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ORIGINAL ARTICLE

Students' understanding of the function-derivative relationship when learning economic concepts

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Abstract The aim of this study is to characterise students' understanding of the function-derivative relationship when learning economic concepts. To this end, we use a fuzzy metric (Chang 1968) to identify the development of economic concept understanding that is defined by the function-derivative relationship. The results indicate that the understanding of these economic concepts is linked to students' capacity to perform conversions and treatments between the algebraic and graphic registers of the function-derivative relationship when extracting the economic meaning of concavity/convexity in graphs of functions using the second derivative.

Keywords Function-derivative relationship · Fuzzy logic · Learning economics concepts · Mathematical understanding · Mathematics-economics relationship

Introduction

Understanding economic concepts involves mastering the mathematical concepts and skills on which they are based, and several studies have reported a relationship between mathematical skills and the ability to learn economics (Arnold and Straten 2012; Ballard and Johnson 2004; Butler, Finegan and Siegfried 1998; Gery 1970). Thus, some economic concepts can be considered as the contextualisation of certain mathematical concepts and relationships, and this can obscure what economics students really need to understand (Hey 2005). In this context, the mathematical relationship between a

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function and its derivative is not only implicit in many economic concepts, but also one of the most important mathematical relationships in the study of economics, and this therefore affects students' learning (Stamatis 2014). Several economic concepts are based on this relationship, such as the demand curve (function) and the concept of price elasticity of demand (derivative), the total product (function) and marginal product (derivative), the total cost (function) and marginal cost (derivative) and the indifference curve (function) and marginal rate of substitution (derivative). The relationships amongst these economic concepts involve the notions of growth, contraction, concavity and convexity that are inherent in the relationship between a function and its derivative; thus, learning difficulties of these economic concepts may be related to limitations in understanding the relationship between a function and its derivative (Ariza and Llinares 2009). In particular, understanding the relationship between a function and its derivative is essential to make sense of the marginal analysis on which these economic concepts are based. Most of the time, these concepts have been described in the curriculum by algebraic rules that do not provide enough sense to their contextual meaning. So, some researchers have emphasised the need of reforming the calculus teaching in economic studies in a more conceptual perspective (Gamer and Gamer 2001) emphasising the relation between function and derivative which appears as a key question in the understanding of the meaning of economic concepts.

This thus raises the need to characterise the role of understanding the function-derivative relationship in learning these economic concepts. The aim of the research presented here was to contribute information on this aspect. Given this goal, we proposed the following research question:

- What are the characteristics of microeconomics students' understanding of the function-derivative relationship when learning economic concepts?

The role of mathematical concepts in learning economic concepts

To attempt to answer this research question, we adopted a cognitive approach to elucidate the role that mathematical concepts play in learning economic concepts. For Duval (1995), access to mathematical knowledge is achieved through the use of a variety of representation registers. By *representation registers,* we refer to a system of symbols employed to represent a mathematical idea or object that allows two actions: transformation within the same register, treatment and conversion, which is a total or partial transformation to another register. Duval (1995) considered *conversion* to be a crucial process in the understanding of mathematical objects which are subsequently used to model economic situations. From this perspective, the mathematical objects represented should never be confused with the system of representation, and it is thus necessary to build cognitive bridges connecting various registers.

Moreover, a key idea from a cognitive perspective of knowledge construction is the concept of the schema, understood as the way in which students relate and organise knowledge. Piaget and Garcia (1983–1989) defined a schema as a coherent set of processes, objects and other schemata which is developed in three stages: Intra, Inter and Trans. These stages are characterised by the ability of students to establish relationships between the elements which constitute ideas. The *Intra* stage is characterised by the fact that students do not recognise all the elements of the schema; they use them in isolation and find it difficult to establish the associations between them. The *Inter* stage is characterised by the students' dawning recognition of the relationships between the elements that constitute the concepts, and there is therefore more possibility of enhancing their deductive capability. The transition from one stage to another usually occurs as a result of reflection on the relationships that have been established between the different elements of the concept. In the *Trans* stage, students construct a cognitive structure which establishes meaningful relationships between the different elements of the concept. The coherence of a schema constructed by students enables them to decide how to use the concept, considering its limitations and constraints. As a means to operationalise this theoretical perspective, Dubinsky (1991) and Arnon et al. (2014) defined the genetic decomposition of a concept as a structured set of mental constructs that describes how the concept is acquired in the mind of an individual. Genetic decomposition must be understood as a hypothetical route through which a student can come to understand the concept.

In this study, we proposed a genetic decomposition of the relationship between a function and its derivative in the use of economic concepts (Table 1), based on the previous research on how students develop an

Table 1 Proposed genetic decomposition of the function-derivative relationship in learning economic concepts

Schema 0:	E1. Conversion of linear economic functions A>G
From algebraic to graphic	E2. Conversion of nonlinear economic functions A>G
	E3. Average rate of change (ARC) between two points
	E4. Estimating the limit of the ARC
Schema 1:	S0- Schema 0
Meaning and use of the 1st derivative	E5. Conversion of linear economic functions G>A
	E6. 1st derivative (linear functions): relationship between algebraic and graphic expressions of the function and its derivative
	E7. Conversion of nonlinear economic functions G>A
	E8. 1st derivative (nonlinear functions): obtaining algebraic and graphic expressions of the function and its derivative
Schema 2:	S1- Schema 1
Meaning and use of the 2nd derivative	E9. 2nd derivative (convexity): explanation of the economic concept of the convex form of a function in both registers and its relationship with the 2nd derivative
	E10. 2nd derivative (concavity): explanation of the economic concept of the concave form of a function in both registers and its relationship with the 2nd derivative
	E11. Derivative at a point>derived function: step from the derivative at a point to the derivative of a convex economic function
	E12. Derivative at a point>derived function: step from the derivative at a point to the derivative of a concave economic function

understanding of this relationship (García, Llinares and Sánchez-Matamoros 2011; Habre and Abboud 2006; Sánchez-Matamoros, García and Llinares 2013; Haciomeroglu, Aspinwall and Presmerg 2010; Zandieh 2000). In particular we considered the following:

i the meaning of the functional relationship between variables (prerequisites)
ii the idea of rate of change (variability of the relationship)
iii the meaning of the rate of change (concavity and convexity)

We assumed that these three elements are articulated through three stages: first, by calculating mean and instantaneous changes using algebraic formulas; second, by using the relationship between the first derivative and the original function in the graphic and algebraic registers; lastly, by using the relationship between the second derivative, first derivative and the original function, and the application of these relationships in different economic concepts. We broke this proposed genetic decomposition down into 12 elements organised into three nested schemata corresponding to the three stages described above (Table 1). We consider that the understanding of the relation function-derivative in economic concepts could be reached throughout a one-by-one nested schemata. The first schema indicates that the first approach to derivative in economic concepts is developed in the algebraic register by the use of tools related to calculus of variation between some economic variables. However, these calculi should be also done in the graphic register, so we have named this first schema as “From algebraic to graphic”. Their four elements represent the basic tools to calculate variations between some variables (E3 and E4) and the transition to the graphic register (E1 and E2).

