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ORIGINAL ARTICLE

The development of a culture of problem solving with secondary students through heuristic strategies

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Abstract The article reports the results of a longitudinal research study conducted in three mathematics classes in Czech schools with 62 pupils aged 12–18 years. The pupils were exposed to the use of selected heuristic strategies in mathematical problem solving for a period of 16 months. This was done through solving problems where the solution was the most efficient if heuristic strategies were used. The authors conducted a two-dimensional classification of the use of heuristic strategies based on the work of Pólya (2004) and Schoenfeld (1985). We developed a tool that allows for the description of a pupil's ability to solve problems. Named, the Culture of Problem Solving (CPS), this tool consists of four components: intelligence, text comprehension, creativity and the ability to use existing knowledge. The pupils' success rate in problem solving and the changes in some of the CPS factors pre- and post-experiment were monitored. The pupils appeared to considerably improve in the creativity component. In addition, the results indicate a positive change in the students' attitude to problem solving. As far as the teachers participating in the experiment are concerned, a significant change was in their teaching style to a more constructivist, inquiry-based approach, as well as their willingness to accept a student's non-standard approach to solving a problem. Another important outcome of the research was the identification of the heuristic strategies that can be taught via long-term guided solutions of suitable problems and those that cannot. Those that can be taught include systematic experimentation, guess–check–revise and introduction of an auxiliary element. Those that

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cannot be taught (or can only be taught with difficulty) include the strategies of specification and generalization and analogy.

Keywords Problem solving · Heuristic strategies · Culture of problem solving · Intelligence · Creativity

Introduction

Efficient teaching of mathematics is supported when pupils are taught to solve problems (see, e.g. National Council of Teachers of Mathematics [NCTM] 2000). The theory of didactical situations in mathematics (Brousseau 1997) states that for every problem, there is a set of knowledge pre-requisite to its successful solution. However, not all of this pre-requisite knowledge is available to the pupil when solving mathematical problems. In other words, the learning of mathematics broadens the students' repertoire of strategies as well as the knowledge available to the pupil. The teacher's role is to create an environment where this development may occur.

In many cases, the pupils may not have all the needed tools to solve unfamiliar problems and this is when they can employ heuristic strategies. Heuristic strategies are perceived here in the sense of Pólya's (2004) and Schoenfeld's (1985) views. We propose, that to a certain extent, pupils can be taught to use some heuristic strategies in problem solving and thus solve problems efficiently. The study reported in this paper supports this conviction.

The main aim of this paper is to communicate the results of a longitudinal research study conducted in three Czech classrooms. Pupils were exposed to the use of particular heuristic strategies in mathematical problem solving for a period of 16 months.

Theoretical background

We presume that problem-solving skills form a basis for successful mathematics education. Finding solutions to problems carefully selected helps develop, refine and cultivate mathematical thinking. In many a curriculum document, problem solving is integral to all learning–teaching process and is not a particular element in mathematics education only. This is illustrated in the following two different curricula. A mandatory curricula for the Czech Republic states that "These problems should underlie all thematic areas in all primary and lower secondary education." (Jeřábek et al. 2013, p. 26). Problem solving is also one of the proficiencies in the Australian curriculum: mathematics (Australian Curriculum and Assessment Reporting Authority [ACARA] 2014). Another example is from the NCTM which states "Problem solving is an integral part of all mathematics learning, and so it should not be an isolated part of the mathematics program." (NCTM 2000, p. 52). The advantages of this approach in mathematics curricula are supported by Hensberry and Jacobbe (2012). We assert that this approach to mathematics education improves pupils' ability to think critically and strengthens their ability to use school mathematics outside the classroom.

Sullivan and McDonough (2007) present two groups of factors which must combine if pupils' engagement in mathematics learning is to improve: one of the groups consists of factors related to the content and learning style (former knowledge, relevant curriculum, interesting problems and appropriate teacher's approach), the other group consists of factors such as goals of teaching, willingness to learn as well as including perseverance in problem solving.

Problem solving has been in the spotlight in mathematics education research for the past decades and has been explored from a variety of perspectives (Silver 1985). Examples of such perspectives can be traced to the 1970s such as 'means–ends' analysis (Simon and Newell 1971), a cognitive and metacognitive approach (Jacobse 2012; Schoenfeld 1985; Tiong et al. 2005; Yimer and Ellerton 2006). A more recent piece of work by Jonassen (2011) proposes problem-solving learning environments, in which problems are precisely classified and linked to explicit heuristic strategies.

Such learning environments are described as 'socio-constructivist learning environments' (Arslan and Altun 2007) in accordance with the concept of 'socio-cultural norms' as used by Sullivan et al. (2003). They draw attention to the fact that the usually recommended learning environments may be alien to some groups of pupils and the teacher will have to be active in overcoming these barriers.

In our research, we focus on the use of heuristic strategies in problem solving. There are many studies focusing on pupils' ability to acquire various heuristic strategies and measures of such ability (Herl et al. 1999; Meier 1992; Schoenfeld 1982; Szetela 1987; Szetela and Nicol 1992; Wu and Adams 2006; Zanzali and Nam 2000).

Whether a pupil will be able to learn a selected heuristic strategy does not depend on the learning environment only, but also on the pupil's dispositions, mainly their attitude to problem solving. Students' attitudes to problem solving were investigated in earlier studies. McLeod (1989) claimed that the internal factors influencing the process of problem solving do not depend on an appropriate heuristic strategy only, but also on the stage the solver is currently at. Positive changes in pupils' attitudes can be observed through problem solving. Pajares and Kranzler (1995) also refer to the generally accepted belief that a pupil's self-efficacy in problem solving has a positive impact on their ability to solve the problem and also helps decrease mathematics anxiety. The same holds vice versa, where successful problem solving is reflected in the pupils' expectations which may in turn result in increased self-efficacy (Babakhani 2011).

Let us consider a situation in which a solver is trying to solve a problem but there are (objective or subjective) obstacles which keeps them from achieving the goal. At this point, the solver will use heuristic strategies as defined by Pólya (2004) and further developed by others (Larson 1983; Michalewitz and Fogel 2000; Schoenfeld 1985).

Elia et al. (2009) studied how excellent performers in mathematics solve non-standard mathematics problems. When discussing future research, they propose "It could be interesting for future studies to examine if the pattern between heuristic strategies and problem solving success changes when students receive systematic strategy training in non-routine problem solving." (p. 616). They suggest that more research in the area is needed with a higher number, a greater variety of problems and a larger sample. The influence of the pupils' age, level in mathematics and the learning environment needs to be taken into account. Another question arising from this research is the difficulty to measure a pupil's use of strategies.

