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ORIGINAL ARTICLE

Influence of proportional number relationships on item accessibility and students' strategies

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Abstract Proportional reasoning is important to students' future success in mathematics and science endeavors. More specifically, students' fluent and flexible use of scalar and functional relationships to solve problems is critical to their ability to reason proportionally. The purpose of this study is to investigate the influence of systematically manipulating the location of an integer multiplier—to press the scalar or functional relationship—on item difficulty and student solution strategies. We administered short-answer assessment forms to 473 students in grades 6–8 (approximate ages 11–14) and analyzed the data quantitatively with the Rasch model to examine item accessibility and qualitatively to examine student solution strategies. We found that manipulating the location of the integer multiplier encouraged students to make use of different aspects of proportional relationships without decreasing item accessibility. Implications for proportional reasoning curricular materials, instruction, and assessment are addressed.

Keywords Proportional reasoning · Assessment · Task characteristics · Student strategies

Introduction

Extensive evidence points to the need for mathematics instruction to tap into students' informal understandings in order to conceptually develop formal mathematical ideas (Ahl et al. 1992; Freudenthal 1973, 1991; Treffers 1987). Contextual problems are a

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common means of helping students access their informal mathematical ideas (Lamon 1993; Moore and Carlson 2012). However, to successfully use context in this manner, we must ensure that these problems are accessible to students and have the potential to promote connections to deeper or more formal mathematics (Jackson et al. 2013; Stein et al. 2000). There is thus a need for research to identify what characteristics make contextual tasks accessible to students as a point of entry and useful for educators in analyzing and pressing students' thinking.

We have selected to investigate contexts within the domain of proportional reasoning due to its influence on students' future success in mathematics and science classes (Heller et al. 1989; Johnson 2015; Lesh et al. 1988; Ramful and Narod 2014), careers (e.g., Hoyles et al. 2001), and life in general (e.g., Capon and Kuhn 1979). The purpose of our work is to further investigate numerical task characteristics that could influence students' strategies and their ability to access initial proportional reasoning situations.

Proportional reasoning is a complex topic with a multitude of relationships and understandings that students must acquire in order to meaningfully utilize ratios across various mathematics and science situations (Heller et al. 1989). Given its multifaceted nature, it is not surprising that many students do not truly develop proportional reasoning or struggle to fluently apply this reasoning to other topics during their school experiences (e.g., Brahmia et al. 2016; Cohen et al. 1999; Gabel 1984). In addition to complexity as potential cause of student difficulties, previous research has demonstrated that proportional reasoning instruction and curricular materials have tended to focus on procedural knowledge and lack depth in terms of developing students' understanding of important multiplicative relationships (Dole and Shield 2008; Heller et al. 1989). Fortunately, due to its importance in students' future success, proportional reasoning is also an area where extensive research has been conducted related to understanding students' thinking and development of key ideas (see Lamon 2007 for a summary). In particular, there is research around the characteristics of contextual proportional reasoning tasks that influence their difficulty (Fernández et al. 2011; Karplus et al. 1983b; Lamon 1993; Tourniaire and Pulos 1985). We perceive this subject as a rich domain in which to investigate characteristics that influence the accessibility of contextual problems.

Theoretical framework

Our focus on using students' thinking as the basis for formal mathematics instruction is rooted in progressive formalization, an aspect of the Realistic Mathematics Education philosophy (Freudenthal 1973, 1991; Treffers 1987). In progressive formalization, students initially apply their existing mathematical knowledge and intuition to solve a problem or to mathematize the situation (Freudenthal 1991). Students continue to solve problems by refining and formalizing their understanding under the guidance of their teacher. Through this process they reinvent progressively more formal mathematical ideas and connect them to established conventions.

Related to progressive formalization, hypothetical learning trajectories (HLT) (Simon 1995; Simon and Tzur 2004) articulate the goal(s) for instruction, ideas about how students develop understanding of the topic, and tasks designed to foster students' development of the articulated goal for instruction. HLTs provide a structure for reasoning about progressively formalizing students' understanding. Lastly, but perhaps most relevant to our present work, the construct of key developmental understandings (KDU) can be used to assist in identifying the important goals for instruction articulated in a HLT (Simon 2006). Articulation of a KDU provides an overarching target to which we can relate our research findings and can then be used to inform the hypothetical learning trajectory. We perceive students' fluent and flexible use of the scalar and functional relationships within proportional reasoning situations as a KDU that should be a point of focus from the very beginning of formal proportional reasoning instruction (Lamon 2007; Lobato et al. 2010; Simon and Placa 2012). Below, we further articulate the terms *scalar* and *functional* relationships and discuss how students would demonstrate evidence of this KDU.

Scalar and functional proportional relationships

Proportional situations are those involving an equivalent relationship between ratios, such that $\frac{a}{b} = \frac{c}{d}$.. Because of this definition, two different multiplicative relationships can be seen within any proportion. Imagine the situation "Callie bought 6 cookies for \$3. How many cookies can Callie buy for \$12?" as represented by the proportion in Fig. 1. One can solve this problem by scaling up both elements of the original ratio by a factor of 4 to find 24 cookies for \$12. We will refer to this as the *scalar relationship* because we are scaling up both quantities in the ratio by a scale factor to create a new equivalent ratio. Alternatively, one might recognize that the number of cookies is always two times the number of dollars spent (or each cookie is 50 cents) to determine that the number of cookies should be 2 × \$12 or \$24. We refer to this as the *functional relationship* because one quantity (cookies) is defined in terms of the other (dollars) multiplicatively.