Once the students are able to calculate variations in an algebraic sense and set them up in the graphic register, in the second schema, the relation between the first derivative of one function and the function has to be understood in both registers. That is the reason why the first schema is considered the first element of the second one. We denominate to schema “Meaning and use of the first derivative” and contains four elements: E6 (linear function context) and E8 (nonlinear function context) to understand the relations between first derivative and function in both registers, and E5 and E7 to coordinate the conversions of the relations between variables of the function from the graphical register to the algebraic one (on the contrary sense that in the first schema). Finally, we consider that the relation function-derivative is definitively reached by the role of second derivative: relations between second derivative, first one and function in both registers (the derivative is considered as a function on which it is possible act). Thus the first element of the third schema is the previous one (which integrates the first one). This schema has been named “Meaning and use of second derivative”. Their elements are the relations between second, first derivative and function (E9 for a convex function and E10 for a concave one) and finally the transition from the derivative at one point to the derivative function in a convexity context function (E11) and in a concavity one (E12)

This three nested schemata composed by these 12 elements has been considered for the construction of the questionnaire which is exposed in Table 2.

The idea is that every item of the questionnaire is related to every of the elements of the genetic decomposition.

On the basis of this genetic decomposition, the aim of this research was to provide information about the extent to which economics students' understanding of the function-derivative relationship determines how they solve situations in economic contexts.

Method

Participants and context

Study participants consisted of 110 students enrolled in the optional subject of "Microeconomics" offered on the Degree in Business Studies at the University of Alicante (Spain). All participants had previously studied the subjects of Mathematics and Economics I, and were thus familiar with the calculation of derivatives, integrals and partial derivatives in economic concepts.

Instruments

Data collection instruments consisted of a questionnaire comprising 5 tasks with 12 items related to economic concepts in which the derivative appeared implicitly or explicitly. Each of the questionnaire items corresponded to one of the elements of the proposed genetic decomposition (Table 2). Next, 25 students were interviewed using semi-structured clinical interviews.

Using a table of data, task 0 presented an economic situation related to the demand and supply functions and students were asked to perform a conversion between the graphic-tabular and algebraic representation modes. The first conversion activity involved linear relationships, and the second, nonlinear relationships. From the mathematical point of view, this task presented relations between variables (price-demand, and price-supply); the construction of two functions which modelled these relations is required in both registers. The aim of this task was to identify the students' ability to perform a conversion between registers (Fig. 1). Task 1 presented an economic situation using an algebraic register, with the aim of analysing students' ability to coordinate the percentage changes between two variables and the interpretation of the quotient ratio as a measure of the change when calculating percentage changes (incremental quotients). In a mathematical sense, this task demanded from students the calculus of the average rate of change between two points and in one point (Fig. 2). Task 2 presented graphic representations of linear functions related to the marginal cost and marginal product functions, which are derivative functions of the total cost and total product functions, respectively. The task involved obtaining the original functions graphically and algebraically. From the mathematical perspective, this task presented the graph of one function without any reference to its algebraic expression, and the graph of its derivative and in both are required algebraic expressions (Fig. 3). Task 3 presented graphic representations of nonlinear functions related to total cost and total

Table 2 Relationship between the 12 items in the 5 tasks and the 12 elements of the genetic decomposition

Schema 0:	E1. Conversion of linear economic functions A>G	Item 0.1	Task 0
From algebraic to graphic	E2. Conversion of nonlinear economic functions A>G	Item 0.3	
	E3. ARC between two points	Item 1.1	Task 1
	E4. Estimated limit of the ARC	Item 1.2	
Schema 1:	E5. Conversion of linear economic functions G>A	Item 0.2	Task 0
Meaning and use of the 1st derivative	E6. 1st derivative (linear functions): relationship between algebraic and graphic expressions of the function and its derivative	Item 2.1	Task 2
	E7. Conversion of nonlinear economic functions G>A	Item 0.4	Task 0
	E8. 1st derivative (nonlinear functions): obtaining the algebraic and graphic expressions of the function and its derivative	Item 2.2	Task 2
Schema 2: Meaning and use of the 2nd derivative	E9. 2nd derivative (convexity): Explanation of the economic context of the convex form of a function in both registers and its relationship with the 2nd derivative	Item 3.1	Task 3
	E10. 2nd derivative (concavity): explanation of the economic context of the concave form of a function in both registers and its relationship with the 2nd derivative	Item 3.2	
	E11. Derivative at a point>Derived function: step from the derivative at a point to the derivative of a convex economic function	Item 4.1	Task 4
	E12. Derivative at a point>Derived function: step from the derivative at a point to the derivative of a concave economic function	Item 4.2	

product. To solve the task, students had to use the concepts of the 1st and 2nd derivatives in the graphic and algebraic registers. From a mathematical point of view, this task consisted in graphing the function (as in the previous task) only in the graphical register and the graph of its first derivative functions and both algebraic expressions are required. In addition, this task demanded the algebraic expression of the second derivative and its relation with the concavity or convexity of function (Fig. 4). Task 4 presented a linear indifference curve function in the graphic register with information related to a particular point. In a mathematical sense, students were asked to calculate the derivative function from the derivative at a point, and to determine the concavity/convexity of the function using the 2nd derivative. The aim of this task was to analyse whether students related the function and its derivative in new economic concepts when starting from the derivative at a point (Fig. 5).

After completing the questionnaire, 25 of the 110 participating students were interviewed in order to obtain more detailed information about how they had solved the different tasks. Students were selected for interview on the basis of several criteria: a) their availability, b) their questionnaire answers, selecting those students whose questionnaires contained incomplete answers or conceptual errors, or who had used original solutions, and c) including a varied range of success. For each task, a prior script of questions was established based on the characteristics observed when analysing the answers given in the questionnaire, but once the interview had started, these questions were modified or widened depending on students' verbal responses. The interviews lasted for 30 min and were transcribed to facilitate analysis.

