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ORIGINAL ARTICLE

Visual and analytical strategies in spatial visualisation: perspectives from bilateral symmetry and reflection

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Abstract This inquiry presents two fine-grained case studies of students demonstrating different levels of cognitive functioning in relation to bilateral symmetry and reflection. The two students were asked to solve four sets of tasks and articulate their reasoning in task-based interviews. The first participant, Brittany, focused essentially on three criteria, namely (1) equidistance, (2) congruence of sides and (3) 'exactly opposite' as the intuitive counterpart of perpendicularity for performing reflection. On the other hand, the second participant, Sara, focused on perpendicularity and equidistance, as is the normative procedure. Brittany's inadequate knowledge of reflection shaped her actions and served as a validation for her solutions. Intuitively, her visual strategies took over as a fallback measure to maintain congruence of sides in the absence of a formal notion of perpendicularity. In this paper, we address some of the well-known constraints that students encounter in dealing with bilateral symmetry and reflection, particularly situations involving inclined line of symmetry. Importantly, we make an attempt to show how visual and analytical strategies interact in the production of a reflected image. Our findings highlight the necessity to give more explicit attention to the notion of perpendicularity in bilateral symmetry and reflection tasks.

Keywords Spatial visualisation · Bilateral symmetry · Reflection · Perpendicularity · Visual strategies · Analytical strategies

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Introduction

Spatial visualisation

It is widely recognised that visualisation is an important process or skill in making sense of mathematical information (Arcavi 2003; Zimmerman and Cunningham 1991). It may be involved in thinking about a number pattern made of an arrangement of dots, in picturing a Venn diagram, in considering an arrangement of people in a row, in making a mental picture of a graph such as a parabola from its equation, in setting the relationship among the parameters in a word problem or in spotting the location of a place from a given map. Visualisation is not restricted to problems involving geometry or situations involving a diagram but may equally be important in solving a word problem (Hegarty and Kozhevnikov 1999). Researchers have been addressing the issue from a number of perspectives, for instance, in terms of measurement of spatial skills (Ekstrom et al. 1976; Yilmaz 2009), training to improve spatial skills (Ben-Chaim et al. 1988) and measuring the effect of dynamic geometry environments (Guven 2012).

There are subtle differences in the way the term visualisation is used in common language as compared to what it means in psychology and mathematics education. For most people the term 'visualisation' essentially means imagining or picturing a situation. In the field of psychology and mathematics education, visualisation specifically involves the generation and manipulation of images as is explicitly outlined in the ensuing review. The terms 'visualisation' and 'spatial visualisation' tend to be used interchangeably. Additionally, the concept of imagery (as described later) is often used in conjunction with visualisation. Although the conception of visualisation in mathematics education has been greatly influenced by theoretical constructs from the psychology literature, there are subtle theoretical divergences between psychologists and mathematics educators (Gutiérrez 1996).

Psychologists (e.g., Carroll 1993; Kozhevnikov et al. 1999; McGee 1979) tend to interpret visualisation as the manipulation of mental images and do not pay explicit attention to external (non-mental) representations. Thus, in measuring spatial visualisation as in the paper folding test (Ekstrom et al. 1976), they ask respondents to manipulate mental images. A comparison of the conceptualisations of visualisation provided by psychologists and mathematics educators serve to illuminate further this difference. McGee (1979) described spatial visualisation as "the ability to mentally manipulate, rotate, or twist, or invert a pictorially presented stimulus object" (p. 383). For Carroll (1993), visualisation involves the "processes of apprehending, encoding and mentally transforming spatial forms" (p. 309). More recently, Kozhevnikov et al. (1999) gave the following definition: "the ability to manipulate or transform the image of spatial patterns into other visual arrangements" (p. 4).

On the other hand, some mathematics educators (Arcavi 2003; Zimmerman and Cunningham 1991) tend to consider the interaction between both mental images and external representations (diagrams, graphs, symbols, etc.) as part of visualisation. For example, Arcavi (2003) describes visualisation as

> the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating

information, thinking about and developing previously unknown ideas and advancing understandings (p. 217).

