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ORIGINAL ARTICLE

Constructing shared mathematical meanings in the classroom with digital artifacts that simulate real-world phenomena

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Abstract
This paper describes and examines students' shared construction of meanings while learning about quadratic functions via digital artifacts that simulate real-world phenomena, like the motion of a ball on an inclined plane, focusing specifically on the role of the teacher in that construction process. The study follows the interactions of twenty 15-year-old students and their teacher during the completion of three sequential digital tasks, analyzing how these interactions promote the students' ability to construct the mathematical meanings of the quadratic function. The study was guided by the theory of semiotic mediation, which treats artifacts as fundamental to cognition and views learning as the evolution from meanings connected to the use of a certain artifact to those recognizable as mathematical. The data analysis showed students progressing from the description of the real-world phenomenon toward a construction of the meaning of the quadratic function that models the phenomenon. While marking the "critical moments" in this progress, we analyze the communication strategies used by the teacher to facilitate it. Our research findings showed evidence that certain types of questions and the strategy "re-voicing" can be particularly effective in prompting students' construction of mathematical meanings.

Keywords Quadratic function · Shared construction of meanings · Digital artifact · Simulation of real-world phenomena · Semiotic mediation · Teacher intervention

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Introduction

Although students' use of digital artifacts to learn mathematical concepts has been extensively studied in mathematics education research (e.g., Artigue 2002; Goos and Bennison 2008; Larkin and Calder 2016; Hoyles 2018; Swidan 2019), less is known about the integration of digital artifacts in the classroom setting to stimulate the students' construction of mathematical meanings (e.g., Trouche 2004; Faggiano et al. 2017). Integrating digital artifacts into their teaching practices can present teachers with a number of complex challenges. One of these challenges is understanding the artifact's potential and its relationship with the mathematical content (Falcade et al. 2007; Swidan and Yerushalmy 2014; Faggiano et al. 2018). Another hurdle for teachers is deciding how to use these digital artifacts to achieve the set pedagogical goals (Arzarello and Robutti 2010; Mariotti 2013; Soldano and Arzarello 2016).

These and other challenges have been partially discussed in the literature devoted to examining the use of digital artifacts to simulate real-world phenomena. Researchers have focused mainly on student engagement (e.g., Bray and Tangney 2016), student interaction with the digital artifacts (e.g., Geiger et al. 2010; Sokolowski et al. 2011), and the effects that digital artifacts have on student mathematical achievements (e.g., D'Angelo et al. 2016). However, less attention has been paid to the students' shared construction of meanings in the classroom context, during learning processes in which digital artifacts were used to simulate real-world phenomena (Lynch 2006), and to the teacher's role in facilitating that construction.

In this paper, we address this gap in knowledge by examining the students' shared construction of mathematical meanings while using digital artifacts that simulate real-world phenomena, focusing particularly on the behavior of the teacher. We describe and discuss the results of a teaching experiment in which a digital artifact that simulates the motion of a ball on an inclined plane was assessed for the extent to which it can promote tenth-grade students' ability to construct a mathematical meaning of the quadratic function. Employing the semiotic mediation approach (Bartolini Bussi and Mariotti 2008), we analyze the evolution of the students' personal meanings toward shared mathematical meanings, noting how that evolution is elicited by the students' interaction with digital artifacts and by the teacher's communication strategies during the collective discussions.

Understanding how digital tools can be used to teach complex mathematical concepts, and what sorts of skills and strategies are required from teachers in order to use them effectively, carries important theoretical and pedagogical implications. Theoretically, the study sheds new light on the role of the teacher as a "cultural mediator" between the students' everyday conceptions and the "culture" of mathematics (Radford 2008). Pedagogically, the study represents a valuable example of how digital artifacts can be used to model real-world phenomena, demonstrating the specific teacher practices that promote the construction of mathematical meanings.

Theoretical framework

To examine how students construct mathematical meanings of the quadratic function when they interact with digital artifacts that simulate real-world phenomena, we use the semiotic mediation approach (Bartolini Bussi and Mariotti 2008). Accordingly, the relationship between the artifact and the knowledge is expressed by culturally determined signs, and the relationship between the artifact and the learners while they accomplish a specific task is expressed via speech, gestures, symbols, and tools. Our choice to use the semiotic mediation approach as a theoretical tool in the current study enables us to thoroughly analyze the interaction between the mathematical objects deployed in the artifact and the subjective meanings ascribed to them by the students.

Bartolini Bussi and Mariotti proposed the Theory of Semiotic Mediation (TSM) to model a learning process that exploits the educational potential of artifacts. The TSM aims to describe how meanings related to the use of a certain artifact can evolve into meanings recognizable as mathematical. Social interactions and semiotic processes are assumed to play key roles in the learning, particularly in situations in which learners are encouraged to use the artifact to solve a given task. In the context of artifact use, this approach describes the relation between personal meanings and mathematical meanings as a double semiotic relationship. On the one hand, the TSM focuses on the use of the artifact to accomplish a task, recognizing that knowledge is constructed while solving the task. On the other hand, it analyzes artifact use in the process, distinguishing between the personal meanings constructed by individuals based on their experiences with and use of the artifact to accomplish a task (top part of Fig. 1), and meanings that an expert recognizes as mathematical (bottom part of Fig. 1) when observing the students' use of the artifact in the process of completing a task (left triangle in Fig. 1).

The TSM distinguishes among three kinds of signs that are generated as a result of practical activity with the artifact: artifact signs, pivot signs, and mathematical signs. Artifact signs, which refer to the artifact and its use, are produced via the social use of the artifact to accomplish a task (upper right vertex in Fig. 1). These signs can be shared and may evolve into mathematical signs that refer to the mathematical context. Mathematical signs are associated with the mathematical meanings shared in the

