heavy algebraic tools (see the earlier analysis of the first draft of unit 3) and instead closer towards proportional reasoning. This was Meng Tin's way of coming to terms with keeping her activeness goal for unit 3 while foregrounding proportionality. In so doing, she also better coped with her initial 'connection' problem that unit 3 was somehow a break from proportionality talk, by maintaining the table as an anchoring of proportionality across these units more smoothly. The lead-on now to unit 4 became more seamless as the first draft of unit 4 contained tasks that were originally put in the table form.

There is no space for detailed analysis of the rest of the IM. In brief, the technique of using the table as an anchoring of proportionality was carried through the rest of the IM, and where this compromised with her other goals of design (such as activeness in unit 3), Meng Tin adjusted but utilized the table as a holder of proportionality talk in her working steps.

Summary response to research question 2

There were minor tweaks that were refinements of principles described in the earlier analysis. The major change was in the insertion of the table as 'anchor' to hold and give visible explicitation to proportionality throughout the IM. It was not straightforward, as doing so partially compromised the activeness principle but at the same time strengthened the connection principle (which was curtailed in unit 3 of the first draft). This is yet another example of teaching as an enterprise that involves sometimes-complementary-and-sometimes-conflicting-goals situations.

This change in the IM-design reflected a shift in Meng Tin's take on the role of 'proportionality as a big idea' as an overarching instructional goal in her planning to teach this topic. As she interacted more with teachers and TEs during the PD sessions, and across a number of drafts, she became clearer about the subject matter and pedagogical considerations surrounding proportionality in relation to the topic at hand. She became more convinced (and perhaps more confident) that proportionality can indeed serve as a big idea to tie much of the strands in the topic together. That the teachers and the TEs were focused concretely on the actual tasks she constructed in the IM drafts (as evidenced from the consistent references, with emphases added by the authors, on 'the worksheet[2]' by all in the PD sessions) helped directly in her re-conceptions as she attempted further drafts of the IM. She was also looking for an 'objectification' of proportionality that she could put into the IM. It would serve as a common reference point for her as a teacher and for her students. She found it in the table representation which she referred to as an anchor of proportionality.

Discussion

The direct answers to the research questions are given in the earlier respective 'Findings from phase 1: first draft of the IM' and 'Findings from phase 2 and phase 3: subsequent drafts of IM'. In this section, we take a step back to reflect on the implications of the findings, especially in its relation to the methodology that is adopted in this study.

Earlier, we claimed that the research that is reported in this paper presented a new approach: the use of teacher-designed IM not just as a record of what teachers plan to do for teaching but as a means to access teachers' underlying instructional goals as (not after) they interact with ideas during PD sessions. In this section, we reflect,

[2] 'Worksheet' is the language Singapore teachers tend to use for IM.

in the light of the case of Meng Tin's design, on how this approach can provide affordances for advancing education research and professional development.

When we carefully examine how teachers design their own IM, we capture the complexity of teachers' goal-considerations and avoid the pitfalls of simplistic stereotypes of how teachers think about and plan their instructional work. Take, for example, the 'teacher-centred' versus 'student-centred' binary talk. As Meng Tin's design shows, she was neither simply teacher-centred nor simply student-centred when thinking about instruction. She was 'teacher-centred' when it comes to deciding on the tasks that students should engage in and how they experience them chronologically, but she was 'student-centred', as evidenced by principles such as activeness and gradation, in terms of being sensitive to how students would engage with the tasks. In other words, in crafting the set of tasks, she was both teacher-centred and student-centred. Or, take the usual caricature of Asian mathematics classrooms as merely 'drill-and-practise' (e.g., Leong & Kaur, 2021). The think-slow principle and the deliberate scaffolding of challenging items contrast against the portrait of mindless repetitions that are usually associated with drill-and-practice. Studying teacher-designed IM gives us window frames to look into how various underlying goals behind design principles, and their conflicts, play out as teachers grapple with instructional contents that really matter to them. It helps us avoid trivializing the actual work of teaching into one-dimensional portrayals; rather, it presents us with a concrete but complex picture of how they weave various considerations into one set of IM.

