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ORIGINAL ARTICLE

Characteristics of the shifts from configural reasoning to deductive reasoning in geometry

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Abstract

The study aimed at characterising the shift from configural reasoning to proof construction in geometry. One hundred eighty-two preservice primary teachers solved two geometry problems in which they had to generate a proof from the information provided by a geometrical configuration. Results indicated that proof construction was linked to the way pre-service teachers coordinated the different apprehensions, as identified by Duval (1995), mediated by the existence of strategic knowledge. Strategic knowledge is undertood as the ability to see some specific geometrical statements as premises of a geometrical proposition that can be used to deduce intermediate statements or the conclusion. We argue that pre-service teachers need to be aware of the connections between specific geometrical facts when they construct a proof by linking visualisation to formal reasoning. We conclude with implications for teacher education programmes.

Keywords Configural reasoning · Deductive reasoning · Discursive and operative apprehension · Geometrical thinking · Proof · Visualisation

Introduction

The relation between visualisation and knowledge of geometry highlights the difficulty some students experience when they have to use previously acquired geometry knowledge to build deductive proof (Battista 2007; Chinnappan and

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Lawson 2005; Clemente et al. 2017; Gal and Linchevski 2010; Prusak et al. 2012; Weber 2001). We focused on characterising the shift students operate when passing from associating configurations with mathematical statements, to building a proof when solving proof problems in geometry. In the case of geometry poof problems that provide a configuration, successful solving processes use true statements, valid modes of reasoning and appropriate modes of representation (Stylianides et al. 2016). Here, we considered deductive proof to be an appropriate argument supported by valid reasoning. A number of different factors intervene in the transition from configuration information (the association of the geometrical facts to the configuration) to the construction of proofs (Duval 1998; Gal and Linchevski 2010; Llinares and Clemente 2014; Prusak et al. 2012; Stylianides et al. 2016). This is the case for example when students associate certain properties or definitions to the given geometric configuration (discursive apprehensions in terms of Duval), and when the geometric facts are related via an "if…then…" deductive chain of inference. In this sense, the solving of proof problems is initially based on giving configural meaning to geometric shapes by means of discursive apprehensions (Duval 1998) and then turning these geometrical facts into the premise of a geometrical proposition (Arzarello et al. 2008). Our knowledge of how students identify which theorem to use to deduce intermediate statements of a conclusion is still fairly limited (Miyazaki et al. 2017). Therefore, the aim of our study was to investigate the characteristics of the shift from configural reasoning to the construction of proof in geometry.

Background and theoretical framework

Existing studies on the shift from configural reasoning to generating a logical deductive chain of reasoning have addressed different factors:

- The role of students' knowledge and the levels of understanding of a deductive proof structure as they attempt to build proofs and to use theorems to deduce intermediate statements or a conclusion (Chinnappan 1998a, b; Chinnappan et al. 2012; Miyazaki et al. 2017; Prusak et al. 2012), and
- The continuity and discontinuity between configural reasoning and deductive proof, both from the epistemological and cognitive perspectives (Arzarello et al. 2008; Duval 1995; Torregrosa and Quesada 2007).

Chinnappan et al. (2012) identified three predictors of performance in proof construction: (1) geometry content knowledge, (2) general problem-solving skills and (3) geometry reasoning skills, with geometrical content knowledge representing the major determinant of success. Nevertheless, general problem-solving skills and geometry reasoning skills significantly contributed to the activation and utilisation of geometry knowledge during the resolution process. According to the authors, geometrical knowledge includes knowledge about geometric relationships and the visual representation of such relationships in addition to geometric concepts. When associating geometrical facts with a configuration in order to generate goal-directed new information, it is necessary to possess solidly organised knowledge to be able to access and retrieve the required theorems (Prawat 1989; Lawson and Chinnappan 1994). From this perspective, the way in which students have organised their geometrical knowledge (e.g. the different links between concepts and theorems) could help or hinder the activation of knowledge needed to construct a proof.

However, students need to understand the structure of proof to find and use proper theorems to deduce new information in a goal-directed manner (Miyazaki et al. 2017). One factor that seems to influence the capacity to retrieve some pieces of geometrical knowledge and its activation in a deductive chain is the ability to consider a geometric fact not only from a configural point of view, but also as part of a sequence of deductive relationships. This implies recognising that a geometric fact can play different roles (epistemological status) in the solution process allowing students to progress from visual reasoning to deductive reasoning (Duval 1998). In other words, to be able to construct a proof, students should be able to organise premises, conclusions and theorems appropriately, linking singular facts to universal propositions (Duval 2007; Miyazaki et al. 2017) illustrating the relationships between the argumentative and the discursive side of a proof (Arzarello et al. 2008; Clemente and Llinares 2015). These relationships reveal the epistemological and cognitive gap between argumentation and proof.

To understand the shift from configural reasoning to a deductive chain in the solving of geometrical proof problems (and the gap between epistemological and cognitive aspects), Duval (1998) suggests considering different types of apprehensions. Duval defines operative apprehension as the student modifies a figure enabling to identifying sub-configurations. He defines discursive apprehension as the student associate configurations or sub-configurations with mathematical statements. Configural reasoning occurs when operative and discursive apprehensions are coordinated: it highlights the relationships between visualisation and geometrical knowledge interlinking figural and conceptual aspects (Torregrosa and Quesada 2007; Zazkis et al. 1996). Coordinating operative and discursive apprehensions may lead to the "idea" for solving the problem generating a logical deductive chain of reasoning, linking premises with conclusions (Torregrosa and Quesada 2007; Torregrosa et al. 2010).