Proportion-based problems involving ratios and rates[1] can be solved using both scalar and functional relationships. However, the number relationships therein may favor use of one relationship over the other. In Fig. 1, a whole number multiplier can be used with both the scalar and functional relationships (×4 and ×2, respectively). But, if the number relationships changed to "*Callie bought 6 cookies for \$3. How many cookies can Callie buy for \$8*?" $\left(\text{i.e.,} \frac{\$3}{6} = \frac{\$8}{\ }\right)$, the scalar relationship × $2\frac{2}{3}$ may become more difficult to utilize due to the lack of a integer multiplier. However, the functional relationship—the number of cookies is two times the dollars—is still relatively easy.

On the other hand, it may be the relationship between the *units* (i.e., \$ to \$ or cookies to cookies for scalar) is a more relevant factor in terms of accessibility. For example, the \$3 to \$8 relationship may be more accessible for students because the units on the quantities are the same. A question of interest to the research community, curriculum designers, and classroom teachers would be, what influence does manipulating the location of an integer multiplicative relationship in favor of either a scalar or functional perspective, have on item accessibility and student strategies?

[1] We use Lobato et al. (2010) definition of rate as a "…set of infinitely many equivalent ratios (p.13)".

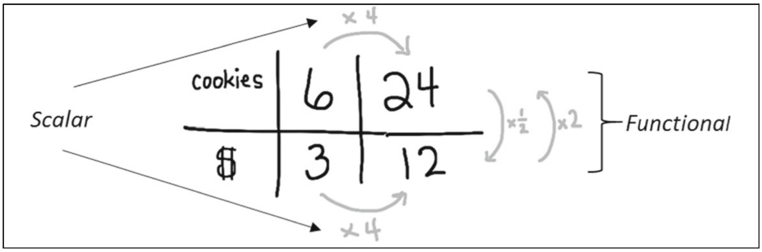


Fig. 1 Scalar and functional solution paths for "Callie bought 3 cookies for $6. How many cookies can Callie buy for $24?"

Task characteristics

We know from research, such as Cognitively Guided Instruction (CGI), that the structure of contextual tasks can influence students' thinking and strategies (Carpenter et al. 2004). There have been multiple investigations of the influence of proportion task characteristics on students' strategies (e.g., Karplus et al. 1983a) and ability to solve problems (e.g., Fernández et al. 2011). The major areas of investigation related to task characteristics that influence students' proportional reasoning are as follows: number relationships, familiarity with contextual situation, units of measure, and item type (e.g., missing value or comparison problems). Our focus is on identifying numerical task characteristics that influence the accessibility of initial, informal proportional reasoning tasks and therefore need to isolate the variables manipulated. We thus chose to focus on number relationships as the primary variable of interest and held the other three areas constant by utilizing: (1) a consistent, familiar context (food items: dollars) (Ben-Chaim et al. 1998; Heller et al. 1989; Saunders and Jesunathadas 1988); (2) discrete, visually distinct units of measure (Behr et al. 1992; Lawton 1993; Tourniaire and Pulos 1985); and (3) a missing value format (Ahl et al. 1992; Modestou and Gagatsis 2010; Tourniaire and Pulos 1985). The choice to hold these particular characteristics constant was based on research (cited above) indicating these selections would decrease item complexity and therefore increase students' access to the items (i.e., they were intended to make the item as easy as possible so the focus could be on the outcome of the manipulation of the number relationships). We provide further description of the research related to number relationships below.

Number relationships The influence of number relationships on students' proportional reasoning tends to refer to two different but related aspects of the scalar and functional relationships. One aspect focuses on whether the scalar factor or multiplicative comparison relationship within the ratio is an integer or noninteger. The evidence from the literature related to this aspect indicates that integer multipliers (e.g., ×4) are more accessible than noninteger multipliers (e.g., ×2.25) (e.g., Fernández et al. 2011; Schwartz and Moore 1998; Tourniaire and Pulos 1985). The second aspect focuses on *location*, referring to whether the number relationships are designed to press for a focus on the scalar or functional relationship by intentionally making one an integer and the other a noninteger relationship. When examining the influence of the location of the integer multiplier, the focus can be on item accessibility and/or student strategies. Item accessibility refers to whether the location of the multiplier—to press a particular relationship—can make the item easier or harder for students to solve. Student strategies have multiple interpretations in the research literature, but the perspective of our current investigation is focused on whether the location of the integer multiplier encourages students to make use of that particular relationship over the other. We first examine the research related to item accessibility, followed by the research related to students' strategies.

