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ORIGINAL ARTICLE

Learning to teach upper primary school algebra: changes to teachers' mathematical knowledge for teaching functional thinking

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Abstract A key aspect of learning algebra in the middle years of schooling is exploring the functional relationship between two variables: noticing and generalising the relationship, and expressing it mathematically. This article describes research on the professional learning of upper primary school teachers for developing their students' functional thinking through pattern generalisation. This aspect of algebra learning has been explicitly brought to the attention of upper primary teachers in the recently introduced Australian curriculum. Ten practising teachers participated over 1 year in a design-based research project involving a sequence of geometric pattern generalisation lessons with their classes. Initial and final survey responses and teachers' interactions in regular meetings and lessons were analysed from cognitive and situated perspectives on professional learning, using a theoretical model for the different types of knowledge needed for teaching mathematics. The teachers demonstrated an increase in certain aspects of their mathematical knowledge for teaching algebra as well as some residual issues. Implications for the professional learning of practising and pre-service teachers to develop their mathematics knowledge for teaching functional thinking, and challenges with operationalising knowledge categories for field-based research are presented.

Keywords Teacher professional learning · Content knowledge · Pedagogical content knowledge · Algebra · Pattern generalisation · Functional thinking · Middle years of schooling

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Introduction

The word "algebra" can evoke unpleasant memories of a difficult abstract topic in secondary school mathematics (Greenes et al. 2001; Lee and Freiman 2004). The Australian Curriculum has brought algebra to the attention of primary teachers explicitly by including the content strand "Number and Algebra" right from the early years (Australian Curriculum Assessment and Reporting Authority [ACARA] 2009). Research on integrating algebra into elementary mathematics curriculum is considered as an emergent endeavour (Carraher and Schliemann 2007) but has highlighted the importance of younger students developing *algebraic thinking* (Cai and Moyer 2008; Carraher et al. 2006; Radford and Peirce 2006).

A meta-analysis of algebra research studies emphasised the effectiveness of focussing on developing students' *conceptual* understanding of algebra rather than only on procedural knowledge (Rakes et al. 2010). Yet how do teachers obtain the knowledge needed to "teach a more powerful and general mathematics for understanding" (Blanton and Kaput 2008, p. 361) when they are likely to have only been schooled in narrow instrumental approaches to algebra and symbol manipulation techniques? Effective algebra teaching and learning has been highlighted as "a major policy concern around the world" (Hodgen et al. 2010). The challenge is to "create a body of knowledge that is learnable and useable by teachers" (Stacey and Chick 2004, p. 18). The provision of teacher professional learning for teaching algebra at earlier levels of schooling is needed, both for beginning and experienced teachers (Lins and Kaput 2004), as their content knowledge and awareness of students' difficulties in learning algebra are of increasing importance (Saul 2008). Kieran (2007, p. 744) also argued that in research on algebra teaching, "little attention has been paid thus far to the study and development of teachers' pedagogical content knowledge." The aim of this study was to investigate upper primary school teachers' development of mathematical knowledge for teaching important aspects of algebra—functions, relations, and joint variation as conceptualised by Kaput (1999).

Reform efforts in the USA over the past few decades have developed functions-based approaches to learning school algebra that initially emphasise relationships between variables and multiple representations of functions as an alternative to the more traditional equations-based approaches (Kieran 2007). Several countries, including Australia, have incorporated a hybrid of these approaches in their curriculum (Sutherland 2002). One key aspect of a functional approach to learning algebra, and highlighted by the Australian Curriculum at upper primary levels, is the exploration of variables through the generalisation of growing patterns and number sequences to develop functional thinking; this aspect of algebra was the focus of this study (ACARA 2009). As a pre-cursor to the study described here, the author surveyed 105 Years 5 and 6 practising teachers and found that for the most part their knowledge of functional thinking was below the level expected for teaching this aspect of upper primary school algebra. A majority of the teachers themselves expressed their concern about their ability to teach algebra at this level (Wilkie 2014). The manipulation of symbolic expressions and the solving of problems involving variables, which are part of an equations-based approach, appear at secondary levels in the Australian curriculum. Functions are used to model many real-world applications and functional thinking is foundational to algebra and calculus mathematics courses in the later years of schooling. Such knowledge underlies innovation and economic success across many science and engineering domains, and there is the need for expertise in this area of mathematics (e.g. Mullis et al. 2004). The study's purpose was to explore ways to support the professional learning of Australian teachers in their endeavour to implement the new Australian Curriculum for algebra, to promote their students' conceptual development of functional thinking, and to prepare students effectively for learning algebra at secondary levels of schooling.

Research has shown that effective professional development for teachers in mathematics seeks to improve their content knowledge and connects to teachers' daily professional practice (Ball 1996; Garet et al. 2001). The study described here focussed on ten individual teachers' development of mathematical knowledge for teaching algebra using functional approaches in the context of participation with colleagues in a design-based research project. It investigated their learning to teach pattern generalisation and functional thinking using a variety of strategies highlighted by the research literature as effective for developing conceptual understanding. The process of teachers improving different aspects of their own content knowledge was of interest as this is seen as fundamental to their ability to teach algebra. Evidence of changes to their pedagogical content knowledge was also considered important and was sought through changes in their participation during interactive aspects of the professional learning, with peers and with students. The findings from the research that are reported in this article address the following research question: How is teacher professional learning evident through shifts in the different aspects of the teachers' mathematical knowledge for teaching functional thinking? The following section describes the context for the project by situating the aspect of algebra that is the focus of the study and by providing an overview of the research on the different types of knowledge needed for teaching it.

Related research and context

Teaching algebra right from the early years has emerged as a central theme in current mathematics education reform efforts (Carraher and Schliemann 2007; Greenes et al. 2001), and differing views have been expressed on what algebra actually *is*, and what defines algebraic thinking (Kaput 2008; Kieran 2004, 2007). There is consensus, however, that *generalisation* is foundational, the cornerstone of mathematical structure (Kruteskii 1976). The expression of generalisations using conventional symbol systems and actions on generalisations are two core aspects of algebra (Kaput 2008). Functional thinking has been defined as a type of "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalisations of that relationship across instances" (Smith 2008, p. 143). Functional thinking relates to understanding the notion of *change* and of how varying quantities (or "variables") relate to each other. It underpins the use of pronumerals (alphabetic letters) to represent variables symbolically and the understanding that functions are a powerful way of expressing many types of relationships in the world around us.

Theoretical framework of the different types of knowledge for teaching mathematics

Attempts have been made over several years to build on Shulman's (1986) well-known definitions of different types of knowledge, but there is still no universal agreement on a widely accepted framework for the types of knowledge needed for teaching mathematics (Petrou and Goulding 2011). In efforts to conceptualise, develop, and test measures of teachers' mathematical knowledge for teaching (e.g. Ball 1996; Ball and Bass 2000; Hill et al. 2008; Hill et al. 2004; Hill et al. 2007), Ball et al. (2008) proposed a model with three types of content knowledge and three types of pedagogical content knowledge:

Content (subject matter) knowledge
- Common content knowledge (CCK)
- Specialised content knowledge (SCK)
- Horizon content knowledge (HCK)

Pedagogical content knowledge
- Knowledge of content and students (KCS)
- Knowledge of content and teaching (KCT)
- Knowledge of curriculum (KC)

This theoretical framework by Ball et al. (2008) was chosen for the study because it was the result of empirical research specifically on mathematics teaching and because of its conceptualisation of several different dimensions of knowledge. Although none of the prevalent theoretical models are considered easy to operationalise in empirical research (Zhang and Stephens 2013), this study used the Ball et al. (2008) model for trying to elaborate the different dimensions of knowledge for teaching functional thinking and teachers' development of them. Pattern and number sequence generalisation is an important aspect of algebra for upper primary students in the Australian context but there appears to be little in the research literature on learning to teach it. Depaepe et al. (2013) review of empirical mathematics educational research in peer-reviewed journals on teachers' pedagogical content knowledge found no studies at all on their development of knowledge for teaching algebra at primary levels of schooling.