Two of the frequent limitations of studies in mathematical problem solving has been reported as the size of the sample and the length of the experiment. This was confirmed not only by Arslan and Altun (2007) but also more recently by Hensberry and Jacobbe (2012). Similarly, Lester et al. (1989) claimed that the greatest positive influence on effective problem solving can be achieved only through long-termed strategic impact of the teacher on their pupils, this has also been supported by Higgins (1997).

The use of heuristic strategies

We understand problem solving as a cognitive process that can be carried out in three approaches depending on the solver's engagement, abilities and skills (see Fig. 1).

The first approach is referred to as *trial*. It is the most primitive way of dealing with a problem. It only requires a solver's external motivation. The solver does not ask if they solved the problem correctly. Their only goal is to 'solve the problem', usually once without any need to check that the solution is correct.

The second approach is referred to as *straight-forward way* and is based on the application of some acquired knowledge. The solver knows the solving process that is required, is able to recognize how they should use it and applies it.

The third approach is referred to as *using a heuristic strategy* (sometimes only *strategy*). The solver does not have the needed knowledge or does not know how to use it and therefore cannot solve the problem in a straight-forward way. The solver is motivated to solve the problem. It is heuristic strategy that allows them to solve the problem.

Choice of heuristic strategies for the experiment

Our understanding of heuristic strategies corresponds to the works of Pólya (2004) and Schoenfeld (1985). These strategies were modified to suit our needs and supplemented by several other strategies. Originally, there were 11 strategies. On the basis of short-term, 4-month long, experiments in schools (Novotná et al. 2014, 2015b), the following six strategies were selected for the long-term experiment:

- Systematic experimentation
- Guess–check–revise
- Working backwards
- Introduction of an auxiliary element
- Specification and generalization
- Analogy

We did not include the following strategies as they were seen as too difficult: omitting a condition, decomposition into simpler cases, generalization and specification, problem reformulation and using of invariant. Short-term experiments indicated that these strategies were difficult to 'implant' into the pupils' cognitive structures.

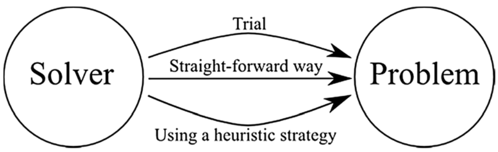

Fig. 1 Approaches to problem solving

Two-dimensional classification of the use of heuristic strategies

Our analysis of pupils' solutions of problems in the short-term experiments indicated the following phenomenon: different pupils, when solving the same problem while using the same strategy, use different modes. One pupil, for example, used drawing for the solution, another pupil drew a different picture, another pupil introduced a variable and another one solved it purely arithmetically. We realized that an important aspect in problem solving is not only the heuristic strategy employed but also the mode of its use. This approach is not unique in mathematics. Expressing a given fact by different means allows pupils to grasp the core of the issue studied and may give many pupils a better insight into other problems. In this respect, we find the writing of Nelsen (1993, 2000) seminal; he expresses, for example, algebraic identities and proofs, using graphical representations. The benefits of using various representations and the ability to use them are also mentioned in a number of curricular documents (e.g. ACARA 2014; NCTM 2000).

This led us to create a two-dimensional classification of the use of heuristic strategies in problem solving. If the solver is acquainted with this classification, they can ask themselves whether the mode that came to their mind first is indeed the most suitable. Let us suppose we have a problem assigned, for example, as follows: "Decide which fraction is greater: $\frac{124}{125}$ or $\frac{125}{126}$?" The first mode of solving this problem coming to the solver's mind will probably be the arithmetical one. However, unless we use the calculator, the straight-forward way of solving the problem may be difficult. Experience shows that some solvers find it more natural to solve this problem using the strategy of analogy, in which they use an illustrative picture to see in which case we 'take away' a smaller part of the whole (see Fig. 2).

The question to the answer is not only "What strategy to use?" but also "How to use the selected strategy?" These two dimensions allow the solver to extend their repertoire of the ways of solving a problem. Let us note that a task can be solved via a number of ways within one strategy and one mode. The example being the above mentioned traditional fraction task.

Let us now introduce the two dimensions of the classification.

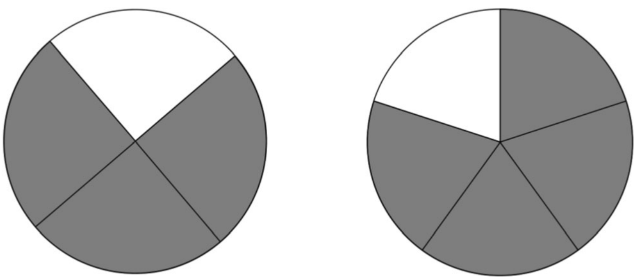

Fig. 2 Which fraction is greater? $\frac{3}{4}$ or $\frac{4}{5}$?

Heuristic strategies

Heuristic strategies are the first dimension of the use of heuristic strategies in problem solving. In the following paragraphs, we describe the six strategies that were investigated in the long-term experiment.

Systematic experimentation Systematic experimentation belongs to the family of experimental strategies. The principle of this strategy is the process of drawing closer and closer to the solution. The solver starts by selecting some value (either the first possible or some closer to the solution) and then works systematically. After each experiment, the solver checks whether the result is the sought solution. If not, they continue. The value for the next experiment depends on a given order (not on the solver's decision). This strategy is based on the fact that the solver is aware that the sought solution is in a chain of values with an underlying system and if this chain is followed, the solution will be discovered.

Guess–check–revise Guess–check–revise counts also as one of the experimental strategies. The principle of the strategy is drawing closer and closer to the solution when the solver makes a random choice within their first attempt. In the second step, they check whether the original guess was correct. If not, they try to assess how wrong they were. In the third step, they revise and correct, which generates a new guess and the whole process starts from the beginning again. The objective is to reach the goal using a finite number of directed iterations.

Working backwards Working backwards works as a frequently used strategy in mathematics when we know the final state and the initial state and try to proceed from the end to the beginning. The solution of the problem is based on 'turning' the found solution round—this is often used in problems from geometric construction. In our view, we do not perceive the solution based on the use of operations inverse to the basic arithmetical operations (the so-called number snakes) as working backwards—this type of problem has a straight-forward way to the solution and tests the knowledge of the inverse operation.

Introduction of an auxiliary element The basic idea of this strategy is that the introduction of an auxiliary element makes the solution much more easily accessible to the solver. We define an auxiliary element as an object not included in the question of a problem and which we insert into the problem, hoping it makes the solving procedure easier. In case of geometrical problems, this will usually be a straight line, a line segment, a point or a figure; in case of arithmetical or algebraic problems, it will be a number or a function.