Several researchers have stated that students have more difficulty with the functional relationship (e.g., Lamon 1993; Simon and Placa 2012; Steinthorsdottir and Sriraman 2009). Steinthorsdottir and Sriraman (2009) examined a developmental trajectory for proportional reasoning through a 12-week investigation with girls in grade 5 (aged 10–11) and placed flexible use of the functional relationship at the final stage. Similarly, Lamon (1993) in clinical interviews with grade 6 students found that strategies involving scalar relations were more readily accessible to students than those involving the functional relationships. Tjoe and de la Torre (2014) found that grade 8 students with low mathematical proficiency performed significantly worse on an item pressing the functional (external) relationship than a similar item pressing the scalar (internal) relationship. These studies potentially indicate an increase in difficulty for items that press the functional relationship. However, it may be necessary to differentiate between students' ease of solving problems and their ability to conceptualize the meaning of the proportional relationships in these same problems. In other words, is it harder to solve a problem that presses for use of the functional relationship versus the scalar relationship or is the difficulty in conceptually understanding the constant multiplicative relationship? Simon and Placa (2012) describes the importance of differentiating between students ability to use unit rate (per-one) reasoning to solve problems and functional reasoning which focuses on understanding multiplicative relationship between co-varying quantities, "…, we do not assume that the per-one notion of intensive quantities brings with it other important ideas, such as the invariant multiplicative relationship between co-varying quantities (p. 39)." In interviews with grade six students, Carney and Crawford (2016) found the majority of students did not conceive of the constant multiplicative relationship when solving problems designed to press functional understanding. Therefore, it is worth examining the accessibility of items that press for the scalar versus functional relationship in a manner that isolates this variable from students' conception of the relationship.

Related to the question of a potential difference in item accessibility for problems that press the scalar or functional relationship, is whether those problems encourage students to focus on and make use of that relationship. Karplus et al. (1983a) examined the ways grade 6 and 8 students made use of the scalar (termed between) and functional (termed within) relationships in solving comparison proportional reasoning problems that pressed the scalar, functional, or both relationships through manipulation of the location of an integer multiplier. In situations where only one integer multiplier existed, the presence of the integer multiplier appeared to encourage students to make use of that particular relationship. In other words, students did not tend to use a particular relationship consistently but instead used whichever relationship allowed for the use of the integer multiplier. This is consistent with the research related to the influence of an integer multiplier. Therefore, providing an integer multiplier for one relationship (e.g., functional) and noninteger for the other relationship (e.g., scalar) may influence students to make use of that relationship. However, this is in contrast to the general assumption in the literature that the functional relationship is harder than the scalar relationship. In addition, there has been little research specifically focused on investigating if students tend to consistently make use of one relationship over another when solving missing value problems or if their strategy shifts based on the location of the integer multiplier. Based on this review, a question of interest is what influence does manipulating the location of an integer multiplicative relationship to press either a scalar or functional perspective have on item accessibility and student strategies?

We developed models to investigate the influence of manipulating the location of the integer multiplier—to press either the scalar or functional relationship—on item accessibility and student strategies. These models are presented in Fig. 2. The two models related to *item accessibility* are focused on determining whether items designed to press the scalar relationship are more accessible than items designed to press the functional relationship (IA Model 1) or if they have similar levels of accessibility (IA Model 2). The two models related to *student strategies* are focused on determining whether students tend to use particular solution strategies with particular item types (SS Model 1) or if students tend to use one solution strategy consistently across the two item types (SS Model 2).

The student strategy models are focused on whether the first step in a students' solution strategy makes use of the scalar or functional relationship. They do not include how students conceive of these relationships, i.e., from a composed unit or multiplicative comparison perspective (Lobato et al. 2010). While we see students' conceptions of these relationships as a very important area of study related to our identified key developmental understanding, we also find it valuable to parse students' use of the scalar and functional relationships in their mathematical processes from students' conception of these relationships.

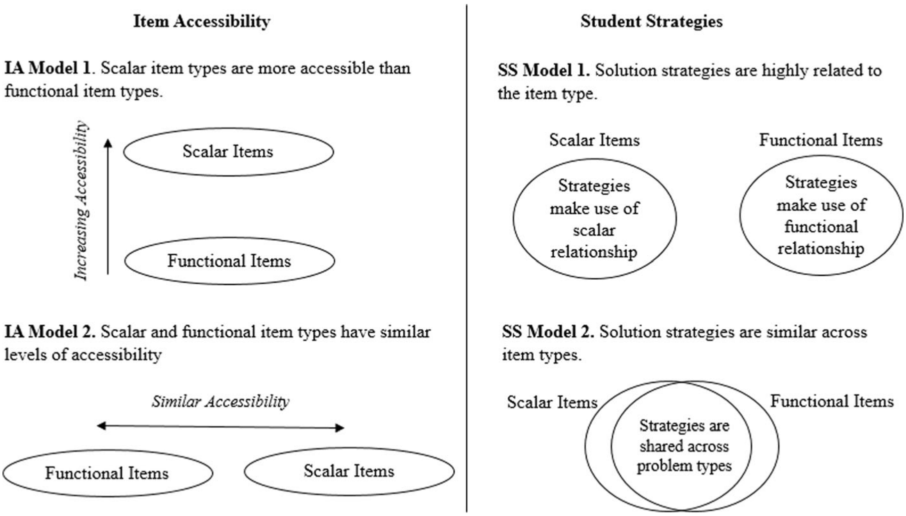


Fig. 2 Models for investigating item accessibility and student solution strategies

The next section describes the assessment framework we created to empirically examine these models, followed by the rationale for using Rasch analysis to examine item accessibility through item difficulty measure scores.