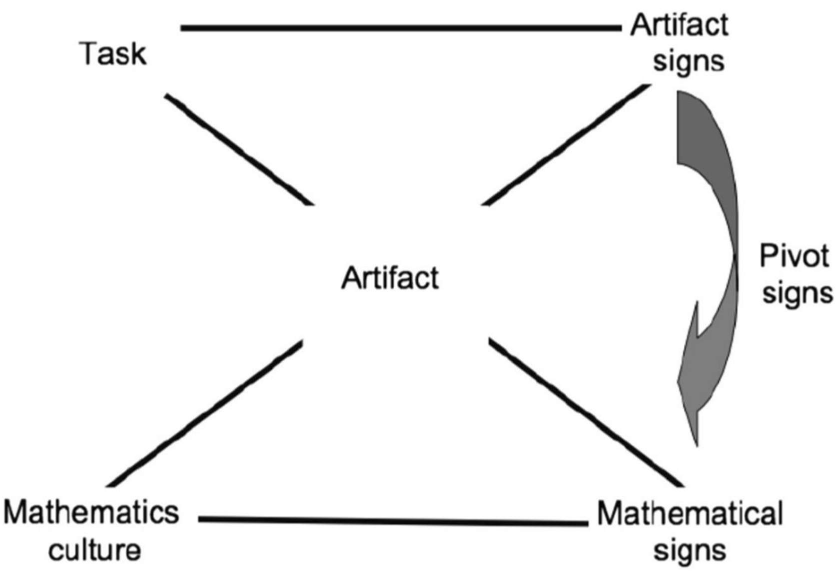


Fig. 1 Semiotic mediation model (Bartolini Bussi and Mariotti 2008)

institution to which the classroom belongs (right side of Fig. 1). The third type of signs—pivot signs—play a pivotal role in the complex process of the evolution of artifact signs into mathematical signs. According to Bartolini Bussi and Mariotti, a distinguishing feature of these signs is their shared polysemy—that is, they may refer not only to the activity being performed with the artifact, but also to the natural language being used to describe the artifact and to the relevant mathematical domain.

The TSM assumes that any artifact can have valuable semiotic potential with respect to particular educational goals. The semiotic potential of an artifact is thus defined as follows (Bartolini Bussi and Mariotti 2008):

> On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. (p. 754)

The double semiotic relationship the authors describe above, referred to as the *semiotic potential* of an artifact, is defined separately for each artifact, i.e., with respect to the particular design and set of pedagogical goals associated with each artifact. The determination of an artifact's semiotic potential, therefore, is an essential and foundational part of the design of any pedagogical plan that will rely on that particular artifact's use. The notion of the semiotic potential of an artifact constitutes a key feature of our study, wherein it was exploited to develop the a priori analysis of the characteristics of the artifact that facilitate the emergence and evolution of signs.

It is worth mentioning that, with respect to the aim of this paper, the semiotic mediation approach is also a useful tool with which to examine the crucial role of the teacher. Indeed, according to the TSM, the teacher, who is aware of the artifact's semiotic potential, can exploit it to help students overcome the cognitive roadblocks they must negotiate when evolving from their personal meanings toward the mathematical meanings (Mariotti 2009, 2013). Teachers can do this both when designing the task to be accomplished with the artifact and, in particular, during collective discussions and in-class communication with the students. For example, teachers can utilize the artifact and pivot signs produced by the students in class and, through them, guide the students toward the mathematical meanings.

The mathematical concept of the quadratic function

The mathematical concept under consideration in the teaching intervention analyzed in this paper is the quadratic function. In what follows, we focus on the different aspects and representations (geometric, symbolic, numeric) of the quadratic function and on the connections among them that must be grasped to fully understand this important but complex mathematical concept (National Council of Teachers of Mathematics 2000; Duval 2006).

Geometrically, a quadratic function is a parabola, namely, a conic section that is a curve formed by the intersection of a plane and a cone, when the plane is at the same slant as the side of the cone. The parabola is a U-shaped curve that can be defined as a locus of points such that the distance to a given point, called the focus, equals the distance to a given straight line not through the focus, called the directrix.

Algebraically, the quadratic function of a *single variable* is a second-degree polynomial that can be represented by the algebraic expression $f(x) = ax^2 + bx + c$; $a \neq 0$. Graphically, it is a parabola, and the coefficients a, b, and c affect the shape and the position of the parabola in the Cartesian system. The symbolic representation of the quadratic function allows the identification of the key features of the parabola, namely, the focus, the directrix, the vertex (minimum or maximum of the curve), and the intersection points with the axes.

Numerically, the quadratic function can be represented by a set of ordered pairs of numbers such that the first differences in the second numbers of the pairs are not constant, as they are in the linear function. Furthermore, the first differences of the quadratic function values constitute an arithmetic sequence of numbers such that the difference between any two successive values in the first difference sequence—i.e., the second differences of the quadratic function—is constant.

We claim that students should be able to connect the different aspects and representations of the quadratic function. In particular, in this paper, we focus on the role of the coefficient "a" which determines parabola dilation (see Fig. 2). Moreover, it is worth noting that the constant value of the second differences of the quadratic function is equal to $2a$. Hence, in the particular case when the function has the form $f(x) = ax^2$, that is when the vertex is in the origin, in order to find the value of the coefficient a by the numerical representation, it is sufficient to divide the second differences by 2.

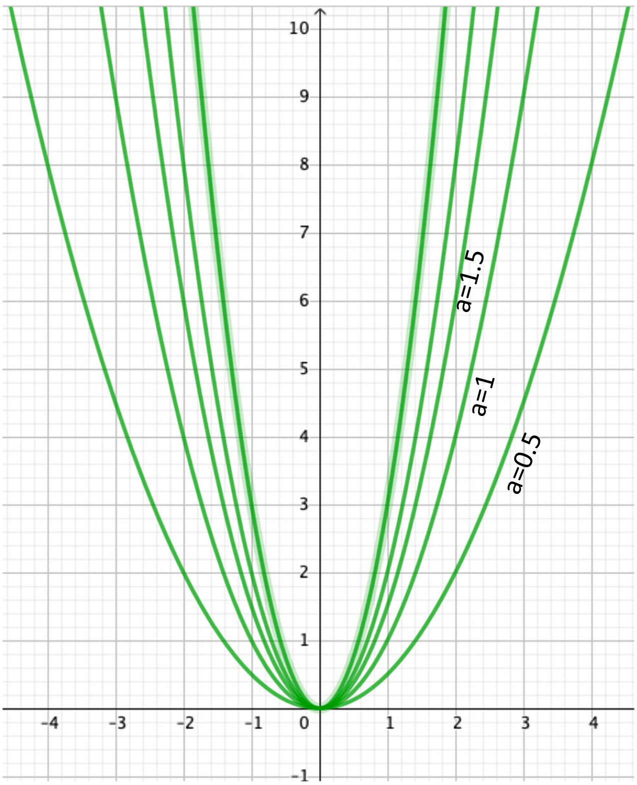


Fig. 2 Effect of coefficient ***a*** on the dilation of the graph

Method

Goals, methods, and research question

The teaching experiment presented and discussed in this paper consisted in the design, development, and analysis of a teaching sequence according to the semiotic mediation approach, and the research hypothesis that the use of digital artifacts simulating real-world phenomena could promote, with the appropriate guidance from the teacher, the shared construction of mathematical meanings. The students participating in this study were given three sequential tasks, which were designed to first be tackled in small groups, and then discussed collectively with the entire class.