In this paper, to understand the shift from configural reasoning to the construction of a proof, we focused on how students come to consider the geometrical facts identified in a configuration, or given as hypotheses in the problem, as the premises of deductive sequences. The key research question was:

- What features intervene in the transition from configural reasoning to the construction of a proof when pre-service teachers solve geometrical proof problems?

Method

Participants

A total of 182 pre-service primary teachers participated in this study (hereon referred to as the students). These students had taken a geometry course (60 h;

4 h per week for 15 weeks). The activities in this course were designed to develop the ability to visualise and understand geometric properties of plane figures such as polygons, triangles, quadrilaterals and parallelograms, and learn to relate geometrical facts (comparing, manipulating and transforming a mental picture). The aim was to generate deductive processes, to justify some construction processes and explore and prove properties of triangles and quadrilaterals. This course led pre-service primary teachers to recall geometry knowledge from their primary curriculum and some properties from their secondary curriculum—such as congruence and similarity of triangles—and thus support their explanations of why geometrical properties are correct and deduce them from already-learned properties. The last part of the course focused on proof. Problems such as those shown in Fig. 1 were submitted to each student who had to solve them. Advanced solutions were subsequently discussed by the whole group. The goal was to make explicit how, during the problem's resolution, geometrical facts were associated with the configuration and how they could be linked to other geometrical propositions to generate a logical deductive chain (as a way of recognising the relationships between elements of a deductive proof). This geometry course was the students' first encounter with proof in the teacher education programme.

The problems

At the end of the course, students individually solved two geometrical proof problems as part of their assessment (Fig. 1). The problems were similar to those solved during the course. Each problem included a geometric configuration and information about the configuration, and asked students to prove the congruence of two segments. In one of the two problems, it was possible to identify different sub-configurations that could be used for the solution generating different resolution trajectories. To identify the geometrical content susceptible to be used in the problem resolution, three mathematics teacher educators identified the geometric facts that could be used to solve the problems from different approaches. The prior analysis of the problems was justified by the coherent link between these problems and the teacher education programme's curriculum and research goals. Table 1 shows the geometric facts used by students in their different problem-solving approaches. These items of knowledge come from the data provided in the

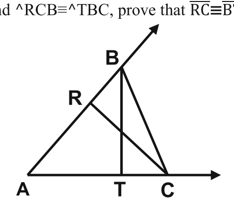
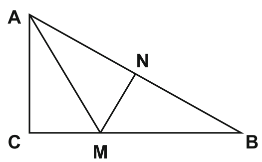
Problem 1	Problem 2
Given the triangle △ABC in the figure, where $\overline{AB} \equiv \overline{AC}$ and ^RCB≡^TBC, prove that $\overline{RC} \equiv \overline{BT}$	In the figure, $\overline{AM}$ is a bisector of ^CAB, △ACB is a right-angled triangle at C and $\overline{MN} \perp \overline{AB}$ at N. Prove that $\overline{CM} \equiv \overline{MN}$
	

Fig. 1 Geometry problems used

Table 1 Geometric facts used in the different problem-solving approaches. *GKi* geometric knowledge item

Problem 1	Problem 2
Association of geometric elements with the configuration: • GK1- Triangle- identification of a sub-configuration • GK4- Opposite angles at the vertex are equal Geometric elements that could be used to infer additional information (using "if ... and ... then ..."; "if …. then …"): • GK2- Properties of an isosceles triangle (two congruent sides, and therefore, two congruent angles- considering the two directions in which properties of isosceles triangles flow (as example of "if …then …"). In a triangle, angles opposite to the two congruent sides are congruent and the sides opposite to the two congruent angles are congruent • GK3- If you take two congruent angles and deduct the same part you will be left with two congruent angles • GK9- Criterion of triangle congruence A-S-A (as example of "if …and … then …")	Association of geometric elements with the configuration: • GK1- Triangle- identification of a sub-configuration • GK5-Definition of an angle bisector (ray by vertex of angle that splits angle into two congruent parts) • GK6-Definition of perpendicular lines • GK7-Definition of right-angled triangle Geometric elements that could be used to infer additional information (using "if ... and ... then ..."; "if … then…"): • GK8- The sum of the interior angles of a triangle is equal to 180° (If we know two angles in a triangle, we know the third) • GK9- Criterion of triangle congruence A-S-A (as example of "if …and … then …")

problem (or from the configuration given by discursive apprehension) and from the information on configuration, that can be inferred using prior geometrical knowledge.

It is possible to identify different sub-configurations in each problem (Fig. 2) that can be used to generate different solution trajectories. Each sub-configuration reflects possible factors triggering or inhibiting the relevant configurations: splitting in embedded sub-figures (in problem 1, sub-configuration SC3; and the sub-configuration in problem 2), and double use of one sub-figure or overlapping figures (in problem 1, sub-configuration SC1 and SC2) (Duval 1995). The identification of these sub-configurations through operative apprehension would allow students to recognise some geometric facts more easily than others. In problem 1, there are several possible figural modifications of the given figure, and three