Assessment framework

Based on our focus on students' initial proportional reasoning and the above literature around item difficulty and student strategies, we used a single familiar context (food items: dollars) involving visually distinct units of measure with a missing value format. This avoids the conflation of multiple task characteristics influencing item accessibility experienced in other research (as described in Karplus et al. 1983b). We manipulated the location of the whole number multiplier in order to press for use of either the scalar or functional relationship so we could examine influence of these attributes on item accessibility and explore their impact on students' use of particular relationships to solve problems. We investigated a specific aspect of the domain of proportional reasoning with the intent of better understanding how initial proportional reasoning may develop in order to inform the creation of tasks and activities for an HLT.

The operationalization of our assessment framework is presented in Table 1. The manipulation of the whole number multiplier to press either scalar or functional understanding is presented along the left-hand side of the table. We differentiated between items that involved application of scalar or functional understanding in situations where the missing value involved generating an equivalent ratio larger than the original ratio or smaller than the original ratio. Along the top of the table the manipulation of the magnitude of the multiplier is represented (i.e., 2 with picture, 4, 3,

Table 1 Assessment framework for systematic manipulation of the magnitude and location of the integer multiplier

Number Relationships			Integral Multiplier: 2 with pic	4	3	7	8
Scalar		x	Cookies 5 _ $ 4 8	Cookies 5 20 $ 2 ?	Brownies 3 ? $ 7 21	Cookies 5 35 $ 6 _	Brownies 2 16 $ 5 ?
Scalar		÷	Cookies _ 10 $ 6 12	Cookies _ 12 $ 2 8	Brownies ? 12 $ 5 15		Brownies ? 24 $ 5 40
Functional	Increasing	x	Cookies 4 6 $ 8 _	Burger 4 6 $ 16 _	Brownies 15 ? $ 5 7	Burgers 4 5 $ 28 ?	Burgers 2 5 $ 16 ?
Functional	Increasing	÷	Cookies 4 _ $ 8 10	Burgers 4 _ $ 16 20	Brownies 3 ? $ 9 12	Burgers 6 ? $ 42 63	Burgers 4 ? $ 32 40
Functional	Decreasing	x	Cookies 6 8 $ _ 16	Burgers 5 8 $? 32	Brownies 5 6 $ _ 18	Burgers 3 4 $? 28	Burgers 3 4 $? 32
Functional	Decreasing	÷	Cookies 6 16 $ _ 8	Burgers _ 4 $ 12 16	Brownies ? 7 $ 12 21	Burgers ? 8 $ 35 56	Cookies _ 8 $ 24 64

7, 8). Multiple assessment forms were created from the assessment framework. Form development is further described in the "Methods" section.

Rasch analysis

Researchers (e.g., Andrich et al. 1997; Callingham and Bond 2006; Long et al. 2011) have argued for the use of Rasch methodology in mathematics education due to its usefulness in examining test performance in relationship to a cognitive model (Bond and Fox 2013). Most often, assessments created to fit the Rasch model consist of items designed to assess a single (unidimensional) theoretical construct (Wilson 2004) although multidimensional Rasch models are available. The estimates of student ability and item difficulty obtained from a Rasch analysis situate test takers' understanding and item difficulty along a common equal interval scale when the data adhere to Rasch model requirements (Bond and Fox 2013). As a result, student ability and item difficulty can be interpreted in relation to one another through probabilistic language.

The simplified version of the dichotomous Rasch model is

$$L = \ln\left(\frac{P}{1-P}\right) = B_n - D_i$$

where L is the natural logarithm of the ratio of the probability of success (P) to the probability of failure. B_n is a student's ability, and D_i is an item's difficulty. The equation states that the log-likelihood for a student to answer an item correctly is a function of the difference between the item difficulty and the student ability. The greater the positive difference ($B - D$), the more likely a student is to respond correctly to an item. The greater the negative difference, the more likely a student is to respond incorrectly to an item. In situations involving dichotomous scoring (0 = incorrect, 1 = correct), a student ability that is equal to the item difficulty indicates a 50 % probability that the individual would respond correctly to that item.

The results from applying Rasch models lend themselves towards use as an investigatory tool for student cognition (Callingham and Bond 2006; Long et al. 2011). For example, examination of the hierarchical relationship among item types on a common interval scale lends itself to validation efforts (Wolfe and Smith 2006a, b) with respect to a priori cognitive models and the empirical item hierarchy. For example, we wanted to determine whether items pressing for use of the scalar relationship would be easier than items with the same multiplier magnitude pressing for the functional relationship. Comparison across item types and examination of patterns in the Rasch item difficulty scores will allow us to make that comparison. In addition, as mentioned previously, when data meet Rasch model requirements, the model transforms ordinal observations into an equal interval scale, meaning differences in items are represented as an interval relationship versus the traditional ordinal ranking resulting from totaling scores or calculating a percent correct (Merbitz et al. 1989; Wright and Linacre 1989).

Previous research involving proportional reasoning assessments has often used a total score or percent correct to examine the relationship between task characteristics and accessibility (e.g., Boyer et al. 2008; Fernández et al. 2011; Van Dooren et al.

