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ORIGINAL ARTICLE

Teaching students to apply formula using instructional materials: a case of a Singapore teacher's practice

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Abstract
It is easy to dismiss the work of "teaching students to apply formula" as a low-order priority and thus trivialises the professional knowledge associated with this practice. Our encounter with an experienced teacher—through the examination of her practices and elaborations—challenges this simplistic assumption. There are layers of complexities that are as yet under-discussed in the existing literature. This paper reports a case study of her practices that reflect a complex integration of relevant theories in task design. Through examining her praxis around the theme of "recognise the form", we discuss theoretical ideas that can potentially advance principles in the sequencing of examples for the purpose of helping students develop proficiency in applying formula.

Keywords Mathematical formula · Instructional design · Cognitive load theory · Variation theory

Introduction

We are part of a bigger project team that aims to distill the distinctives of mathematics teaching in Singapore classrooms. This team focuses on Singapore mathematics teachers' use of instructional materials in their work of teaching.

It is now becoming our common experience—after numerous rounds of data collection—to initially not detect anything particularly noteworthy about a teacher's instructional work at the point of interview or lesson observation; but only to find later as we revisited the data through careful analysis—the detailed method of analysis will

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be explicated in a later section—that what appeared as uneventful instructional practices at first glance were actually localities of rich and intentional constructions by competent teachers. In this paper, we report one such case: of how a teacher designed her instructional materials to help students "recognise the form" (repeatedly, in her words) within the context of learning to apply formula.

In the initial stages of analysis, we attempted to "fit" the profile of her instructional work into analytic frames that we were familiar with. We experimented with variation theory—as she mentioned repeatedly during interviews that she used "a variety of questions" in the design of her instructional materials; cognitive load theory was also referenced as "mak[ing] things easy" for her students was ostensibly a main goal of her teaching work.

But it became clear to us that a single theoretical framing by itself did not do justice to all the richness and nuances as captured in the data—which was the reason behind the exploratory shifting of one theoretical lens to another. We think that, while her instructional work yielded a surface structure that may appear to be an "application" of one of these theoretical models, the distinctive lies in the practical *integration* at a deeper level of a number of ideas derivable from these models. Before we present the analysis of her practice, we review some of the constructs and theories related to the case study.

What is a mathematical "formula"?

From anecdotal evidence, it seems the popular conception of doing mathematics is "applying formula". It is not uncommon in the literature to equate the experiences of school mathematics as learning formulas and applying them to standard exercise items (e.g., Flores et al. 2015; Grouws et al. 1996; Stipek et al. 2001). But what exactly is a mathematical "formula"?

Within the formal discipline of mathematics, the language of "formula" is, however, less common. We surmise the following reasons: (1) within the axiomatic approach commonly adopted in the development of mathematical content in formal mathematics courses, "formula" has no place (or, its imprecise meaning does not fit into) into the categories of axioms, definitions, lemmas, theorems, and corollaries; (2) in academic mathematics programmes, the goal is to help students focus on the rigours of logical development instead of merely using the end product of the reasoning process—be it an algorithm, a rule (such as L'Hopital's Rule), or, more generally, a theorem.

Thus, we see that "formula" is not a term that is formally defined within the academic discipline of mathematics; it likely emerged from a popular depiction of working within school mathematics, roughly equivalent to steps to follow based on a prescribed procedure. While formula is also usually associated with an algebraic equation (such as $A = L \times B$, as in area of rectangle equals the product of its length and breadth), we take here a broader interpretation to include other mathematical results (such as geometrical theorems) that are not usually captured in equation form. To add, when used in this way, there is no distinction with respect to the epistemic status of the formula—for example, $a^0 = 1$ and $a^m \times a^n = a^{m+n}$ are both regarded as formulas, even though the former is a definition and the latter a theorem.

Defined this way, formula is ubiquitous in school mathematics: area of triangle is half base times height, Pythagoras Theorem, Sine of an angle is “opposite over hypotenuse”, angle at centre is half the angle at the circumference, method of solving linear equations in one variable by elimination, among many others. As such, teaching students to be fluent in the correct application of formulas remains a big part of mathematics teachers’ instructional programme. In fact, the metaphor of “teaching application of formulas” has become the representative imagery of traditional mathematics teaching in the literature (e.g., Flores et al. 2015; Stipek et al. 2001).

Skill of applying formulas

But just because formula-application remains a common practice in mathematics classrooms does not in itself justify its value in students’ learning of mathematics. In fact, for some time now, the rhetoric has been to de-emphasise the (rote) application of formulas and focus instructional work on the concepts involved in the development of—and underlying the use of—the formulas instead (e.g., Crooks and Alibali 2014; Rittle-Johnson and Alibali 1999; Skemp 1976). Implicit in this stance is that there is little value in fluency with respect to the correct use of formulas compared to having deep knowledge of “why the formula works”. This argument is also framed in the literature as procedural knowledge versus conceptual knowledge talk (e.g., Baroody et al. 2007; Byrnes and Wasik 1991; Crooks and Alibali 2014).

However, there has been a buildup of recent work (e.g., Rittle-Johnson et al. 2015; Mann and Enderson 2017) that proposed a strong correlation between procedural knowledge and conceptual knowledge, even a causal direction from the former to the latter. Translated to students’ learning, when students work on procedures towards fluency, they may not merely be developing procedural knowledge—in the process, they may also co-develop the related conceptual knowledge. Rittle-Johnson et al. (2015), in reviewing the research conducted in this area, asserted that, “Overall, both longitudinal and experimental studies indicate that procedural knowledge leads to improvements in conceptual knowledge, in addition to vice versa. The relations between the two types of knowledge are bidirectional” (p. 591). This coheres with the anecdotal experiences of many—including the authors of this paper—that our learning experience in mathematics often begins with application of formula; and after a while, when the procedure is automatised, we begin to shift our attention to particular steps in the formula and uncover the conceptual underpinnings of them. In other words, we can “get to” concepts through first training for fluency in applying formulas. Moreover, fluency in a formula application allows us to see its connections (a form of conceptual understanding) when used in conjunctions with other formulas. It also attends—conceptually—to the required conditions and constraints under which the application of the formula is valid (Klymchuk 2015).

In a departure from previous depictions of quality mathematics instruction in the USA as one that pares down on procedural practices (e.g., National Council of Teachers of Mathematics 1989; 2000), the model of mathematical proficiency proposed by Kilpatrick et al. (2001) placed “Procedural Fluency” as one of five strands that need to be developed in mathematics classrooms. It is defined as “knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (p. 121). This clearly squares with our notion of being fluent in applying formula. In addition, they stated:

> Procedural fluency and conceptual understanding are often seen as competing for attention in school mathematics. But pitting skill against understanding creates a false dichotomy. … [T]he two are interwoven. Understanding makes learning skills easier … . By the same token, a certain level of skill is required to learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding (p. 122)

A similar positioning of "skill in applying formula" can be discerned in the literature on mathematics problem solving. In Schoenfeld's (1985) language, he classed the ability to carry through the correct application of known formulas as "Cognitive Resources" necessary for successful problem solving, alongside the other components—Heuristic, Control, and Belief System.