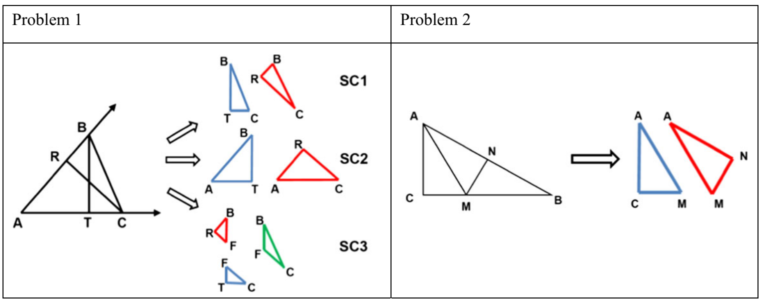


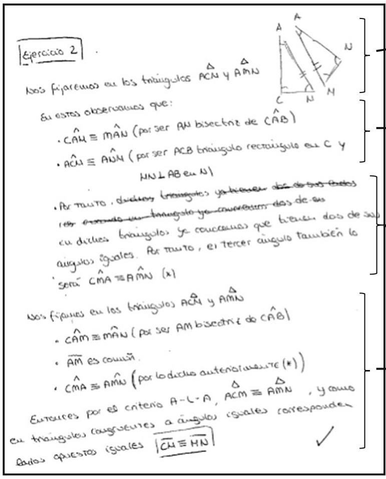
Fig. 2 Possible sub-configurations when solving the problems (the F label of problem 1 was included by some students when identifying the sub-configuration SC3)

operations to work them out. In problem 1, the figure shown initially did not have the F label, but some students named the intersection of sides $\overline{BT}$ and $\overline{RC}$ when identifying the sub-configuration SC3.

Analysis

The data under analysis corresponded to the students' written arguments. In each individual answer to the problems, we identified how students constructed their arguments based on operative and discursive apprehensions and their coordination. We conducted the analysis in two phases. In phase 1, we identified (Fig. 3):

i) Evidence that the students had identified a relevant sub-configuration through operative apprehension. For example, in problem 1, when the students indicated that they were considering the triangles △RCB and △TBC (sub-configuration a); △ATB and △ARC (sub-configuration b); or △CFB, △RFB and △TFC (sub-



Identification of triangles △ACM and △AMN through operative apprehension

They recognise the information of the problem in the configuration, giving it a configural meaning
- ^CAM is congruent with ^MAN because AN is a bisector of CAB
- ^ACN is congruent with ^ANM because △ACB is a right-angled triangle at C and MN is perpendicular to AB at N

Use of a fact not given to general additional configural information: the sum of the interior angles of a triangle is 180º.Therefore, as we know two angles, we can obtain the third angle

Use of a non-provided geometric fact (congruence of triangles A-S-A) to relate facts known as premises in a deductive process of type "if...then..."

Student 16's answer translated:

Considering the triangles △ACM and △AMN
We can see that:
^CAM≡ ^MAN (because AN is a bisector of ^CAB)
^ACN≡ ^ANM (because △ACB is a right-angled triangle at C and MN is perpendicular to AB at N)
Therefore, in those triangles we already know that two of their angles are the same. So the third angle will also be the same ^CMA≡ ^AMN (*)
If we look at triangles △ACM and △AMN
^CAM≡ ^MAN (as AM is a bisector of ^CAB)
AM is common
^CMA≡ ^AMN (following what was said above) (*)
Therefore, using the criteria of A-S-A, △ACM≡△AMN and as in congruent triangles with equal angles, their opposite sides are equal CM≡MN

Fig. 3 Student 16's answer to problem 2 (A16P2E2) configuration c); similarly, in problem 2, when they had taken into account triangles △ACM and △AMN.

ii) If, the students recognised a fact in the configuration and gave it a configural meaning, for example, if in problem 1 and for sub-configuration c, they recognised that angles ^RFB and ^TFC were congruent because they were opposite vertex F. Similarly, in problem 2, if they indicated that the bisector of the angle ^CAB created two equal angles in ^A; or that angle ^N is a right angle because $\overline{MN}$ is perpendicular to $\overline{AB}$.

iii) If the students used external knowledge beyond the data to infer additional information. For example, in problem 1, if they used the following facts:

- If sides $\overline{AB}$ and $\overline{AC}$ are congruent in triangle △ACB it follows that △ACB is an isosceles triangle, and therefore, angles ^TCB and ^RBC will also be congruent;
- If angles ^FBC and ^FCB are congruent in triangle △CFB (a given fact since these angles are the same as ^RCB and ^TBC), it follows that it is an isosceles triangle, and therefore, sides $\overline{BF}$ and $\overline{CF}$ will also be congruent;
- If you take two equal angles (^TCB and ^RBC) and deduct the same part (^RCB and ^TBC), you will be left with two congruent angles (^BAT and ^CAR).

Similarly, in problem 2, if they used the fact that the interior angles of a triangle add up to 180°, therefore, it is possible to know the third angle in a triangle when we know two angles.

iv) If the students used data and previously obtained information as premises in a geometric proposition. For example, if they used the criterion of congruence of triangles A-S-A once they had recognised the information given in the configuration as premises for this criterion of congruence.

We identified the geometric facts and the relationships used as students coordinated their discursive and operative apprehensions (geometrical properties, and geometrical propositions as the criteria of congruent triangles). Next, we grouped the various steps of the solution process into two stages in order to identify configural reasoning and the generation of a deductive process when considering some items of knowledge as premises for a proposition or geometric theorem. In this way, we were able to identify when the students were relating the geometric facts in the configuration to a previously learned theorem to construct the proof. We could therefore identify the points at which items of knowledge could play different roles, from the configural to the logical status in a deductive process, in each of the solution trajectories followed.