2005). However, total scores and percentages present a potential shortcoming in that equal differences between different sets of data points do not represent equal amounts of the construct under investigation due to the ordinal nature of the data (Wright and Linacre 1989). As such, we opted to use Rasch methodology over the potentially more easily understood total score or percent correct based on its ability to transform the data into an equal internal scale if the data meet model requirements. This transformation then allows the valid application of parametric statistics that assume at least an interval scale. However, it may be important to note, for those less familiar with Rasch methodologies, that increases or decreases in item difficulty result in respective decreases and increase in percent correct (i.e., as item difficulty increases the number of students who answer that item correctly decreases).

Research questions

To investigate the development of students' fluent and flexible use of the scalar and functional relationships within proportional reasoning situations, we examined the influence of the location (i.e., pressing the scalar or functional relationship) of the integer multiplier on item accessibility and students' use of particular mathematical relationships. Our overall research question is, what influence does manipulating the location of an integer multiplier to press either the scalar or functional relationship, have on item accessibility and students' strategies? More specifically, we sought to address the following two questions:

1. Are items designed to press the scalar relationship more accessible than items designed to press the functional relationship (IA Model 1) or do they have similar levels of accessibility (IA Model 2)?
2. Do students tend to use the mathematical relationship associated with an integer multiplier (SS Model 1) or do students tend to consistently use a particular mathematical relationship regardless of the location of the integer multiplier (SS Model 2)?

Methods

Our intent was to design an instrument that assessed students' informal proportional reasoning. Therefore, we wanted to assess students at the beginning of the school year, prior to formal instruction in proportional reasoning. While the assessment items were not designed through the lens of the Common Core State Standards[2]—examination of the standards indicated the assessment framework primarily addressed aspects of the content from the grade 6 standards.

[2] The Common Core State Standards have been widely adopted in the United States and provide guidance to teachers and school districts related to the mathematics content taught at each grade level.

Instrument

Four different forms of the assessment were created from the items presented in Table 1. There were a total of 12 items per form with the first six items the same across all four forms and the remaining 24 items were distributed with six items per form. The six items that were consistent across the four forms were selected to represent an anticipated range of item difficulties and different types of items with three problems each for the scalar and functional perspectives. The remaining 24 items were distributed across the forms with the intent of providing a relatively equal spread in anticipated item difficulties and types.

The items all maintained a consistent format and spacing. There were six items per page. The problems all had a blank line for students to indicate their answer and a space to show their work (see Appendix for example of format from the first page of the assessment).

Participants

We opted to use students in grades 6–8 (approximately ages 11–14) to ensure we had a broad range of abilities within the sample. Older students in our sample should have received instruction around proportional reasoning. However, review of previous state standards and contact with teachers in our study indicated that instruction was based primarily on algorithmic implementation of cross-multiplication, with little or no instruction emphasizing a scalar or functional perspectives.

The teachers of the students in our sample were participants in a 1-day proportional reasoning professional development workshop in the summer of 2014. They came from two different regions within our state, representing a mix of urban, suburban, and rural school districts.

Instrument administration

Teachers were asked to volunteer to administer the assessment as close to the start of the school year as possible (within the first 1–3 weeks) prior to any formal proportional reasoning instruction. There was no time limit for the assessment but we informed teachers we anticipated it would take students about 30 min. We requested that students not be allowed to use calculators. In the directions, we asked teachers to remind student to show or explain their thinking for each problem. Teachers then used the prepaid postage mailing envelopes to return the assessments. A total of 473 assessments were returned. Students responded to one of the four assessment forms with the following number of students for each grade: grade 6, 313; grade 7, 45; grade 8, 103; and no grade indicated, 12.

Quantitative data analysis

Initial data analysis involved application of *dichotomous scoring* (0 = incorrect, 1 = correct) using the Rasch model in the WinSteps version 3.70.0.5 (Linacre 2010). Each form of the test was first analyzed independently with a focus on examination of item fit for that form. Fit indices ranging from 0.7 to 1.3 for Infit and Outfit

MNSQ were considered acceptable (Bond and Fox 2013). Items that are not consistent with the Rasch model requirements fall outside these indices and were flagged for further qualitative investigation by one of the authors. For example, further investigation of responses to misfitting items indicated mis-scoring of the item or the presence of the correct answer but the coder missed it because it was not placed on the answer line provided. Once these abnormalities in the data were corrected, the data from the four forms were combined and analyzed through concurrent calibration.

The Rasch model sets the mean of the item difficulties to zero (SD = 1.14) (for identification purposes related to estimation of the model parameters) and the student mean, estimated in relation to the item mean, was 0.48 (SD = 2.00), indicating the sample was slightly more able than the items were difficult. While the student separation reliability of 0.72 (analogus to KR20 in classical test theory—see Smith (2001)) was not as good as the item separation reliability of 0.95 (on a scale of 0–1), the intent of this aspect of our research is to better understand item characteristics. Our high item separation reliability statistic indicates a spread in item difficulties on the logit scale and supports comparisons between item scores (Wolfe and Smith 2006a, b).