We therefore argue that, mathematically, there is value in teaching students to be proficient in applying formula correctly. This does not mean that we advocate the *exclusive* goal of teaching application of formulas in mathematics classrooms. But it means that learning the application of formula has its rightful place—when balanced with other goals of teaching, such as mathematical reasoning, and conceptual understanding—in mathematics classrooms.

Theories that inform task design for application of formula

Due to the mathematics education community's emphasis on *conceptual development* of formulas, and the concomitant paring down on "mere" application of formula, there has been comparatively less research directly related to quality teaching with respect to application of formula. Specifically within the area of designing task sequences to help students gain proficiency with the application of formulas—the focus of our case study in this paper—there are two main research streams: the work of cognitive psychologists with an interest in conditions for improving students' mathematics achievement, and the development work of variation theorists.

Cognitive load theory

Within the research tradition of cognitive psychologists in this area, the studies adopted mainly an experimental research design, where the manner in which mathematical tasks were used were varied as treatment conditions, and the effects of treatment measured using standardised mathematics tests on targeted topics of learning. In this area of work, the empirical findings reported in these research publications supported the importance of practice as a major independent variable in the acquisition of specific mathematical skills, including the fluent application of formulas (e.g., Carroll 1994; Ward and Sweller 1990; Zhu and Simon 1987).

On the theoretical side, a number of studies in this tradition are grounded in cognitive load theory (e.g., Pawley et al. 2005; Phan et al. 2017; Paas et al. 2010). It is a “theory that was explicitly developed as a theory of instructional design based on our knowledge of human cognitive architecture” (Sweller et al. 2011, p. v). Beginning from the seminal paper by Sweller (1988), the theory has undergone significant development. It is thus beyond the scope of this paper to review the full development of the theory. We make here a brief reference to the theory insofar as it relates to our focus on sequencing of tasks to help students acquire fluency in formula application. The assumption of the theory is that “cognitive load” that is imposed on the working memory can affect the ease in which the learner processes information and hence the quality of skill proficiency. “Working memory load may be affected either by the intrinsic nature of the learning tasks themselves (intrinsic cognitive load) or by the manner in which the tasks are presented (extraneous cognitive load)” (van Merrienboer and Sweller 2005, p. 150). The application to task design, in its most basic tenet, is that while relatively little can be done to reduce the intrinsic cognitive load—as it is essential to learning the targeted skill, research can be directed at intra-task construction and inter-task sequencing in order to reduce the unnecessary extraneous cognitive load that can cause overload and thus hinder learning.

The synthesis of research in this line of inquiry by Atkinson et al. (2000) provided an overview of findings that can inform instructional principles with respect to the design of example sequences to aid acquisition of skills. On intra-example features, useful instructional principles include the need to integrate different modes of information, such as diagram, text, and symbols, in a form that is easily accessible to students; however, when the example is too complicated, there is tendency for cognitive overload. In such cases, the example presentation should be accompanied with explicit methods of directing students’ attention to pertinent features of the task and solution(s).

On inter-example features, the findings favoured the use of more than one example to illustrate a target formula for application; however, excessive varying of examples along multiple dimensions can lead to cognitive overload. The recommendation was that, for a set of examples illustrating a common formula application, a common problem structure such as a unifying cover story be used. As to the sequencing of practice examples and demonstrated examples, the interspersing of examples throughout practice produced better outcomes than lessons in which a blocked series of demonstrated examples is followed by a blocked series of practice examples.

Variation theory

Variation theory, on the other hand, has its roots in the phenomenographic tradition. Within this tradition, the starting point in examining learning is not in the theoretical cognitive workings of the individual, but in how one *experiences* with phenomena with respect to a particular object of learning: “The structural aspect of a way of experiencing something is thus twofold: discernment of the whole from the context on the one hand and discernment of the parts and their relationships within the whole on the other” (Marton and Booth 1997, p. 87). Further, the “the discernment of a feature amounts to experiencing a difference between two things or between two parts of the same thing” (Marton and Pang 2006, p. 199). In other words, in order to be aware of a particular feature—within the context of education, a targeted aspect of an object of learning—

one needs to discern that feature (otherwise, that feature would recede into the background of one's experience, since the limitations of our human experience is such that we cannot be aware of all features in the same degree at the same time). A necessary condition for discernment is the *variation* of the particular feature along a dimension of change. "These aspects are more likely to be discerned as a dimension of variation. A learner is more likely to experience what something is if it is contrasted with what it is not" (Runesson 2005, p. 84). Moreover, the variation is to occur against a backdrop of constancy for its variation to be more easily discerned. This aspect of the theory was emphasised in Marton and Pang (2006) as they repeatedly purport "patterns of variance and *invariance*" (emphasis added) as a condition necessary for learning.

In recent studies that employ Variation Theory as a guiding theory (e.g., Pang et al. 2016; Sullivan et al. 2016; Vale et al. 2017, the axioms translate into principles of designing tasks, including sequencing of examples, to help students become aware of aspects of learning through systematic variation of tasks along desired dimensions. For illustration of its application, we summarise a teacher's use of such a pattern of variance and invariance as reported in Cheng and Lo (2013): The focus was on "the method to calculate the perimeter of compound rectangles" (p. 9). A sequence of exercises was designed using the principle of "a pattern of variation and invariance". There were 3 items of the same diagram of a rectangle with a rectangular indentation at the top left corner. Figure 1 shows the three diagrams corresponding to the three exercises—the common task is to calculate the perimeter of the "compound rectangle". The identified feature to be discerned—and hence the focus of deliberate variation—is that the combined lengths of the two horizontal segments at the top of the rectangle equals the length of the horizontal base. Thus, the lengths of the two segments were varied across the three items—4 and 3, to 6 and *a*, to *b* and *a*, while keeping invariant the shape of the figures and the length of the base as 7. To variation theorists, a critical role of a teacher is to design sequences of items of such a nature as to systematically "open … a space of variation that enable[s] the learners to discern those particular aspects of the object of learning" (Runesson 2005, p. 77).

Within this tradition of research but taking on a more disciplinary-based standpoint, some researchers (e.g., Watson and Mason 2006; Zaslavsky and Zodik 2007) broadened the inquiry into the "example space" in which mathematics teachers drew upon for their instructional practice. In the study conducted by Zodik and Zaslavsky (2008), they found common principles that teachers adopted when choosing examples, which included: start with an easy example, draw attention to relevant features, and include uncommon cases. Our study here can be seen as taking a further step in learning about how teachers utilize the example space and integrate these principles in actual construction of instructional materials.