In phase 2 of the analysis, the answers to each problem were assigned a 3-vector V[(1),(2),(3)] according to the criteria indicated in Tables 2 and 3 (Lin and Yang 2007). This analysis aims at identifying when a student performs an incorrect identification based on a superficial similarity; when a student makes a correct

Table 2 Criteria for generating vectors V [(1),(2),(3)] associated with the solution to problem1

V	Description		Activated geometrical knowledge
(1)	Identification of a relevant sub-configuration (SC1, SC2, SC3)	0: Does not identify/make an incorrect identification based on a superficial similarity 1: identified	-Triangle (GK1)
(2)	Identification of items of knowledge that could be used as hypotheses for applying a theorem (premises in a deductive chain). Two types of items: - Obtained directly from the facts of the problem and linked to a specific sub-configuration: "SC1" "SC2" "SC3" H1: $\overline{BC}\equiv\overline{BC}$ H1: $\overline{AB}\equiv\overline{AC}$ H1: ^RFB ≡ ^TFC H2: ^RCB ≡ ^TBC H2: ^BAT≡^CAR H2: ^FCB ≡ ^FBC - Obtained from prior geometric knowledge: "SC1" "SC2" "SC3" H3: ^TCB ≡ ^RBC H3: ^ACR ≡ ^ABT H3: $\overline{BF}\equiv\overline{CF}$ H4: ^RBF ≡ ^TCF	0: Does not identify/make an incorrect identification based on a superficial similarity 1: Identify H1 and H2 2: Identify H1, H2 and H3 ("SC1" and "SC2"), identify H1, H2, H3 and H4 ("SC3")	-Isosceles triangle: congruent angles and therefore two congruent sides // Two congruent sides and therefore two congruent angles (GK2) -Angle (if you take two equal angles and deduct the same part you will be left with two congruent angles). (GK3)
(3)	Conclusions reached: "SC1" "SC2" "SC3" C1: △RCB≡△TBC C1: △ATB≡△ARC C1: △RFB≡△TFC (mentions use of criterion A-S-A) to derive… C2: $\overline{RC}\equiv\overline{BT}$ (mentions the use of criterion of congruence of triangles)	0: No conclusions reached/correct conclusions based on incorrect fact/ uses correct data and reaches a correct conclusion but based on an incorrect deduction. 1: Get C1 and C2	-Congruence of triangles (criterion A-S-A) (GK9)

statement but bases it on an unfounded logical claim; and when a student invokes prior but incorrect knowledge, but provides a correct solution, for example, when students reach a correct conclusion but based on an incorrect deduction. In this latter case, the associated vector is V [1,2,0]. By assigning a 3-vector to each problem-solving process, we can track down how the geometrical facts are used and how they are linked.

An example of how the analysis process was carried out is given below. Figure 3 shows the result of the process followed in phase 1 of the analysis. Next, we placed it in a 3-vector (using Table 3, as it is problem 2). For that, we first checked whether the student identified a relevant sub-configuration assigning a score of 1 in the first grid box if the answer was yes. We then looked for evidence in the written answers that the student had recognised the problem's data in the configuration

Table 3 Criteria for generating vectors V [(1), (2), (3)] associated with the solution to problem 2

V	Description		Geometrical knowledge activated
(1)	Identification of a relevant sub-configuration	0: Does not identify 1: Identify	-Triangle (GK1)
(2)	Identification of items of knowledge that could be used as hypotheses for applying a theorem (premises in a deductive chain). Two types of items: -Obtained directly from the facts of the problem: H1: $\overline{AM}\equiv\overline{AM}$ H2: ^ACM ≡ ^MNA H3: ^CAM ≡ ^MAN -Obtained from prior geometric knowledge: H4: ^AMC ≡ ^AMN	0: Does not identify/or makes an incorrect identification based on a superficial similarity 1: Identify H1, H2 and H3 2: Identify H1, H2, H3 and H4	-Bisector (ray that cuts the vertex of an angle and divides it into two congruent parts) (GK5) -Definition of perpendicular lines (GK6) -Definition of right-angled triangle (GK7) -Angle (the sum of the internal angles of a triangle is equal to 180°) (GK8)
(3)	Conclusions reached: C1: △ACM≡△AMN (mentions the use of the criterion A-S-A) to derive… C2: $\overline{CM}\equiv\overline{MN}$ (mentions the use of the criterion of congruence of triangles)	0: No conclusions reached/correct conclusions based on incorrect fact/uses correct data and reaches a correct conclusion but based on an incorrect deduction. 1: Get C1 and C2	-Congruence of triangles (criterion A-S-A) (GK9)

(evidence of discursive apprehension). In this case, we assigned 1 to the vector's second coordinate. When the student also considered some geometrical facts or previously learnt proposition, we assigned a score of 2 in the vector's second coordinate. Finally, we verified the conclusions in this case, via the correct application of the criterion of congruence of triangles A-S-A; and we scored 1 in the vector's third coordinate (Table 4).

Table 4 Vector grid generated in phase 2 of the analysis, using student 16's answer to problem 2

Student	V	Description (problem 2)	Score	Geometrical knowledge activated	Vector
A16P2E2	(1)	Identify: △ACM and △AMN	1	GK1	V [1,2,1] Passage from configural reasoning to deductive process
	(2)	Obtain: H1: $\overline{AM}\equiv\overline{AM}$ H2: ^ACM ≡ ^MNA H3: ^CAM ≡ ^MAN H4: ^AMC ≡ ^AMN	2	GK5 GK6 GK7 GK8	
	(3)	Obtain: C1: △ACM≡△AMN C2: $\overline{CM}\equiv\overline{MN}$	1	GK9	

Results

Our findings are grouped into two sections. The first section describes students' solution trajectories and shows how geometrical knowledge intervenes in configural reasoning. The second section shows how students used geometrical facts as premises of a proposition to deduce intermediate statements or the conclusion (i.e. what has been asked to be proven).