Specification and generalization Within this strategy, we choose a specific value or position or we select a specific case, in the first stage. We solve the problem. If we can generalize the result of the problem, we formulate a hypothesis about the result of the original problem. We either leave the hypothesis on a plausible level or prove it (if the solver's abilities are sufficient). If we cannot make the generalization, we continue the solving process by another specification.

Analogy The use of this strategy is based on the fact that the solution of a problem 'analogical' to the original problem is simpler. The solver discovers the way to the solution and then applies the solution procedure to the original problem. What is important here is that the solver poses this analogical problem on their own. This is usually done by substituting objects in the problem by other objects. Some of the qualities of the objects remain intact, others are changed. For example, a sphere is substituted by a circle, greater numbers are substituted by smaller numbers, and an 8×8 chessboard is substituted by a 3×3 chessboard. The new objects in the analogical problem are called 'user-friendlier'.

The use of the individual heuristic strategies in solving problems is illustrated further in the section titled "Methodology". Some heuristic strategies are described in detail (Eisenmann et al. 2015; Novotná et al. 2015a, b).

Modes of use of heuristic strategies

The second dimension in our classification is the manner in which the heuristic strategy is performed. We speak of so-called *modes*:

- Arithmetical mode—is based on a numerical solution without introducing a variable
- Algebraic mode—one or more variables are introduced and the problem is solved using equations
- Graphical mode—is based on pictures and other graphical illustrations.

Culture of problem solving

In order to be able to describe a pupil's ability to solve problems, we introduced the so-called Culture of Problem Solving (CPS) construct. The phrase 'culture of problem solving' is found in several pieces of work (e.g. Clarke et al. 2007; Reiss and Törner 2007), where the word culture is not strictly defined and can be understood as a more cultivated approach to the studied phenomenon. Such authors as Clarke et al. (2007) link the word culture to the word inquiry—culture of inquiry. When forming the phrase CPS, the word culture was understood as a system of various meanings, activities and patterns of behaviour that can be met within problem solving at schools.

When composing the components of the structure, we drew on a previous works (e.g. Herl et al. 1999; Meier 1992; Schoenfeld 1982; Szetela 1987; Szetela and Nicol 1992; Wu and Adams 2006; Zanzali and Nam 2000). The work of Wu and Adams (2006) was the most relevant. Their problem-solving profile, conceived as a tool for changing a pupil's ability to solve problems, focuses on two components of the structure we developed: reading/extracting all information from the question (p. 97) and mathematics concepts, mathematization and reasoning (p. 100).

The ability to solve problems depends on a whole range of the solver's internal attributes. In cooperation with psychologists specialized in problem solving in education, we selected the following four attributes that we decided to focus on for the needs of our research. We are convinced these attributes have impact not only on the success rate in problem solving but also on the ability to learn to use heuristic strategies when solving problems:

1. Intelligence
2. Reading comprehension
3. Creativity
4. Ability to use the existing knowledge in mathematics

There are no doubts about the indispensability of including *intelligence* in the structure of CPS. As Wenke et al. (2005) state, from the inception of the concept of 'intelligence', the ability to solve problems has featured prominently in virtually every definition of human intelligence. In addition, intelligence has often been viewed as one of the best predictors of the problem-solving ability.

The second component was pupils' *ability to read texts with comprehension*. Obviously, this is one of the key competences without which successful problem solving would be impossible, as pointed out by a number of authors (Pape 2004; Schoenfeld 1992) and verified by Hite (2009). The inclusion of this component is based on Polya's four stages of solving a problem (Pólya 2004). The first stage is understanding the problem. The basis of solving any problem is understanding its structure which is connected with the ability to read the assignment of the problem with comprehension. This means that having read the assignment, the solver is able to grasp the relations in the problem, identify the initial and output variables of the problem and handle the input data in an appropriate way. Let us note that no pupils with diagnosed function disorders took part in the experiment.

The third component was *creativity*. The key role of creativity in problem solving is discussed by Bahar and Maker (2011) or Sriraman (2005). Nadjafikhah et al. (2012) speak of creative problem solving. "At the school level, creativity in mathematics is generally related to problem solving and or problem posing." (p. 290). Chamberlin and Moon (2005, p. 38) state that "Creativity refers to the domain-specific thinking processes used by mathematicians when engaged in non-routine problem solving." Sak and Maker (2005) assume that creativity is displayed in problem solving. Creativity is measured in four components—fluency, originality, flexibility and elaboration.

The fourth component was the pupils' *ability to use their existing knowledge*. This ability was considered pre-requisite to successful solving of non-routine problems. When solving such kind of problem, knowledge is not sufficient; the solver must be able to use it.

We measured the pupils' progress in three of these—all the pupils were tested before the teaching experiment and after it with the exception of an intelligence test. The intelligence was measured only at the beginning of the experiment, as the test of intelligence chosen for the experiment does not anticipate any change over the period of the teaching experiment. The aim of testing this CPS component was to measure pupils' intelligence as one of the factors influencing their ability to solve problems—not to specify the change of the indicator within the effects of the teaching experiment.

With respect to an individual pupil, we find the use of CPS in teaching important in three areas. The knowledge of a pupil's CPS can help the teacher select appropriate problems that the pupil will be able to solve successfully. Also, it may help eliminate a pupil's weaknesses that may be an obstacle to solving problems. Knowing the pupil's CPS may help the teacher decide which heuristic strategies and in what depth can be handed over to the pupil.

Research questions and hypotheses

The following questions were posed at the beginning of our experiment:

1. Which strategies do pupils choose spontaneously before starting the experiment?
2. Which strategies are pupils able to acquire with the teacher's guidance and use actively?
3. How successful are pupils using the strategies at the end of the experiment?
4. Do some of the components of the pupils' CPS change as a consequence of participation in the experiment?
5. How does a pupil's ability to solve problems and use heuristic strategies relate to the individual components in CPS?

At the beginning of our experiment, we formulated the following two hypotheses:

1. After the long-term experiment, pupils will be able to use some heuristic strategies in problem solving actively and to a greater extent.
2. After the long-term experiment, pupils will have markedly better results in all components of CPS.

Methodology

Pupils and teachers

The study was conducted in three classes from three schools in the Czech Republic. The teaching experiment was conducted from September 2012 to February 2014. It was conducted with 62 pupils aged between 12 and 18 years; drawn from

- Grammar school[1] in Hořovice, 24 pupils aged 12–14 years, teacher Eva (58 years old)
- Secondary school in Ústí nad Labem, 18 pupils aged 14–16 years, teacher Martin (49 years old)
- Grammar school in Prague, 20 pupils aged 16–18 years, teacher Jan (32 years old)

[1] Secondary school is a common school for children aged 11–16. Grammar school is a selective school with attendance from 11 years of age (eight-grade grammar school) or at 16 years of age (four-grade grammar school). Approximately 37 % of children in the Czech Republic attend grammar schools.