T0. A vegetable fibre is traded in a competitive world market and the world price is 9 €/kg. The EU can import unlimited quantities at this price. The following table shows the domestic supply (Qs) and domestic demand (Qd) in the EU for this product as a function of its price P.

Price P	*Qs in the EU*	*Qd in the EU*
3	2	34
6	4	28
9	6	22
12	8	16
15	10	10
18	12	4

0.1) Draw a graph of the market equilibrium.

0.2) Obtain the algebraic expressions of demand and supply, explaining how you obtained them.

0.3) and 0.4) Answer the same questions as in the previous two items (respectively) for the new table of demand shown below, maintaining supply constant ***(note that the new demand function is nonlinear)****.*

Price P	*Qd' in the EU*
3	8
6	4
9	2.666666666666
12	2
15	1.6
18	1.333333333333

Fig. 1 Questionnaire task 0

Analysis

Analysis of the answers was performed in two stages. In the first stage, the answers to the questionnaire (and responses in the interview if students had been interviewed) were scored according to the elements and relationships used to solve tasks and the explanations given during the interview. The score assigned referred to the degree of acquisition of the corresponding element in each item. We established five levels of acquisition (0, 0.25, 0.5, 0.75, 1) for each element considered in the genetic decomposition of the function-derivative relationship schema in learning economic concepts. "0" is scored to those items in which there is no answer at all. A score of 0.25 is assigned to those answers which are wrong but the student tries to answer and writes some explanation. The score of "0.5" means the minimum to consider that the student has understood the item; for example, the written answer is right but not the

T1.

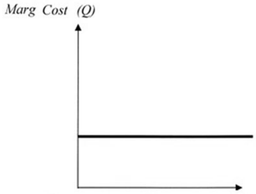
1.1) Calculate and interpret the rate of change in the quantity demanded and in the price from the initial point P=6 to the end point P=9 for the demand functions Qd=10 – P and Qd=81/P. What economic concept would you associate with this rate?

1.2) Calculate and interpret the rate of change in the quantity demanded with respect to price at point p=6 for both demand functions. What economic concept you would associate this calculation?

Fig. 2 Questionnaire task 1

T2.

2.1) For the following marginal cost function, draw and explain a graphical representation of the total cost curve, and also obtain possible algebraic expressions for both functions (consistent with the graphic form presented), explaining how you obtained them



2.2) For the following marginal product function (MPg (L)), draw and explain a graphical representation of the production function, and also obtain possible algebraic expressions for both functions (consistent with the graphic form presented), explaining how you obtained them

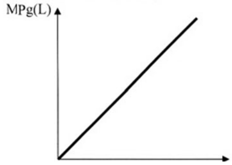
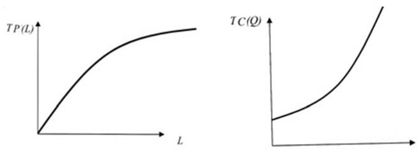


Fig. 3 Questionnaire task 2

explanation or justification; this score could also mean that the student answers rightly half an item (for example in item 0.2 in which it is required the algebraic expressions of two functions). A score of "0.75" means the answer is correct with a complete explanation but with some slight mistake in algebraic calculus or graphical representation. Finally, a score of "1" is assigned to those answers which are correct and the explanation is clear, without any type or mistake or misunderstanding. For example, for the item 2.1, a score of "0" is assigned if there is no answer, "0.25" would be scored for wrong answers like graph a constant function for the total cost, "0.5" for a linear graph of the total cost function with a constant scope and an expression like "2Q", but without

T3.

The total cost, TC(Q), and total product functions (or production function, TP(L)) that an entrepreneur faces in the short term using two units of capital are defined by the following figures:



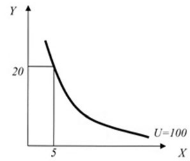
3.1) Obtain the graph and algebraic expression of marginal cost MC(Q). Determine [using the 2nd derivative, the graphs and the algebraic expressions for the total cost TC(Q) function shown and its MC(Q)] whether the TC(Q) function is concave or convex, explaining what either form would imply.

3.2) Answer the same question regarding the total product TP(L) function shown and its marginal product MP(L).

Fig. 4 Questionnaire task 3

T4.

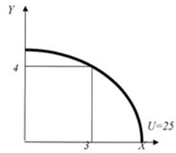
4.1)A consumer presents the following indifference curve regarding the consumer goods X and Y:



- Calculate the marginal rate of substitution (MRS) at the point shown.

- Draw a graph of the first derivative of this function (X on the x-axis and MRS on the y-axis) and explain why it is convex using the concept of the 2nd derivative (you may use algebraic expressions that correspond to the graphic forms of the functions if you think appropriate).

4.2)A consumer presents the following type of indifference curve towards the goods X and Y:



Answer the same questions as in the previous section.

Fig. 5 Questionnaire task 4

explanation of the relation between this proposed function "2Q" and its derivative which would have as algebraic expression "2" and graphically would be like the representation done by the item. We assigned "0.75" to those answers in which the graph of the total cost function is correct (like in the previous score), and in addition, there is a explanation of the relation between both functions, but some slight algebraic mistake, like writing "Q+1" as total cost expression and "2" as marginal cost expression ("2" is not the derivative of "Q+1") . Finally, We assigned a scored of 1 if the answer reflects both correct algebraic expressions (for example "3Q" for total cost function and "3" for marginal cost function) with a correct graphical representation of it and with the explanation that "3" is the first derivative of "3Q".