Student solution strategies analysis

Student strategies for items 2–6 (see Table 2) were coded by solution strategy. These problems were selected because they were administered to all students in the sample and represented a range of item difficulties and number relationship

Table 2 Item description, difficulty, context, and number relationships for the five items selected for strategy analysis

Relationship	Item #	Difficulty(SE)	Problem Context	Number Relationship Structure
Scalar	2	-.80(.13)	Hillary found a cookie deal with 5 chocolate chip cookies for $2. How much will it cost to buy 20 chocolate chip cookies?	Cookies | 5 | 20 $ | 2 | ?
	3	2.22(.14)	Angie found a brownie deal with 24 brownies for $40. How many brownies can she get with $5?	Brownies | ? | 24 $ | 5 | 40
Functional	4	-.85(.13)	Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12?	Brownies | 3 | ? $ | 9 | 12
	5	.09(.12)	Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost?	Burgers | 4 | 5 $ | 28 | ?
	6	.65(.12)	Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers?	Burgers | 5 | 8 $ | ? | 32

structures to allow for investigation of students' solution strategies on scalar and functional item types.

We analyzed students' correct solution strategies for these five problems. Our coding involved identifying whether students' first step in their solution strategy made use of the scalar or functional relationship. Demonstrating evidence of the use of the scalar relationship involved (a) iterating or partitioning the initial ratio—typically through doubling or halving—to determine the quantity of the missing value, or (b) determining the scale factor that scales the initial ratio to the quantity of the missing component. Either method involved calculations among quantities with the same units. Demonstrating evidence of use of the functional relationship involved identification of the multiplier between quantities with different units, typically by dividing (or multiplying) one component of the initial ratio by the other. This was followed by either iterating the resulting unit ratio to generate the unknown value or applying the functional relationship in a single step to generate the unknown value. Evidence for the 'other' category involved use of (a) cross-multiplication, (b) providing the correct answer with no associated work, or (c) situations where the initial solution strategy was indeterminate. The correct answer with no associated work was the predominant code within this category. Table 3 provides the coding rubric with multiple exemplar strategies for items 2 and 6. As evidenced by the multiple examples provided in Table 3, there were different paths that followed students' initial first step in their solution. These paths were primarily additive or multiplicative in nature. For the purpose of answering our research question related to students' strategies, further breakdown of the students' solution strategies was not necessary. However, our future work will further examine the hierarchy among these strategies.

Results

In this section, we describe and interpret the results of our investigation into item accessibility and student strategy use as related to the two item types; scalar and functional. We first examine item accessibility through the Rasch item difficulty scores. We then examine student strategy use through the distributions of the frequency of their use by item type.

Scalar vs. functional item difficulty

To examine potential differences in item accessibility between scalar and functional item types, we first present the item difficulty measures across all the forms within the perspective of the assessment framework (see Table 4). Beyond the increasing difficulty measures for the first row of the scalar items, we could discern no specific pattern at the item level related to: size of multiplier, whether the missing quantity involved an increasing or decreasing ratio, or item type.

Our research question focused on examining potential differences in item accessibility by item type. Figure 3 presents box-plots of the item difficulty measures by item type. The box-plots demonstrate the variance in the functional items was less than the variance in the scalar items but do not seem to indicate a difference in item

Table 3 Coding rubric for students' first step in their solution strategy

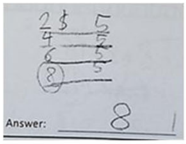
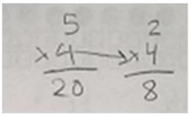
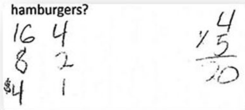
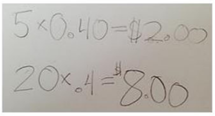
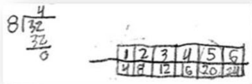
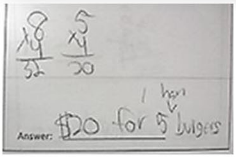
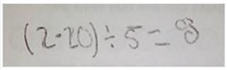
Initial Focus	Strategy Description	Exemplar Solution Strategy by Problem Type	
		Scalar Cookies | 5 | 20 $ | 2 | ?	Functional Burgers | 5 | 8 $ | ? | 32
Scalar	- Iterates or partitions the initial ratio – typically through doubling or halving - to determine the quantity of the missing value. - Determine scale factor that scales initial ratio to the quantity of the missing component.	 	 8 ÷ 5 = 1.6 32 ÷ 1.6 = 20
Functional	Identifies the multiplier between quantities with different units, typically this involved dividing (or multiplying) one component of the initial ratio by the other - Then iterates resulting unit ratio to generate the unknown value, or - Applies the result in a single step to generate the unknown value.	5 ÷ 2 = 2.5 $\frac{2.5}{\$1} = \frac{5}{\$2} = \frac{10}{\$4} = \frac{20}{\$8}$ 	 
Other	Makes use of the cross-multiplication procedure, provided no work, or the initial solution approach was indeterminate.		32*5 = 8 * x

Table 4 Item difficulty measures (and standard errors) presented within the original assessment framework

Number relationships	Difficulty (SE) by integral multiplier						
			2 with pic	4	3	7	8
Scalar		×	−2.46 (0.17)	−0.80 (0.13)	0.27 (0.25)	0.68 (0.27)	0.84 (0.24)
		÷	−1.04 (0.29)	2.44 (0.27)	2.29 (0.24)		2.22 (0.14)
Functional	Increasing	×	−1.36 (0.30)	−0.27 (0.24)	1.13 (0.26)	0.09 (0.12)	−0.41 (0.25)
		÷	−0.61 (0.24)	−0.82 (0.29)	−0.85 (0.13)	0.07 (0.25)	0.56 (0.26)
	Decreasing	×	−1.29 (0.26)	0.65 (0.12)	−0.65 (0.29)	−0.44 (0.27)	0.26 (0.27)
		÷	0.58 (24)	−1.89 (0.34)	0.27 (0.28)	0.20 (0.25)	0.33 (0.25)