Jan and Eva's (the teachers' names were changed) pupils were from general, comprehensive classes. Martin's pupils attended a class with a particular focus on mathematics education. They were divided into this class at the age of 11 years based on performance in a mathematics test (about every third pupil from the school). The pupils from the different classes were performing at more or less the same level as far as school mathematics is concerned.

All three teachers can be described as engaged; they invest energy into their teaching and have attended in-service teacher training courses. Their teaching styles are similar. All teachers were paid for their work in the experiment.

They introduced their pupils to the use of the heuristic strategies outlined earlier through the solving of problems for the entire period of 16 months.

Development of problems

When planning the teaching experiment, we were aware of the limitations that teaching problem solving in a one period block presents (Fan and Zhu 2007; Higgins 1997). The research in this area shows that if problems are included in only one block, pupils perceive it as an isolated unit which is not connected to other parts of mathematics. This can often result in purely formal knowledge of problem solving, which is contrary to what we wanted to achieve in heuristics. Thus, we created 160 problems illustrating the use of individual heuristic strategies in such a way that teachers could integrate them naturally into their planned teaching units during the whole experiment.

Pilot work

The problems were debugged in short-term experiments conducted in other schools, before starting the teaching experiment, and also simultaneously with the teaching experiment in its first year. Over the period of 4 months, teachers in another 12 classes attempted to teach their pupils two selected heuristic strategies. That involved a total of 290 pupils. The goal was to gain experience from the trial that would guarantee a smooth process in the main, long-term experiment and to create a set of high-quality suitable problems illustrating each of the heuristic strategies. Further details of these short-term experiments are described by Novotná et al. (2014, 2015b).

Example of a problem

Each heuristic strategy was represented with approximately 20 problems. Each problem offered a straight-forward way to a solution, also a solution using a given strategy (strategy that is efficient for solving the particular problem) and at least one more solution using a different heuristic strategy. The task to follow illustrates this approach.

Problem: An employee's monthly salary is 15,755 CZK. During the year, he got a pay rise of 2,100 CZK per month. In which month did he receive his pay increase if his annual income was 195,360 CZK?

Solution:

1. Straight-forward way:

   The problem can be solved with an equation (algebraic mode). Let x be the number of months when the employee was earning a higher salary.

$$\begin{aligned} 12 \times 15755 + 2100x &= 195360 \\ 2100x &= 6300 \\ x &= 3 \end{aligned}$$

Answer: The employee's monthly salary was higher since October.

Another possibility is the following calculation that determines how many months the employer's salary was higher (arithmetical mode).

$$\frac{195360 - 12 \times 15755}{2100} = 3$$

2. Using heuristic strategy:

   Some pupils find it difficult to set up the equation or do not have the potential to 'see' the arithmetic calculation. The way out of this impasse may, for some pupils, be the use of one of the following heuristic strategies.

   (a) Guess–check–revise:

   If the salary increase had been applied from January, the annual income would have been 214,260 CZK. This is more than the employee's real annual income. Let us now look at what would have happened if he had received the pay rise in July. If this were the case, he would have earned 201,660 CZK. This is still more than his real annual income. Let us try and give him a pay rise in November. Then, his annual income would have been 193,260 CZK. That is too little. So, if we give him the pay rise 1 month earlier, that is, in October, his annual income would be 195,360 CZK. This is the sum that corresponds to the original problem.

   (b) Systematic experimentation:

   We proceed with a systematic list of all possibilities recorded in Table 1.

   The potential of this strategy becomes the most efficient if a spreadsheet is used.

   Spreadsheet is a pedagogic medium here, allowing the teacher to realize a scale of arithmetic operations with their pupils and therefore improving the insight into the studied problems (Calder et al. 2006; Haspekian 2014).

Table 1 List of all possibilities

Pay rise since	Number of months with original salary	Number of months with higher salary	Annual income
January	0	12	214,260
February	1	11	212,160
	...	...	...
October	9	3	195,360
	...	...	...
December	11	1	191,160

(c) Analogy:

Let us formulate the problem with 'user-friendlier numbers'. Its solution is easy and indicates the solution to the original problem.

Assigning an analogical problem: An employee's monthly salary was 10,000 CZK. During the year, he got a pay rise of 5,000 CZK. Since which month was his monthly salary higher if his annual income was 150,000 CZK?

The solution is easier to see here:

$$\frac{150000-12 \times 10000}{5000} = 6$$

Answer to the analogical problem: The employee's monthly salary was higher since July.

Let us now 'copy' this calculation and transfer it into the original problem:

$$\frac{195360-12 \times 15755}{2100} = 3$$

Answer: The employee's monthly salary was higher since October.

Lesson design in the main teaching experiment

The substantial difference between usual lessons the pupils were used to and the teaching experiment was the way in which problems were solved with the pupils. The process was always as follows:

The teachers presented the problem to their pupils (mostly in written form, on a worksheet). The pupils were allowed to work and after some time (when at least one half of the pupils had solved the problem), they were asked to present the solution to the others. Then, they checked the rest of the class had understood the solution that was presented and invited the pupils to explain their own solution if was an alternative solution. If the expected solution did not appear among the solutions presented, the teacher demonstrated it to the pupils. The pupils were always encouraged to look for more ways of solving a particular problem and record their problem-solving procedure. This means the solving of the problem did not finish when the result had been found. In discussions, the pupils were asked to justify their procedures chosen. This was critical as it does not only develop pupils' communication skills but also improve their ability to solve problems. Sometimes, the problems were set for homework. However, the way of working with the solutions at school was consistent.

Obviously, as a consequence to this approach to problem solving, there was more time devoted to solving and less time devoted to instruction. Also, the overall number of problems solved during the experiment was lower than in usual lessons.

One more difference to the usual lessons was that the teachers were teaching their pupils to recognize the used heuristic strategies and distinguish among them. Their aim was to teach pupils to use the strategies in problem solving.

Teacher engagement

The cooperation of the participating teachers with the members of our team was intense and systematic. Each of the three teachers had one partner from the research team. The close cooperation covered a 2-year period.

The teacher always collected the pupils' worksheets with the solutions and evaluated them. The worksheets, individual problems and strategies used along with the individual pupils' responses were then discussed at regular meetings with one member of the research team. These meetings were usually held once every 2 weeks. The teachers also sent a brief report by email once a week. The members of the research team had access to the pupils' worksheets during the whole experiment. They used those for enriching the existing problems with new procedures that were spontaneously developed in the lessons. Moreover, the worksheets served as feedback with respect to the success rate of the solutions.