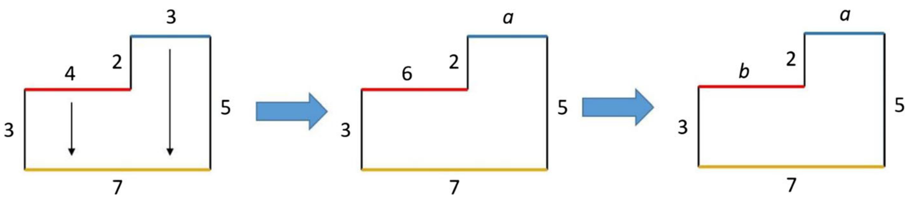

Fig. 1 Sequence of three items

Back to Teacher Beng Choon

Interestingly, the sequences of items that Beng Choon prepared for her students reflected the principles of design derivable from both cognitive load theory and variation theory. By this, we do not claim that she had explicit knowledge of these theories. It is possible that she built these principles independent of these theories. This is acknowledged by Marton and Pang (2006):

> We are not saying that such patterns of variation and invariance cannot be brought about by teachers who are ignorant of the framework because it is impossible to teach without using variation and invariance, and many teachers often intuitively create the necessary conditions for mastering the specific object of learning they are dealing with. (p. 217).

But Beng Choon's design work provides us a concrete case of an integration of design applications in both cognitive load theory and variation theory—an integration we have yet to come across in the literature. Her goal of "recognise the form" brings together elements from both of these theoretical streams, in a way that is rooted to the practical considerations of teaching in a Singapore mathematics classroom. The deep analysis in this case study also affords us a portrait of the complexity of managing other goals of actual classroom teaching in ways that studies located purely within each of the theoretical tradition rarely provides.

Thus, instead of trying to 'fit' Beng Choon's design work into preset theoretical moulds, we formulated a method of analysis that can account for her own constructions of the design process and how she viewed their utility in her lessons in the classroom. Theoretically, it was a refinement of the method of progressive widening of analytical lens as developed by the first author (Author 2008) and applied in another study (Authors 2019) that was similar in nature to the current one. In-depth grounded approach was first carried out within a rich region of analysis to obtain preliminary conjectures. These conjectures were then tested and refined as the analyses broadened to increasingly wider regions of data. The methodical details are given in the proceeding sections.

Method

Like other participants in the bigger project, Beng Choon was identified as an experienced and competent teacher. "Experienced" is defined as having taught the same mathematical course at the same level for a minimum of five years; and "competent" selection is based on recognition by the local professional community as a teacher who is effective in teaching mathematics.

As mentioned briefly at the start of the paper, the choice of Beng Choon as a subject of deeper study was largely due to her own reference to help her students "recognise the form" within the context of learning to apply formula. In addition, a number of other factors about Beng Choon's practices lends itself to a rich unpacking of her work—a characteristic feature of case study: (1) During interviews, she was able to articulate comprehensively her goals for many tasks. This allows us to uncover her intents behind the activities we recorded in her classroom; (2) she produced a full set of handouts for students' use in class (hereafter referred to as "Notes") before the start of the module and supplemented these along the way in the form of additional practice items. In other words, her work yielded a rich set of instructional materials on which to ground our study; (3) she constantly made references among her goals, her actual activity in class, and her use of instructional materials. This enabled us to study the interactions among these major pieces of her instructional processes.

The class that Beng Choon taught was a Year 9 Express class. In Singapore, students progress to secondary level based on the scores they obtain at the end of Year 6 in the Primary School Leaving Examination (PSLE) conducted nationwide. Using the PSLE score, pupils are streamed into three ability streams. The streams are known as Express, Normal Academic (NA), and Normal Technical, and the percentage of students in each of these streams is roughly 60, 25, and 15 respectively. Unlike the NA course which stretches over a duration of 5 years, students in the Express stream cover the same content in 4 years.

The module that Beng Choon taught was "Differentiation". The contents—as stipulated by the Ministry of Education (2012)—that she covered during our study were (i) derivative of f(x) as the gradient of the tangent to the graph of y = f(x) at a point; (ii) derivative as rate of change; (iii) use of standard notations f'(x), f''(x), $\frac{dy}{dx}$, $\frac{d^2y}{dx^2} = \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$; (iv) derivatives of x^n, for any rational n, sin$x$, cos$x$, tan$x$, e^x, and ln$x$, together with constant multiples sums and differences; and (v) derivatives of products and quotients of functions. The module was taught over nine lessons, two of which were 30-min lessons while the rest were 60-min ones.

Data

Under instructional materials, Beng Choon used mostly the set of Notes she designed. During the course of the lessons, she also consolidated students' learning by assigning items for short practices. These related materials form the first primary source of data.

The next source of data is the interviews we conducted with Beng Choon. We conducted one pre-module interview before her lessons and three post-lesson interviews after each of three lessons she selected—Lessons 04, 08, and 09. All interviews were video recorded. We designed an interview protocol with two sets of questions and probes respectively for the pre-module interview and post-lesson interviews.

The pre-module interview was conducted to find out what Beng Choon's instructional goals were and how she designed and planned to utilise her instructional materials to fulfill her goals. Some prompts in the pre-module interview were as follows:

- Please share with me what mathematical goals you intend to achieve for this set of materials that you will be using.
- Are there any other specific instructional materials that you are going to prepare for this module?

The post-lesson interviews were conducted to find out if she had met her instructional objectives with the instructional materials she designed and planned to use. Some of the questions were as follows:

- Did you use all the materials that had you intended to use for the lesson?
- How did the materials help you achieve your goals for this lesson?

The third source of data is Beng Choon's enactment of her lessons in the module. We adopted non-participant observer roles during the course of our study—one researcher sat at the back of the class to observe Beng Choon's lessons. This is so that the researcher will be able—during the post-lesson interviews—to make relevant and specific references to her teaching actions when pursuing certain threads during the interviews. A video camera was also placed at the back of the class to record Beng Choon's actions. All nine lessons were video-recorded.

Analysis of data

We proceeded with the analysis along four stages. Figure 2 summarises the procedure.

Stage 1: identification of units of analysis of the Notes We took the sections of the Notes as prepared by Beng Choon (e.g., "Basic Formula", "Chain Rule") as the basic units of our preliminary analysis. We broke down the units further to examine the specific mathematical contents targeted in each section. For example, under the "Basic Formula" unit, we studied each exercise item listed in the unit to posit the intended instructional goal for each of them. We matched the comments in Beng Choon's interviews according to the references she made to these units and exercise items. Together with the coded content, we were better able to verify the instructional goals intended for each unit.