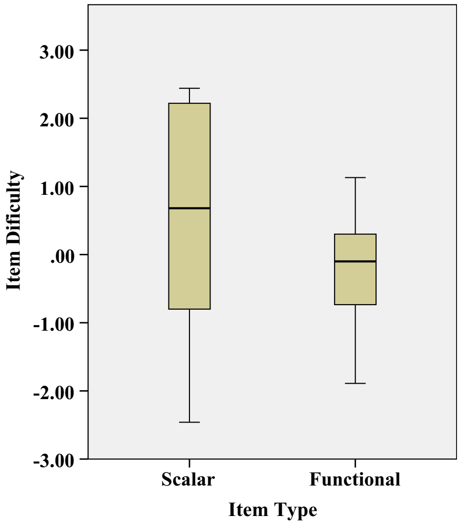


Fig. 3 Box-plot of item difficulty measures by problem type

difficulties. To confirm the visual examination of the data, an independent-samples *t* test was conducted to determine whether the scalar and functional item type item difficulty measures were significantly different. There was no significant difference in the scores for scalar (M = 0.49, SD = 1.69) and functional (M = −0.22, SD = 0.77) item types; $t(10) = 1.21$, $p = 0.26$. Levene's test indicated unequal variances ($F = 7.99$, $p = 0.01$), so degrees of freedom were adjusted from 27 to 10. These results suggest there is no difference in difficulty between missing value items with single digit multipliers that press for the scalar versus the functional relationship. There is also some indication that the scalar items had more variance in their item difficulties when compared to the functional items.

Analysis of student strategies

To examine potential differences in strategy between scalar and functional item types, we examined students' initial solution strategy. We selected the two scalar and three functional item types that all students in the sample solved ($n = 475$). The selected items and coding rubric were previously provided in Tables 2 and 3, respectively. Table 5 and Fig. 4 provide the frequency and percent of each solution strategy by item type for items 2–6, respectively.

The percentage of students who used the scalar or functional relationship on each item clearly indicates that students' first step in their solution strategy was strongly influenced by item type. On scalar items students preferred to use the scalar relationship as the first step in their solution process and on functional items students preferred to use the functional relationship as the first step in their solution process. These results provide strong evidence that location of the whole number multiplier (to press for either a scalar or functional strategy) does drive students' solution strategies for our particular context.

Table 5 Frequency of students who made use of a scalar, functional, or "other" approach as the first step in their solution strategy

Item	Correct			Incorrect or missing
	Scalar	Functional	Other	
2	243	3	64	164
3	89	1	36	348
4	8	194	40	232
5	6	192	110	166
6	6	159	46	263

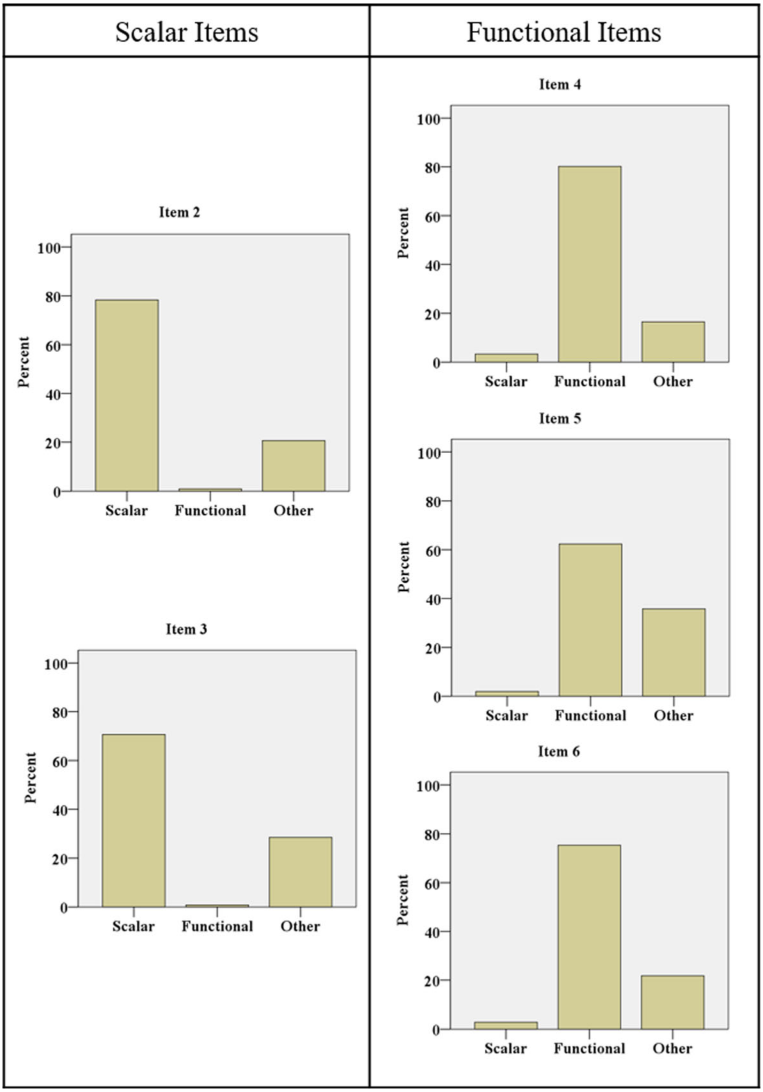


Fig. 4 Histogram of percentage of students who made use of a scalar, functional, or "other" approach as their first step in their solution strategy