Fennema and Franke's (1992) conception of knowledge for teaching as being developed *in context* also contributed to the design of the study. They emphasised the interactive and dynamic nature of teachers' knowledge development through interactions with the mathematics itself and with students in the classroom environment—two key activities incorporated into the design of the professional learning for the teachers in this study. Fennema and Franke (1992) additionally incorporated teachers' *beliefs* in their model, representing a bi-directional interaction between knowledge and beliefs. This article does not explicitly discuss changes in the teachers' beliefs as these have been discussed elsewhere using Clarke and Hollingsworth's (2002) model for professional growth, in which beliefs play an important role in the salient outcomes of professional learning for teachers (Wilkie and Clarke 2015).

This study considered various dimensions of the content knowledge and pedagogical content knowledge it is believed that upper primary teachers need to have for teaching functional thinking. It then investigated each teacher's development of these through professional learning—changes evident in their own algebra learning, in their teaching of algebra with their classes, and in their discussions with colleagues. The Ball et al. (2008) framework is used in this article firstly to outline the literature on teachers' mathematical knowledge for teaching functional thinking, and secondly to discuss findings from an in-depth case study of ten individual teachers over 1 year. Some issues with trying to operationalise the different types of teacher knowledge in the Ball et al. (2008) model for field-based research are also shared.

According to the Ball et al. (2008) framework, CCK relates to the mathematical knowledge used in everyday life by adults and "is used in the work of teaching in ways in common with how it is used in many other professions or occupations that also use mathematics" (Hill et al. 2008, p. 377). SCK enables teachers to "accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al. 2008, p. 378). It is "entirely mathematical, but it is not mathematical work done by many non-teaching adults" (Hill et al. 2007, p. 133). In this study, SCK has been used to describe the knowledge for generalising patterns, writing functional rules, and thinking that implies a conceptual rather than procedural understanding of functional relationships. It was categorised as *specialised* as opposed to *common* content knowledge arguably because these aspects of algebra were not viewed as part of the everyday mathematical work done by non-teaching adults. It was also categorised as SCK rather than KCS or KCT because it was viewed as being about the mathematics itself—an ability to generalise a growing pattern and create a rule for the functional relationship does not imply knowledge of how students learn to do so, or about which teaching strategies and representations are helpful for supporting that learning. HCK is "about having a sense of the larger mathematical environment of the discipline" and "an awareness of connections to topics that students may or may not meet in the future" (Jakobsen et al. 2013, p. 4). Although HCK was not explicitly explored in this study, the knowledge of representing variables symbolically in equations with pronumerals (alphabetic letters) described in higher levels of the secondary school algebra curriculum could be viewed as this type.

In the model of Ball et al. (2008), pedagogical content knowledge is divided into three categories. The first type, KCS, is defined as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (p. 375). Teachers with this knowledge attend to how students typically learn a concept, and to common mistakes and misconceptions. It implies an understanding of students' thinking and what makes the learning of particular concepts easy or difficult, but does not include "knowledge of teaching moves" (p. 378) which is conceptualised as a second type termed KCT. KCT includes knowledge about how to choose representations and examples, how to *build* on students' thinking, and how to *address* student errors effectively. An elaboration of how the specific aspects of these two types of knowledge were couched in the study is presented in the next sub-section. The third type of PCK is conceptualised as KC and matches Shulman's (1986) previously mentioned *Curricular knowledge*. Petrou and Goulding (2011) highlighted that for teachers in those countries which have official curriculum documentation and assessment systems, KC includes not only an awareness of possible resources and materials to use in their teaching but also of the mandated content from the relevant curriculum. In this study, KC has been used to describe the knowledge of how to apply the relevant content descriptions from the Australian curriculum to appropriate learning activities for students, which gives insight into whether the curriculum content itself is understood.

Types of knowledge needed for teaching functional thinking

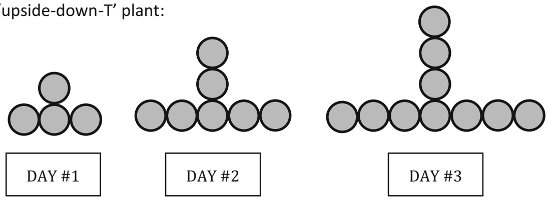
Having conducted an extensive literature review on algebra, Kieran (2007) concluded that existing research has barely begun to explore the dimensions of knowledge that teachers need for teaching school algebra, particularly related to the development of students' algebraic thinking, their misconceptions and difficulties, and their likely different approaches (conceptualised as KCS in the model of Ball et al. (2008)). Yet she pointed out that there are abundant findings in the literature on the algebra *learner*, which can be utilised for research on the algebra *teacher*. The following three sub-sections seek to connect these two bodies of research by reviewing the literature on learning algebra and trying to categorise different aspects, related to developing functional thinking, using the different types of knowledge from the framework.

Specialised content knowledge (SCK)

An understanding of linear functional relationships is considered an appropriate starting point for developing functional thinking, and this can extend naturally from counting experiences involving repeated addition (Smith 2008). The generalisation of growing patterns provides "a powerful vehicle for understanding the dependant relations among quantities that underlie mathematical functions" (Moss et al. 2008, p. 156). Different approaches to pattern generalisation are described in the literature, and because learners typically use these strategies at varying levels of mathematical sophistication, a key aspect of SCK needed for understanding the algebra concepts at upper primary levels is being able to use these approaches. The two main approaches are *recursive* and *explicit* generalisation. Stacey (1989) referred to recursive generalisation as "near generalisation", and this involves finding the next item in a growing pattern using step-by-step drawing or counting. She termed explicit generalisation as "far generalisation" which involves finding the general rule or equation (p. 150). Confrey and Smith (1994) described these two approaches as co-*variation* and *correspondence* respectively. Co-variation highlights the relationship between successive items in a pattern—also known as a *local rule* (Mason 1996)—and correspondence describes the relationship between two quantities or variables (usually the item/term position number in the pattern/sequence and a quantifiable aspect of the item/term itself—also known as a *direct* or *closed* or *relational rule*). It is this correspondence approach that enables the description of relationships between variables and which Usiskin (1988) considered an essential aspect of school algebra. Radford and Peirce (2006) also emphasised that it is not only the noticing of a commonality between items in a growing pattern but grasping that it applies to *all* possible items and being able to *express* it directly that constitutes algebraic thinking. Figure 1 provides an example of co-variation (recursive) and correspondence (explicit) approaches to generalising a geometric growing pattern.

Figure 2 represents the same growing pattern in a table of ordered pairs.

The leaves of a growing 'upside-down-T' plant:



Co-variation (recursive): The plant on day #1 has 4 leaves and each day the plant has 3 new leaves – the total numbers of leaves each day are 4, 7, 10, 13 …
Correspondence (explicit): Each plant has the same number of leaves on each of its three stems as its item number – the total number of leaves is 3 times the item number plus 1 for the leaf in the centre ($t = 3n + 1$)

Fig. 1 Two approaches to describing functional relationships in a growing pattern

This study was designed to provide teachers with an understanding of the two types of generalisation, the differences between them, and how to use them to generalise a growing pattern for themselves as part of their grappling with the mathematics itself.

Knowledge of content and students (KCS)

For the purposes of this study, KCS was elaborated as knowing about the process by which students progress through using the two types of generalisation to the stage where they can represent an explicit rule symbolically with pronumerals. It includes knowledge of common issues students have in doing so, such as difficulties moving beyond perceiving and describing patterns recursively to generalising them explicitly to find functional rules or algebraic representations (e.g. Confrey and Smith 1994; English and Warren 1998; Stacey 1989). Recent research has found that the process of looking for a relationship between quantifiable aspects of a geometric growing pattern supported students' ability to learn to think functionally and ability to write symbolic representations of a generalisation (e.g. MacGregor and Stacey 1995; Markworth 2010; Warren and Cooper 2008). An issue related to students' difficulties in moving beyond recursive strategies is that experiences only with one-dimensional patterns (creating a growing pattern without relating each item to its position in the sequence) or number sequences can obscure the variable on which the pattern or sequence depends (the item number), thus keeping its structure as a function well-hidden (Kaput 2008). This can lead inadvertently to students' focussing only on the relation between consecutive items

Item position number	Item (e.g., total number of leaves)
1	4
2	7
3	10
4	13
5	16

Co-variation: *"When the item position number increases by 1, the item increases by 3."*

Correspondence: *"Three times the item position number and add 1 equals the item."*

Fig. 2 Two approaches using a table of ordered pairs (adapted from Smith 2008, p. 147) (recursive rather than relational thinking). Knowing that students need experience of relating both variables from a growing pattern or sequence to be able to learn to generalise explicitly can be considered KCS; knowing the types of representations and questioning that can help students to do so relates more to KCT, but they are obviously closed integrated.