Stage 2: composition of chronological narratives For some of these selected units with rich related data on Beng Choon's enactment and interview comments, we crafted chronological narratives (CN) for each of them. In each CN, we integrated a number of data sources—pre-module interview transcriptions, post-lesson transcriptions, tasks in

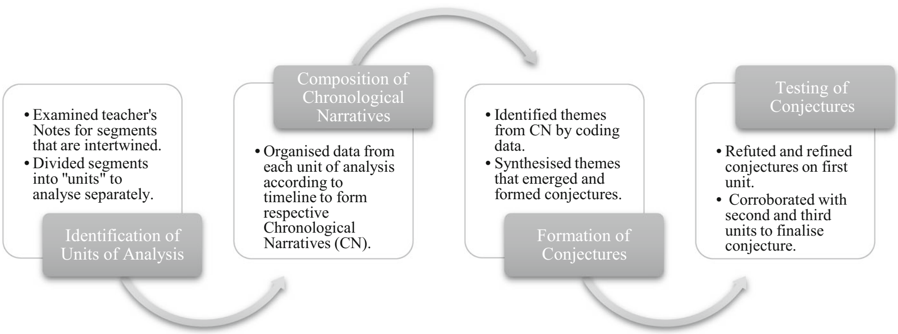

Fig. 2 Analysis procedure

her Notes, and her classroom vignettes. The CN of Unit A for the Basic Formula, for instance, was composed by first examining the text in the pre-module interview. We found that she commented at length about how she planned to help students “recognise the form” using her Notes. This intention cohered with the contents of the Notes in the “Basic Formula” unit. We also proceeded to locate the video recordings of the related lessons she conducted for evidence to corroborate her use of the instructional materials in this unit. She started with the unit in the middle of Lesson 01 and continued to develop it in Lesson 02. During Lesson 02, she assigned students to work on some items to consolidate their learning. Thereafter, she explained the answers to the practice items and reinforced the unit by highlighting common errors. In other words, the CN for the unit is a coherent chronology that is drawn from these data sources: interviews, Notes, and videos of her lessons. The CN for this unit is summarised in Table 1.

Stage 3: formation of conjectures related to “recognise the form” We begin specifically to look for themes related to how Beng Choon help students “recognise the form” by closely examining the CN for Unit A. This CN was chosen as a first-entry study because it is one where Beng Choon articulated that her main goal was to help her students “recognise the form”. This CN became an intensive source of analysis for emerging themes related to this central theme. After several rounds of discussions, conjecturing, and refuting, we arrived at a point where there was stability in agreement among the members of the research team—where the conjectures could be substantiated from all the data sources.

Stage 4: testing of conjectures In the final stage of analysis, we repeated this process in Stage 3 on another unit of analysis so as to develop our conjectures, refute previous ones, or substantiate/revise those generated earlier. After going through further refutations and refinements of conjectures, we managed to refine the conjectures into a form

Table 1 Overview of the chronological narrative (CN) of unit A

No.	Event/activity	Data
1	Pre-module interview	• Explained that the examples she crafted were from Levels 1 to 3 on a scale of 1 to 5—whereby 5 is the most difficult—so as to build confidence among students • Explained that she planned to teach students to “recognise the form” of functions so that they could “apply the formula”
2	Lesson 01 Notes Basic formula (p. 2)	• Utilised four examples in Task 1 to teach students to apply the basic formula for differentiation • Rewrote $y = \frac{1}{x}$ as $y = x^{-1}$ and $y = \sqrt{x}$ as $y = x^{\frac{1}{2}}$ for Items (c) and (d) in Task 1 respectively so that students can see that they have the same form as $y = x^n$ • Emphasised that students have to rewrite expressions to obtain the form of $y = x^n$ so as to apply the basic formula $\frac{d}{dx}[x^n] = nx^{n-1}$ • Reiterated that students have to “recognise the form” by rewriting $y = \frac{2}{\sqrt{x}}$ as $y = 2x^{-\frac{1}{2}}$ for Item (b) in Task 2
3	Lesson 02 Notes Basic formula (p. 3)	• Recapped the basic formula $\frac{d}{dx}[x^n] = nx^{n-1}$ at the beginning of the lesson by eliciting students’ responses • Proceeded to $\frac{d}{dx}[af(x)] = a\frac{d}{dx}[f(x)$ and $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$ • Within exercise items (b)—(d) in Task 3, she emphasised the importance of rewriting the expressions in the form of $y = x^n$

that contribute to theory generation. In the next section, we present our findings on the processes of analysis under Stages 3 and 4 by first detailing the CN on Unit A, followed by the other CN on Unit B.

Chronological narrative on unit A: basic formula of $\frac{d}{dx}(x^n) = nx^{n-1}$ For context, the unit before this was an introduction to differentiation. She started by presenting the graph of $y = x^2$, and students were asked to find the gradient at various points on the curve. She then drew the students' attention, through observing a pattern, that the gradient at these points are twice that of the x-value of the points. This observation was generalised into "the gradient function of x^2 is $2x$", followed by a formal introduction to the various notations of the gradient function. She continued by stating, without further exploration, that if the same procedure was used for the functions x^3, x^4, …, the respective gradient functions were $3x^2$, $4x^3$, … . Clearly, she was leading the students towards the formalisation of the basic formula which was the highlight in the next unit of her Notes. In fact, she wrote the formula on the board at the end of this section of the lesson.

Beng Choon then proceeded to show how it can be applied to the items given in the next unit (the focus of the current inquiry) of her Notes, as shown Fig. 3:

For item (a), she pointed to the formula that was written on the board and reminded the students that "it was done already". One of the students readily volunteered the answer, "three x-squared." Beng Choon wrote the answer on the board and did not say anything more. She did not seem to think that any student would have difficulty with this item. From the point of view of design of the instructional materials, we are interested in her selection of this as the first item in this unit. From her response, we surmise that she intended the first item to be a very direct and obvious application of the formula to build confidence in her students, especially in this early stage of the topic (that is, the first lesson). That this was the general approach taken by Beng Choon is attested by the following:

> When I took them [as students in the beginning of last year], I checked [that they were] always failing [in mathematics] so you find that they lose confidence, … In fact the very first test I gave them, after the test they all said, "Oh man, if only I can get [a score of] two digits." … I was quite shocked [at their lack of

(a) $y = x^3 \rightarrow \frac{dy}{dx} =$

(b) $y = 5 \rightarrow \frac{dy}{dx} =$

(c) $y = \frac{1}{x} = \rightarrow \frac{dy}{dx} =$

(d) $y = \sqrt{x} \rightarrow \frac{dy}{dx} =$

Fig. 3 Extract of Beng Choon's Notes on application of the basic formula

> confidence] … So that's why, for them … my approach to them is that it's actually *not that difficult*, helping them to *build up their confidence*. [Pre-module interview, emphases added]

Notice that she wanted to present mathematics to students that render it "not that difficult". Elsewhere, within the same interview, she reiterated that she intended to "make things easy" for her students. It appears that, especially at the beginning of the set of exercise items, she wanted students to feel comfortable—without excessively heavy cognitive load—and confident to move along.