A stronger claim we would like to make is: the study of the IM not only tells us how teachers *plan* their teaching; it reflects a trajectory, down to the grain-size of actual tasks, of how they would actually *teach* in the classroom. To explain the connection between planning and actual classroom instructional work, we need to go back to the intercontextuality of teacher-designed IM. There is an intercontexuality between IM *as designed* and IM *as utilized* in class (Fig. 1). As illustrated by Meng Tin, teachers within this context do not design IM merely as an academic exercise or as a rough lesson guide; they do so with a view of practically keeping closely to its content and development during in-class instruction. There is no reason for not following closely the IM that one designs for the very purpose of implementation. Thus, a study of the IM gives us a good approximation of the lesson image of the teacher and hence how the lesson would be conducted. Bidirectionally, as was our argument from the beginning of this paper, this means that IM can also serve as a workable boundary object to influence actual classroom instruction through the teachers' design, thus closing the PD-Classroom gap as identified by Leong et al. (2019c).

In fact, the findings of the case of Meng Tin's design provide supporting evidence that when the PD work is focused on the contents of teacher-designed IM, there is readiness in teachers to incorporate reform ideas into their classroom instructional work through adaptations in their IM. Here, we offer some reasons for this phenomenon: (1) Contents in IM provide a certain concreteness to innovative/reform ideas that abstract talk on principles do not. This concreteness enables teachers to think and plan in a way that is consonant to the concrete reality of classroom work. A contrastive illustration may explain this better. Supposedly, instead of discussions around IM items, the PD sessions with Meng Tin were centred around general pedagogical principles and advanced mathematical concepts relating to proportionality. Even if these principles and concepts were imbued with actual examples (not the ones in the IM drafts), Meng Tin still had to overcome two non-trivial hurdles for these ideas to be practicable for her: these ideas and examples offered by the TEs do not by themselves present answers to how she can modify the tasks she crafted, and there were no easy ways to fit them into the goal-architecture that she had built into her draft. This latter point leads naturally to the next reason. (2) PD discussions about drafted IM tasks and sequences do not call for an overhaul of the basic structure and development of the IM as conceptualized by the teacher. In other words, the drafted IM is a kind of agenda set by the designing teacher and so long as it is not fundamentally challenged, within the safety of this set agenda, teachers are generally more willing and open to accept tweaks to improve it. That designing teachers were given freedom to change (or not at all) strengthened this sense of safety and openness. (3) When reform ideas are discussed in a way that is closely linked to the designed tasks in the IM, teachers find it easier to find 'objects' surrounding the tasks that they can practically incorporate into their IM re-drafts and hence their instructional work in class. (In the case of Meng Tin, she found an 'anchoring' object.) (4) The IM drafts come with problems/gaps that the designers experienced when crafting them. In the case of Meng Tin, she admitted the connection problem in unit 3. These are problems that the teachers come with when they present the drafts and they come ready for answers. Thus, when the discussion during the PD session yields coping strategies that are seen as improvements, they are more readily adopted.

The preceding paragraph should not be taken to imply that incorporating innovative ideas into IM is a straightforward matter. As Meng Tin's case show, it called for a re-weighing of her own commitments to teaching mathematics and it involved adjustments, even compromises, to her existing design goals. The foregrounding and explicitation of proportionality in the table form meant that she had to pare down on her activeness principle while strengthening her connection principle. She took into consideration the points offered in the PD sessions and made the changes. This shows that the approach of studying IM-design provides yet another affordance: the analysis of changes in the IM-drafts can account for the complexities involved when teachers make changes (or do not make changes) to their instructional practices in tandem with inputs from PD.

This approach also shows us the realistic outcomes of PD that are based on IM-redesign: it does not usually change the overall goal-architecture of teachers' instructional practices (just as Meng Tin's overall development structure remained largely intact), but it does change some implements (such as 'anchoring' objects) that can nevertheless carry reform ideas in a significant way. Our hunch is that this kind of PD is effective in yielding incremental (not transformational) changes but when taken together over a number of such interactions can result in new habits of mind for teachers.

Conclusion

Teachers as designers of mathematics IM is not a common phenomenon when viewed internationally—although it is common among Singapore secondary mathematics teachers. Therefore, international research in this area is only in its beginning stages. This study provides encouragement that this is potentially a fruitful line of research in the coming years, both in the areas of teacher PD and in research methodology. For the former, how teachers design their own IM opens up complex interactions of design principles, such as activeness, implicit-explicit, gradation, think-slow, and challenge (among others), that are currently under-reported. As to the latter, there is potential in developing the analysis of teacher-designed IM as a methodology for studying instructional planning and shifts in teachers' learning.

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Declarations

Competing interests The authors declare no competing interests.

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