Markworth (2010) developed an empirically substantiated instruction theory about the process of upper primary students' developing functional thinking using generalisation of geometric growing patterns. She refined a conjectured local instruction theory based on findings from repeated teaching experiment cycles with classes of Year 6 students. The six stages of her subsequent learning progression were adapted slightly in this study to include references to recursive and explicit generalisation (language used in the state curriculum with which the teachers in the study were familiar) and to develop a numerical rubric for scoring the level of generalisation. Consultation with colleagues led to the addition of a level involving the application of functional thinking. The modified learning progression framework is presented in Table 1.

This learning progression framework was used in the study as a rubric to analyse teachers' SCK before and after their participation in the project, and also as a resource for the teachers to learn about students' likely progression and to assess their students' written work (KCS). Knowledge of this learning progression would also obviously provide knowledge about the types of representations, examples, and teaching strategies that help students progress, which is considered KCT. Knowing a student's difficulty or error at a particular stage (KCS) as well the next stage of the progression would also help a teacher in addressing it, which is another aspect of KCT.

Knowledge of content and teaching (KCT)

In this study, KCT was related to knowledge of the exploratory use with students of concrete materials with item cards to represent both variables in a geometric growing patterns, appropriate graphs of the patterns, and symbolic representations of functional rules, along with questioning techniques that elicit students' noticing of functional features of these different representations. It was also related to teachers knowing how to respond to a student error in generalising or issue with being able to progress to

Table 1 A learning progression for development of functional thinking with growing patterns (Wilkie 2014, adapted from Markworth 2010, p. 253)

1. Extend a growing pattern by identifying its physical structure, features that change, and features that remain the same (*figural reasoning*).
2. Identify quantifiable aspects of items that vary in a geometric growing pattern.
3. Articulate the linear functional relationship between quantifiable aspects of a growing pattern by identifying the change between successive items in the sequence (*co-variation* or *recursive generalisation*).
4. Generalise the linear functional relationship between aspects of a growing pattern by:
   4.1 Describing the relationship between a quantifiable aspect of an item and its position in the sequence (*correspondence* or *explicit generalisation*);
   4.2. Using symbols or letters to represent variables; or
   4.3. Representing the generalisation of a linear function in a full, symbolic equation.
5. Apply an understanding of linear functional relationships between variables to further pattern analysis and multiple representations.

explicit strategies. Several teaching approaches to developing students' functional thinking through examples, representations, and questioning are described in the literature. Confrey and Smith (1994, p. 32) proposed "the interaction of context, multiple representational forms, and technological tools". This focus on teachings with different representations of functional relationships is widespread in the literature, particularly by those who advocate a reform-oriented perspective of school algebra (Kieran 2007). Teachers help students to learn about functional relationships within a familiar or real-life context. They use diagrams, language, graphs, equations, and tables of values to create and represent solution processes and therefore think about functions in diverse yet legitimate ways. Warren and Cooper (2008) advocated the use of concrete materials for representing growing patterns and specific questioning by the teacher to help students notice the explicit relationship between an item in a pattern and the item's position number. Moss et al. (2008) advocated a number of strategies that facilitated Year 4 students' ability to generalise patterns, including the following: building geometric patterns with pattern blocks, using position cards to highlight the item position number, and the use of function machines to explore relationships between variables.

Knowledge of curriculum (KC)

Although there are documents, such as the *Principles and Standards for School Mathematics*, that contain comprehensive detail on functional approaches to learning algebra in the middle years (National Council of Teachers of Mathematics [NCTM] 2000), this study explored the teachers' KC in terms of their knowledge of the national and state curricula relevant to them. During the project, the schools were in a period of transition from a state to a national curriculum. The state curriculum was the Victorian Essential Learning Standards (VELS) and referred to upper primary students constructing and using "rules for sequences based on the previous term, recursion, and by formula." It also referred to students being able to "identify relationships between variables and describe them with language and words" (Victorian Curriculum and Assessment Authority 2007, February). The *Australian Curriculum: Mathematics* in the sub-strand "Patterns and Algebra" described the expectation of students being able to describe, continue, and create patterns (Year 5) and sequences (Year 6), and to describe the rule that creates a sequence (ACARA 2009). Familiarity with these content descriptions of algebra and knowledge of appropriate activities to help students learn them were considered as a type of KC.

Research on teachers' actual knowledge for teaching functional thinking

The knowledge teachers *ought* to have for teaching mathematics seems well-documented in the literature (Adler et al. 2005; Ball and Bass 2000; Carpenter et al. 1989; Nathan and Koellner 2007), yet there seems to be considerably less research on the knowledge teachers actually *do* have, particularly pedagogical content knowledge and in the specific domain of algebra (Kieran 2007). A few studies examined elementary/primary pre-service teachers' algebra knowledge including two large British studies (n=154 and n=201, respectively). Less than half of the participants were able to complete correctly the items on generalisation using words and symbolic notation (Goulding et al. 2002). Another study of 58 US pre-service teachers investigated their understanding of algebraic generalisations and ability to link symbolic equations to visual growing patterns. They reported that the participants struggled to interpret what the variables actually represented and to identify the patterns themselves (Rule and Hallagan 2007).

A 6-year large-scale international comparative study titled *Teacher Education and Development Study in Mathematics* (TEDS–M) investigated the knowledge of pre-service primary and lower-secondary teachers from 17 countries (Tatto et al. 2012). Senk et al. (2012) reported on results for the primary pre-service teachers and found that those participants who performed at a higher level demonstrated “some familiarity with linear expressions and functions” yet “had limited success applying algebra to geometric situations” (p. 8). A report on specific test items found that 77 % of primary pre-service teachers correctly predicted the 10th figure of a growing pattern but only 54 % were able to find the rule for the number of people that could be seated around *n* tables. One PCK item on matching equations and growing patterns was completed correctly by only 31 % of them (Australian Council for Educational Research [ACER] Ball et al. 2008). This report indicated that although around half of pre-service teachers were able to employ explicit generalisation (SCK), less than one third demonstrated appropriate knowledge for *teaching* this aspect of algebra (PCK).

Research on professional learning for teaching algebra

In a comprehensive review of research on the learning and teaching of school algebra, Kieran (1992) noted the scarcity of research emphasising the role of the classroom teacher in teaching early algebra. Even in 2007, Carraher and Schliemann (2007) described research on the teaching of early algebra as still emergent, not yet fully known, and far from consolidated.

Warren (2006) researched Australian lower primary school teachers’ development of CK and PCK of algebra from a sociocultural perspective on learning. She conceptualised a professional learning framework that incorporated an initial phase of “expert input and sharing” of mathematical “knowledge in action” via a demonstration lesson (p. 537) which assisted teachers in exploring their current level of knowledge and identifying key mathematical ideas. A subsequent phase involved cyclic collaborative lesson planning in pairs with expert input and feedback via email, teacher implementation in classrooms, reflection in a group, and decision-making about the next sequence of lessons. The study on upper primary teachers described in this article also incorporated an initial demonstration lesson for teachers and cycles of expert input but involved face-to-face interactions during meetings and lessons rather than relying on email. In another study, Steele et al. (2013) utilised a constructivist approach to learning and developed a content-focussed course for US pre-service and practising teachers (mostly at secondary levels of schooling). They investigated the development of mathematical knowledge for teaching patterns and functions and found that the key features of their teaching experiment that supported teacher learning of this aspect of algebra were as follows: focussing on a specific area of the curriculum relevant to teachers, regular re-visiting and refining of the main concept, and teachers attending to tasks first as learners and then as teachers.

Research design

One criticism of some of the different theoretical models for teacher knowledge is that they seem to be founded on individualistic assumptions about knowledge—on the knowledge of an individual teacher—a more cognitive than situated view (Petrou and Goulding 2011). As with research that employs different paradigms based on how student learning is viewed, research on the development of teacher knowledge might pay attention to changes in an individual's knowledge or to changes in "ways of participating and ways in which participants and practices change" (Lave 1996, p. 157). Yet it has been suggested that both perspectives can be used for local sense making rather than relinquishing either paradigm (Sfard 1998). Although this study used the Ball et al. (2008) framework to examine individual teachers' development of different types of knowledge, it also considered their professional learning from a situated perspective: a dynamic experience in the context of participation with colleagues and in classrooms, and involving changes to their interactions and joint practices. The methodology that resonated with examining the development of teacher knowledge from both cognitive and situated perspectives, and with using the literature on the algebra learner to research the algebra teacher, was a design-based methodology.