Moving on to (b), even before she finished writing "$y = 5$" on the board, Student Don voiced, "zero". Here, unlike the case of Item (a) where she simply accepted the answer, she responded, "Why is that so?" which signaled an intention to help students go beyond merely a correct answer. The student offered the expected explanation of applying the formula of "bringing down the power of zero" when you write 5 as $5x^0$, which results in the product giving zero as the final answer. [Note: to prove the formula from first principles in the case for $n = 0$, one will need to do it differently from the more usual starting case of n as a positive integer; the same is true for n as a negative integer and n as rational. Here, it was not the intention of the teacher to prove the formula rigorously, but merely to apply it]. That this was not the purpose of the "why is it so" question was made clearer when she followed on with an alternative explanation, "In fact, when you draw the horizontal line of $y = 5$ [her both hands moving left to right repeatedly, gesturing an imaginary horizontal line] … the gradient right – you can also see that the gradient is zero".

The manner in which she directed students to visualizing $y = 5$ as a horizontal line graphically indicates that she likely intended to connect to the emphasis of differentiation as finding gradient in the previous unit. Here, she saw the opportunity—since $y = 5$ as a horizontal line is easy to recall and visualise—to reinforce this foundational concept and seized upon it. [As an aside, this connection to "finding gradient" becomes significant as later items—Beng Choon termed them "word problems"—were set within the context of finding gradients. The detailed discussion of these items is beyond the scope of this study].

For Item (c), there is a deliberate variation from the x^n surface form. Beng Choon wrote $y = \frac{1}{x}$ on the board and asked the class, "What is the power of x here?" Student Jon offered the answer as -1. She then rewrote the expression as x^{-1}. That this was intended during the design of the Notes was clear in the space allocated on the right side of the "=" sign of $y = \frac{1}{x}$ (see Fig. 1). She then proceeded—asking students' responses at every step—to show the application of the basic formula, writing $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$. She stressed that the formula applies not only for positive integer n; it also applies for negative integer n. We also noticed that, compared to the previous two items, the time spent explaining this item is longer (52 s compared to 35 s for Item (b) and 15 s for Item (a)). This was due not only to more steps involved but also more deliberate points of pause to look at students' expressions—presumably to check that the steps were sensible to the students. In other words, it appeared that Beng Choon judged that this item was becoming less easy for students and so was more willing to slow down for students to keep cognitive resonance with her. This is in line with her goal of "building confidence" in this group of students, as stated earlier.

Unpacking further, we may extend our inquiry into the actual "difficulty" that Beng Choon was seeking to address as she slowed down. She revealed it in the pre-module interview:

> I need to *teach them to recognise*. So that – [for example in] trigo there are a lot of formulas, so how do I know which formula to use? So I have to *teach them to recognise* … So that you know that this is the form [to apply correctly] … . I find that a lot of times I have to *teach them to recognise the form* (emphases added).

From the repetition of the phrase in a short section of the interview, we know that Beng Choon puts priority in her teaching to helping students "recognise the form" that fits the formula-use. Seen through this lens, the deliberate and fore-planned re-writing of $\frac{1}{x}$ to x^{-1} is an example of how a change of the external representational form of a function can help the students connect to and "recognise" it as a case of x^n and hence satisfying the condition, thus triggering the application of the correct formula. Perhaps, the later re-writing of $-x^{-2}$ to $-\frac{1}{x^2}$ was yet again a reinforcement of this skill of toggling between "forms" of representation of such functions, since she identified this "recognition of form" was a difficulty among these students.

This fundamental goal of "teach students to recognise the form" is extended to Item (d), and so was the intention of providing further variation of the form x^n—in this case, away from integer exponents. The whole process in which Beng Choon discussed Item (c) was essentially duplicated for Item (d). She took an even longer time to demonstrate the steps (73 s) as she slowed down further at two points: (i) when determining the power after differentiation; (ii) in obtaining $\frac{1}{2\sqrt{x}}$ from $\frac{1}{2}x^{-\frac{1}{2}}$. For (i), her voice accented both in volume and pitch when she asked, "And the power is?" When some students gave the answer as negative half, she further asked, "Do you know why it is negative half?" For (ii), she guided the students by using the language of numerator and denominator, showing items that should go to either. In short, Beng Choon continued to slow down as she expected Item (d) to be even more "difficult" for the students—as she mentioned in the second interview, "Especially terms that contain square root. From experience, I find that students always have problem with one term with a square root"; at the same time, she wanted to "teach students to recognise"—in this case, the recognition that $\sqrt{x}$ can be written as $x^{\frac{1}{2}}$ and hence belonging to the form of x^n and so valid to apply the results of the formula.

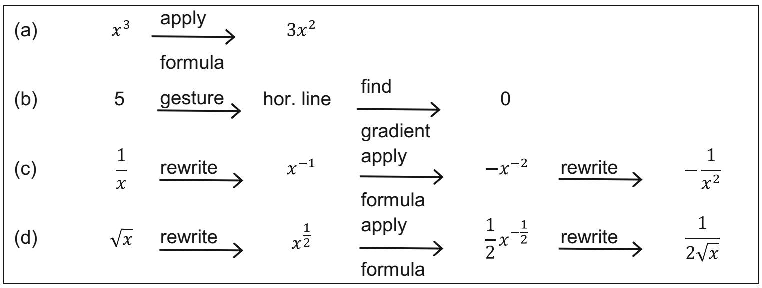

Figure 4 summarises the CN enacted by Beng Choon for each of items as discussed in the preceding paragraphs in this section.

Fig. 4 Summary of the chronological narrative of unit A

The length of the chain of procedures as shown in Fig. 4 shows a gradual increase in "difficulty" in the sequencing of the items, beginning with the easiest (to "recognise the form" in order) to apply the formula directly, and ending with that which she considered most difficult for students. The insertion of (b)—while providing the option also to apply the formula—was meant to connect back to the concept of "differentiation as find gradient". To help students "recognise the form", in the cases of (c) and (d), she deliberately provided space in her Notes to signal the scaffold of rewriting into a form that was easier to see it as a case of x^n and thus fulfil the condition for applying the formula.

Also, the different surface forms—positive exponents, constant, reciprocals, and square root—were varied systematically in order to serve the purpose of drawing students' awareness that they were merely rewritings of the same underlying form of x^n. Beng Choon mentioned this intention repeatedly in the first interview:

> I … give [them] exposure to different types of questions that this topic can come up with. I always look for a *variety* of questions … So I told them that I give [them] a *variety* of questions. But when the question is different, something new ... [they] have to fall back to basic … (emphases added).

Based on the analysis as explicated in the CN of this unit, we advance the following conjectures with respect to Beng Choon's design of the instructional materials to help students develop fluency in formula-application:

(1) To build confidence in students of their ability to apply the formula, the exercise items are sequenced such that the first item requires easiest recognition of form, and gradually increasing in difficulty, with the items considered the most difficult at the end of the set of exercises;
(2) The surface forms of the items are varied in order to help students discern the underlying invariance;
(3) Where appropriate, items are inserted within the set to help students connect to concepts (in this case, differentiation as finding gradient)—that were introduced earlier and which would be required later in the topic;
(4) "Recognise the form" is a prerequisite to applying the formula, and items are crafted to require a rewriting of the initial 'form' (e.g., $\sqrt{x}$) so that it becomes easier to recognise this form (i.e., $x^{\frac{1}{2}}$) that fits the formula.