The teachers were instructed in recording observations of their pupils. Their focus was on the changes that could be observed in the pupils' performance in mathematics, their attitude to problem solving and to learning mathematics in general. These changes were then reported during the meetings with the responsible members of the research team.

The cooperating member of the research team also observed a lesson in the experimental class once every 6 months. Additionally, two lessons were recorded with video. The main purpose of the video recordings was to show the atmosphere in the other two classrooms to other members of the research team and also verify the reflections from the observations.

Initial and final tests

Design of the initial and final tests

All the pupils sat a test at the beginning of the teaching experiment and at the end of it. The test consisted of eight problems for each of the classes. The tests were different for each of the classes, with respect to the pupils' age level and knowledge. Fifteen different problems were used in the tests; some of them were used just in one, others in two or even all three tests. The initial and final tests were identical. The test problems were not presented to the pupils during the 16-month-long experiment and were not discussed even after the initial test.

The test problems were chosen after a multistage pre-test with pupils of the same age, not involved in the long-term experiment. The majority of the problems could be efficiently solved using two different heuristic strategies and all the problems could be solved in a straight-forward way. Table 2 shows the number of problems that could be solved using the given strategy.

Examples of problems

To illustrate, let us present herewith one problem for each age group, with a brief description of the solution using the strategy that is regarded as an efficient way of solution. We also add a note in which other possibilities of solving the problem are mentioned.

Table 2 The number of problems that can be solved using individual strategies

Strategy	Ages 12–14 years	Ages 14–16 years	Ages 16–18 years
Systematic experimentation	3	3	3
Guess–check–revise	3	3	3
Analogy	3	2	3
Introduction of an auxiliary element	2	3	2
Working backwards	2	2	2
Specification and generalization	2	2	2

Ages 12–14 years (teacher Eva):

Problem: A shopkeeper bought a book at one seventh of the original price and sold it for three eighths of the original price. What was the shopkeeper's percentage profit?

Solution: Let us now specify the problem and let us presume that the original price was, for example, 56 CZK. This means the shopkeeper bought it for 8 CZK and sold it for 21 CZK. His profit is easy to calculate.

$$21-8 = 13$$

The percentage profit is:

$$\frac{13}{8} \times 100 = 162.5$$

Based on our specification, we got one result. If we choose several different prices, we can easily verify that the choice of the original price has no effect on the result. This allows us to generalize the result.

Answer: The shopkeeper's profit was 162.5 %.

Note: This problem can also be solved efficiently using the strategy of analogy (formulating the problem with 'user-friendlier numbers'). The problem can also be solved in a straight-forward way (using an equation).

Ages 14–16 years (teacher Martin):

Problem: Peter comes home and asks his brother: "First I lost one quarter of my marbles and then I lost one quarter of those that I had left. Now I have only 18 marbles. How many did I have to begin with?"

Solution: Let us solve this problem using the working backwards strategy. This means we start from the end. At the end of the game, Peter had 18 marbles, which was $\frac{3}{4}$ of the marbles he had had before his second loss. Peter entered the second game with $\frac{4}{4}$ of the marbles, i.e. with 24 marbles. These represent $\frac{3}{4}$ of the marbles with which he had played the first game. Peter started the first game with $\frac{4}{4}$ of the marbles, i.e. that is with 32 marbles, which was the original number.

Answer: Peter had 32 marbles.

Note: This problem can also be solved efficiently using the guess–check–revise or systematic experimentation strategies. It can also be solved in a straight-forward way (using an equation).

Ages 16–18 years let (teacher Jan):

Problem: Let *ABC* be an arbitrary triangle. Points *P*, *Q*, *R* and *S* divide sides *AC* and *BC* into three congruent parts (see Fig. 3). What is the ratio of the areas of figures *PQSR* and *ABC*?

Solution: Let us use the strategy of introduction of an auxiliary element. Let us divide triangle *ABC* with the help of auxiliary elements—line segments parallel to sides *BC* and *AC* into nine congruent triangles as is shown in Fig. 4. Trapezoid *PQSR* is covered by three of them. Let us point out here that there are other possible auxiliary elements useable in the solution (Nováková and Novotná 2015).

Answer: The area of quadrangle *PQSR* is $\frac{1}{3}$ of the area of triangle *ABC*.

Note: This problem can also be solved efficiently using the strategy of specification and generalization, namely by choosing a special case when triangle *ABC* is a right triangle with the right angle at point *C*. We will get the result by comparing the areas of right triangles *ABC*, *PRC* and *QSC*, which are also similar in the case of an arbitrary triangle. The problem can also be solved in a straight-forward way (using the knowledge of how to determine the area of a triangle and a trapezoid).

Test evaluation

All the problems from the test were analysed and assessed in detail. Each solution was coded by a member of the research team with respect to the following phenomena:

- Approaches to problem solving (straight-forward way or using a heuristic strategy)
- Modes of use of heuristic strategies
- Success rate in problem solving (successfully/unsuccessfully)
- ‘Blank sheet’ (the pupil did not even try to solve the task)
- Non-evaluable response
- Misunderstanding the question

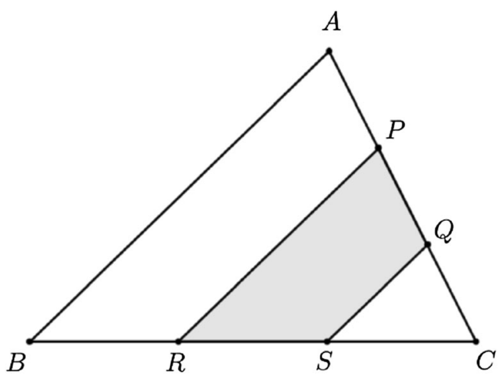

Fig. 3 The trapezoid in a triangle

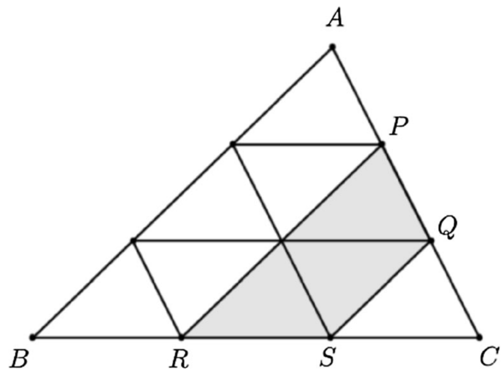

Fig. 4 Auxiliary elements in triangle *ABC*

Culture of problem solving

The first three components of CPS (intelligence, reading comprehension, creativity) were assessed by psychologists. A psychologist came to all three participating classes before the teaching experiment and after it. The psychologist spent two units of teaching collecting data, which were then used for evaluation of these components of CPS. The method of measuring the component of ability to use the existing knowledge in mathematics was developed by the research team. The methodology of the assessment of the above CPS components are described in detail by Eisenmann et al. (2014).