In this study, the teachers and researcher experienced the project as a collective effort and where teacher learning and student learning were two joint goals (Gravemeijer and van Eerde 2009). The three key aspects of this methodology are instructional design and planning, ongoing analysis of classroom events, and retrospective analysis (Cobb 2000). Teachers and researcher inquire together "into the nature of learning in a complex system" with the intent of producing "useable knowledge" (Baumgartner et al. 2003, p. 7)—principles and "explanations of innovative practice" (p. 8). This study explored a variety of strategies from the research literature for learning and teaching functional thinking through cycles of teaching experiments involving the researcher, each teacher, and their class of students. Interactions between materials, teachers, and learners were enacted through continuous cycles in order to produce meaningful change in contexts of practice (Baumgartner et al. 2003). Borko (2004) advocated the use of design experiments for exploring the learning of individual teachers during a professional development program. This study also sought to investigate theoretical perspectives on how teachers might develop their knowledge through a program incorporating collaborative professional experimentation (Clarke and Hollingsworth 2002).

Since there was little research found in the literature on practising teachers' actual knowledge for teaching functional thinking, an initial survey of 105 upper primary teachers was undertaken beforehand to inform the design of an in-depth collective case study of 10 teachers (who were included in the initial survey). The detailed results of the initial survey are reported elsewhere (Wilkie 2014) and are also used in this article to make comparisons with the initial and final responses of the case study teachers, who attempted the same survey questions before and after their participation with a gap of nearly a year in-between (and with no discussion of the items with them during their participation). Given their initial responses and difficulties with a number of the items, it was considered worthwhile to give teachers the opportunity to re-visit them and to assess any changes in their responses as one type of evidence of their professional learning.

Cycles of interaction with colleagues occurred throughout the year and included collaborative planning, implementing, evaluating, and revising five sequential lessons with the researcher and their year level teaching teams (Hiebert and Stigler 2000). An outline of the lesson sequence and the learning focus for each is presented in the Appendix, as is a sample task from lesson #4. The researcher provided “a set of exemplary instructional activities and materials” sourced from the literature. Given her experience teaching algebra across primary and secondary levels and familiarity with the relevant research literature, the researcher was well placed to accept the role of “expert” in this study. She conducted the initial training session with teachers and the demonstration lessons for all 10 classes. She team-taught with each of the teachers in their classrooms for three of the four subsequent lessons and gradually decreased her proportion of co-teaching according to each teacher’s wishes. Teachers and researcher co-analysed students’ participation, work samples, and inferred learning to revise the learning tasks in each lesson and to plan the next lesson. Both “careful review of the data and a reflection on the process of the teaching experiment” were considered important (Gravemeijer and van Eerde 2009, p. 514).

The participants

The 10 teachers who participated in the study were from two schools, one with low SES and the other with middle SES. Eight teachers were female and two were male. At School A (low SES), the three upper primary teachers (with pseudonyms Gemma, Fiona, and Trisha) each taught a composite Year 5/6 class and were all experienced teachers who had taught at upper primary levels previously. At School B (medium SES), three teachers had Year 5 classes (Molly, Gavin, and Yvette) and four teachers had Year 6 classes (Paula, Tom, Heather, and Sarah). Of these teachers, two were early career (second year out and less than 5 years) and one had several years’ teaching experience but was new to upper primary levels. With the exception of two teachers, one at each school who were noticeably vocal about their concerns with teaching algebra and maintained a preference for observation and working with a small group of students, all of the other teachers willingly experimented with co-teaching various aspects of the lessons throughout the year. The meetings throughout the year and the group interviews were conducted in the teachers’ existing teaching teams.

Data collection and analysis

The study incorporated a “descriptive and interpretive” approach to data collection and analysis, with the triangulation of multiple sources of data (O’Toole and Beckett 2010, p. 43). Hill et al. (2007) advocated the use of multiple approaches for examining teacher knowledge and to avoid sole reliance on traditional test formats. This study investigated changes to different facets of individual teachers’ knowledge through the use of initial and final surveys. It also explored how the teachers’ conversations and practices revealed evidence of shifts in the different types of their knowledge through observation of their interactions during meetings and lessons throughout the year. Initial data included the previously mentioned survey of 105 teachers about their mathematical knowledge for teaching functional thinking. At that time, the mathematics leaders and principals of the schools involved in a larger professional learning project were informed of the algebra sub-project and indicated their teachers' likely interest in participation. Many schools responded and two were selected (bounded by the researcher's university teaching commitments and the nature of the in-depth research.) Data from the subsequent collective case study of 10 upper primary teachers from these two schools over a 1-year period form the basis for the discussion in this article. Figure 3 provides an overview of the data collected during the teachers' participation and which were used to investigate evidence of changes in the teachers' mathematical knowledge for teaching.

A sequence of five lessons in each teacher's class with meetings in-between attended by the researcher was timetabled. The team meetings were audio-recorded and included data on teachers' discussion of the following: students' responses to activities and work samples; classroom experiences and practices; exploration of content using instructional materials; evaluation of the previous lesson; and planning for the next lesson. Learning tasks for the students were also designed to solicit informative written data on students' mathematical thinking and interpretation, and were analysed by teachers and discussed in the meetings (Cobb 2000). Observations and reflections from each of the three additional lessons with each of the 10 teachers and each team meeting were documented in detail in a researcher's journal. Teachers were also provided with a proforma for voluntary post-lesson reflection.

The teachers were interviewed both individually and in teaching teams. Question samples from the individual and group interview schedules are in the appendix of Wilkie and Clarke (2015). The teachers also attempted the questionnaire again after their participation for the year. To increase the validity of results for the survey of 105 teachers, a pair of researchers (including the author) underwent a cyclic process of scoring 10 responses and discussing their evaluations to reach consensus for just over half of the 105 teachers' responses. Detailed examples of teacher responses and subsequent scoring are presented in the "Discussion" section. Qualitative data from the open-ended survey items (pre- and post-) and transcripts of the interviews (individuals and groups) were analysed using content and interpretive analysis of line-by-line coding with QSR NVivo qualitative analysis software (Creswell 2007). Data (audio

Data collection	Research methods	Sources of data
INITIAL	Initial survey	– Survey questions (n = 105, including 10 teachers in this study)
MAIN	In-depth collective case study Cycles for 5 lessons: **Demonstration lesson** ← ↓ **Planning with researcher (expert) and colleagues** ↓ **Lesson with students** ↓ **Debrief with researcher and colleagues** → (back to Demonstration lesson) *(Teacher's own experimentation)*	For 10 participants in 3 teaching teams: Yrs 5/6 (*3 teachers)* from one school, and Yr 5 (*3 teachers*) and Yr 6 (*4 teachers*) from another school: – Audio-recordings of meetings (planning and debrief) – Student work samples from 10 classes for 5 lessons – Photographs from lessons – Researcher's journal
FINAL	Individual interview	– Audio-recordings of 10 interviews
	Final survey	– Survey questions (n = 10)
	Group interviews in teaching teams	– Audio-recordings: 3 groups

Fig. 3 An overview of the data collection process for the research

and textual) analysis from the meetings and lessons was undertaken repeatedly throughout the case study and at its conclusion to enable emerging ideas to re-shape perspectives, improve instrumentation, and allow for additional data gathering (Miles and Huberman 1994).

Discussion

In considering teachers' development of knowledge for teaching functional thinking from both cognitive and situated perspectives, the following discussion draws on data from the 10 individual teachers' responses (pre- and post-questionnaires, observations of lessons, and interviews) and from interactive experiences with each other (teaching team meetings between lessons and group interviews). Changes to the different aspects of each teacher's knowledge evidenced from their initial and final responses to the growing pattern generalisation task on the survey are discussed. Evidence of their developing knowledge in the participatory contexts of regular teaching team meetings, interactions with their students in lessons, and group interviews are also discussed within the same framework of knowledge categories taken from Ball et al. (2008) model. Some issues with trying to operationalise these categories for field-based and participatory research are illustrated.