Solution trajectories from the identified sub-configuration

The coordination of discursive and operative apprehensions linked to a specific sub-configuration led to defining possible solution trajectories, thus to the use of a sequence of knowledge items. A solution trajectory is the sequence of geometric facts (knowledge) and their relationships. Table 5 displays the sequence of knowledge items defining the solution trajectories identified for each problem. The solution to problem 1 was started by 164 of the 182 students (116 from sub-configuration *SC1,* 39 from sub-configuration *SC2*, and 9 from sub-configuration *SC3*), and the solution to problem 2 by 165 of the 182 students.

The way in which the geometrical facts and propositions were organised into a deductive step as well as the arrangement of deductive steps seems to indicate that some type of mental association exists between prototypical figures and mathematical concepts determining students' performance. Differences in the success levels appear to support the existence of a mental association between certain prototypical figural configurations and geometric concepts that apply to them in the proof process.

From configural reasoning to deductive reasoning

Table 6 shows the vectors identifying the features of the solution trajectories. Figures 4 and 5 present the graphs of frequency showing the shifts from configural reasoning to deductive reasoning. The trajectories reveal that prospective teachers use and infer geometrical facts from their discursive and operative apprehensions, but they also use previously learned geometrical facts. The relationships between geometrical facts (from configural reasoning and from remembered geometrical facts) seem to govern the shift to deductive reasoning. We identified three groups of trajectories that characterised these transitions.

Table 5 Percentage of sub-configurations identified for each problem and possible sequences of mobilised knowledge items. GKi = items of geometric knowledge active in the solution trajectory

Problem	Sub-configuration	Solution trajectories (Geometry Knowledge)	Students	Total	% About $n = 164$	% about $n = 182$
P1	SC1	GK1 + GK2 + GK9	116	164	70.7%	90.1%
	SC2	GK1 + GK2 + GK3 + GK9	39		24.8%	
	SC3	GK1 + GK2 + GK3 + GK4 + GK9	9		5.5%	
P2	Only one	GK1 + GK5 + GK6 + GK7 + GK8 + GK9	165	165		90.6%

Table 6 Classification of the solution trajectories adopted by students

Group	Vector	P1				P2
Unable to identify relevant sub-configuration	V [0,0,0]	18 (9.9%)				17 (9.4%)
No deductive process generated		Relevant sub-configuration				
		SC1	SC2	SC3	TOTAL	
	V [1,0,0]	16	3	1	20 (11%)	11 (6.1%)
	V [1,1,0]	50	7	5	62 (34.1%)	47 (25.8%)
Passage from configural reasoning to logical deductive chain	V [1,2,0]	4	3	0	7 (3.8%)	13 (7.1%)
	V [1,2,1]	46	26	3	75 (41.2%)	94 (51.6%)
TOTAL		116	39	9		
		182				182

The *first group* (V [0,0,0]) corresponds to students ($n = 18$ in problem 1 and $n = 17$ in problem 2) who failed to identify a relevant sub-configuration and did not employ configural reasoning.

The *second group* is made up of students who associated at least one geometric fact with the configuration without generating a deductive process. This is performed.

- when the student only identified the initial triangles as a relevant configuration (V [1,0,0]), and.
- when the student generated discursive apprehensions through direct associations of geometric elements to the configuration from the data (V [1,1,0]), but were not capable of generating any deductive processes.

The *third group* of students moved from configural reasoning to deductive reasoning. In this case, we identified two subgroups:

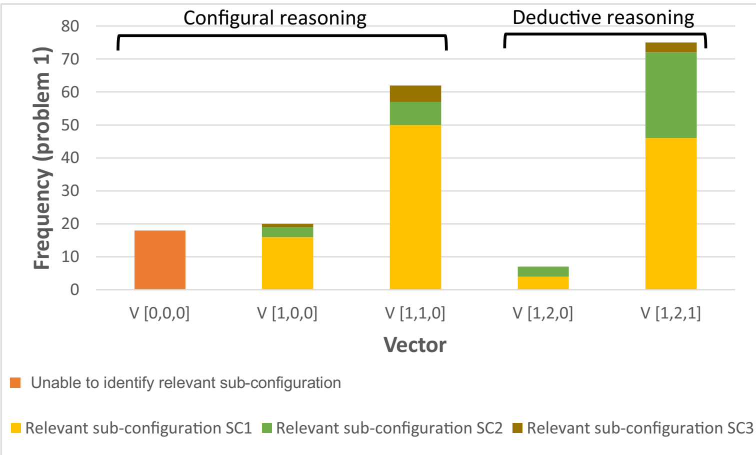


Fig. 4 Bar chart of 3-Vectors associated to the resolution trajectory of problem P1

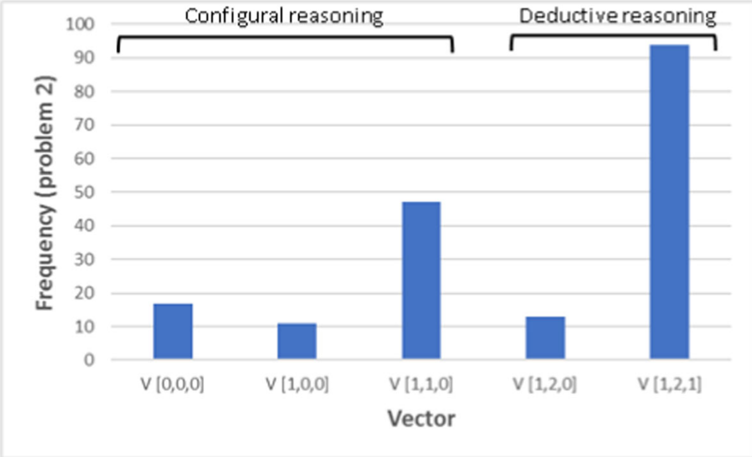


Fig. 5 Bar chart of 3-Vectors associated to the resolution trajectory of problem P2

- those who tried to generate new information (V [1,2,0]), but failed to identify any of the items of knowledge as premises for "if…then…" propositions (in these problems this was the criterion of congruence of triangles A-S-A), and
- those who considered the geometric facts associated with the configuration as premises in an already known proposition (congruence of triangles, A-S-A) enabling them to generate a deductive process (V [1,2,1]).