Intelligence

Pupils' intelligence was tested by the Váňa intelligence test (VIT). This test is regarded as valid in the Czech Republic and respects the specifics of the Czech school environment. This test was selected because of its verified correlation with pupils' school performance (Hrabal and Hrabal 2002). It is a verbal intelligence test, suitable especially for group testing. It is used for assessment of the level of development in cognitive abilities of individuals, especially in research situations, requiring collection of basic data about the pupils. The result is given by the number of points achieved.

Reading comprehension

The pupils were set a short text (one paragraph) which they were asked to summarize in four lines without changing the meaning and content. In other words, the goal was to pinpoint the key information. The pupils' text was assessed on the scale of 1 to 5, where 1 was excellent and 5, poor result.

Creativity

Creativity is, for the purpose of our study, understood in the context of divergent thinking. In accordance with Sternberg (2005), we do not perceive creativity as a single attribute but a set of attributes, and with respect to the study, we selected a set of strategies to focus on. However, as enquiry into creativity is not our basic goal, we limited it to the structure adapted from Meador (1997).

The level of creativity was measured using Christensen–Guilford test (Kline 2000), covering divergent thinking. The pupils proposed as many uses of common objects as possible. What is important here is how logical and practicable the answers were.

The Christensen–Guilford test measures four dimensions:

- Fluency—how many relevant uses the pupil proposes
- Originality—how unusual these uses are
- Flexibility—how many areas the answers refer to
- Elaboration—quality and number of details in the answer

Qualitative evaluation of each dimension was translated into points, and the total score indicated an index of creativity. The higher the index, the more creative the pupil is purported to be.

Ability to use the existing knowledge

This component of CPS was assessed on the basis of a set of problems developed by the research team. It was tested in a written form. Dyads of problems were used for this testing—the first problem to find out whether a pupil has a particular piece of knowledge and the other whether the pupil can use or apply it. The pupils sat tests consisting of four such dyads before and after the experiment. The problems in the initial and the final tests were different. They tested presence of knowledge (the first problem in the dyad) and ability to use this knowledge (the second problem in the dyad). This knowledge was related to what the pupils were expected to have learned in the period of 6 months before sitting the test. Each of the problems was evaluated either as correctly (R) or incorrectly (W) solved. Thus, each pupil was assigned, for every pair of tests (final/initial), two ordered quartets of numbers corresponding to the frequency of codes RR, RW, WR and WW. These data were then processed using the formula

$$\mathrm{AUEK} = (\mathrm{RR}_{\mathrm{fin}}\text{-}\mathrm{RR}_{\mathrm{in}}) + 3 \times (\mathrm{RW}_{\mathrm{fin}}\text{-}\mathrm{RW}_{\mathrm{in}}) + 2 \times (\mathrm{WR}_{\mathrm{fin}}\text{-}\mathrm{WR}_{\mathrm{in}}) + 2 \times (\mathrm{WW}_{\mathrm{fin}}\text{-}\mathrm{WW}_{\mathrm{in}})$$

and each pupil was assigned an index of change indicating the ability to use their existing knowledge. Obviously, the greater the number, the more this pupil's ability had improved. The difference in the brackets shows the change in individual cases. The difference $\mathrm{RR}_{\mathrm{fin}}\text{-}\mathrm{RR}_{\mathrm{in}}$ describes the change in the number of present and correctly used knowledge items between the initial and the final tests. The different coefficients of combination represent the weight assigned to the differences in order to distinguish between three levels. Of most interest is the situation when the knowledge is (formally) present but the pupil cannot use it while solving the problem. The least interesting is the situation when the knowledge is present and the pupil can use it while solving the problem. The other two situations where the presence of knowledge is not proven are assessed identically since this is not an indicator we would be monitoring.

Results and discussion

Success in problem solving

In this section, we focus on two phenomena arisen from analysis of initial and final tests: correctness of problem solving and the frequency of a blank sheet.

As far as the correctness of problem solving is concerned, we measured the frequency of occurrence of correct solutions of problems in the initial and the final tests. The number of correct solutions increased in all the participating classes. The following figures in Table 3 express the proportion of correctly solved problems from all problems set to pupils in the initial/final tests (with respect to the large number of the assigned problems, it seems more convenient to express this value using percentage figures).

These results allow us to answer one of the research questions, namely "How successful will pupils be when using the strategies at the end of the experiment?"

The success rate in problem solving was in all cases higher at the end of the experiment (the problems in the initial and final tests were identical). It can be expected that in the period of 16 months, the success rate would also increase naturally without the experiment as a consequence of the pupils' personal development as well as their development of knowledge and skills (not only in mathematics). The tools used in the experiment do not allow us to separate completely the impact of the teaching experiment and the pupils' natural development. However, we are convinced that we can observe a positive impact of the teaching experiment on the pupils' increased success rate. We explain this at the end of the following subsection, where we draw attention to the relation among pupils' success rate, the use of heuristic strategies, the frequency of use of graphical mode in solving problems and the frequency of blank sheets.

The other monitored aspect was the frequency of blank sheet, that is, the situation in which the pupils did not even make an attempt at solving the problem. The results are presented again in the form of relative frequency (see Table 4).

It was expected that the frequency of unsolved problems dropped in all classes. However, it is hard to say whether this was a consequence of the experiment or other influences. Still, it can be assumed that the long-term experience of using suitable heuristic strategies played a role in the pupils' decision to at least attempt a solution. This phenomenon is connected to the change in pupils' attitude to problem solving in general (this will be discussed in one of the following subsections).

Table 3 Frequency of correct solutions of problems in the tests (%)

	Initial	Final
Eva	44	58
Martin	54	84
Jan	45	76

Table 4 Frequency of blank sheet in the tests (%)

	Initial	Final
Jan	7	4
Eva	17	8
Martin	16	4

The use of heuristic strategies

The overall frequency of the strategies used in the initial and final tests is described in Table 5 as follows: The figure 31 in the third row and second column indicates that in all 8 problems in the initial test some heuristic strategy was used in 31 cases in Jan's class. The relative frequency 19 % in the third row and fourth column is the result of the calculation of $\frac{31}{20\times8}$, where 20 stands for the number of pupils in Jan's class.