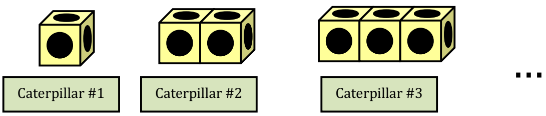
Changes to the teachers' specialised content knowledge (SCK)

To investigate evidence of change in teachers' SCK using a written format, the survey contained open-response items on generalising a geometric growing pattern and writing a functional relationship between variables (caterpillar task). The teachers were asked to provide four possible correct student responses to parts iii and iv from a growing pattern task, shown in Fig. 4.

The teachers' four responses were each analysed and assigned a score based upon the previously described learning progression (Table 1). The numbers of responses at each level on the initial and final surveys are presented in Table 2 along with comparison to the larger cohort of teachers in the initial survey.

The main shift for the teachers over the course of the study was from a written description to a symbolic expression of an explicit generalisation. Of the 10 teachers, 4 demonstrated a higher level of generalisation ability on the final survey. For one teacher, there was a more dramatic change from a score of 1 (drawing the pattern) to

One day my little niece saw a clump of wriggling spotted caterpillars on the branch of a tree. Later she made her own collection of caterpillars with linking blocks and stickers:



i) How many stickers are needed for the next caterpillar's spots (Caterpillar #4)?
ii) How many stickers are needed for Caterpillar #7?
iii) How many stickers are needed for Caterpillar #17?
iv) For any caterpillar number you are given, how do you find the total number of stickers needed for its spots?
(*Task adapted from Markworth, 2010*)

Fig. 4 The caterpillar task (source: Wilkie 2014)

Table 2 Teachers' highest levels of SCK (scores) for caterpillar task on the initial and final surveys

Score	Description	Illustrative example	% teachers initial survey (n=105)	Num. case study teachers initial survey	Num. case study teachers final survey
1 (low SCK)	Extend pattern by drawing or adding up stickers on caterpillar	"1+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+1" "5+4+4+4+4+4+4+4+4+4+4+4+4+4+4+4+5"	6 %	2	1
2	Create sequence or table of quantifiable aspects of caterpillar	"Caterpillar # 1 2 3 4… Stickers 6 10 14 18…" "Adding up—skip counting 4 s. 6, 10, 14, …"	10 %		
3	Explain recursive generalisation by referring to change in caterpillar structure (co-variation)	"Start with 6 spots and add 4 more for each new caterpillar" "Keep adding 4 for each caterpillar"	2 %		
4.1	Explain explicit generalisation in words or with a calculation (correspondence)	"Number of caterpillars multiplied by 4 then add 2 for each end" "2 blocks on each end have 10 stickers. Take away those 2 blocks from number of blocks then multiply by 4" "$17\times4+2$"	39 %	5	3
4.2	Represent generalisation using symbols/letters	"$4\,n+2$" "$(5\times2)+(c-2)\times4=$answer" "$6+4(n-1)$"	31 %	3	6
4.3 (high SCK)	Represent generalisation with full, symbolic equation	"$(a\times4)+2=b$, where a=no. of blocks, b=no. of stickers" "$4x+2=y$"	2 %		
Unscored response	Incorrect or inappropriate response	"$(6\times17)-(2\times17)$" "Look for the pattern and develop an algebraic equation"	8 %		
No response			3 %		

a 4.1 (a correct written description of an explicit rule). For those teachers whose highest score remained the same level on the rubric, it was noticeable that their responses had increased in the number of references to the actual structure of the caterpillar and was more likely to include an explanation of the connection between it and their generalisation. This seems to indicate a greater conceptual understanding of explicit generalisation and using the inherent structure of a growing pattern item to find a relationship between the variables.

Interestingly, none of the teachers yet demonstrated the use of a full symbolic equation with the use of pronumerals for *both* variables, which could be viewed as relating to their HCK since this algebra content only appears in secondary levels of the curriculum. (Although primary students are likely to experience or explore the use of symbols as placeholders for a number in an equation, this is different to the use of pronumerals representing two variables in a functional relationship). In the last few lessons of the teaching sequence, a small group of students in each class did in fact write correct full symbolic equations. It was noted that several of the teachers preferred to "leave" their more advanced students to the researcher to teach, preferring to interact with the other students. It is possible therefore that some of them may not have noticed their students' explorations with pronumerals in an equation.

The initial survey of 105 teachers showed that the majority were unable to represent generalisations symbolically. Even after the intervention described here, there was little evidence of improvement in this area even though several students experimented with pronumerals towards the end of the lesson sequence and some wrote full symbolic equations. One teacher commented, "I really don't have any remembrance of learning algebra, I totally wiped it" and later explained that although she found the use of tables of the different quantities "helpful" in lessons, she still struggled with "making that leap to a rule and also knowing the 'n'" (Heather). Both students and teachers demonstrated issues with the correct use of pronumerals to represent numbers or quantities—mistakenly thinking of them as being shorthand for the names of objects. This well-known misconception appears to take time to address despite efforts to develop the students' and teachers' understanding throughout the lesson sequence. Yet it is a critical aspect of conceptual knowledge in algebra and to be able to support the learning of upper primary students who are ready to explore a higher level of algebra learning, their teachers need to know the mathematics at their level of teaching (SCK) and at levels beyond (HCK).

In the planning meetings throughout the year, the teachers focussed on their own knowledge of pattern generalisation by grappling with different growing patterns that were to be later presented to their students in lessons. One teacher said that she needed to try the proposed task first to see if she could do it: "I need a bit of a play… before I feel really confident with what I'm teaching" (Fiona). In the group interviews, the teachers also referred to the importance of "do[ing] background work first" to ensure they understood how to generalise a pattern themselves (Trisha). Initially, the teachers simply sought to answer the handout questions for the coming lesson; later in the lesson sequence, some teachers also tried to find *more* than one solution because they had noticed in lessons that their students would come up with a number of differently expressed rules for the same pattern. One teacher shared her discomfort when this happened during a sharing time she was leading. She explained that some students said:

‘We came up with this way.’ ‘Oh’. They’re both right and they’re both differently written, but they’re both right. So, yeah, I thought that was quite—that was also a surprise, but a challenge trying to be on top of it. (Fiona).

Later meetings found teachers asking each other about their different visualisations and explanations of their pattern generalisation. Some but not all teachers seemed to gain confidence in using pronumerals to express generality (level 4.2 in Table 1) yet they were all able to express their explicit generalisations as a description (level 4.1), and their engagement with the more difficult patterns was observed in the later stages of their participation. It appeared that repeated opportunities to experience the mathematics for themselves before teaching their classes, but in a participatory context with the researcher and each other, gave teachers access to opportunities to increase their SCK for generalising growing patterns.

Changes to the teachers’ knowledge of content and students (KCS)

An important aspect of KCS for functional thinking in algebra was framed in this study as familiarity with the process of students learning to generalise patterns and the progression from *continuing* patterns to using *recursive* (co-variation) and then *explicit* (correspondence) strategies. For the caterpillar task, the teachers were asked to provide examples of four possible correct student responses. These were analysed to consider teachers’ knowledge of the *range* of different types of possible student strategies, as an indication of their awareness of moving from recursive to explicit generalisation and the different levels of understanding in the learning progression (Table 1). The changes to the range of responses provided by teachers on the initial and final surveys are given in Table 3.

There was a noticeable increase in the number of levels of pattern generalisation demonstrated by five of the teachers. Interestingly, only one of these teachers included a co-variation (recursive) approach as a possible response; in her initial survey response, she had only been able to continue drawing the pattern, whereas in her final survey response, she demonstrated generalisation using both approaches. The remaining four teachers demonstrated drawing of the pattern and different levels of *explicit* generalisation, missing recursive strategies altogether. This unexpected result might be due to the open-ended wording of the question or to the strong focus on explicit

Table 3 Teachers’ number of different types of student responses for the caterpillar task

Number of different types of responses (using learning progression levels)	% teachers initial survey (n=105)	Num. case study teachers initial survey	Num. case study teachers final survey
One type	23 %	3	2
Two types	45 %	4	2
Three types	21 %	3	6
Four types	1 %		
Unscored response	8 %		
No response	3 %		

generalisation in the cycles of lessons. To determine if teachers can identify both types of generalisation, the survey question would be better worded to ask for at least these two types and to examine teachers' familiarity with the likely order of progression.