As described under the "Method" section, we proceed to Stage 4 of the analysis by examining the CN of Unit B on Chain Rule. The focus in the next section will be less on the detailed description of the how the narrative was produced, but the evaluation of the conjectures—examining, in particular, whether they are refuted, substantiated, or in need of refinement.

Chronological narrative of unit B: Chain Rule In between unit A and unit B, Beng Choon taught two other formulas: $\frac{d}{dx}[af(x)] = a\frac{d}{dx}[f(x)]$, and $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$. Again, a set of exercise items followed the introduction of each of the formulas. Also, "word problems" were given to help students maintain the connection of differentiation to finding gradient

She started unit B with a motivational activity: How would you obtain $\frac{dy}{dx}$ for $y = (2x + 1)^2$? After briefly recalling that one can expand and then apply the formulas learnt earlier, she proceeded with $\frac{dy}{dx}$ for $y = (2x + 1)^{10}$. She used this development to motivate that a different technique was needed. She then demonstrated how she would do the differentiation on the first item on the Notes: $y = (2x^2 + 1)^{10}$. Figure 5 is a reproduction of Beng Choon's work on the whiteboard for this segment of the lesson.

She then demonstrated the same procedure for $y = (3x^2 + 2x)^7$. Following this segment, she told the students that the technique she used for both items was officially called Chain Rule and she formalized it as $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ on the whiteboard, which was also printed in the Notes.

As in other units in her Notes, this introduction of the formula was followed by a set of exercise items, as shown in Fig. 6.

For (a) and (b), she reproduced the items as they appeared in the Notes on the board, and left spaces in between for the working. She asked the students to try on their own—as she walked from table to table to monitor and guide. She added, "The first two are easy, yes – super easy". After 51 s, students were asked to present their working on the board. Beng Choon then pointed out a mistake on the student's working of (b)—he missed out a negative sign—and did not elaborate further before asking students to proceed with the other items. This segment of her follow-up explanation lasted 65 s.

For (c), she started by saying, "Again, it is easier to make it to the form of something to the power …", before showing on the board that $\frac{d}{dx}\left[\frac{6}{(2-x)^2}\right] = \frac{d}{dx}\left[6(2-x)^{-2}\right]$. Like in (a) and (b), she left a space on the board after this line—to signal to students that she expected them to attempt the working by themselves. She proceeded after the space to do a similar verbal emphasis, "Again, if I have square root, what is the power?"

$$y = (2x^2 + 1)^{10}$$

let $u = 2x^2 + 1$, then $y = u^{10}$

$\frac{du}{dx} = 4x$ $\frac{dy}{du} = 10u^9$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 10u^9 \times 4x$$

$$= 10(2x^2 + 1)^9 \times 4x$$

$$= 40x(2x^2 + 1)^9$$

Fig. 5 Reproduction of whiteboard working

(a) $\frac{d}{dx}(2x-1)^4 =$

(b) $\frac{d}{dx}(2-3x^2)^6 =$

(c) $\frac{d}{dx}\left[\frac{6}{(2-x)^2}\right] =$

(d) $\frac{d}{dx}\sqrt{x^2+2x+2} =$

(e) $\frac{d}{dx}\left(2-\sqrt{x}\right)^4 =$

(f) $\frac{d}{dx}\left(\sqrt{x-\frac{1}{x}}\right) =$

(g) Calculate the coordinates of the point on the curve $y = \sqrt{x^2-2x+5}$ at which $\frac{dy}{dx} = 0$.

Fig. 6 Extract of Beng Choon's Notes on application of Chain Rule

followed by rewriting for (d): $\frac{d}{dx}\sqrt{x^2+2x+2} = \frac{d}{dx}(x^2+2x+2)^{\frac{1}{2}}$, before leaving off to check on students' seatwork.

After 91 s, she asked one student to present the solution for (c). At the same time, after observing the seatwork of the students for these items, she noted in class, "Okay, quite a number of you are confused already". She immediately proceeded to go through the student's solution for (c). The student's working started with this: $-12(2-x)^{-3}(-1)$. She noted that the student "skipped [a step]" and proceeded to insert a line prior to this: $6\,(-2)(2-x)^{-3}(-1)$, explaining each factor of this expression – "6 I leave it, … he brings down -2, …". The reason for Beng Choon to further break down by inserting the step becomes clearer as she continued to explain, "the power becomes -3, … now I notice some of you – you start to differentiate here [pointing to "$(2-x)$"] already, which is wrong … take care of the power first … then differentiate "$2-x$" to get -1". Some students were beginning to misapply the formula by collapsing the two separate steps—do $\frac{dy}{du}$ followed by multiplication of $\frac{du}{dx}$ – into a single mesh, as in $6(-2)(-1)^{-3}$. Beng Choon's insertion of the additional step was meant to help students see each distinct step in the application and, in particular, to highlight the collapsing mistake.

She proceeded to show a similar procedure for (d), again warning against the tendency to collapse: "… the term [pointing to (x^2+2x+2)] – do not change anything to the term [pause], alright do not change anything to the term …". Due to her slowing down to break down the steps and repeatedly emphasising on not confusing the steps, her explanation for (c) and (d) took 490 s, significantly more than the teacher explanation component for (a) and (b).

The process she adopted for (e) and (f) is similar to (c) and (d). She rewrote the expressions into a form where the exponents become clear, before applying the Chain Rule, again stressing the separation of steps between $\frac{dy}{du}$ and the multiplication of $\frac{du}{dx}$. The time taken for the teacher explanation was 583 s.

For (g), Beng Choon again introduced this as a "word problem". The solution process is longer than the earlier items; as such, we will not detail each step. Instead, we focus on those segments that relate directly to our conjectures. First, we noted that Beng Choon used the language of "notice that the gradient is zero" twice in her explanation although the text of the item merely states "$\frac{dy}{dx}=0$", neither does the solution process require such a connection between $\frac{dy}{dx}$ and "finding gradient". It appears to us that Beng Choon had intended to make connection at every opportunity. When we asked during the post-lesson interview if this was indeed her intention, her reply was, "Yeah, yeah, yeah. I always have to check- yeah, I always have to check back [to this connection made in my earlier lesson], because when they learn something now … they may have forgotten something [taught] before". Second, when she obtained the expression for $\frac{dy}{dx}=(x-1)(x^2-2x+5)^{-\frac{1}{2}}$, she proceeded to do something she had not done in the workings for (a) – (f), which was to explain and rewrite the expression as $\frac{x-1}{\sqrt{x^2-2x+5}}$, giving the reason for doing so as "you will find it is easier [to see] if you write it this way". This statement became clearer when she proceeded to help students see the justification for how $\frac{dy}{dx}=0$ would result in $x = 1$. It is by re-writing the expression into a rational expression that it is easier for students to recognise the fraction in the reasoning: that if a fraction equals zero, the numerator equals zero (and the non-zero denominator doesn't play a part at all). Although the study here is about recognising the form in order to apply a formula, it is interesting to note that helping students to recognise the form—in this latter case, the form of a fraction—was also seen as useful by Beng Choon to teach a deductive reasoning process. Figure 7 summarises the CN in this unit.