In accordance with our theoretical framework, we first performed an a priori analysis of the three tasks that the students were to be given, to determine the semiotic potential of the artifacts that they employ. We then used the a priori analysis as a comparative framework through which to analyze the students' learning process, noting the ways in which it corresponded to—and deviated from—our expectations. To this end, we examined the transcripts of video-recorded classroom sessions and identified a series of "critical moments" of shared construction within the learning process. We defined "critical moment" as an activity in which the students sought to endow different kinds of signs with mathematical meanings, and to build upon these meanings so as to progress toward the intended didactic goal of the session. Based on this definition, we asked:

– What were the critical moments of these students' learning process, and how were these moments facilitated by the guidance of their teacher?

Participants

The study participants consisted of an entire class of 20 tenth-grade (15-year-old) students from a scientifically oriented school in Italy and their teacher, who is experienced with the principles of the semiotic mediation approach. At the time the study took place, the participants had already learned the concepts associated with the linear function, but not yet those of the quadratic function. They were familiar with the concept of finite differences and its representation. Linear functions had been taught to them based on their high school textbook, which characterized this type of function as correlations with equal first differences. Nonetheless, the students were already familiar with the concept of the parabola from their physics course, in which they learned about the parabola in connection with the trajectory of a moving object. In that context, the students generally referred to the geometrical shape of a body's trajectory when they referred to the parabola, but some students were also able to connect the shape to a symbolic expression. In addition, the students were also familiar with conventional function graph software (e.g., GeoGebra), which they had already used in the context of their school's formal mathematics curriculum.

Overview of the artifacts and the tasks

A sequence of three tasks was designed based on the assumption that exploring different characteristics of the same phenomenon may lead students to construct the mathematical meaning of the quadratic function in its different aspects. In particular, the aim of the teaching intervention was to exploit the semiotic potential of artifacts that simulate real-world phenomena in order to give meanings to different representations of a quadratic function and establish connections among them. For our purposes, we choose to use the motion of a ball on an inclined plane, which is indeed expressed by a quadratic function.

Below, we describe the three tasks (see Fig. 3), and in the next section, we present the analysis of the semiotic potential of the artifact in relation to the tasks.

The participants' first task was to obtain the mathematical model of a ball rolling on an inclined plane after viewing a short video (https://youtu.be/-c5GiXuATh4), i.e., artifact 1, about the well-known Galileo experiment. The video shows that both the elapsed time and the distance traveled by the ball varied while the angle of plane inclination was constant. Insofar as it demonstrates the rolling of a ball on an inclined plane, the video is effectively an artifact with an important role in the process of the students' construction of mathematical meaning. Indeed, it functions as a mediator between the students' general understanding of the physical phenomenon of a rolling ball on an inclined plane and its mathematical model, which is described by a quadratic function (Fig. 4a).

The participants' second task entailed their interactive exploration of the same situation analyzed in the first task. In this case, however, the artifact was a dynamic digital environment that allowed students to simulate the rolling of the ball while simultaneously observing the values of the distances moved by the ball while it is in motion via a numerical representation of the distance-time relationship. The corresponding values of the first differences of the distances with respect to the elapsed time are also displayed. Finally, students could also vary the plane's inclination, after which they were asked to conjecture and then verify how plane inclination affects ball motion. In addition, they were asked to find an equation that describes the motion of the ball. The second artifact was chosen specifically for its potential to focus on: (a) the role of the second differences in describing the phenomenon, (b) the effects of the variation of the plane inclination on the coefficient a of the symbolic expression, and (c) the dilation of the graph representing the phenomenon.

Task 1

Your task is to watch the **Galileo Inclined Plane Experiment** clip and answer the following questions.

A) What drew your attention when you watched the clip? Write as many observations as you can.
B) Can you make a conjecture about which one of the observations will change if the plane inclination changes? Why?
C) Can you make a conjecture about which one of the observations will change if the time unit is reduced? Why?

Task 2

Your task is to explore how the change of the plane's inclination may affect the movement of the ball.... You are asked to open **the first applet,** and change the plane inclination by dragging point B in the applet.

A) Can you make a conjecture about how the change of the inclination will affect the ball movement?
B) Change the plane inclination by dragging point B in the applet to verify or refute the conjectures you raised in (A). Have your conjectures changed? If so, why? If not, prove your conjectures.
C) Can you find equations that describe the ball movement? Why or why not? Justify your answer.

Task 3

In this task you are required to explain and justify your answers using graphic and numeric representation. You are asked to open **the second applet** and to observe the changes that occur on the spaces travelled as you vary the inclination angle of the plane.

A) Can you explain the shape of the curves when the plane inclination changes?
B) How do the differences between the y-values of the points on the graph change when varying the plane inclination? Why?
C) Add an additional column to the table of values to compute the differences of the distance differences. What do you observe? Why? Are your conjectures always true? Can you prove it?

Fig. 3 Written tasks given to the students

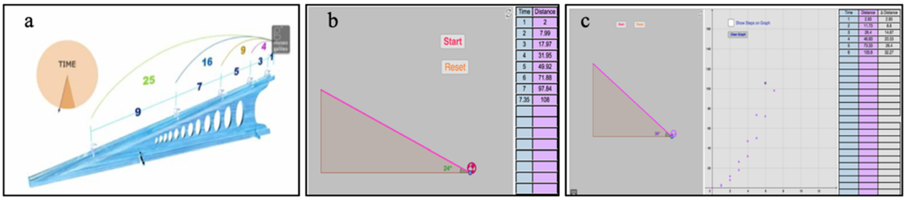


Fig. 4 The three artifacts used in the study. **a** The video of Galileo experiment. **b** The simulation of the ball movement on the inclined plane. **c** The simulation and its graphic representation

To accomplish the third task and successfully derive an equation describing ball motion on the inclined plane, students used the same artifact they had used in the second task. In this case, however, a different version of the artifact was used, in which an additional kind of mathematical representation was displayed, namely, the graph of the distances traversed by the ball with respect to time. Students were asked to derive the properties of the mathematical model that describes the ball's motion.

In addition to the three artifacts described above, the students were free to use other artifacts, such as GeoGebra, Excel sheets, calculators.

Semiotic potentials of the artifacts

In line with the aim of the teaching intervention and task design, the first artifact was used to draw each student's attention to the distance-time relationship of the rolling ball. Likewise, the second artifact was designed to familiarize the students with different ball rolling scenarios by allowing them to alter certain characteristics of the phenomenon while keeping the others constant. The students' engagement with the second artifact was thus intended to demonstrate to them the fact that the quadratic relation of ball motion is unaffected by changes in plane inclination. Lastly, a third artifact, which emphasized the graphical representation of the distance-time relationship, was also used. Interaction with the third artifact acquainted the students with the different representations of the quadratic function to help them establish the connections between them.