Discussion

The focus of this research was to investigate the influence of manipulating the location of an integer multiplicative relationship to press either a scalar or functional perspective on item accessibility and student strategies with the primary purpose of informing initial proportional reasoning instruction. Our process involved developing and testing models for item accessibility and student strategies.

Item accessibility

The two models related to item accessibility (see Fig. 2) centered on determining whether items designed to press the scalar relationship are more accessible than items designed to press the functional relationship (IA Model 1) or if they have similar levels of accessibility (IA Model 2). Our results indicate they were equally accessible in terms of item difficulty, providing support for IA model 2 for our particular proportional reasoning context. These results indicate it is not harder to solve problems that press for the functional relationship. We do not see these results in contradiction to the research by Steinthorsdottir and Sriraman (2009) and Lamon (1993) regarding increased difficulty around the functional relationship. Instead it is likely, as indicted by Simon and Placa (2012), that the increased difficulty is related to conceptually understanding the constant multiplicative relationship as opposed to solving the problem from a procedural perspective.

Student strategies

The two models related to student strategies (see Fig. 2) centered on determining whether particular solution strategies are associated with particular item types (SS Model 1) or if the type of solution strategies used are consistent across the two item types (SS Model 2). In particular, we wanted to know if manipulating the location of the integer multiplier could be used to encourage students to focus on either the scalar or functional relationship. Our results indicate students' first step in their solution strategy was strongly influenced by item type, thus supporting SS Model 1 for our particular proportional reasoning context. This is consistent with the research on the influence of an integer multiplier (Fernández et al. 2011; Schwartz and Moore 1998; Tourniaire and Pulos 1985). When considered in relation to the findings on item accessibility, these results indicate that while the items are roughly equivalent in difficulty, pressing students to make use of a particular relationship by manipulating the location of the integer multiplier does encourage them to make use of that particular relationship.

Implications

Two potential instructional and curricular implications result from these findings. First, they indicate development of curricular materials that intentionally manipulate the location of an integer multiplier will encourage students to focus on the different mathematical relationships that exist in proportional situations, while maintaining a similar level of accessibility. Second, while this needs further investigation, it appears the difficulty around the functional relationship may be related to understanding the relationship instead of procedural ability to solve the problems. Therefore, it is likely students lack of understanding of the constant multiplicative relationship could be masked by procedural competence with these types of problems. It is important that teachers are aware of this and provided the knowledge and curricular materials to assess students' understanding of the meaning of functional relationship, separate from their ability to make use of that relationship.

While our present research does not focus on students' conceptual understanding, this is the next step in our research around students' initial proportional reasoning. In particular, how students conceive of the scalar and functional relationships—from a composed unit or multiplicative comparison perspective (Lobato et al. 2010)—is an important extension to the present work. In the meantime, the current results support the notion of developing materials that intentionally press both relationships from the start of proportional reasoning instruction, as called for by others (e.g., Schwartz and Moore 1998; Simon and Placa 2012). These types of materials, in conjunction with classroom discussion around the different solution strategies related to the scalar and functional relationships, could assist students in developing strong arithmetic and conceptual understanding of the two relationships.

Limitations

There are factors that may have impacted our findings, such as the use of a discrete, easy to visualize context and missing value problem types. It is possible these factors influence the level of accessibility and/or students' strategies. Future research could focus on intentionally manipulating the contextual situation to determine if particular contexts are useful for encouraging students to focus on either the scalar or functional relationship.

Conclusions

Our initial focus was on using contextual problems to tap into students' intuitive mathematical ideas with the goal of progressively formalizing understanding over time. Simon's HLT's (1995; 2004), and more specifically KDU's (2006), articulate a framework for enacting progressive formalization. However, it requires domain specific articulation of "…hypotheses about the process of students' learning (Simon and Tzur 2004, p. 91)." Research, such as what we have described here, can provide the necessary details to guide the development of an HLT to assist both in curricular development and to help teachers successfully implement this type of instruction. By systematically investigating factors that influence task accessibility, we provide teachers and curriculum designers with information on students' thinking within a particular domain and key points to consider when modifying or creating tasks to scaffold students throughout instruction.

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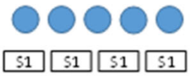
Appendix

1.) Callie bought 5 sugar cookies for $4. How many cookies can she buy with $8?  Answer: ________________	2.) Hillary found a cookie deal with 5 chocolate chip cookies for $2. How much will it cost to buy 20 chocolate chip cookies? Answer: ________________
3.) Angie found a brownie deal with 24 brownies for $40. How many brownies can she get with $5? Answer: ________________	4.) Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost? Answer: ________________
5.) Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12? Answer: ________________	6.) Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers? Answer: ________________

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