This procedure allowed us to assign each student a tuple of 12 values. In the second stage, we used fuzzy logic to identify different levels of acquisition of the function-derivative relationship schema in economic concepts. The fuzzy technique was selected in order to overcome the limitations involved in assigning students to different levels of schema acquisition using qualitative analysis. The concepts of fuzzy set and fuzzy topology (Chang 1968; Zadeh 1965) provide a new approach to characterise the extent of understanding. A fuzzy set is defined by assigning an interval value (0, 1) to each element of a universe of reference. This value represents the degree of belonging to that set. This notion introduces the notion of "blurriness" to the idea of belonging to a set, and can model many real phenomena in which objects do not have a defined membership criterion. In the present study, the membership function indicated the extent to which a student understood the function-derivative relationship schema in learning economic concepts, considering the means by which a set of Microeconomics problems had been solved. To obtain the membership function, we used the notion of fuzzy metric space described by George and Veeramani (1994), considering the standard fuzzy metric induced by the Euclidean metric d, of the set X, which is given by the formula

$$F_d : (x, y, t) = \frac{t}{t + d(x, y)}$$

This definition means that the fuzzy metric value depends on a contextual parameter "t", which allows consideration of the uncertainty that characterises the context of the analysis. In our study, the value of t was determined in various stages. First, we assumed that a student Q, with zero in all elements of the schema, should obtain a degree of membership score less than or equal to 0.25. This assumption is supported by the fact that all student participants had demonstrated knowledge of the necessary prerequisites for solving the problems. Second, once the degree of membership of the student Q had been established, we obtained a value of "t" for each of the fuzzy sets or schemata considered (schema 0: from algebraic to graphic, schema 1: meaning and use of the 1st derivative, and schema 2: meaning and use of the 2nd derivative) (t= 0.66). From this value of the parameter "t", each student was assigned a fuzzy score or distance in each schema, where the fuzzy distance of schema 2 (which integrated the previous two) would correspond to a given level of acquisition of the function-derivative relationship schema in economic concepts.

From the conceptual description of the INTRA level, we know that students at this level do not establish relationships between schema elements, whereas at the INTER level, they begin to establish relationships between these elements and to construct the meaning of the relationship between a function and its derivative in the graphic register. In this case, a student at the INTER level would be able to use the concept of the 1st derivative in the graphic register and coordinate it with the algebraic register, for both linear and nonlinear functions. Meanwhile, a student at the TRANS level would consider the relationship of the concept of the 2nd derivative, which would enable him or her to pass from the derivative of a function at a point to the derivative of a (concave or convex) function and explain how to use the meaning of the concavity/convexity of the economic function. From this conceptual analysis, two fuzzy scores were determined as boundary points between the intra-inter levels (F_d=0.27) and the inter-trans levels (F_d=0.36).

Results

The analysis procedure employed enabled us to assign a fuzzy score to each student, which in turn allowed us to characterise the degree of acquisition of the *function-derivative relationship in learning economic concepts* schema (Table 3). In this section, we describe the characteristics of the understanding students demonstrating of the function-derivative relationship when solving the economic situations presented in each of the three levels.

Table 3 Students at each level of acquisition of the function-derivative relationship schema in learning economic concepts

LEVEL	Number of students	Percent
INTRA: F_d<0.27	72	64.45
INTER: 0.27≤F_d <0.36	33	30.00
TRANS: 0.36≤F_d	5	4.55
TOTAL	110	100

Characteristics of the INTRA level of acquisition of the schema

A total of 72 students obtained a fuzzy score of less than 0.27, and were assigned to this level. These students were able to calculate the rate of change between two economic variables using the average rate of change (ARC) between two points and by estimating the limit of the ARC. They were also beginning to understand conversion to the graphic register to show the functional relationship between economic variables. At this level, students encountered difficulties in interpreting the relationship between the function and its derivative in the graphic register as a means to understand the measure of the rate of change given by the derivative of the function in economic concepts. Consequently, they had difficulty in converting functions from the graphic to the algebraic register. In addition, these students had difficulty understanding the meaning of the rate of change between economic variables using the relationship between the function, the first and the second derivative, in the graphic register (schema 2).

For example, in item 1.1 of task 1 (Fig. 6), student St.45 related changes in the two variables using a percentage quotient. These percentages were obtained as the difference between the values of functions at the two given points. However, to obtain the elasticity point in item 1.2, this student modified the expression used in the previous item to obtain the elasticity point using the derivative of the demand function with respect to price (ΔQd/ΔP). Thus, the student calculated the value of instantaneous change or change at a point. This approach indicates that the student calculated the rates of change between two points (element E3) and the average rates of change at a point or estimated limit (element E4) using the concept and the algebraic expression of the elasticity point.

Conversions from the algebraic or numeric register to the graphic register do not pose a problem for students at this level. For example, for items 0.1 and 0.3 (Fig. 7), student St.46 correctly indicated all the points given by the task and obtained two linear functions that intersected at the point of equilibrium.

This student also correctly represented the case of the nonlinear function demand, obtaining a different point of equilibrium. This is a characteristic of students at this level, that they are able to represent linear and nonlinear functions graphically from the algebraic register. That is, they can convert linear functions (element E1) and nonlinear ones (element E2) from algebraic or numerical registers to the graphic register. However, students at this level encounter difficulties in establishing relationships