Two teachers demonstrated a wide range of responses in both their initial and final survey responses and another two teachers demonstrated one type of response both times. One teacher unexpectedly *decreased* his number of *types* of responses but he actually showed two *different* ways to visualise the structure of the same pattern and create two different explicit rules. Wilkie (2014) found that only 5 out of 105 teachers in the initial survey had been able to do this. A handful of students from a few of the classes had also each been able to create different rules for the same geometric growing pattern and had even represented the rules (and their inverses) using full symbolic equations, demonstrating a sophisticated conceptual understanding of functional relationships. In later interviews, two teachers commented on the value of seeing students learning to visualise the same pattern in different ways during sharing time by seeing the solutions of others. This aspect of knowing that different rules can be generated from different visualisations of the same growing pattern is difficult to categorise as SCK or KCS since it is related to both types. In this study, it emerged from the teachers' interactions with their students who produced different rules and surprised them, which might be viewed as the development of their KCS. From that, they then learnt to anticipate a variety of visualisations before a lesson, which could be related also to their development of SCK. This example illustrates the challenge of distinguishing between some of the categories from the Ball et al. (2008) model (Petrou and Goulding 2011).

Changes to the teachers' KCS from interactions with their students in lessons seemed to be evidenced by their expressions of surprise about their students throughout the lesson sequence and in their interviews. They appeared to modify their understanding of how students learn algebra after observing their own students' attempts to generalise patterns. Comments about their initial apprehension and subsequent surprise included:

> Initially I went, 'Oh my goodness, my kids are never going to be able to do this.' That's just how I felt and I said a couple of things to the others, 'I'm not sure how they're going to go with this or whatever.' But I guess sometimes they can surprise you. (Sarah)

> I wasn't sure how they would find it, and I was a bit nervous because I thought this could end up being too challenging for them, but it wasn't. (Gavin)

Some teachers expressed surprise that their expectations about individual students were challenged:

> I think the highlight would be and it got me a little bit excited, was my low kids that sometimes struggle through some maths concepts no matter how you present it (you're like 'oh no') they were able to look at the patterns, and they could come up with a formula. (Yvette)

> One of my big things was that a girl who is really weak, she just understood this area because she worked out the patterns and understood the patterns, and then

> everything else made sense. And, you know, she was getting tutoring for remedial maths and you name it, and but she had all my smart kids stripped, in this area. (Tom)

> A couple of children that do well in maths generally—overall, they're my clever ones in most areas—didn't do so well in these lessons. (Fiona)

After the demonstration lesson, each teacher was encouraged to determine each of their student's highest level of generalisation demonstrated on the assessment task using the learning progression (Table 1). The researcher did the same and discussed the scores in the next meeting. The teachers seemed to find this activity engaging and were interested in finding out the researcher's reasons for any differing scores. A few teachers verbalised their surprise that some of their capable students were not able yet to generalise explicitly. They wondered if their scoring had been subjectively influenced by their pre-conceptions of their students' abilities (obviously not an issue for the researcher). Another teacher had initially scored a student at the recursive stage but was shown that the student had in fact used explicit strategies. This surprised him as he had predicted she would struggle with algebra. It seemed that the teachers were confronted with their prior conceptions about the process of learning algebra and about their students' knowledge, and this increased their KCS. Opportunities to observe their students "in action" with generalising patterns, seeing their responses, and then discussing with colleagues enabled them to adjust their knowledge, and possibly their preconceived ideas about algebra learning.

Over time, it was noticeable that the teachers were considerably engaged with comparing their classes' and individual students' responses with each other during meetings, particularly those that perplexed them. It was this aspect of seeing, analysing, and discussing their students' responses to several patterns over time that seemed to indicate a noticeable shift in teachers' KCS and increase in confidence. One teacher described her experience of questioning a student to understand their particular visualisation of a pattern because their explicit rule was different but worked nonetheless:

> And I had that experience this year too, where somebody came up with a certain way of getting the rule and I knew the rule was right, but I had no idea how they were visualising it, and I'm like, 'Now, explain to me what you see.' And it took quite a few goes, then I'm like, 'Oh, I see.' (Fiona)

She explained that "the kids are going to come out with different findings, different reasonings, that it's more open than [she] would have believed." She reflected later, "As a teacher, [what] I've tried to really change myself is to getting them to explain why, and not so much focus on the answer, but on the process."

The teachers also shared anecdotes about their students' affect, for example a Year 5 student who exclaimed that a particular lesson was one of his "best maths sessions ever" and challenged his teacher to give him any item number from a pattern and he would find the matching total number of blocks: "now you can ask me straightaway and I'll know" (Molly). Another teacher shared that her class had ignored the bell for morning break and they "just sat there and were listening" to each other at sharing time (Yvette). She reported, "A lot of them went home and said, 'Mum, I can do *Algebra*.'"

Such discussions led to teachers hypothesising about aspects of the lesson structure, tasks, and teaching strategies that influenced their classes' engagement and responses, demonstrating the interrelatedness of KCS and KCT. The increase in the confidence of teachers to interpret their students' work was also mirrored by the students in tackling each new task over time and some of the teachers commented on how the students knew what to expect and seemed to focus automatically on generalisation of the pattern right from the start of the lesson because they understood the process: "we're ready for this, what have you got?" (Gemma).

Changes to the teachers' knowledge of content and teaching (KCT)

In this study, it was difficult to research the teachers' change in KCT, partly because the choice of representations and examples used in the tasks for the lesson sequence—a key aspect of KCT—was made by the researcher as part of the provision of professional learning for the teachers. But the ways in which the teachers worked within this provision through their choice of questions to help students learn was considered as evidence of their developing KCT. The teachers were observed to increase the frequency of their interactions with students during lessons and appeared to become more confident, particularly in their knowing what types of questions to ask students about their approaches, in their use of algebra terminology, and in seeking to help students learn to generalise explicitly. Although two teachers still preferred to observe more than participate in the lessons, the others were willing to "go out on a limb" and experiment with teaching different aspects of the lesson, such as the introduction to the task, selecting students at various levels of responses to share, and leading the sharing time at the end. Eight of the teachers reflected on the perceived value of students sharing their different generalisation strategies with the class, to "seeing the penny drop at times" for some students during this time (Gavin). Some also expressed surprise at the students' noticeable engagement and explained their interest in changing their teaching practice by incorporating sharing time in other lessons: "You run out of time because they all want to share!" (Molly).

The teachers were also asked in their pre- and post-questionnaires to provide examples of appropriate learning experiences, which could be viewed as part of KCT. Yet the question was worded to be a response to their understanding of the relevant content descriptions in the Australian curriculum and their ability to apply them, so it was categorised as KC. Again, this example highlights the interrelated nature of the different dimensions of a teacher's knowledge, which makes it challenging to design written tasks or collect observational data based on specific categories of knowledge.

Another key aspect of KCT, which involves being able to address student errors, was investigated in the questionnaires. The teachers were asked to examine a student's response to the caterpillar task and to suggest what they might subsequently do or say to the student:

> Caterpillar #37 will have 37 times 4 spots for the top, bottom and sides of the caterpillar, which is 148.

They were not told that the student response was in fact *incorrect*. The teachers' responses were scored using a tailored rubric (columns 1–3) and are presented in Table 4.