Thus, in Arcavi's definition, there is emphasis on both mental images and external representations. However, such a difference is not always universally explicit in mathematics education. For example, Clements and Battista (1992) define spatial visualisation as the "comprehension and performance of imagined movements of objects in two- and three-dimensional space" (p. 444). Additionally, Yakimanskaya (1991) characterises spatial visualisation as the ability to generate and manipulate images. Integrating the different aspects of visualisation, Gutiérrez (1996) proposed the following definition: "as the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or prove properties" (p. 9). His definition encompasses four dimensions: mental images, external representations, process of visualisation and abilities of visualisation.

Summarising, the point of commonality among these definitions include the creation of mental image from visual/spatial information and the manipulation of the mental image. We use this point of commonality as our working definition of spatial visualisation. In the current study, we report on the ways in which students create and manipulate mental images in reflecting objects on a line of symmetry.

Imagery

Intertwined with the concept of spatial visualisation is the notion of imagery. According to Kosslyn (1990), imagery is used "when we reason about the appearance of an object when it is transformed, especially when we want to know about subtle spatial relations" (p. 75). Indeed, Yakimanskaya (1991) considers images as the basic operative units of spatial visualisation. Kozhevnikov et al. (1999) also distinguish between *visual imagery* (which refers to the visual appearance of objects such as its form, colour, or brightness) and *spatial imagery* (which refers to the representation of spatial relationships between objects and imagining spatial transformations). In the current study, we were particularly interested with spatial imagery. To further illustrate the heterogeneity and complexity encapsulated by the concept of imagery, we refer to one of the most frequently cited categorisations of imagery in mathematics education. Presmeg (1986a) differentiates between five types of imagery, one of which, namely kinaesthetic imagery (i.e. imagery typically involving students using their fingers to trace a shape or pattern) was particularly apparent in the data collected for this study. Based on the observation of our participants' interaction with the symmetry tasks in this study, we adhere to the definition of mental image articulated by Gutiérrez (1996) as: "any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements" (p. 9).

In this inquiry, we focus on the concepts of bilateral symmetry and reflection to develop a fine-grained understanding of the underlying spatial visualisation mechanisms. For ease of reading, we shall use the term, 'symmetry' and 'bilateral symmetry' interchangeably. Understanding the ways in which visual and analytical strategies interact in the solution of mathematical problems has been an ongoing challenge for mathematics educators (Zazkis et al. 1996). Some steps have been taken to explain such an interaction. For instance, Hoyles and Healy (1997) showed how students attempted to synthesise the visual anticipation of the solution and their analytic symbolic representations in a microworld environment which allowed dynamic actions. In their analysis of the role of visual reasoning, Hershkowitz et al. (2001) suggested that spatial visualisation can be an analytical process itself.

The current topic under study, symmetry and reflection, has conceptual foundations that can be investigated through visual and analytic strategies. Previous accounts of students' difficulties with symmetry have explained the nature of errors but have not provided a detailed account of what might be the sources of those errors. For example, Hoyles and Healy (1997) underlined the occurrence of visualisation (which they implicitly referred to as 'visual appreciation' or 'visual image' (p. 41)) as their participants solved reflection tasks in a microworld environment. However, visualisation was not a central feature of their study. We attempt to explain the sources of errors in symmetry and reflection tasks from the perspective of spatial visualisation, an orientation that has not been explicitly taken into consideration. In the ensuing section, we review studies seminal to the subject.

Reflections and bilateral symmetry

Research conducted since the 1980s has consistently shown that the concept of symmetry is problematic for many students. One of the first extensive studies conducted in this domain is by Küchemann (1981) who identified five essential variables that influence students' ability to perform bilateral symmetry, namely the slope of the line of symmetry, the slope of the object, the complexity of the object, the existence or absence of intersection between the object and the line of symmetry and the presence or absence of a grid in the problem. These variables were further analysed in depth by Grenier (1985) in her dissertation study to investigate patterns of errors (described as orthogonal procedure, covering or prolonging, parallelism procedure, horizontal point sliding or vertical point sliding). Studies involving reflection conducted with French, German, British and Japanese middle school students seemed to have produced consistent results in terms of patterns of errors observed (Denys 1985). In fact, recent studies (e.g. Bulf 2010) continue to highlight the influence of the variables identified by Küchemann (1981) in students' performance.