Let us now turn our attention to the individual heuristic strategies used at least once in the solutions of the problems. As the representation of the singular strategies that can be used to solve the problems in the initial and the final tests is about the same in all the classes (see Table 2), we will comment on the results of all pupils globally in the following Table 6. This is legitimate also due to no significant differences among the involved classes' school performance in mathematics, the teaching experiment in all the participating classes was conceived similarly and the teachers' teaching styles were similar. The figures in the second and the fourth columns give the overall frequency of occurrence of the individual strategy in the initial/final tests, the figures in the third and fifth columns, the number of successfully used strategies (correctly solved problems). Due to the relatively low rate of occurrence, we only use the absolute frequency in this case.

We can observe an increase in the use of all heuristic strategies.

Experimental strategies (systematic experimentation, guess–check–revise) and the strategy of working backwards were chosen by the pupils spontaneously also at the beginning of the experiment, that is, before the teaching experiment started.

The considerable increase in the use of heuristic strategies was in cases of systematic experimentation and introduction of an auxiliary element. A multiple increase in the frequency of use could also be observed in cases involving strategies of analogy and specification and generalization. However, we cannot interpret this result as significant due to the low absolute frequency of use of these strategies.

The pupils were almost always successful at the end of the experiment when using the strategies of systematic experimentation, analogy, working backwards and

Table 5 Frequencies of the used strategies

	Absolute frequency		Relative frequency (%)	
	Initial	Final	Initial	Final
Eva	22	55	11	29
Martin	7	31	5	22
Jan	31	45	19	28

Table 6 Frequency of the use of and success rate in the use of individual strategies

Strategy	Initial test		Final test		Number of problems that can be solved using the given strategy
	Overall frequency	Frequency of successfully used strategies	Overall frequency	Frequency of successfully used strategies	
Systematic experimentation	19	16	37	32	9
Guess–check–revise	20	10	27	19	9
Analogy	2	2	9	9	8
Introduction of an auxiliary element	6	4	33	19	7
Working backwards	10	6	13	13	6
Specification and generalization	1	1	8	7	6

specification and generalization. The high success rate of experimental strategies is also reported in the case of more difficult problems (the authors label them as 'Insight problems'), which is documented by Martinsen and Kaufmann (1991).

As far as the strategy of introduction of an auxiliary element is concerned, about one half of the pupils were successful in the final test. Based on observations of the pupils, the following can be seen as the reason—it is relatively easy to start drawing auxiliary elements in geometrical problems. Some students may find it difficult to complete the solution of the problem with success, using the auxiliary elements as suggested. The teachers also report that a relatively large number of problems need to be solved with the pupils with this strategy, similarly as in the case of the strategies of analogy and specification and generalization.

Based on an analysis of records of observations of the pupils, the following pupils' attitudes to the individual strategies could be identified:

- Systematic experimentation and guess–check–revise can be used with a great variety of problems and its use is simple.
- Working backwards is in some problems the easiest way of finding the solution.
- When using the strategy of introduction of an auxiliary element in geometry, it is helpful to make illustrative pictures and mark in these pictures as much as possible.
- When using the strategy of analogy, it works well to pose a simpler problem with 'user-friendlier numbers'. This helps the solver realize how to solve the original problem.

Similar positive pupils' evaluation of inclusion of heuristic strategies is stated by Arslan and Altun (2007); the pupils involved in their research experiment claimed that "the studies with non-routine problems improved their thinking" (p. 58).

As far as the modes of using heuristic strategies are concerned, there was an increase in the use of graphical mode between the initial and the final tests. The results are presented in Table 7. It indicates what proportion of the problems was solved using the different modes.

Table 7 Frequency of using individual modes (%)

	Arithmetical mode	Algebraic mode	Graphical mode
Initial test	65	21	3
Final test	47	30	18

Let us remark here that the possibility of solving the problems using different modes was more or less balanced in all age categories. The sum of relative frequencies does not equal 100 %, as in problems coded as blank sheet and non-evaluable response, it is impossible to classify the mode of solution.

The growth in the use of algebraic mode is caused by the fact that, at the time of the teaching experiment, Martin's pupils appropriated the use of equations and used it frequently in two problems in the final test. The increase in the use of graphical mode in the final test is the same in all three classes, namely in two situations: when solving problems using the strategy of introduction of an auxiliary element and when solving the problem in a straight-forward way, namely using an illustrative drawing (as shown in Fig. 2 in the problem with comparing fractions). We assume the increase in the use of graphical mode may signal a deeper understanding of the problem in the sense of visual thinking, as used by Nelsen (1993, 2000).

Let us conclude this subsection by looking at the previously mentioned relationship between the use of heuristic strategies and pupils' success rate in the tests. The increase in pupils' success rates between the initial and the final tests can be perceived as the effect of the following interlinked facts: We can observe an increase in the frequency of the use of heuristic strategies in the final tests (moreover, the pupils were almost always successful when using four out of the six taught strategies at the end of the experiment). We can also observe an increased success rate in the straight-forward way of solution as a consequence of the use of a graphical mode. A decrease in blank sheets could also be observed. These facts account for some of the relations between the two-dimensional classification of the use of heuristic strategies and problem solving.

Culture of problem solving

Unlike with the previous results from initial and final tests, we present here the indicators of the different components of CPS for each class separately. The reason is that these can be interpreted with respect to age.

Intelligence

The following Table 8 shows the range of pupils' VIT index in the individual classes and their arithmetic mean:

Jan's and Eva's pupils were above average, Martin's pupils average (average is given by the VIT index 90–109).

The analysis of the tests proves the following correlation of the VIT index to use of heuristic strategies in the experimental sample: The pupils who used strategies of analogy and specification and generalization in the final test had a higher VIT index (above 120). It can be assumed that these strategies are a suitable tool for solving

Table 8 VIT index

	Minimum	Maximum	Arithmetic mean
Eva	100	145	129
Martin	90	125	106
Jan	109	141	121

problems for pupils with higher VIT indexes. The internal structure of these strategies suggests the need of more pre-requisite knowledge than required by other strategies (Eisenmann and Přibyl 2014).

Reading comprehension

The following Table 9 shows the average value of the reading comprehension index for each of the classes at the beginning and at the end of the experiment, where 1 was excellent and 5, poor result.

The best results in this component of CPS are in Eva's class. The level of all the other pupils is about the same. The ability to read texts with understanding improved in all cases. This improvement is about the same and inconsiderable and cannot be interpreted as an unequivocal outcome of the teaching experiment. Our educational experiment never aimed at reading comprehension improvement and the increase shown is therefore to be accounted on the natural maturing in pupils.