The semiotic potential of each artifact was analyzed a priori with respect to the given tasks to identify the appearance and evolution of signs during the teaching-learning activity. Table 1 shows how the characteristics of the artifacts are connected with different mathematical aspects and how the signs embedded in the artifacts may allow artifact signs to appear during the students' interaction and to evolve toward mathematical signs. The a priori analysis of the semiotic potential of the artifacts, in accordance with the TSM, will constitute the basis on which the teaching-learning activity will be analyzed in order to answer our research questions, looking for the emergence of artifact and pivot signs and their evolution toward mathematical signs.

Procedure

The study took place at a high school in Turin (Italy), where we observed the teacher as she led two 1.5-h sessions. At the beginning of each session, the students were required to work in small groups, each of which shared a worksheet (containing the task) and a computer.

Table 1 Semiotic potentials of the three artifacts

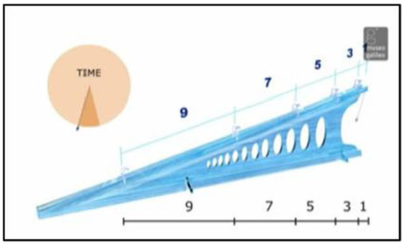
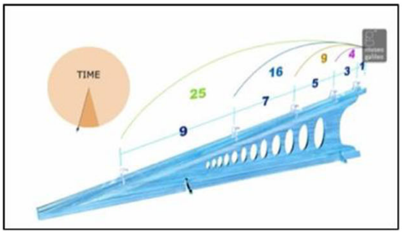
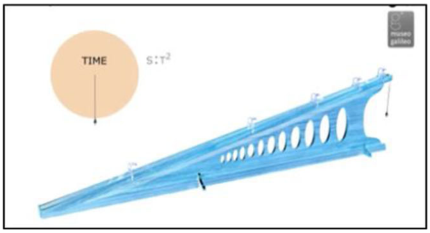
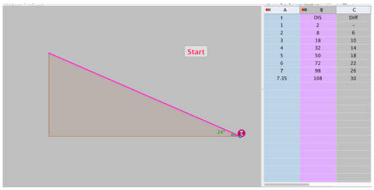
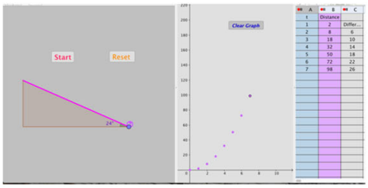
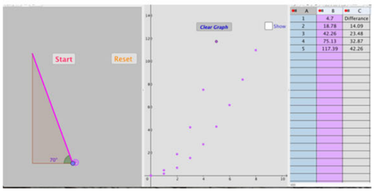
Mathematical aspects	**Semiotic potential of artifact 1**
Sequence of the segments traversed by the ball in each period of time constitutes an arithmetic sequence.	 The metal bells suspended above the inclined plane, the segments above the plane, and the numbers (segment lengths) may help the students notice that the numbers (which represent the distances moved by the ball in each time period) are all odd. In addition, students may notice that the differences between the lengths of two adjacent segments are constant, indicating that the numbers constitute an arithmetic sequence.
Distances moved by the ball from a position of rest are proportional to the squares of the times. Hence, the relation between the distances and the times is expressed by a quadratic function.	 The differently colored numbers and the corresponding arches, each representing the distance moved by the ball from a position of rest, may help the students realize that the distances moved are square numbers. The arrangement of the row of differently colored numbers and the row of odd numbers on the screen may help the students realize that the differences between pairs of adjacent square numbers constitute an arithmetic sequence, i.e., the second differences are constant.
The ball's motion is expressed by a quadratic function.	 The expression $S{:}T^2$, which appears briefly at the beginning of the video clip and then disappears, may enable students to notice the relation between the distances traversed by the ball and the squares of the times.

Table 1 (continued)

Mathematical aspect	**Semiotic potential of artifact 2**
The ball's motion is expressed by a quadratic function.	 The values shown in the table (times, distances moved by the ball from the position of rest, and the differences between two adjacent distances in the first, second and third columns, respectively) may allow the students to discern the relation as a quadratic function. In particular, the default value of the angle was set at 23.6 degrees so that the numbers in the middle column would resemble those used in the video viewed by participants for the first task. In addition, this angle also ensured that the values of the numbers in the middle column would be easily recognizable as being double those numbers in the video. Moreover, the default value may allow the students to recognize that the differences between two adjacent numbers in the third column are constant.
Changing the angle of the plane does not affect the type of relation, which is still expressed by a quadratic function $f(x) = ax^2$.	Changing the angle of the plane and observing that the differences between any two adjacent numbers in the third column are identical and remain constant may help the students recognize that the ball's motion is described by a quadratic function.

Mathematical aspect	**Semiotic potential of artifact 3**
The first differences increase over time.	 The graphical representation may enable students to discern the increment of the differences between the y-values of any two adjacent points.
Varying the angle of the plane causes a change in the coefficient a, which affects the dilation of the graph.	The graphic depiction of the resultant family of parabolas, which can be displayed simultaneously, may allow the students to discern the effect that plane inclination has on parabola dilation.  Discussing parabola dilation may help the students recognize that changing plane inclination angle affects the first differences of the segments traversed. Comparing the parabolas of the family may allow students to recognize that the variation of the plane affects the change of the a coefficient.

While the students worked in small groups, the teacher walked around the classroom and answered only students' technical questions, such as how to insert a formula to calculate the second differences in the simulation and how to reset or clean the graph. After the small group work, the teacher held a discussion with all of the students. She initiated the discussion by asking the students to share with their cohorts the new mathematics ideas (if any) that had been generated as a result of their participation in the small group work. In the first session, the students worked on the first two tasks in groups for 50 min, after which 40 min was devoted to the general discussion guided by the teacher. In the second session, the students worked on the third task for 35 min while the general discussion lasted for 55 min.

Data collection

We video-recorded the sessions in their entirety, including the general discussion led by the teacher. During the group work, we filmed all of the student groups (and their computer screens) as they worked together to solve the task, and we filmed the interventions by the teacher.

Data analysis

The videos were transcribed in Italian, the students' mother tongue, and that transcription was then translated into English. The findings presented below are the result of two rounds of data analysis. In the first, we read the video transcripts multiple times to identify the critical moments during the learning process with the strongest potential to foster the students' construction of the mathematical meaning of the quadratic function (Swidan 2019). For example, the recognition that the distances the ball traversed do not represent a linear relationship is considered a critical moment in the learning process. The meaning given to the distances—not linear—may help the student to explore the relationships between the distances, which may ultimately lead to the intended didactic goal—the quadratic relationship. In the second round of data analysis, we investigated the evolution of the students' personal meanings into mathematical meanings by distinguishing among the different signs, i.e., artifact signs, pivot signs, and mathematical signs. For example, terms that referred directly to the artifact and its use, such as "pink line," were coded as artifact signs. Words that are part of the natural language, such as "doors"—a term used to refer to the metal bells, which resembled miniature doors, in the Galileo apparatus (Fig. 4a) and that may also refer to the mathematical meaning of interval endpoints—were coded as pivot signs. Mathematical terms such as "first differences," "second differences," and "graph" were coded as mathematical signs.