Another line of research looked at the ways in which a microworld environment prompted students to develop the ideas of transformation geometry (Guven 2012; Hollebrands 2003; Hoyles and Healy 1997). Some researchers have also paid attention to the relation between teacher knowledge of symmetry and instruction. Particularly, Leikin et al. (2000) pointed out how problems involving inclined line of symmetry and the parallelogram "remained the most troublesome figure" (p. 26) for pre-service teachers. Edwards and Zazkis (1993) described how pre-service teachers' naive ideas about reflection influenced the way they performed such a transformation formally in a computer microworld. They observed that their participants had a tendency to focus on 'visually-salient' features of objects such as edges and centre points in performing reflection. They also defined a persistent 'reflection bug' (where the object is moved to the line of symmetry and then reflected) to describe the local conceptualisation of transformation that students hold rather than a global mapping of the plane. Recent studies in geometry (e.g. Gal and Linchevski 2010) have also drawn our attention to the role of visual perception in understanding students' manipulation of geometrical objects. Along the same lines, psychologists (Enns and Kingstone 1995) differentiate between local and global perception, two constructs which were particularly important in analysing the data in this study. We describe these constructs in our conceptual framework.

Research questions

Although much is known about the type of variables that affect students' ability to perform symmetry and reflection, the source of the conceptual difficulties have not been thoroughly investigated from a visualisation perspective. In this paper, we use constructs from the area of spatial visualisation to analyse the processes behind students' strategies and errors (intuitive or learned) in performing bilateral symmetry and reflection tasks. We address the following research questions:

1. What are the sources of conceptual difficulties associated with inclined lines of symmetry?
2. How are visual strategies enacted in bilateral symmetry and reflection tasks?
3. How do visual and analytical strategies interact in the production of the image from the object?

Conceptual framework

We developed our conceptual framework by interpreting students' actions in symmetry and reflection tasks that they are generally called upon to perform in school mathematics, as is the case in the present study. We used three main constructs to analyse the data: (1) visual strategy and analytical strategy, (2) visual-mental folding and visual-mental reflection as mental operations and (3) local and global perception.

1. Visual strategy and analytical strategy

Presmeg (1986b) considers a visual method 'as one which uses visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are employed' (p. 298). In this study, we refer to a *visual strategy* when attention is given to the use of imagery that is related to shape, location, position, and orientation. Such a strategy may also include kinaesthetic imagery, i.e. imagery involving physical movement such as the movement of the finger, the head, etc. (Presmeg 1986b), as will be shown in the data analysis.

Now we explain what we mean by analytical strategy. From a mathematical perspective, reflection (as a transformation) constitutes an isometry since it preserves length, shape and angle. The two main properties that are useful to reflect an object on a line of symmetry are (i) perpendicularity between the line segment joining corresponding points on an object, its image and the line of symmetry and (ii) equidistance between object, line of symmetry and image. We use the term *analytical strategy* whenever explicit reference is made to any of these properties in performing a reflection, finding the line(s) of symmetry or in the construction of a symmetrical object.

2. Visual-mental folding and visual-mental reflection

In Piaget and Inhelder's (1956) sense, a mental operation essentially refers to the mental ability of imagining a situation and determining the likely outcome without actually carrying out the scenario in action. The two mental operations described below were defined on the basis of the observations that we made as the participants in this study interacted with the tasks. We distinguish between two types of mental operations where spatial images are involved in reflection and symmetry tasks:

(i) *Visual-mental reflection*: The visual/mental action of anticipating the image of an object from a line of symmetry. This process occurs when a printed object on paper (either plain or grid paper) is to be reflected on a given line(s) of symmetry. For example, if we are to reflect a flag on the x-axis, we can anticipate the image of the reflected flag without actually drawing it. There must be a mental operation that allows us to make this anticipation. This mental operation is termed visual-mental reflection.