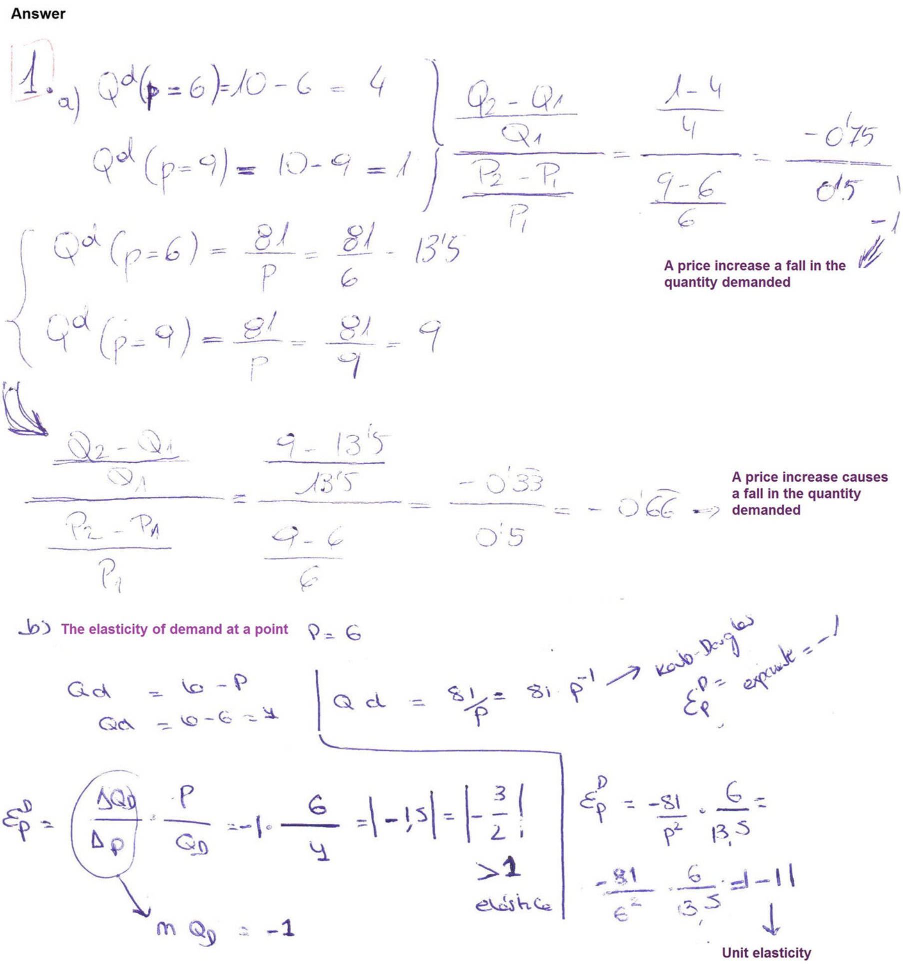


Fig. 6 Student St.45's answer to task 1

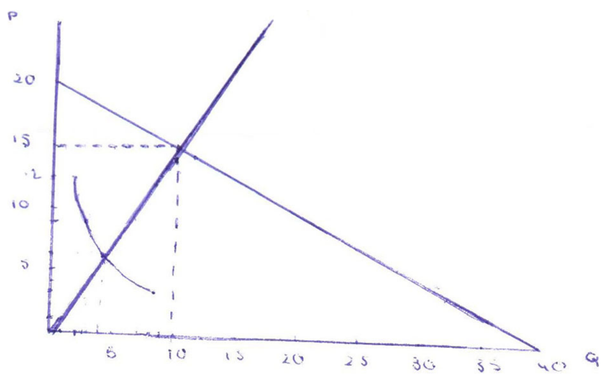


Fig. 7 Student St.46's answers to items 0.1 and 0.3 in task 0

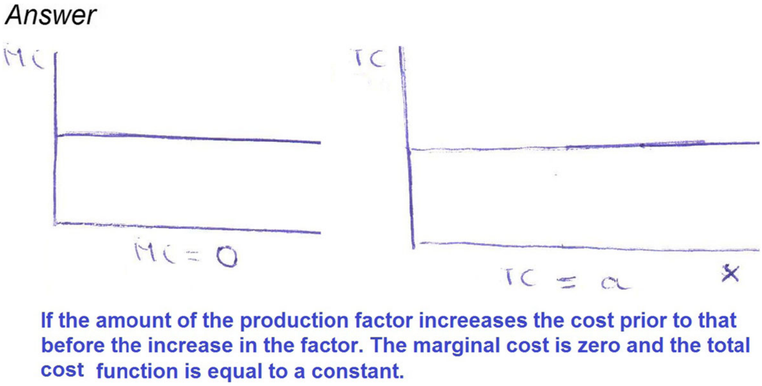


Fig. 8 Student St.16's answer to item 2.1 in task 2

between algebraic and graphic expressions of the linear function and its derivative (element E6) and between the nonlinear function and its derivative (Element E8) which characterise schema 1 of the genetic decomposition. For example, students St.16 and St.22 (Figs. 8 and 9) were unable to obtain the function from the graph of the derivative, highlighting their difficulty in understanding the economic significance of the derived function (items 2.1. and 2.2).

For item 2.1 (Fig. 8), student St.16 wrote an incorrect algebraic expression for the total cost, giving the expression CM=Q with an explanation that demonstrated a lack of understanding of the relationship between the economic functions of marginal cost and total cost. These two economic concepts are related by being a derivative and a function in the sense that marginal cost is the function that measures the rate of change of the total cost function. Student St.16 gave an incorrect graphic representation of the original function, representing it as equal to the derivative.

If the explanation and the algebraic expressions given by student St.16 are considered together, we can infer that this student implicitly knew that marginal cost (MC) is the derivative of total cost (TC), since the derivative of a constant "*a*" is effectively zero. However, this student confused the meaning of zero with that of "*constant*" in the graphic register, and therefore appeared not to understand the graphical relationship between the marginal cost and total cost functions, not even when using the algebraic register and the tool already learnt, *calculating a derivative*. This lack of understanding of the function-derivative relationship in the graphic register also occurred in the case of nonlinear functions. Figure 9 shows the answer given by student St.22, from which we can infer that this student did not understand the relationship between a function and

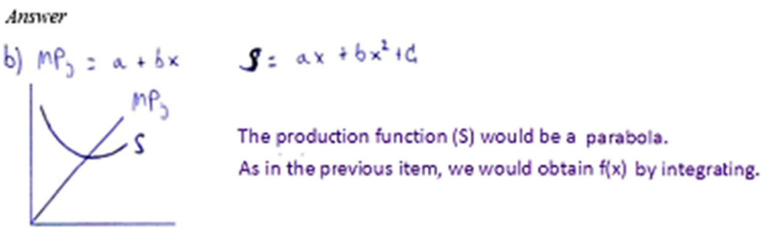


Fig. 9 Student St.22's answer to item 2.2 in task 2

its derivative graphically in the context given by the relationship between the marginal product function and the total product function when the marginal product was a slope with a positive gradient. In particular, student St.22 was unable to see that this form implies the growth of the original function. However, this student answered the question in the algebraic register correctly, establishing the relationship between the two functions to obtain the algebraic expression of the original function by integration.

For students to understand the economic meaning of the marginal cost function, they must relate the graph of the first derivative with that of the original function, coordinating both expressions in the algebraic register. That is, they must (a) *perform conversions of linear and nonlinear functions from the graphic to the algebraic registers* (elements E5 and E7) and (b) *obtain first derivatives in both registers and their relationship with the original linear and nonlinear functions* (elements E6 and E8).

Characteristics of the INTER level

At this level, 33 students obtained a fuzzy score for schema acquisition which ranged between 0.27 and 0.36. These students showed a good understanding of the meaning of change between economic variables (schema 0). Furthermore, they were able to convert graphs of linear and nonlinear functions to the algebraic register. This enabled them to relate the function and its derivative in economic concepts such as total product-marginal product and total cost-marginal cost (schema 1). Lastly, these students identified the relationship between the function and its derivative and calculated the second derivative, but were unable to generate explanations of their meaning and failed to construct these relationships for the economic concepts of indifference curve-marginal rate of substitution (schema 2).