To ensure the reliability of the data analysis, we (the paper's two authors) independently coded the data in both rounds, after which we discussed the analytical processes that each of us performed.

Results

In what follows, we present the results according to the critical moments we found throughout the learning process.

Recognition of the regularity of the distances

After the students completed the first task, the teacher began a discussion by asking them to list what observations were evoked in them while watching the video. Some of these observations were related to the ball's speed increments as it rolled down the inclined plane, while others described the distances between the unevenly spaced metal bells located on the inclined plane. In the conversation that follows, the teacher interacts with a student who commented on these distances:

1. Matteo: The distance between the two doors was always two.
2. Teacher: Was it always two or always increased by two?
3. Matteo: The distance between the previous and the next one was always increased by two.
4. Teacher: Did the distance between the two doors always increase by two? But two of what?
5. Matteo: From the video we see that the distance from the first to the second was three, from the second to the third it was five, from the third to the fourth it was seven and from the fourth to the fifth it was nine.

At this stage of the experiment, the students mainly focused on describing the relationship between ball speed and elapsed time. In contrast, Matteo directed his attention to the metal bells via the sign "the doors," and assigned a personal meaning to the distances between the metal bells: "the distance was always two." His comment indicates that he was referring to the differences between the numbers displayed in the video, which signify the distances between pairs of successive bells. Matteo's sign the doors is therefore a pivot sign because it refers, on the one hand, to the metal bells in the video and, on the other hand, to the distance the ball moves in each unit of time.

The teacher, however, re-directed Matteo's attention to the distances (traversed by the rolling ball) between the metal bells, asking Matteo, "was it always two or always increased by two?" To further prompt Matteo to clarify the meanings he ascribed to these distances, the teacher re-voiced Matteo's statement by using the same signs "door" and "distance" and at the end of the utterance asking, "two of what?" In response to the teacher's questions, Matteo referred to another artifact sign, namely, the numbers displayed in the video that signify the distances between each pair of adjacent bells. This time he correctly associated the number between each pair of adjacent bells with the distance traversed by the ball in the corresponding interval: "distance from the first to the second was 3", "from the second to the third it was 5", "from the third to the fourth it was 7" and "from the fourth to the fifth it was 9". Matteo's statements suggest that he associated this set of signs with the mathematical meaning of covariation.

Recognition that the relationship is not a straight line

The students found it particularly difficult to determine the equation that describes the ball's motion. To help them overcome these hurdles, the teacher drew the students' attention to the table of values in the simulation (Fig. 5). The students explained that the numbers in the third column of Fig. 5 were the differences between the lengths of the distances traversed by the ball.

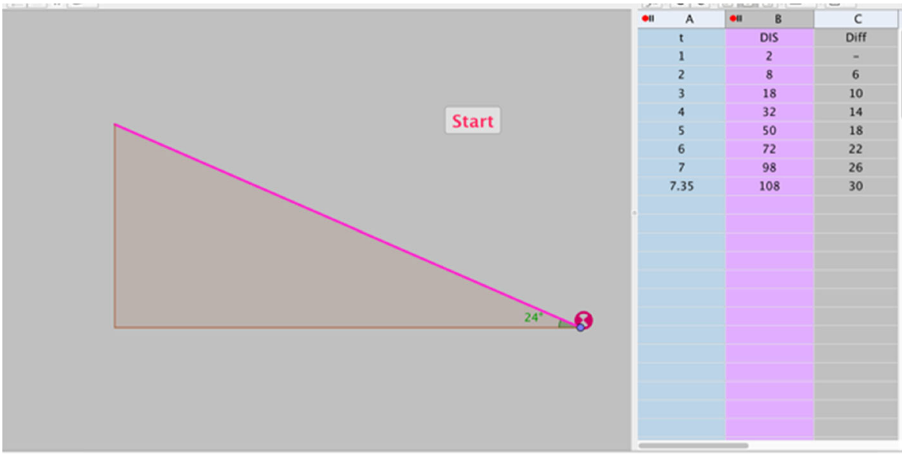


Fig. 5 Second artifact

Then, the teacher again asked the students how to describe the ball's motion on the plane. The following excerpt represents a fundamental step in the evolution of the signs among the students:

6. Teacher: So how do I write a relationship between this space and time? Is it a straight line?
7. Chorus: No!
8. Teacher: Why?
9. Maria: Because otherwise we would have had all the same differences
10. Teacher: Because we would have had all the same distances. And what [do we have] in this case?
11. Chorus: In this case there is no [same distances]
12. Teacher: In this case there are not and therefore it is not a straight line.

In asking the students to focus on the table of values, the teacher made them aware that this phenomenon cannot be described by a linear relationship. Doing so, she used the "relationship" as a pivot sign and referred to artifact signs "time" and "space" as two variables. The students focused on the artifact sign, the column of the first differences, and observed that these numbers are not equal. From this observation, they concluded that the distance-time relationship is not linear. Although in line 12 the teacher rephrased the students' answer, she used the mathematical sign "straight line" to confirm the mathematical meaning assigned by the students to the relationship, i.e., that it is not a straight line.

Emergence of the notion of the parabola

The excerpt above shows that the students rejected the possibility that the ball's motion can be described by a linear distance-time relationship. The teacher's subsequent aim in the discussion, therefore, was to prompt the students to identify the relationship as a quadratic one. Since she had seen another group (Andrea's group) using GeoGebra, she asked them to report on their experiences to the whole class. Andrea explained that to determine what type of relationship correctly describes the rolling of a ball on an inclined plane, he and his group had plotted several points from the table of values in the second artifact by using the straight-line tools included in GeoGebra. Upon entering the first two points, the students in Andrea's group discovered that the line did not pass through the other points, thus determining that the relationship was not a linear function. Moreover, Andrea conjectured that the points in the table of values were points of a parabola, and therefore, he and his group had tried to plot a graph by using the GeoGebra command "Conic through five points":

13. Teacher: What did you do?
14. Andrea: We created a list of points on the spreadsheet and we created a parabola.
15. Giulia: A parabola is obtained. But we stopped because we cannot find the relationship. We are at this point.