Table 4 Teachers' responses to student misconception in the caterpillar task and illustrative responses

	PCK rubric score	Description	Illustrative example	% teachers initial survey (n=105)	Num. case study teachers initial survey	Num. case study teachers final survey
Not recognised as incorrect	N/A	Interprets student response as correct	"That it is correct"	2 %	1	
Unclear if recognised as incorrect	1	Vague, incorrect or irrelevant explanation	"Can you prove it? Tell me how you came to that answer"	18 %	2	1
	2 (low KCT)	Reference to relevant terms or concepts but key ideas missing or not communicated clearly	"How could you prove to me that your formula/answer is correct? Without drawing 37 blocks? Could you try your calculations for a smaller number of blocks and find out if they work?"	8 %		2
Recognised as incorrect	1	Vague, incorrect or irrelevant explanation				
	2 (some KCT)	Reference to relevant terms or concepts but key ideas missing or not communicated clearly	"Maybe get them to visualise and remember that the blocks are connected so you don't always count the sides. Also ask how many sides does a cube have?"	10 %	2	
	3	Reference to the missing end stickers	"I think you may have missed the ones at the end"	33 %	2	3
	4 (high KCT)	Reference to missing end stickers *and* to appropriate strategy for student e.g., look at structure of caterpillar, look at smaller caterpillar to check	"Firstly, praise that they are close, but need to get some blocks to check. So get connector blocks and join 37 or a small group such as 10+ together, count how many sides and then how many spots. Hopefully from the model they will see the 2 extra spots on the ends"	10 %	2	4
	No response			20 %	1	

Six of the 10 teachers demonstrated an improvement in their diagnosis of the student's incorrect response (technically KCS) and an appropriate teaching strategy to address it (KCT). One teacher shifted from interpreting the student response as correct to making a sophisticated response (score of 4). Two teachers remained on the same level (one had already demonstrated a high level of knowledge) and strangely enough two teachers' level of response decreased! The demonstrated changes to the teachers' KCT about addressing a student error were minor shifts, indicating that developing a repertoire of likely student misconceptions (KCS) and knowledge on how to address them effectively (KCT) takes more extensive involvement and experience with teaching pattern generalisation than this study provided. It also illustrated the interconnected development of KCS along with KCT, with each building on the other bi-directionally through professional learning and through interactions with students. A shift in the teachers' KCT was also considered as evident in the increasing specificity of their comments when debriefing about each lesson afterwards because they demonstrated more awareness of why the choice of representations and examples used in the lessons were effective for helping students develop their functional thinking. Initially, their comments tended to focus on quite general observations, such as the engagement of the students, the benefit of having students verbalise their thinking, the hands-on nature of the tasks, the grouping of students for different tasks, and the story context of the tasks. For example, one teacher said, "Getting children to share their generalisations was really good because you can develop their generalisation further by having a discussion with them" (Paula). Another said that "we had the hands-on and they remembered stuff" (Yvette). Gavin also said "what helped was the fact that there was a lot of hands-on, as well. It helped them to conceptualise… When I was at school, algebra, it was just all up there, and I really struggled." In later debriefs on lessons, the teachers seemed to reference more specific concepts related to algebra, such as the students' recall of particular ideas and terminology like "variables", examples of how different activities supported students' ability to find the explicit rule, judgements of the level of difficulty of various patterns, suggestions on how to improve future lessons, and respond to specific issues highlighted students' written responses (using KCS to develop more KCT). Gemma explained "You could say, you know, 'I've got a kid that did this. I don't understand.' So we'd talk about it, or you'd pick a student out and say, 'Oh, this person did this. Let's explore what was going on there.'" One teacher described how she continued a lesson after recess and was able to help the students understand how different ways to visualise a geometric structure might lead to different but equivalent (explicit) rules. It seemed that observing their students—and developing further KCS—then led to increased KCT through articulating these experiences in their discussions with colleagues and the researcher, and developing further ideas collaboratively to build on their students' thinking.

Changes to the teachers' knowledge of curriculum (KC)

Obviously, an important aspect of KC for the teachers in this study is familiarity with the content of the Australian (or the previous state) curriculum about pattern generalisation. One teacher in the case study asked during the first planning meeting if pattern generalisation was actually in the curriculum. The initial survey of 105 teachers highlighted that a considerable proportion of the teachers were not teaching the prescribed curriculum content on pattern generalisation. It found that less than 40 % demonstrated the expected KC for teaching upper primary students by identifying appropriate learning experiences for students related to the "Algebra and Patterns" strand in the curriculum. Both state and national curriculum refer to finding the rule for patterns and sequences, but it is possible that the minimal detail in the documentation may increase the likelihood that teachers overlook this aspect of algebra in their teaching. One teacher in the study noticed partway through the year that her class had recognised a growing pattern generalisation task on a Year 5 national assessment practice test (NAPLAN)—"Oh! This is algebra!"—and had been pleased with their ability to attempt it (Gemma). Perhaps seeing such examples of tasks would make this aspect of algebra more visible in the curriculum to teachers.

For the professional learning of the teachers in the study, explicit discussion about relevant content on pattern generalisation, different types of representation, and use of pronumerals using the *Principles and Standards for School Mathematics* (NCTM 2000) was included. It provided much more comprehensive and detailed information on the types of experiences, knowledge, and skills students need to develop in algebra at this stage compared to the Australian Curriculum. Although they were only asked about their own national curriculum on their final survey (to maintain the same questions from the initial survey), 9 out of the 10 teachers were then able to describe appropriate learning activities for the curriculum content, in more detail and with more use of relevant terminology, demonstrating the usefulness of international resources to supplement the Australian curriculum and increase teachers' KC.

Implications and conclusion

This study explored changes to practising teachers' mathematical knowledge for teaching pattern generalisation to upper primary students using Ball et al. (2008) different categorisations of knowledge, and from both cognitive and situated perspectives on knowledge development. This concluding section discusses possible implications for the professional learning of teachers (practising and pre-service) based on findings about each type of knowledge, limitations of the research, and future directions for research on functional approaches to algebra learning. It contributes to the field of mathematics education by building on research efforts to understand more specifically the types of knowledge teachers (like those in the study) might need to develop about upper primary algebra and what professional learning experiences might effectively support their ability to teach it—an area of research underrepresented in the literature (Depaepe et al. 2013; Kieran 2007). In addressing the criticism that studies on teachers' development of knowledge typically neglect to define and conceptualise clearly PCK and its components (Depaepe et al. 2013), this study has attempted to provide detailed definitions of the different categories of knowledge from the chosen theoretical framework, both CK and PCK, and how they were conceptualised for studying teachers' development of knowledge for functional thinking. It provides insights into how teachers' professional learning might be usefully explored using a theoretical model of teacher knowledge types as an analysis tool—a need highlighted by Kieran (2007)—and highlights some of the difficulties encountered in operationalising it in field-based research.

Previous research about pre-service teachers at the end of their teacher education found that their SCK about explicit generalisation and use of symbolic equations was problematic (Australian Council for Educational Research 2010; Goulding et al. 2002; Rule and Hallagan 2007; Senk et al. 2012). This study demonstrated that planning pattern generalisation lessons for their classes throughout the year gave the teachers repeated opportunities to learn the mathematics for themselves—to notice the structure of the items in each pattern and to express their generalisation as an explicit rule. The main development in their SCK that the teachers demonstrated was moving from descriptive rules to symbolic expressions. There is some indication that they did not understand clearly that a functional relationship involves *two* variables which are both represented with pronumerals since none of the teachers demonstrated the use of a full symbolic equation (although some students in each class progressed to this stage). The teachers needed to pay attention to the *total number of blocks* (t) being expressed in terms of the other variable (e.g. for the caterpillar problem, the full equation $t=4n+2$ and not just the expression $4n+2$). These findings suggest that professional learning programs on algebra for teachers consider focussing on explicit generalisation and on the meaning and use of pronumerals to represent variables in functional relationships. Increasing teachers' HCK is important for supporting the learning of students who are conceptually ready for more advanced content from secondary levels of the curriculum. This might include focusing on more than just pattern generalisation to make connections with other contexts for functional relationships, for example, graphical representations on a Cartesian plane (content from the Australian curriculum at upper primary levels), function machines with the use of input and output as two variables, and real-life scenarios which involve the relationship between two variables.

The findings of the study suggest that teachers' knowledge of the process by which students learn to think functionally (KCS) is challenging to develop, even in the context of classroom-based teaching experiments in which they can observe their students over time. It does seem to develop to some extent by collaborative professional learning that uses an empirically substantiated learning progression framework to help teachers focus on the likely stages of learning to generalise patterns. Developing an understanding of the different levels of sophistication of strategies for generalisation and how these relate to increases in functional thinking seemed to help teachers engage with the approaches used in lessons to elicit these strategies from their students (KCT). And re-visiting these approaches over time through cycles of lessons and shared discussions gave the teachers opportunities to see the different strategies appear in their students' work and to observe their development in being able to generalise increasingly difficult patterns. Repeated experiences in analysing their students' work together and learning how to interpret different levels of generalisation seemed to give teachers increased confidence in recognising students' thinking, which is an important aspect of KCS.