For example, in answering item 2.1, student St.20 (Fig. 10) particularised the graph given for the function of marginal cost to the case of MC=3, and thus obtained the expression of the total cost function as TC=3Q. The relationship

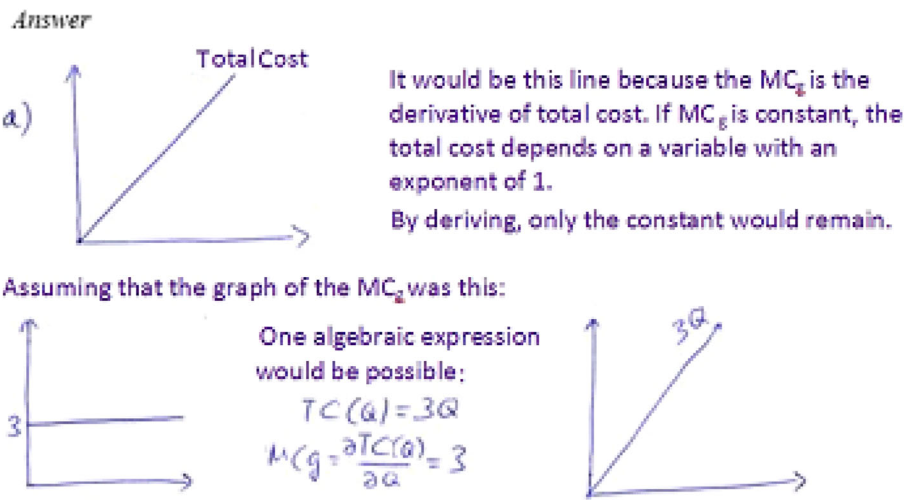


Fig. 10 Student St.20’s answer to item 2.1 in task 2

between these two algebraic expressions and the graphs were established by deriving the total cost function. This student explicitly identified the marginal cost function (MC) as the derivative of total cost (TC).

In answering item 2.2 (Fig. 11), this student used the calculation of the integral to obtain the expression of the original function, correctly represented by an upward parabola. Student St.20 understood that the marginal product (MP) and the total product (TP) are the derivative function and the function, respectively, and performed a conversion from the graphic to algebraic register by writing MP(L)=L. This algebraic expression of the marginal product (MP) enabled the student to calculate the algebraic register (treatment) through integration, and subsequently convert it again to the graphic register and represent an upward parabola graphically as the TP function.

This approach is characteristic of students at the Inter level, indicating that they understand the relationship between the first derivative and the function in the graphic register with the help of the algebraic register, for linear functions (element E6) and nonlinear ones (element E8) in the context of the pairs of economic concepts marginal cost-total cost and marginal product-production function. The answers given by St.20 to items 0.2 and 0.4 in task 0 demonstrate the ability of students at this level to perform conversions from the graphic register to the algebraic register, overcoming in this respect the limitation of Intra level students, who are only capable of performing conversions from the algebraic register to the graphic register. For example, when answering item 0.2 in task 0 (Fig. 12) with linear functions, student St.20 used the point-slope equation to obtain first the gradient of the slope and then the algebraic expression of the function from one of the points in the table of values given. In this student's answer to item 0.4 of task 0 (Fig. 13), we see that St.20 drew the graph using the values given, based on a knowledge of the generic algebraic expression for nonlinear functions using the first point given (3.8) to obtain $Q_D(P)=24/P$. This student was interviewed to confirm that he was aware that the function given in item 0.4 was nonlinear, allowing him to take the algebraic expression of the form $Q=X/P$ as the original function. This was confirmed in the interview by his response that it was necessary to employ a downward parabola since this was a nonlinear function.

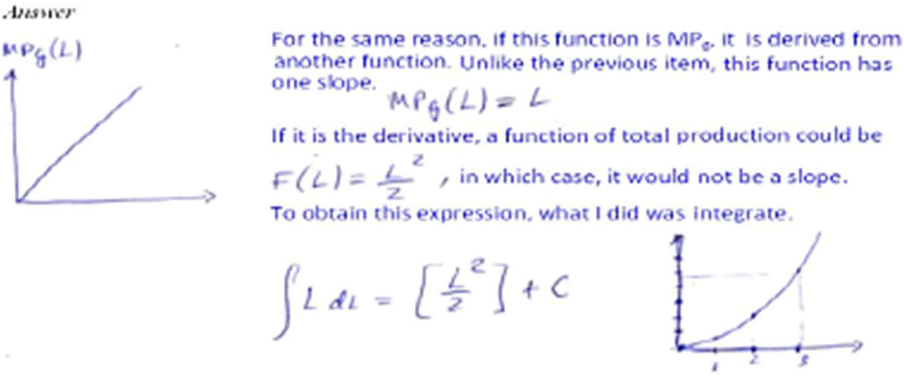


Fig. 11 Student St.20's answer to item 2.2 in task 2

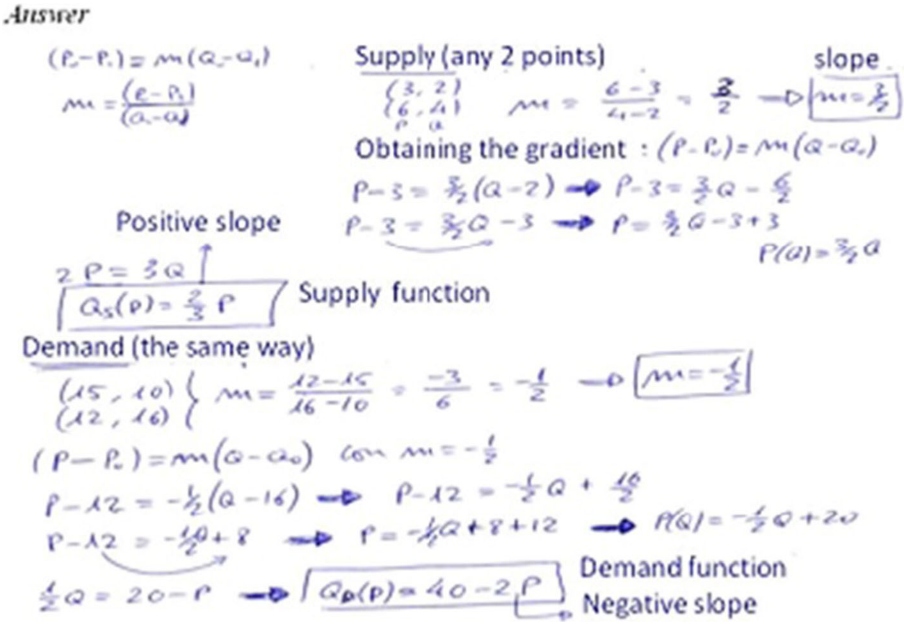


Fig. 12 Student St.20's answer to item 0.2 in task 0

Interviewer: At first, you put that Q=X/P. Why?