We include here brief clarifications of the terms/symbols used in Fig. 7. The "double arrow" used in (c)–(f) is to highlight how Beng Choon intentionally slowed down to emphasise the two steps in the application of the Chain Rule—focus on seeing y in terms of u for $\frac{dy}{du}$ then seeing u in terms of x for $\frac{du}{dx}$. This process differs from the more straightforward one-step application of the formula in (a) and (b). The term "double rewrite" was used for (f) because, unlike in the earlier items, the re-writing into exponents occur at two places in the expression—from $\sqrt{u}$ to $u^{\frac{1}{2}}$ (where $u=x-\frac{1}{x}$), and from $\frac{1}{x}$ to x^{-1}. This could be judged by Beng Choon to be more difficult for students and therefore placed towards the end of this set of exercises.

We do not know why there was no re-writing for (f) after the application of the Chain Rule; we surmise that she did not want to impose extraneous cognitive load on the students, since her focus for this set of exercises was correct application of the Chain Rule. As such, she dealt with only rather straightforward post-application re-

(a)	$(2x-1)^4$	$\xrightarrow{\text{apply formula}}$	$4(2x-1)^3(2)$	$\xrightarrow{\text{rewrite}}$	$8(2x-1)^3$		
(b)	$(2-3x^2)^6$	$\xrightarrow{\text{apply formula}}$	$6(2-3x^2)^5(-6x)$	$\xrightarrow{\text{rewrite}}$	$-36x(2-3x^2)^5$		
(c)	$\frac{6}{(2-x)^2}$	$\xrightarrow{\text{rewrite}}$	$6(2-x)^{-2}$	apply formula and $\rightrightarrows$ separate steps	$6(-2)(2-x)^{-3}(-1)$	$\xrightarrow{\text{rewrite}}$	$12(2-x)^{-3}$
(d)	$\sqrt{x^2+2x+2}$	$\xrightarrow{\text{rewrite}}$	$(x^2+2x+2)^{\frac{1}{2}}$	apply formula and $\rightrightarrows$ separate steps	$\frac{1}{2}(x^2+2x+2)^{-\frac{1}{2}}(2x+2)$	$\xrightarrow{\text{rewrite}}$	$(x+1)(x^2+2x+2)^{-\frac{1}{2}}$
(e)	$(2-\sqrt{x})^4$	$\xrightarrow{\text{rewrite}}$	$(2-x^{\frac{1}{2}})^4$	apply formula and $\rightrightarrows$ separate steps	$4\left(2-x^{\frac{1}{2}}\right)^3(-\frac{1}{2}x^{-\frac{1}{2}})$	$\xrightarrow{\text{rewrite}}$	$-2x^{-\frac{1}{2}}\left(2-x^{\frac{1}{2}}\right)^3$
(f)	$\sqrt{x-\frac{1}{x}}$	$\xrightarrow{\text{double rewrite}}$	$(x-x^{-1})^{\frac{1}{2}}$	apply formula and $\rightrightarrows$ separate steps	$\frac{1}{2}(x-x^{-1})^{-\frac{1}{2}}(1+x^{-2})$		
(g)	$\sqrt{x^2-2x+5}$	$\xrightarrow{\text{rewrite}}$	$(x^2-2x+5)^{\frac{1}{2}}$	$\xrightarrow{\text{apply formula}}$	$\frac{1}{2}(x^2-2x+5)^{-\frac{1}{2}}(2x-2)$	$\xrightarrow{\text{rewrite}}$	$\frac{x-1}{\sqrt{x^2-2x+5}}$

Fig. 7 Summary of the Chronological Narrative of Unit B

writing, as were in the case for Items (a)–(e), and in the case of (g) where the rewriting has the utility of helping students recognise the fraction form for the deductive step $\frac{x-1}{\sqrt{x^2-2x+5}} = 0 \Rightarrow x-1 = 0$.

Similar to the increasing lengths and complications of the working chains as seen in Fig. 4, Fig. 7 strengthens Conjecture (1) that Beng Choon intentionally built into the sequence of items a principle of gradated levels of difficulty. Also, the manner in which she varied the items in the set of exercises to help students "see" the underlying au^n structure amidst the different surface forms strengthens Conjecture (2). As for Conjecture (3), the way Beng Choon deliberately connected to "gradient" in Item (g) supports it.

Conjecture (4) would require further refinement based on the analysis of this Unit. Although "recognise the form" was implicitly built into the design of the exercises as seen from Beng Choon's consistent use of rewriting (see Fig. 7) to a form that explicitised the exponents, a closer examination reveals that there were at least two "levels" of "forms" involved that students would need to recognise in order to correctly apply the Chain Rule formula: (1) recognise that y is linear combination of u with exponents; and (2) (nested within u) recognise that u is a linear combination of x with exponents. [Using Item (c) to illustrate, student has to first recognise that y is $6u^2$ (linear combination of u^2) which is in the form that triggers the application of earlier formulas, then recognise u is $2 - x$ (linear combination of x) which is again in the form that is rendered easier for application of earlier formulas. Chain Rule essentially takes the product of these two results]. This need for two-level recognition was not crafted ostensibly into the Notes in this unit. Rather, it was in the enactment in class that she made it explicit, and in a flexible manner—she did not emphasise the two-level recognition in the first two items, but only from Item (c) when she detected that this separation of recognition was needed to address some students' tendencies to wrongly collapse the two concomitant steps. In other words, the Notes were crafted in such a way as to allow her to flexibly emphasise the type and level of "forms" she wanted to help students to recognise, depending on the emerging needs of the students during the lesson. We therefore refine Conjecture (4): "recognise the form" is a prerequisite to applying the formula. There may be nested forms within a form that correspond to various other formulas. The items are crafted so that the teacher can flexibly attend to different combinations (or levels) of forms during the lesson according to the difficulties students face in correctly applying the formula.

Reframing the conjectures

In this section, we attempt to integrate the various conjectures with respect to Beng Choon's design of items to help students in formula-application. Conjectures (1) and (2) provide the principles that Beng Choon drew upon consistently in the sequencing of items. To her, it is important to consider gradation of item difficulty as a way of taking into consideration the cognitive load the task would pose to the students. The gradation allows students to enter into the set of tasks with minimal cognitive load, and as they become more familiar, the cognitive demand is gradually increased with each succeeding item in the sequence. At the same time, the items—taken together as a set—are deemed by her to present necessary variation of surface forms in order for students to experience a "pattern of variation and invariance". Relating these conjectures to the earlier literature reviewed, we may say that Beng Choon drew upon and integrated principles derivable from cognitive load theory and variation theory in her design of item sequences. [We reiterate here that we do not claim that she was cognizant of the specifics of these theories; but one can apply self-generated ideas that coincide with principles derived from established theories]. Represented in diagrammatic form, we view these two principles as feeding into Beng Choon's deliberate consideration in the construction of her item sequences, as shown in Fig. 8.