In this excerpt, the students independently decided to use GeoGebra, to plot some coordinates from the simulation to graphically represent the relationship. To share Andrea's group endeavor with the whole class, the teacher asked "what did you do?" Answering the teacher's question, Andrea introduced the sign "parabola," which can be considered an "artifact sign" because it referred to the graphical distribution of the points on the artifact. In that framework, however, it was devoid of the mathematical meaning of a quadratic relationship: "But we stopped because we can't find the relationship".

Emergence of the symbolic expression of the parabola

In the previous section, we showed how Andrea used the sign parabola. The next excerpt shows how Sofia, a student from another group, associated mathematical meaning with the parabola sign:

16. Sofia: If it is a parabola, there must be a term to the second.
17. Teacher: Right. If it is a parabola, there will be something to the second. Is there something "to the second"?
18. Francesca: In the upper part of the video there was written s:t^2
19. Teacher: So?
20. Sofia: The sum of all the differences was the square of time, for example, when the space was sixteen, the time was four, when the space was nine, it was three...

In the first utterance, Sofia used the parabola sign introduced by Andrea and, based on her statement "there must be a term to the second," endowed it with the mathematical meaning of a quadratic formula (Fig. 6b). In doing so, Sofia introduced a new sign that, insofar as it is connected to the previous sign (the parabola), can be considered a pivot sign because it refers, on the one hand, to the graph connecting the points and, on the other hand, to the symbolic expression of a quadratic relationship. The polysemous meaning of the new sign introduced by Sofia, a term to the second, allows the sign parabola to evolve into a pivot sign.

In line 17, the teacher used the same signs that Sofia did. The teacher's interventions were intended to coax the students to further explore the artifacts until the symbolic

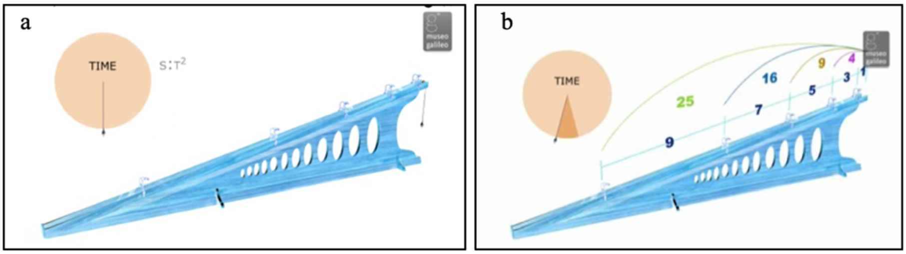


Fig. 6 Locations in the video clip referred to by Francesca (**a**) and Sofia (**b**)

expression emerged: "is there something to the second?" Francesca referred to the artifact sign, "$S : T^2$", which was displayed in the video (Fig. 6a). To involve other students in the discussion and to foster it, the teacher asked the question "so?" Sofia, another student, referred to the video, but she focused her attention on the numbers representing the sums of the distances moved in each unit of time, in the process making sense of the connection between the distances and the squares of the time. To this end, she linked the sign term to the second with the squares of the time. Sofia's consideration of both the symbolic expression of a quadratic relationship and its connection with the real-world phenomenon of a ball rolling on an inclined plane suggests that, for her group, the parabola sign evolved from a pivot sign to a mathematical sign.

Emergence of the coefficients of the symbolic expression

The students agreed with what Sofia said about the quadratic relationship that appeared in the video. Soon after, Andrea pointed out the simulation, emphasizing that the distances, the numbers in the second column, were not the squares of the times, the numbers in the first column. The excerpt that follows shows the emergence of a new idea—the relationship between the time and the distances traversed needs to be expressed by a quadratic formula containing a coefficient:

21. Matteo: We set the angle to be 25°. At 1 it was 2.13, at 2 we got 8.51, and at 3 we got 19.15.
22. Teacher: okay, and so?
23. Matteo: And now if we calculate 19.15 over 3 to the square, it gives us 2.13. And we did it with all the others and we got always 2.13.
24. Teacher: You got 2.13 ... And then?
25. Matteo: $\frac{S}{T^2}$ always gives me a constant value. I do not think it's a random result.

In this excerpt, Matteo referred to the sign $S : T^2$ and conferred on it the meaning "distance over time to the square." After calculating several values of the ratio between the distances and the squares of the times, he noticed that all calculations of the ratio yielded the same value. In this way, he conferred on the sign $S : T^2$ the mathematical meaning of constant value. Also, in this episode, the teacher used questions from the form "and so?", "and then?" to foster the discussion to help the students explain their thinking.

Recognition of the relationship between the angle and coefficient

Next, the students found a way to identify the constant value:

26. Giulia: In this case, it could be $y = 2.13x$^2, but only in this case, in which we have 25°. Perhaps changing the angle, the constant may change
27. Lucia: Yes, because space changes.
28. Teacher: Let us try!

This excerpt demonstrates how, as a result of the calculation of the ratios, the meaning of the sign "constant value" evolved to become a coefficient in the symbolic expression of the relationship. Giulia noticed that the constant may vary as a result of changing the inclination of the plane, eliciting the need to verify the conjecture for other plane inclinations.

The students thus verified that for many values of the inclinations, the ratio between the distances and the squares of the times was constant. Accordingly, Giulia introduced a new sign, the coefficient "k":

29. Giulia: So, it could be $y = kx$^2, where k is a constant that varies based on the variation of the inclination.

In this excerpt, although Giulia associated the value of the coefficient with plane inclination, she did not mention how the inclination affects the coefficient. It is worth noting that this excerpt also illustrates how students do not only interpret signs but may introduce new personal signs coming from their shared cultural environment.

Recognition of the relationship between the coefficient and second differences

At this stage of the experiment, the teacher focused on how the coefficient may vary as the angle of the inclined plane changes. Some of the students argued that the coefficient can be found by looking at the distance value at $t = 1$. As the following excerpt shows, others noticed a relationship between the coefficient and the second differences:

30. Carlotta: We noticed that k is given by the second differences divided by two.
31. Teacher: k is given by the second differences divided by two, why?
32. Carlotta: Because, as the second differences are all constant, then dividing all these by two, we get k.

As the first utterance of the excerpt above shows, Carlotta argued that the coefficient k can be found by taking the value of the second difference, which is constant, and dividing it by two. The teacher asked Carlotta to explain how she reached this conclusion. In her answer, it seems that Carlotta used the sign "constant" to refer to both the "second differences" and to "the coefficient k", thus expressing the relationship between them.