St.20: The new demand function is a downward parabola and its formula is a number divided by P. Then I took any point to obtain the value for X

Inter level students are able to perform conversions of linear and nonlinear functions from the graphic register to the algebraic register (elements E5 and E7, respectively). This characteristic is an important indication that at this level, students use the function-

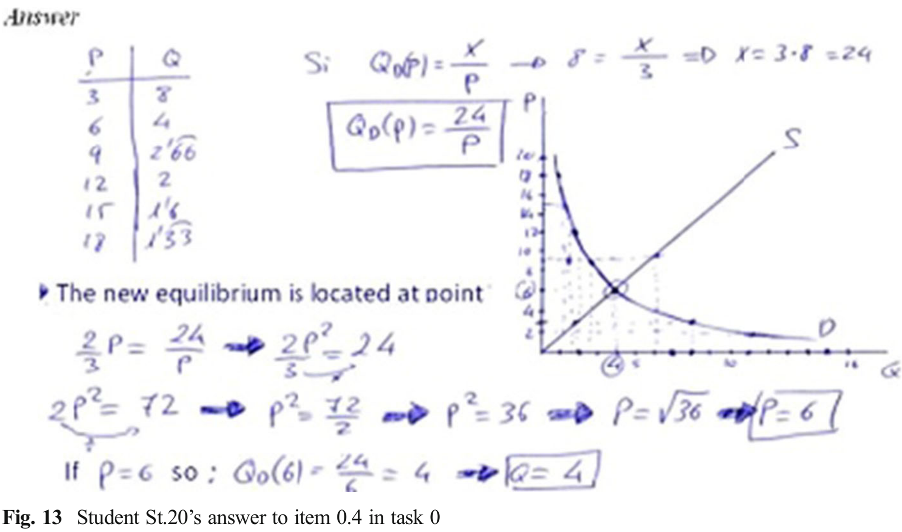


Fig. 13 Student St.20's answer to item 0.4 in task 0

derivative relationship in processes that involve treatments and conversions of functions between the two registers—from the algebraic to graphic registers (E7) and from the graphic to the algebraic registers (E8). However, these students still encounter difficulties with the meaning and use of the second derivative (schema 2).

For example, St.7's answer to item 3.1 in task 3 (Fig. 14) highlights the difficulties that the economic concepts of total cost and marginal cost pose to Inter level students when they involve the second derivative. St.7 initially converted the graphical representation of the total cost (TC) in the task into the algebraic expression TC=x^2+1. Then, he obtained the value of the derivative, MC_g=2x, and again converted it to the graphic register to represent the derivative graphically. This approach demonstrates an understanding of the function-derivative relationship in the case of the total cost-marginal cost functions, performing a sequence of conversions→treatment→conversions to obtain a graph of the marginal cost function. However, although he calculated the second derivative and mentioned convexity, he did not specify which function was convex and nor what this meant, as required by the task. In the case of the total product, TP(L), and the marginal product, MP(L), St.7's answer to item 3.2 in task 3 (Fig. 15) indicates that he understood the function-derivative relationship, but in relation to the 2nd derivative he only performed a calculation of the derivative of the derivative, as happened in the case of total cost-marginal cost.

Characteristics of the TRANS level

A fuzzy score above 0.36 marked the boundary of the TRANS level. Five students at Trans level obtained high scores for items in schema 0 and schema 1, whilst for items in schema 2, they obtained moderate scores for most items and high scores for some. These students calculated the rates of change using the expression of the price elasticity of demand. They understood the function-derivative relationship of the concepts of total product-marginal product and total cost-marginal cost, and used the function-derivative relationship for the concepts indifference curve and marginal rate of substitution. In addition, these students established the relationship between an economic function and its derivative in the algebraic and graphic registers, interpreting the meaning of the convexity/concavity of the function and using the algebraic and graphic registers to understand the relationships between functions and their derivatives.

The aspect that distinguished them from INTER level students was their understanding of the economic significance of the concavity/convexity of a

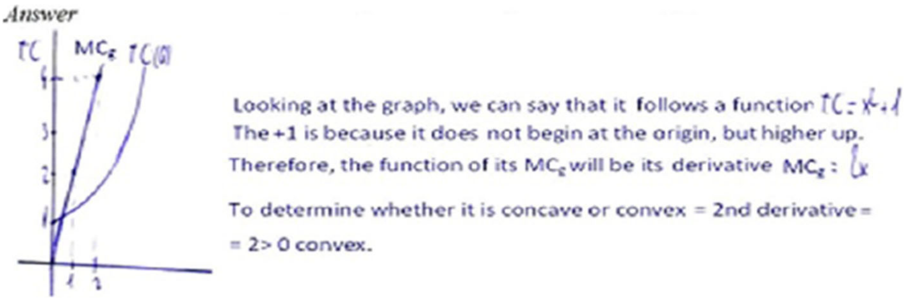


Fig. 14 Student St.7's answer to item 3.1 in task 3

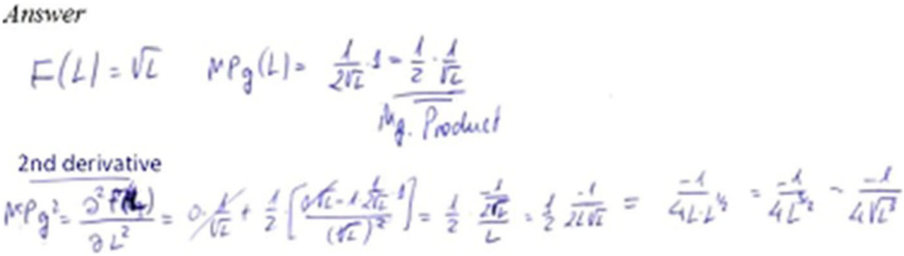


Fig. 15 Student St.7's answer to item 3.2 in task 3

function. The protocol shown in Fig. 16 corresponds to the answers given by student St.14 to item 3.2 in task 3, the goal of which was to relate the concept of convexity/concavity to the calculation of the second derivative and its economic meaning. The student identified the graph with the algebraic expression $x^{½}$, calculated the derivative and represented it graphically (downward parabola). From the algebraic expression of the derivative, he conducted a further derivation and obtained a negative value that enabled him to conclude that the function was concave. A characteristic feature of the answers given by students located at this level is their explanation of the behaviour of the function; "*as 'x' increases, the function increases less and less*", which corresponds to the meaning of the concavity of the function. This characteristic indicates that Trans level students understand the function-derivative relationship in the algebraic and graphic registers and also understand the relationship between the first and second derivatives in the algebraic register. This allows them to relate the concept of convexity/concavity with the calculation of the second derivative and its economic meaning. Thus, these students are able to explain the convex (concave) shape of an economic function in the graphic and algebraic registers and its relationship with the second derivative (elements E9 and E10).