Next, we illustrate the difference between V[1,1,0] and V[1,2,1] for problem 1. Figure 6 displays an answer to problem 1 assigned as V[1,1,0] since the prospective teacher generates discursive apprehensions (link to the sub-configuration SC2, identifying ▵ATB and ▵ARC - codified as GK1-) and makes direct associations of the given geometric elements with the configuration: AB≡AC; ^A is common; and finally ^ABT ≡ ^RCA, but this last affirmation was unsubstantiated.

Figure 7 shows an example of a solution to problem 1 assigned to V [1,2,1], in which the student relates the figural geometric facts with an already known proposition. The student infers from the AB≡AC data that the ΔABC is isosceles; hence, the angles ^ABC and ^ACB are congruent. Through operative apprehension, the student identifies

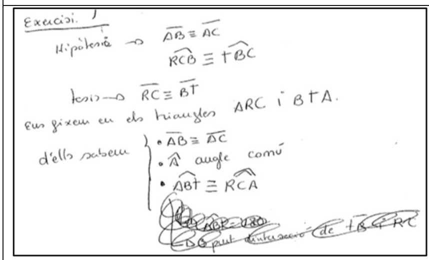
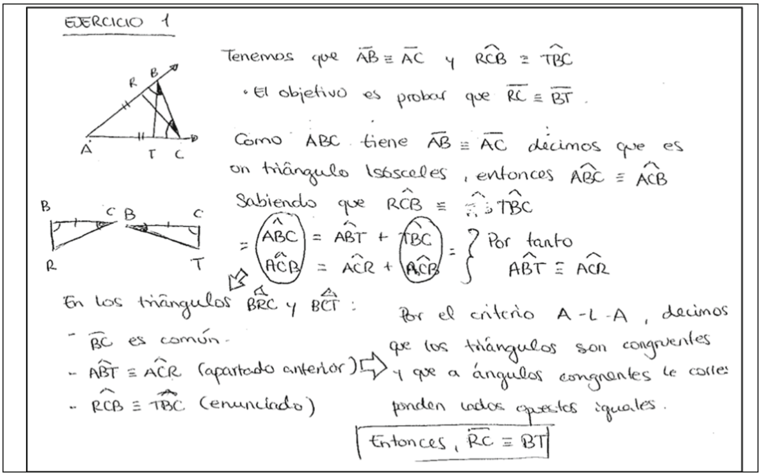
	Student 15's answer translated (A15P1E2):
	*Hypothesis: AB≡AC* *^RCB≡ ^TBC* *Thesis: RC≡BT* *We look at the triangles: ARC and BTA* *We know that: AB≡AC* *^A is a common angle* *^ABT≡ ^RCA*

Fig. 6 Example of problem 1 answer assigned as V [1,1,0]



Student 4's answer translated (A4P1E2):

We have: AB≡AC and ^RCB≡ ^TBC
The goal is to prove: RC≡BT
As ABC has AB≡AC, we can say it is an isosceles triangle, therefore: ^ABC≡ ^ACB
Given that: ^RCB≡ ^TBC
^ABC = ^ABT + ^TBC
^ACB = ^ACR + ^RC
Then: ^ABT≡ ^ACR
In the △BRC and △BCT triangles:
BC is common
^ABT≡ ^ACR (previous section)
^RCB≡^TBC (hypothesis/ given fact)
Based on A-S-A, we can say that the triangles are congruent and congruent angles have congruent opposite sides.
So: RC≡BT

Fig. 7 Example of problem 1′ answer assigned as V [1,2,1]

the triangles ΔRBC and ΔBCT (sub-configuration SC1) and marks the congruence of angles ^RCB and ^CBT (data given by the problem). Next, the geometrical knowledge item GK3 is used (if you take two congruent angles and deduct the same part you will be left with two congruent angles) to infer that the angles ^ABT and ^ACR are congruent. At this point in the solution process, the student changes the status of these geometrical facts from figural to formal. This change is evidenced by the fact that the student uses the geometrical fact associated with the configuration as premises of an "if ... then ..." relationship (the criteria for A-S-A triangle congruence). This student indicates the change with the symbol "→". The student therefore proves the thesis (RC ≡ BT).

Trajectories in problem 1

In problem 1, a total of 18 out of the 182 students (9.9%) failed to identify any relevant configuration (V[0,0,0]). Of the remaining 164 students who identified a relevant sub-configuration, 82 initiated a deductive process by shifting from configural reasoning to a logical deductive chain (V[1,2,0] and V[1,2,1,]), and of these, 75 successfully solved the problem (V[1,2,1], 41,2%).