In terms of the outcomes she intended from the carefully-sequenced items, her main goal was to help students "recognise the (nested) form(s)" because she saw it as a prerequisite to applying the required formulas (Conjecture 4). Where necessary, she would emphasise this goal in the classroom enactment of the instructional materials, including the specific technique of rewriting in order to make the "form(s)" more explicit to the students' awareness. But apart from this goal, she also slipped in connections to other ideas insofar as that they were easily derivable from the recognition of the related form(s) (Conjecture 3). These conjectures and observations were also included in the representation of Beng Choon's overall design conception in Fig. 8.

Discussion

From the analysis of the case of Teacher Beng Choon, we unpack the underlying considerations in designing sequences of items for her students to learn formula-application. In popular conceptions of studying mathematics as primarily "applying formula", the ascent is naturally on the verb "apply". The accompanying image is thus one of students repeatedly and mechanically 'applying' the formula on numerous almost-identical practice items. However, this study uncovers the nuance behind the act of "apply". Prior to students' being able to "apply", they need to "recognise" the form that fits the condition for application. At least, application should include recognition as a pre-requisite, or perhaps the verb

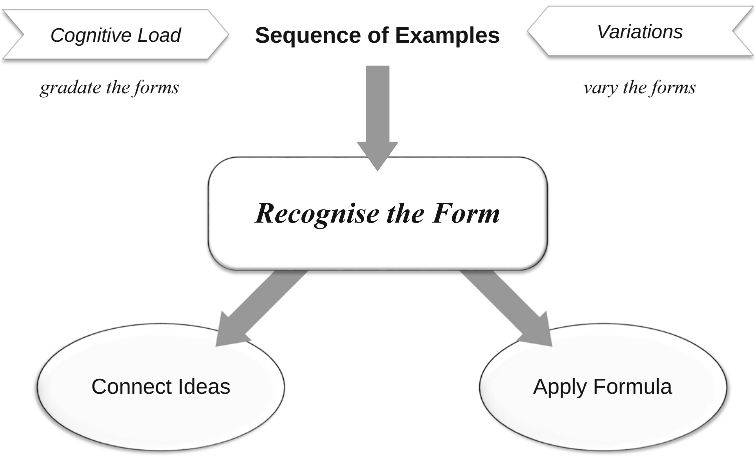

Fig. 8 A model of Beng Choon's design considerations

needs to be modified to account for this realisation, such as "recognise-apply" the formula. This suggestion to include the significance of recognition is not merely a play of words. In the case of Beng Choon, the weight of her design efforts was indeed on recognition—as presented in the findings. [For this reason, the centrality of "recognise" is reflected in Fig. 8]. The systematic variation of surface forms and difficulty in the sequence of items was to help students first to recognise the form in order to correctly apply the formula.

This shift of emphasis has practical implications for task design. If design of item sequences is naively based only on the act of mere "applying", then it is no wonder that items produced have identical surface forms for the purpose of unreflectingly repetitive practice. But once the designer is cognizant of "recognising-applying" as a two-step conjoined process, there is a slant in design orientation away from mere repetition; there would be a deliberate varying of surface forms to target recognition. The leverage on recognition is not only in serving the needs of formula-application. As illustrated in Fig. 8, it pays other dividends. Interestingly, teachers can seize upon the recognition of certain standard forms to connect to related and useful ideas.

To concretise this point, we reflect on its relevance on the design of another "formula": the Pythagoras theorem. The goal is students' fluency in the application of the theorem. If we are merely focused on "applying", we may provide a set of diagrams of random right-angled triangles, each comprising a task that requires students to calculate the length of one side given the dimensions of the other two sides. But once "recognising" comes also to the foreground of designing, we consider how right-angled triangles can be recognised within varied contexts—in different orientations, and when composite with other shapes (such as with other triangles and within circles)—or even non-right-angled triangles to highlight the dangers of mis-recognition. When opportunity presents, we might also make connection to how when two sides of a right-angled triangle are given, the triangle is fixed by congruence (SAS or RHS) and so it is unsurprising that the third side is also determined. This transference of principle to another formula illustrates the practical usefulness of reconceptualising "apply" as "recognise-apply" in task design.

On the theoretical side, we learn about how Beng Choon integrates ideas derivable from cognitive load theory and variation theory into the design of her instructional materials. Although she managed the enterprise from a realistic-pragmatic perspective—that is, with the goal of helping students gain fluency and in a way that aligns with her assessment of the contextual needs of her students—it contributes to our understanding of how an eclectic approach to theories can translate into design principles. Not only did she demonstrate a complementary praxis that ties both these theoretical streams together, our analysis unravels the depths of her craft knowledge (to borrow a term from Hiebert et al. 2002) that she brought to bear in this enterprise of task design. Figure 8 alone in its summarised form illustrates that there are more than meets the eye in what most would cursorily consider easy work for teachers—to prepare "practice items" for students. But the domains of knowledge and the multi-faceted considerations that Beng Choon drew into this work challenge a simplistic view of theory-practice link. The genesis of her craft knowledge is beyond the scope of this study; the point is: the integrated and goal-oriented (towards building students formula-application fluency) nature of her craft knowledge provides us with a starting model to view the enterprise of basic task design.

There is a direct lesson for mathematics teacher education. To date, we are not aware of mathematics methods courses that would prepare teachers to design for students' procedural fluency. Often, this is taken as trivial work left to pre-service teachers to figure out in the field. This study challenges this common assumption; it argues for a place in teacher preparation courses for a more in-depth discussion of this piece of professional work, perhaps beginning with Figure 8 as a theoretical starting point. Pursuing this line of thought further, most would consider teaching formula-application as the most basic work of teaching. We see in this study that even so, there are potential layers of complexities undergirding its design and implementation. Thus, in designing for more ambitious goals of teaching, such as teaching reasoning or teaching problem solving, we can imagine the challenge at task design to be far more onerous. This perhaps partly explains the persistent lack of success at scale for ambitious educational innovations; it also reminds interventionist researchers of the need to carefully examine the cognitive loading into the craft knowledge of teachers when supporting the design of ambitious tasks.

Conclusion

This case study provides us a peek into the way a competent mathematics teacher in Singapore designs sequences of items for the goal of students' fluency in formula-application. The picture that emerges from the study is one that challenges the conventional conception—that it is boring uninteresting work; rather, it reflects the deliberate integration of knowledge strands coherently so that the sequence of items used in the classroom would fulfill the teacher's goal while meeting the contextual needs of the students. The theorisation of the teacher's conception is summarised in Fig. 8. Regardless of whether similar characteristics of task design are shared by other Singapore mathematics teachers—a follow-up study to the one reported here—this conceptualisation can serve as a provisional model for professional development in task design.

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