There were 11 pupils whose reading comprehension improved considerably. In eight cases, the following happened: In comparison to the initial test, there were fewer problems in the final test coded as misunderstanding the question. This code was used to describe of situations when the solver did not understand the question of a problem; consequently, they made a related mistake.

Creativity

The following Table 10 shows the average value of creativity index at the beginning and at the end of the experiment:

The input values correspond to the pupils' age. The youngest pupils—Eva's class—have the lowest score, the oldest pupils—Jan's—have the highest score. This complies with cognitive psychology which states that one of the factors of creativity is linked to intellectual abilities, which change considerably at this age in the sense of growth of measurable gross scores. Another factor—personal characteristics—also develops at this age, especially qualitatively.

Table 9 Average value of the reading comprehension index

	At the beginning	At the end
Eva	2.2	2.0
Martin	3.1	3.0
Jan	3.1	2.8

Table 10 Average value of the creativity index

	At the beginning	At the end
Eva	10	31
Martin	15	35
Jan	20	44

All classes show a significant increase in the creativity index, which doubled or in Eva's case, more than tripled. The psychologists conducting this survey claim that the above-mentioned increases are considerably higher than what natural increase related to the increase of the age of the pupils would indicate. One of the sources of this development may be the teaching experiment. The fact is that one of the factors that facilitates creativity considerably is an intellectually rich and supportive environment (Getzels and Jackson 1962; Lubart 1994; Wittmann 1995).

We assume one of the reasons for the increase in creativity may be attributed to the manner in which pupils worked when solving problems. The pupils were not only asked to find the result, teachers wanted the pupils to solve the problems in numerous ways and to compare the efficiency of the used strategies. The pupils were introduced not only to different strategies for solving problems but also to different modes for their use. The teachers encouraged the pupils to consider problems from various perspectives. Thus, their creativity was stimulated.

A more detailed analysis indicates that the most significant growth can be observed in our research sample in the area of fluency and flexibility (the growth of this index is triple the average).

There is one more interesting fact to point out with respect to pupils' results in the initial and the final tests. The pupils whose creativity improved considerably (the index grew about three times) often selected the strategy of introduction of an auxiliary element in the final test.

Ability to use the existing knowledge

The following Table 11 shows the range of pupils' AUEK index in the individual classes and their arithmetic mean. The AUEK index describes the change indicating the ability to use pupils' existing knowledge. The greater the number, the more these pupils' ability had improved.

The results are divergent. In case of Martin's class, there was no difference. Jan's class got slightly worse while Eva's class considerably better. It is hard to interpret these results. Unlike other CPS components, the ability to use the existing knowledge is to a great degree influenced by non-cognitive factors (the teaching style and method in other subjects, atmosphere in the school and class, extracurricular activities etc.)

Table 11 AUEK index

	Minimum	Maximum	Arithmetic mean
Eva	−1	6	3
Martin	−3	3	0
Jan	−4	1	−0.6

However, there is one correlation between two CPS components: The pupils with higher intelligence index (above 115) have, at the same time, a higher index in the ability to use their existing knowledge.

Changes of pupils' performances and attitudes towards solving problems

Changes in pupils' attitudes to problem solving and mathematics teaching in general were observed as secondary results of the research. The following findings were formulated on the basis of long-term observation of the pupils during the whole experiment and comparison of performance in the initial and final tests:

- The willingness to experiment increases
- Attitude to mathematics improves
- More attention is paid to feedback (checking the result)
- The ability to communicate with others, to justify and explain their solution, react to their opponent's arguments develops
- The ability to record their solutions improves
- The pupils stopped fearing problem solving, they did not put them off if they could not see the solving procedure at once

Impact of the experiment on the participating teachers

Also, the following changes in teachers were observed as a secondary result of the experiment. They were formulated on the basis of long-term cooperative work with the teachers and of analysis of their regular reports from the whole experiment:

- Eva and Martin lowered their demands on accuracy and correctness in their pupils' communication and recording in favour of understanding the problem solving procedures.
- Eva and Jan showed more tolerance to varieties in pupils' solutions.
- Eva and Martin acknowledged a change in their attitude to mathematics teaching to using constructivist and inquiry-based approaches.
- Jan and Eva grew more interested in pupils' thinking processes while solving problems.

One of the essential results in this area is that the teachers started to pose their own problems with the aim of improving the pupils' understanding of various strategies.

Conclusions

This study demonstrates that the pupils used all the taught heuristic strategies more often after engaging in the teaching experiment. Experimental strategies (*systematic experimentation*, *guess–check–revise*) and the strategy of *working backwards* were almost exclusively the only strategies used spontaneously by the pupils before the teaching experiment. The largest increase in the use of heuristic strategies was observed with the strategies of *systematic experimentation* and *introduction of an auxiliary element*. The pupils were almost always successful at the end of the experiment when using the strategies of *systematic experimentation*, *analogy*, *working backwards* as well as *specification and generalization*.

The research shows that if pupils are to be taught to use some heuristic strategies, they ideally solve a relatively high number of problems. This applies to the strategies of *introduction of an auxiliary element*, *analogy* and *specification and generalization*.

The components of CPS considered, all the pupils showed some but moderate improvement in the *reading comprehension* component. The pupils from all classes improved in the component *creativity* considerably. A more detailed inquiry shows the highest degree of improvement in the area of fluency and flexibility. The pupils whose creativity improved considerably (the index grew about three times) often selected the *introduction of an auxiliary element* strategy in the final test.

Based on long-term observations of the pupils during the experiment, it can be concluded that pupils grew more willing to experiment and improved their ability to communicate with others, to justify and clarify their solutions and to react to opponents' arguments. They also improved their ability to record their solutions and pay more attention to feedback.

What we consider to be one of the most important outcomes is the change in pupils' attitude to problem solving in general. The pupils ceased to fear problem solving and did not avoid such problems if they could not identify the solving procedure immediately. This could be observed with roughly one half of the participating pupils. This outcome was formulated on the basis of long-term observation of the pupils during the whole experiment and comparison of performance in the initial and final tests.

On the basis of the knowledge gained in the long-term cooperation with the teachers and mentoring them during the whole experiment, it can be concluded that some of them show now more tolerance to varieties in pupils' solutions, they are keener on their thinking processes while solving problems and admit their attitude to teaching mathematics has changed for more a constructivist, inquiry-based approach. One of the major results in this area is that the participating teachers started to pose their own problems with the aim of improving students' understanding of various strategies.

Future research

It would be interesting to verify the findings and conclusions on homogeneous and larger samples of pupils, for example, to work with three similar classes of 15-year-old pupils. It would also be interesting to try and teach pupils to use those strategies that we did not manage to teach. One of the possible ways might be to start from a carefully created larger set of suitable problems.

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