Recognition of the relationship between the angle, the graph dilation, and the coefficient

As mentioned above, in the second session, the students used a new version of the simulation (that includes the distance-time graph) to work on the third task. The teacher tried to guide the students toward focusing on how the inclination of the plane affects the distance-time graph. Carlotta said that the more the plane is inclined, the closer the curve is to the y-axis and, answering the teacher's request for an explanation, added what follows:

33. Carlotta: If the inclination is bigger, the ball traverses more distance in the same time, so the triangles subtended by the curve have the vertical cathetus which is always bigger and so they always go toward the y-axis.
34. Teacher: (*Drawing a parabola on the blackboard and adding some of the triangles subtended by the curves*) Carlotta is saying that it approaches the y-axis more because the triangles subtended by the curve, this is what we have called these triangles, have the vertical segment which is always bigger, while…
35. Carlotta: The ones on the bottom remain fixed.

Carlotta linked the inclination of the plane with the dilation of the parabola by means of the subtended triangles. These triangles are geometrical objects that the students already encountered during their school lessons on the linear function, and can therefore, in this context, be considered shared mathematical signs. With regard to them, the students learned that the ratio between the two perpendicular sides represents the slope of the straight line. Here, Carlotta used these triangles to support her explanation of why the curve approached the y-axis with increasing plane inclination. Carlotta's utterance— "If the inclination is bigger, the ball traverses more distance… so... the vertical cathetus… is always bigger"—suggests that she recognized the relationship between the inclination of the plane and the distances traversed, which she signified with the vertical sides of the subtended triangles. Thus, it seems that the subtended triangles were used by Carlotta as a mathematical sign to connect the distances traversed by the ball to the graph that was generated and to describe the graph's dilation. To share this knowledge with the whole class, the teacher drew a parabola on the blackboard and added the triangles subtended by the curves as mathematical signs that signified the rate of change.

The teacher then tried to connect the graph with the symbolic expression:

36. Teacher: And from the point of view of the equation, what will happen?
37. Giulia: The coefficient increases. That is, if the coefficient increases as a consequence the distances are bigger, and that if the second differences are bigger, then the number k is always bigger… if the angle increases the differences increase and the coefficient increases too.
38. Teacher: Okay and so?
39. Giulia: As a consequence, the graph always becomes more inclined towards the y-axis because in the same unit of time the values are always greater.

In this excerpt, the teacher initiated the connection between the different signs that are in the artifacts. Giulia, to give the graphs meaning, focused on three different levels of representation. The first was the symbolic expression she exploited to connect the distances to the coefficient. The second was a table of values by means of which she connected the coefficient to the second differences. The third is the simulation of the real-world phenomenon by means of which she connected the angle with both the first differences and the coefficient. This allowed her to conclude how plane inclination affects graph dilation: “the graph always becomes more inclined towards the y-axis because in the same unit of time the values are always greater”.

At this stage, as this excerpt illustrates, the students were starting to use mathematical signs to endow the motion of the ball with meaning. For example, in her argument, Giulia used different mathematical signs, such as “coefficient,” “distances,” “[first] differences,” “second differences,” “angle,” and graph. Moreover, Giulia formed some important connections between these different mathematical signs, showing that she was aware of how any one of the signs affects the others. She also used them to describe a real-world phenomenon.

Discussion

This study was designed to examine the shared construction of the mathematical meanings of the quadratic function when it is learned by using digital artifacts that simulate real-world phenomena. We analyzed the students’ interactions with the digital artifacts and the effect of the teacher’s intervention during the class discussion by using TSM to describe the shared construction of mathematical meanings. In this section, we will discuss the evolution of the mathematical meanings of the quadratic relation by comparing the a priori analysis (Table 1) with the a posteriori analysis and by focusing on the instructional strategies used by the teacher to prompt the evolution of the signs. Lastly, we will state our conclusions and propose directions for future research.

The first assumption we made with the analysis of the semiotic potential of the artifact was that the students would first notice the metal bells on the inclined plane and the square numbers that appear above the plane, after which they would notice the differences between the square numbers and would find the relationships among them. Our first assumption was only partially confirmed. In the first phase of the experiment, the students’ attention was indeed focused only on the metal bells and on the differences between adjacent square numbers. The students used the word door to signify the distance between two adjacent bells. Thus, the pivot sign doors quickly evolved to acquire the mathematical meaning of the “extremes of a segment,” and the distance between adjacent doors took on the mathematical meaning of the “length of the segment.” Furthermore, the meaning given to the arithmetic sequence that signifies the distances moved was that the differences between two adjacent numbers are fixed (line 1). This meaning is important to identifying the quadratic relationship that models the ball’s motion on the inclined plane. It was elicited by the students when they attributed the value 2 to the differences of the distances between adjacent metal bells (line 1). However, this meaning was not connected to the quadratic relationship, perhaps as a result of the teacher’s intervention. The aim of the teacher was indeed to prompt the students to describe the arithmetic sequence as a sequence in which each subsequent element can be obtained by adding a constant to the previous one (line 2). In this case, the teacher missed the opportunity to connect the second differences with the quadratic relationship. To form this connection, one must look at the arithmetic sequence as a sequence in which the difference of two constitutive elements is constant.

Our second assumption—that using the dynamic simulation, the students would construct mathematical meanings of the quadratic relation based on what they can see in the table of values, namely, the distances or the first differences—was also partially confirmed. The students constructed mathematical meanings of the quadratic function, but not according to the processes we anticipated. Students did not use the distances or the first differences of the distances to identify the quadratic relation. The first differences of the distances helped the students recognize that the distance-time relation is not a straight line. This was due to the teacher's interventions, which prompted the students to use the "logic of not" strategy (Arzarello and Sabena 2011), namely, if the first differences of the distances are not constant then the distance-time relation is not a straight line. Their use of the logic of not assisted the students, with the teacher's support, in their investigation, which sought to answer the question, "If it is not a straight line, then what is it?"

To answer this question, the students introduced a new artifact that enabled them to plot a graph after inserting ordered pairs of points. At this moment, a new sign was produced by the students: parabola. The sign parabola is a pivot sign because initially, it was endowed with the meaning of trajectory, possibly because the students who participated in this study were familiar with parabolic trajectories from their physics lessons. Later on, the sign parabola was endowed with mathematical meanings due to the prior knowledge of one of the students who argued that a parabola is related to the second degree.

It seems that the evolution of the sign parabola and its endowment with the mathematical meanings prompted the students to identify the quadratic relation. The identification process of the quadratic relation happened due to two artifact signs in the video: the text $S:T^2$ and the square colored numbers that are displayed upon the inclined plane. Although it appears in the videos just for 3 s, the sign $S:T^2$ attracted the students' attention, possibly because of the algebraic nature of this expression. It is worth noting that the students initially encountered these two signs during the first meeting of the experiment, but they activated them during the second meeting. It seems that the students had paid attention to these signs, even though they did not express their meanings verbally. We concluded that for the students to express the meaning of signs, they have to meet the intellectual need (Harel 1998) for using them.