Considering the relevant sub-configurations that provide the starting point for the different trajectories, of the 116 students that started the trajectory linked with sub-configuration *SC1,* only 50 were able to shift from configural reasoning to a logical deductive chain, and of these, 46 successfully solved the problem (40%, 46 of 116). In this trajectory (GK1 + GK2 + GK9), students identified the triangles △RCB and △TBC of sub-configuration *SC1* (GK1); they also used the «properties of an isosceles triangle» (GK2) on the basis of the problem statement (sides $\overline{AB}$ and $\overline{AC}$ are congruent) to infer that the △ABC is an isosceles triangle, and consequently, that angles opposite congruent sides are congruent (H3: ^TCB ≡ ^RBC). This fact is also used in trajectories linked with sub-configurations *SC2* and *SC3*. At this point in the solution process, these students related this fact with hypothesis H2 (^RCB ≡ ^TBC) and H1 ($\overline{BC}\equiv\overline{BC}$) through a previously learnt proposition, the criterion of the congruence of triangles A-S-A (GK9). Establishing this relationship allowed them to infer that $\overline{RC}\equiv\overline{BT}$. Having established this latter relationship, and once the students considered the geometric facts as premises for one of the criteria for triangle congruence, these geometric facts changed from having a configural meaning (facts linked to the configuration) to being used in a deductive chain that allowed students to construct a proof.

In the sub-configuration *SC2*, of the 39 students who started, 29 initiated their shift from configural reasoning to a logical deductive chain, and of these, 26 succeeded in solving the problem (67%, 26 of 39). In this trajectory (GK1 + GK2 + GK3 + GK9), the students identified triangles △ATB and △ARC of sub-configuration *SC2* (GK1). They also used "properties of an isosceles triangle" (GK2) based on the problem statement "$\overline{AB}$ and $\overline{AC}$ are congruent", deducing that the triangle △ABC is isosceles, and consequently, the angles opposite the congruent sides were congruent. With this new information and the hypothesis H2 (^BAT≡^CAR), they used the «addition property of congruent angles» (GK3) to derive the new information H3 (^ACR ≡ ^ABT). This way of proceeding was also used in the solution trajectory linked with sub-configuration *SC3*. They then considered these two geometric facts, along with H1 ($\overline{AB}\equiv\overline{AC}$), as premises for the «criterion of congruence of triangles» (GK9), which allowed them to leave the configural reasoning and generate the deductive reasoning that solved the problem. Students leave configural reasoning when they change the epistemic status of the knowledge items from the configural (fact relating to the configuration) to their consideration as premises in a deductive process that allows them to construct a proof.

Finally, the trajectory defined by sub-configuration *SC3* was followed by 9 students, of which 3 were able to leave configural reasoning and succeed in solving the problem (33%, 3 of 9). In this solution trajectory (GK1 + GK2 + GK3 + GK4 + GK9), the students identified the triangles △RFB, △TFC and △BFC of sub-configuration *SC3*, allowing them to use the «properties of an isosceles triangle» having two congruent sides/angles (GK2); the use of the «addition property of congruent angles» (GK3) and the «congruence of angles opposite the vertex» (GK4) to obtain H1: ^RFB ≡ ^TFC. These geometric facts, when considered as premises for the «criterion of congruence of triangles A-S-A» (GK9), allowed students to leave configural reasoning and to initiate a process of deductive reasoning making it possible to solve the problem.

These data indicate that the success rate in problem 1 was greater for students who identified the sub-configuration *SC2*, and the success rate was lower for students who identified the sub-configuration *SC3*.

Trajectory in problem 2

In problem 2, a total of 165 of the 182 students identified the relevant sub-configuration and continued on a solution trajectory that involved the activation of six geometric knowledge items (GK1 + GK5 + GK6 + GK7 + GK8 + GK9). Of these 165 students, 107 left configural reasoning, and of these, 94 successfully solved the problem (51.6%). In this solution trajectory, students identified the triangles △ACM and △AMN (GK1). They also employed discursive apprehensions with H3: ^CAM ≡ ^MAN derived from the problem statement ($\overline{AM}$ is a bisector of angle ^CAB). They then used the knowledge item GK8 («the sum of the interior angles of a triangle is equal to 180°», and consequently: if we know two angles of a triangle we know the third), to derive the information H4: ^AMC ≡ ^AMN. The relationship between these three knowledge items H1: $\overline{AM}\equiv\overline{AM}$; H3: ^CAM ≡ ^MAN; and H4: ^AMC ≡ ^AMN through a previously learnt proposition (congruence of triangles A-S-A) allowed them to leave configural reasoning and to start a process of deductive reasoning to solve the problem.

Discussion

The objective of this study was to identify features characterising the transition from the identification of geometrical configurations and its association with mathematical statements to the construction of a proof during the solving of proof problems in geometry. We focused on the interaction between figural and conceptual aspects during the coordination of discursive and operative apprehensions (Duval 1995) when students had to construct a proof. The passage from configural reasoning to the construction of a proof was linked to the way in which students integrated figural and conceptual knowledge in a mental model (Fischbein 1993). The hypothesis underlying this research is that in order to teach geometry in primary education, pre-service primary teachers must know the geometry content in such a way that allows them to go beyond the simple recognition of properties and geometric facts of plane figures (Nason et al. 2012; Stylianides and Ball 2008). Our results provide information that enables us to improve our understanding of what Duval (1998) called a double gap in the transition from configural reasoning to the construction of a proof, "There is a double gap between naïve behaviour and mathematical behaviour. The one is about visualisation and the other is about reasoning. Thus, some specific skills must be developed from the common way of looking at figures and from the natural discursive reasoning. It would be a pedagogical illusion to present mathematical behaviour through the appearance of naïve behaviour (or in continuity with it) because of the visualisation" (Duval 1998, p.49) (emphasis added).

Our findings reveal a barrier between configural reasoning and deductive proof (Figs. 4 and 5) and a link between the identified sub-configuration, the mobilised knowledge items and the success rate (Tables 5 and 6). Based on the data, we advance two possible reasons for this barrier: (1) students need to change the epistemological status of geometrical facts (from a figural to a logical status) to consider them as premises in some previously learnt geometrical proposition to generate a deductive chain (via a logical link "if … then…"); and (2) to perform this change, students need "strategic knowledge" allowing them to choose the geometrical fact (theorem) that has to be applied. These latter claims are developed in the two sections that follow.