Regarding another aspect of PCK—knowledge of content and teaching (KCT)—in general the teachers demonstrated an increase in addressing student mistakes, using various strategies to encourage explicit generalisation, and using algebra-specific terminology when discussing events from the lessons. The teachers' comments over time became less about the general aspects of the lesson and more about students' particular responses to a task, sharing of detailed anecdotes, and articulation of the concepts of pattern generalisation. The actual ability to observe teachers' interactions with students *during* lessons to investigate shifts in PCK was limited due to the team-teaching approach to the cycle of lessons and two teachers' ongoing observational roles. This highlighted the difficulty of researching teachers' development of pedagogical content knowledge in ways that do not interfere with their development of confidence in experimenting with their students. Responses from the initial survey in this study highlighted the concern many primary teachers experience about teaching and learning algebra, making research about their levels of PCK and professional development an ongoing challenge. Several teachers in the case study also described their anxiety: "the unknown can be quite scary, and the word 'algebra' doesn't bring nice thoughts" (Fiona).

The findings about teachers' development of knowledge of curriculum (KC), the third type of PCK, suggest that primary teachers in similar circumstances would benefit from exposure to a variety of curriculum documentation, which provide substantially more detail than the previous state and current Australian curriculum. Utilising international curriculum documentation for professional learning is likely to increase their ability to *understand* the content of curriculum because they are able to learn about an area of mathematics expressed in different ways and using a variety of language. The teachers in the study spent time examining international curriculum documentation about algebra and demonstrated subsequent familiarity with a greater repertoire of appropriate algebra terminology and learning experiences for students.

Attempts to analyse changes in the teachers' knowledge according to the specific definitions of types of knowledge in the Ball et al. (2008) framework raised some challenges in differentiating between them. For example, is being able to represent generalisations of a growing pattern with a symbolic equation SCK or HCK? In the Australian curriculum, this is designated for secondary students, but students in the study were ready for this content and therefore teachers need to know this mathematics for themselves. Does knowing that different rules can be generated from different visualisations of the *same* growing pattern show SCK or KCS? This knowledge emerged from the teachers' interactions with their students who produced various rules for the one pattern, which might be viewed as the development of their KCS. From that, they then learnt to look for different visualisations before a lesson to anticipate their students' responses, which could also be related to their development of SCK. The study also illustrated with multiple examples the interconnected development of KCS alongside KCT—each building on the other bi-directionally through cycles of professional learning and interactions with students. For countries with a mandated curriculum, knowing the content at different levels and applying it to appropriate learning experiences is considered KC but also relates to KCT in the choice of appropriate representations and examples. A levelled curriculum could also be considered a source of HCK since it provides a spectrum of mathematical skills and concepts for learning.

In terms of the structure of the professional learning experience, some of the teachers in the study suggested that they would prefer to compress cycles of interaction into a shorter time period to increase the intensity of their learning and help them remember the concepts from one lesson to the next. In order to attend each of the lessons with each of the 10 classes, such a format was untenable to implement for one researcher. A longitudinal design that incorporates an initial intensive session of collaborative professional development, followed by each teachers' own classroom experimentation over a short time period and later follow-up observations in class would perhaps yield different perspectives on teachers' development of knowledge. It might also provide more useful opportunities to explore specifically their development of KCT in terms of their later choice of representations and examples in their own lessons, which was not possible in this study as the researcher provided these as part of the structured professional learning program.

In addition to reflecting on their professional learning, the teachers were also asked to describe what they now believed important for teaching this area of algebra. Four teachers emphasised the value of sequenced lessons of increasingly challenging tasks that help students learn about variables conceptually by connecting their meaning to quantifiable aspects of growing patterns and figural representations. Five teachers highlighted the benefit of encouraging students to play with the hands-on materials and explore—of the process of acting out how a pattern grows and relating the structure to the variables before trying to generalise. Three teachers commented on students being given opportunities to articulate their thinking and demonstrate their generalisations throughout the lesson and at sharing time, both to each other and to the teacher.

This study highlights that the collaborative professional learning experience of being engaged in exploratory and learning tasks with their classes on pattern generalisation, and use of conceptually relatable figural representations, provided teachers with situated opportunities that helped develop their mathematical knowledge for teaching upper primary school algebra. Learning about the content of various curriculum documentation, attempting and discussing tasks together before teaching them, analysing and discussing their students' work with colleagues who had taught the same lessons, planning subsequent lessons together, and observing and team-teaching during lessons helped to develop both their content knowledge and pedagogical content knowledge.

This study also demonstrated implications for the education of pre-service teachers in terms of learning to teach algebra at upper primary levels. It highlighted the importance of teachers developing their own ability to generalise patterns and to learn to understand the process by which students develop functional thinking through recursive and explicit generalisation. Incorporation of algebra content related to variables and pronumerals from the early secondary years of schooling is important for Australian primary teachers so that they develop confidence in supporting students who are ready to move beyond worded generalisations to full symbolic equations and to multiple representations of functions.

A worthwhile avenue for further research relates to connecting pattern generalisation and graphical representation on a Cartesian plane, content which is also in the curriculum at upper primary levels. Responses by some students in this study indicated the potential for further development of functional thinking and exploration of non-linear functions. Another interesting direction could investigate how encouraging students to find different visualisations of the same pattern might lead to their production of different expressions for the same rule and opportunities for students to explore equivalence conceptually. This might support their later learning of equations-based approaches to algebra. Although algebra "challenges a teacher's thinking, as well as a child's thinking" (Fiona) and there remain certain aspects of their understanding that would benefit from further professional learning, these interactive experiences in the context of teachers' own schools and classrooms seem to have "demystified it" a little more (Gavin and Heather); algebra became "not as threatening" (Molly). To build on the work described here, the author is engaged in ongoing research with lower-secondary students and teachers to continue to investigate ways to address the professional learning needs of teachers in this important area of mathematics.

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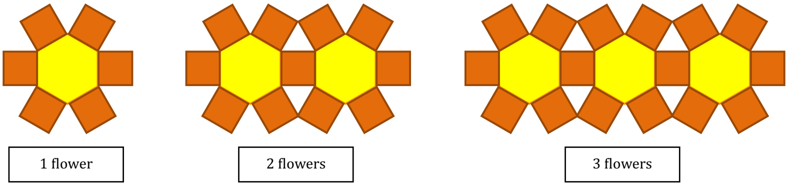
Appendix

Table 5 Outline of five lessons for developing functional thinking

Lesson #	Learning focus	Representations presented or elicited in task
1	Assessing student's prior knowledge of growing patterns and ability to continue, describe, and generalise linear functional relationship in a geometric growing pattern	• Sequential geometric growing pattern • Description of generalisation (in words, with symbolic expression or with full equation)
2	Generalising growing patterns using a correspondence approach by starting with one prototype item from the pattern (3 growing patterns at different levels of difficulty; group discussion then pairs)	• Prototype of one item • Sequential/non-sequential geometric growing pattern • Non-contiguous table of values • Description of generalisation (in words, with symbolic expression or with full equation) • Graph
3	Noticing the structure of unordered items of a growing pattern to encourage different ways of visualising their structure and generalising using a correspondence approach (2 different tasks for students to choose from; working in pairs)	• Selection of prototypes from a geometric growing pattern (non-sequential items) • Non-contiguous table of values • Description of generalisation (in words, with symbolic expression or with full equation)
4	Assessing students' generalisation of a growing pattern, ability to recognise invalid use of proportional reasoning, and to create and interpret scatterplot of the linear relationship	• Sequential geometric growing pattern • Description of generalisation (in words, with symbolic expression or with full equation) • Non-contiguous table of values • Graph
5	Generalising more difficult and non-linear growing patterns (as appropriate) and understanding different ways of visualising the structure leading to different expressions of the functional rule	• Prototype of one item (non-linear functional relationship)/sequential geometric growing pattern (linear but more complex pattern—this task is presented in the Appendix) • Description of generalisation (in words, with symbolic expression or with full equation) • Graph

Pattern block daisy chain

Jill made a daisy chain out of pattern blocks:



a) What do you notice about the structure of the daisy chain and the way it grows each time a flower is added?
b) How many hexagons and squares will Jill need to make a daisy chain with 7 flowers? Explain / show how you obtained your answer.
c) How many hexagons and squares will Jill need to make a daisy chain with 20 flowers? Explain / show how you obtained your answer.
d) For any number of daisies you are given, how do you calculate the total number of blocks Jill will need for her chain? Explain / show how you obtained your answer.
e) Draw a scatter plot for the first ten daisy chains:
f) What do you notice about the graph?

Challenge: *Is it possible to make a daisy chain that uses exactly 100* ***squares****? Explain / show how you obtained your answer.*

Fig. 5 Sample learning task

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