Our third assumption—that the graph, which signifies the distance-time relation of the rolling ball on the inclined plane, will prompt the students to identify the dilation of the graph and associate it with the physical phenomenon—was fully confirmed. To do so, the students leveraged the subtended triangles, a mathematical sign that was part of the classroom culture, to explain the graph's dilation. In this case, the vertical side of the subtended triangles takes on the mathematical meaning of Δf, and at the same time, it takes on a physical meaning—the distance the ball travels in a unit of time (line 28). This physical meaning, and the mathematical meaning of the ratio between the value of the vertical side to the value of the horizontal side of the subtended triangles, did not evolve into the mathematical meaning of rate of change. Although the complexity of learning the rate of change concept is well known in the literature (Thompson 1994; Johnson 2012), the teacher did not contribute enough to the development of the mathematical meaning of the rate of change. Instead, the teacher pushed the students' discussion toward describing how the change in the graph is reflected in the algebraic expression. In fact, this was not the only moment the teacher missed the opportunity to foster the evolution of mathematical meanings among the students. It is known that one of the main characteristics of the quadratic relation is that the second differences are constant. Though the students referred to the second differences several times in different contexts, as described above, the teacher never introduced the possibility of associating the second differences with the quadratic relation.

While the teacher did not give particular attention to the conceptual meaning of the second differences and the rate of change, she did play a crucial role in endowing the parabola sign with its mathematical meaning: by associating it with its algebraic expression, by giving meanings to the coefficient of the quadratic expression, and by associating the physical phenomenon with its mathematical meanings. This was done primarily through the use of two communication strategies: questioning and re-voicing.

The importance of teachers' questions as prompts for students' mathematical thinking has been acknowledged for many years (e.g., Wood 1998; Fraivillig et al. 1999; Stein et al. 2008). Wood (1998), for example, has categorized teachers' questions into two types: funneling and focusing. Funneling occurs when teachers ask questions that guide the students to the desired end or to perform a certain procedure for solving problems (Herbel-Eisenmann and Breyfogle 2005). Focusing questions, in contrast, require teachers to listen carefully to the students' responses and to guide them based on their own understanding rather than how the teacher would reach the desired solution.

In our study, most of the questions asked by the teacher were focusing questions, i.e., questions that aimed at revealing and building upon the students' understanding. Focusing questions in the form of "why-questions" and "what do we have?" proved effective in prompting the construction of mathematical meanings (lines 3, 27, 5). Questions of the form "and so?" encouraged the students to continue describing the meanings they held, but it did not help the students to change their personal meanings in favor of mathematical ones. Only once in our recorded sessions did the teacher use a question that confronted one student's personal meaning with the mathematical meaning of a sign. This question helped the student to change his personal meaning of the sign and give it a mathematical meaning. Since this finding is based on just one case, however, drawing conclusions from it should be considered very carefully.

Another type of questions that were found to be effective in the evolution of mathematical meanings was questions aimed at linking representations. These kinds of questions not only helped the students transition from one representation to the other and to connect them, but they also helped students draw out their thinking, and may help teachers to achieve their didactical goals (Herbel-Eisenmann and Breyfogle 2005). In all the cases where the teacher asked such questions, the students focused their attention on the representation to which the teacher directed them. In these situations, the students interpreted and endowed with meanings the representation to which they had been directed, while retaining the meanings they had already associated with the other representations. Transitions between the multiple representations and the endowment of each of them with mathematical meanings are a crucial activity, necessary to developing students' understanding of the mathematical concept (Duval 2006).

The second strategy the teacher used was "re-voicing." Re-voicing is a communication strategy which involves the re-uttering of another person's speech through repetition, rephrasing, expansion, and reporting (Forman et al. 1998). With re-voicing, "the teacher essentially tries to repeat some or all of what the student has said" (Chapin et al. 2009, p.14). Our findings showed that the teacher used re-voicing with two functions: (a) to share and eventually emphasize one student's idea with the others in the classroom and (b) to add knowledge to what was said. The latter is in tune with what Arzarello et al. (2009) called the semiotic game, through which the teacher intentionally used the signs produced by the students, endowing them with mathematical meanings. The findings also showed that both the functions of re-voicing contributed to the evolution of mathematical meanings. Apparently, sharing the idea of one student with the class encouraged the others to contribute their part to the discussed issue, and these contributions bestowed a multitude of meanings to the sign. This function is in accordance with Enyedy et al.'s (2008) claim that: "One of the primary purposes of re-voicing is to promote a deeper conceptual understanding of mathematics by positioning students in relation to one another, thereby facilitating student debate and mathematical argumentation" (p. 134). The second function, re-voicing and adding knowledge to what was said, confirms the students' statements. Moreover, the added knowledge serves to remind the students of some previous shared meanings.

To conclude, this study highlighted the complexity of endowing mathematical concepts with meanings, as they are learnt in a dynamic environment that simulates a real-world phenomenon. The research findings showed that being aware of the semiotic potential of artifacts is just one step in the process of endowing mathematical concepts with meanings. In addition to the semiotic potential of the artifact, the interactions among the students and the student-teacher interactions in the process of constructing mathematical meanings should be taken very seriously. To the best of our knowledge, the role of the two instruction strategies used by the teacher to construct mathematical meanings (i.e., questioning and re-voicing) has not been extensively discussed in the literature of semiotic mediation. Our research findings showed evidence that certain types of questions and re-voicing can be particularly effective to prompt students' construction of mathematical meanings. We believe that the findings of this study can be the basis for a professional development course for teachers, aimed at the construction of mathematical meanings. Of course, both the interventions in which the teacher successfully assisted the students in constructing mathematical meanings, and those that ultimately proved unsuccessful can be useful elements for discussion within a professional development course.

Finally, as we mentioned early in the "Method" section, this teacher is familiar with the principle of semiotic mediation, and the students are familiar with the teacher's teaching style, which emphasizes classroom discussion and pays special attention to identifying and interpreting different kinds of signs. These characteristics of the teacher and the students are considered one of the study's limitations, since other teachers, with different theoretical and pedagogical backgrounds, are likely to produce different results. To support and expand upon our findings, especially regarding the instructional strategies that prompt the evolution of mathematical meanings, more research is needed, with a more diverse sample of participating teachers and students.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflicts of interest.

Ethical approval We confirm that we followed the ethical principles for doing research. Identical information about the participant is not available in the manuscript, and the names used in the manuscript are pseudonyms.

Informed consent We confirm that the statements of written informed consent from students' legally authorized representatives/parents/guardians as well as the teacher consent have been reached.

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