Changing the epistemological status of geometrical facts

The different trajectories linked to the configuration initially identified by students (Table 5) which characterise the shift from visual to discursive anchorage (Duval 1998) shows that different mental associations can exist between the configurations and the geometrical facts. However, this first discursive anchorage is not sufficient to construct a proof, understood as the organisation of premise, conclusions and intermediate propositions, as shown by the difference between trajectories V[1,0,0], V[1,1,0] and trajectories V[1,2,0] and V[1,2,1] (Figs. 4 and 5). These trajectories show the difference between identifying information from the configuration (or given by the problem) and using it as a premise in a deductive chain, to infer new information via intermediate propositions. In our research, and considering the characteristics of the problems used, the passage from configural reasoning to generating logical deductive chains occurred when students were able to make connections between geometric facts in the configuration and the criteria for the congruence of triangles (they selected the part of their geometrical knowledge they deemed suitable) (Vector [1,2,2]). However, the logical relationship between geometrical facts is not immediate as shown by the vector [1,2,0] (3.8% in problem 1 and 7.1% in problem 2). For the relationship to come about, the geometric fact associated with the configuration must change its status, from having a configural meaning to being used as a premise in a theorem to infer new information (Heinze et al. 2008). The proof is generated when students can make connections between various geometrical facts and consider them as premises in a proposition (in this case, triangle congruence criteria) supporting the development of a logical-deductive chain. We contend that students need to assign different roles to geometrical facts during the solving process to construct a mental model (Miyazaki et al. 2017). For this to happen, Duval (1998) argues that the given information must be processed at both a representational and symbolic level, thus demonstrating that with mathematical behaviour in a problem-solving process, "reasoning starts only from the discursive apprehension and is independent from visualisation. The purely configural change does not give the steps and the organisation of deductive reasoning for the proof, BUT it shows some key points, or an idea which allows them to select the main theorem to be used" (Duval 1998, p.48). This explanation has a bearing on the fact that the shift from configural reasoning to the construction of proof is based on the ability to establish connections between geometrical facts through an already known proposition, and does not only depend on knowing the facts and the propositions ("the bridging process" of Heinze et al. 2008). This aspect highlights the importance of strategic knowledge for constructing proof allowing students to select the part of their geometrical knowledge they judge meaningful for the proof process (in the words of Duval, "... which allows them to select the main theorem to be used" (Duval 1998, p.48)).

The existence of strategic knowledge

Some students were capable of recognising the geometric facts in the figure, but failed to construct a proof (the difference between V [1,2,0] and V[1,2,1]). Analysis of proof construction processes in different domains of mathematics and at different educational levels has shown the need for students to possess “strategic knowledge” to be able to construct proofs successfully (Chinnappan 1998a, b; Chinnappan et al. 2012; Weber 2001) and to recognise the logical relationship between the premises and the conclusion (Heinze et al. 2008). Here, strategic knowledge should be understood as “knowledge about the situation” that allows students to see the proof situation in which they find themselves as a particular case of a more general situation. In our study, this strategic knowledge is the student’s recognition of the usefulness of triangle congruence criteria when dealing with the type of problem they were solving (allowing them to shift from configural reasoning to deductive reasoning). In this way, the change of a geometric fact’s epistemic status, from being linked to a configuration, to being considered as the premise of a theorem that is necessary to initiate a deductive process, can be seen as evidence of this strategic knowledge. This prior knowledge involves mental actions that shift students’ focus as they remember that a particular theorem or geometric fact can be relevant for relating data-hypothesis with the thesis. Here, once students recognise one of the criteria for triangle congruence as a pertinent property, they only used the geometric facts linked to the configuration that had a configural meaning, as the premises of an “if…then…” theorem adopting another epistemic meaning. Regarding our results, the failure to recognise the relevance of the theorem “criteria of congruence of triangles” hindered the students’ ability to leave configural reasoning to construct a proof. This is the key aspect that characterises strategic knowledge: the possibility of recognising the problem as a particular case of a more general situation. This issue opens new research paths on the nature of the difficulties encountered by students in recognising the problem as a particular case of other general situations. Interviews could be conducted with students who fail to perform this shift to obtain insights into the nature of their difficulties and to characterise students’ thought processes as they struggle to pass from configural reasoning to deductive proof.

Some implications for teaching

Our findings contribute to research on the factors that deserve attention and lead to two inferences of didactical nature concerning primary teacher education programmes. Firstly, pre-service primary teachers need to be aware of the use of geometrical facts to establish relationships to construct a proof (as a means of changing the figural value of the geometrical facts). Thus, instruction aimed at enabling students to learn how to construct proofs should focus on developing this strategic knowledge in specific geometrical domains to favour the bridging process in their proof-building (Miyazaki et al. 2017; Heinze et al. 2008). In Duval (1998)‘s words, “some specific skills must be developed from the common way of looking at figures and from the natural discursive reasoning” (Duval 1998, p.49). Secondly, teacher trainers should consider different types of problems (with or without figures in the problems, and with or without names to different configural elements in the configuration) to support different trajectories and students’ understanding of the structure of deductive proofs. Our results suggest that by using problems with different hypothetical sub-configurations it would be possible to create contexts that favour the generation of different trajectories. These different work spaces during a problem’s solution could help pre-service primary teachers become aware of the figural and conceptual meanings of the geometrical facts and the structural relationships between premises and conclusions. These, however, constitute future lines of research.

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