Table 4 presents the characteristics of the levels of acquisition of the function-derivative relationship schema in economic concepts, showing the incorporation of characteristics as students move from one level to the next.

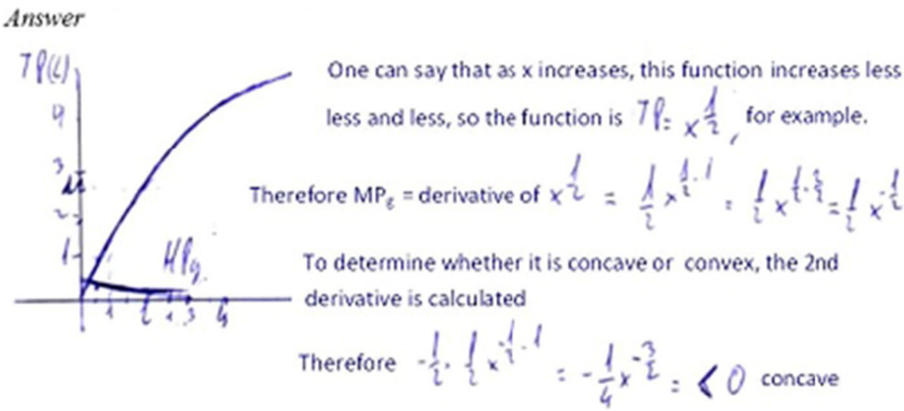


Fig. 16 Student St.14's answer to item 3.2 in task 3

Table 4 Characteristics incorporated at each level

Levels	Characteristics
INTRA	a.Students can calculate average rates of change between two points and at one point or estimate the limit using the concept of elasticity in the algebraic register.
	b.Students can perform conversions of linear and nonlinear economic functions from the algebraic to the graphic register.
	c.Students can only establish relationships between a function and its derivative in the algebraic register.
INTER	b.Students can perform conversions of linear and nonlinear economic functions from the graphic to the algebraic register.
	c.Students can establish relationships between a function and its derivative in the graphic register.
	d.Students can use the concept of the 2nd derivative in the algebraic register.
TRANS	d.Students can use the concept of the 2nd derivative in the algebraic register and can apply the meaning of concavity/convexity.

Discussion and conclusions

Our study has elucidated some of the characteristics of understanding of the function-derivative relationship schema in learning economic concepts. In this characterisation, the graphic and algebraic registers play an important role in determining the levels of acquisition of the schema (Intra, Inter and Trans). The results indicate that some students only use the concept of the derivative in the algebraic register (Intra level), whilst other students make better use of the derivative when it is presented in the graphic register than in the algebraic register (Inter level). An economic understanding of the concept of concavity/convexity and its relationship with the second derivative in the algebraic register (Trans level) suggests a higher level of acquisition of the schema. These data indicate that treatment in a register and conversion between the two registers demonstrate the development of an understanding of economic concepts that involve the relationship between a function and its derivative. In this respect, Haciomeroglu, Aspinwall and Presmerg (2010) showed the importance of using the graphic register and the ability to graphically represent functions and their derivatives. The results reported by Vrancken, Engler and Müller (2011) also confirm the need to introduce tasks that connect different systems of representation, enhancing the visualisation of ideas and the comprehension of concepts. However, our results suggest that many students find it difficult to convert nonlinear functions from the graphic register to the algebraic register and vice versa, due to overuse of linear functions (De Bock, Van Dooren, Janssens and Verschaffel 2007). A contribution of our study is that some students are only capable of establishing the relationship between a function and its derivative in linear cases, and experience great difficulty when functions are nonlinear. Moreover, our results underscore the importance of the ability to convert functions from the graphic to the algebraic register as a characteristic of schema acquisition (Inter level). In this respect, students' ability to convert functions between both registers and in both directions is necessary to establish the relationships between a function and its derivative and thus understand the economic meaning of the first derivative of a function. Traditionally, the economics curriculum has focused on conversions of functions from the algebraic to the graphic register whilst placing much less emphasis on conversions in the opposite direction, which can become an obstacle to the acquisition of understanding of certain economic concepts.

Our research has highlighted the difficulties students encounter in understanding the concept of the second derivative beyond the algebraic register. In this regard, one important characteristic of the Inter level is that few students had difficulties with the algebraic treatment of a concept, but showed a better understanding when the graphic register was the point of reference. This supports the suggestion made by Hey (2005) that the graphic register can contribute to an understanding of the concepts of microeconomics. The graphic register can further understanding of the idea of measuring change based on the relationship between a graph of the derivative and that of the function (Elia 2006; Gagatsis and Shiakalli 2004; Gagatsis, Elia and Mousoulides 2006).

The results of our research indicate that students would not have achieved much success in understanding the relationships between a function and the first and second derivatives without the intervention of the algebraic register or with the sole use of the graphic register, although use of the graphic register allows them to advance towards the Trans level of schema acquisition. The tasks included in our research were based on the graphical relationship between a function and its derivative, and they showed that many students had difficulty solving this. As Yoon and Thomas (2015) pointed out, some students need to introduce algebraic methods in the construction of derivative and antiderivative, even when no explicit algebra is provided. This suggests that integration of the different systems of representation may be a key aspect in understanding the relationship between the notion of functions and derivative functions in economic concepts. García, Llinares and Sánchez-Matamoros (2011) have emphasised the importance of the relationship between the derivative at a point (local perspective) and the first and second derivatives of a function (global perspective) in order to acquire a proper understanding of the relationship between a function and its derivative. In our research, the last two tasks proposed constituted an example of how to analyse the process of obtaining the first and second derivative from the item of data used to obtain the derivative at a point. In our genetic decomposition, this was the most advanced step within the level of schema acquisition.

Although more research is required, our study provides evidence of the complementarity that exists between a mathematical understanding of the relationships established between mathematical concepts and economic concepts. In this respect, the incorporation of fuzzy metrics to study the acquisition of cognitive schemata is a complementary perspective that helps us better understand the relationship between mathematics and understanding of economic concepts.

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