(ii) *Visual-mental folding*: The visual/mental action of imagining the shape of an object being folded to determine the one-to-one geometric or morphological correspondence between the parts of an object. For example, if we are to find the number of lines of symmetry in the letter H, we can visually/mentally fold the letter without physically doing it. The underlying mental operation that allows us to do this is termed visual-mental folding. This process occurs in finding the lines of symmetry of shapes or alphanumeric characters.

We hyphenate the terms visual and mental as it is difficult to dissociate between them using the common tools of research in education. It may be argued that a combination of mental operations is involved in carrying out a reflection or finding out symmetry rather than just the two operations defined above. We chose to restrict our claims to these two operations as they were the ones most apparent from the responses and actions of the participants.

3. Local and global perception

As we interpreted the participants' actions to reflect the objects in the line of symmetry, we could infer that at times they focused on the whole object, while in other instances they targeted their attention on particular parts or key features such as the line segments or end points of the object. This prompted us to consider the salience of visual perception in executing the symmetry tasks. In fact, Kosslyn (1990) suggested that imagery and perception share common features. It is to be noted that previous research on symmetry (Grenier 1985; Küchemann 1981) alluded to global perception but did not give much attention to its influence on students' actions. In this study, we give explicit consideration to the concept of *local* and *global perception* from psychology literature (Enns and Kingstone 1995) in describing the participants' focus of attention as they reflected the object on the given line of symmetry. Psychologists use the term *local perception* to refer to the interpretation of an image when it is visually parsed into units. On the other hand, *global perception* refers to the overall structure of the image being processed. For example, the perception of a car in its entirety is referred to as global perception whereas the perception of the different parts in terms of its lights, windshield and wheels are referred to as local perception. Thus, in reflecting the L-shaped object in task 2.2 in Table 1 (presented in the next section), one may pay attention to the whole object (*global perception*) or focus on the individual line segments that constitute its shape (*local perception*).

Method

This paper presents the results of four task-based interviews that we conducted with two volunteer students from two different schools. One of the motivations for this study was to understand how visual and analytical strategies interact in making a reflection, especially in relation to an inclined line of symmetry. Thus, our focus was to scrutinise processes rather than end results of students' thinking.

Participants' prior knowledge of symmetry

The two participants of the study are identified by the pseudonyms, Brittany (age 15 years) and Sara (age 14 years). Both Brittany and Sara studied symmetry and reflection when they were in primary school. At the start of the interview, we asked them to describe what they knew about lines of reflection and symmetry to make an initial assessment of their conceptions, both analytically and visually.

Brittany's description of symmetry and analytical strategy

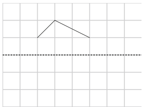
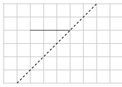
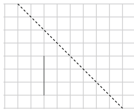
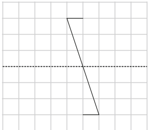
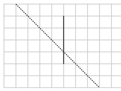
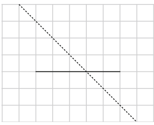
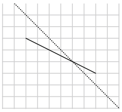
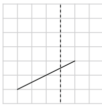
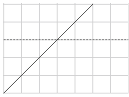
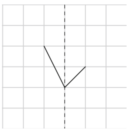
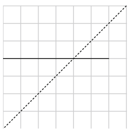
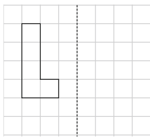
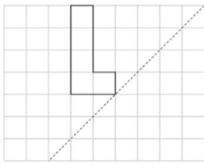
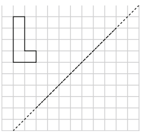
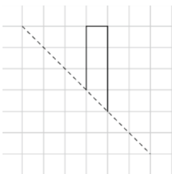
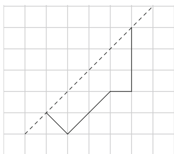
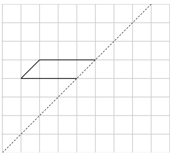
Brittany described reflection in terms of 'exactly opposite' and 'folding'. We highlight the fact that she used the terms 'exactly opposite' rather than just 'opposite'. The image of reflection encapsulated by the term 'exactly opposite' was a particularly strong metaphor for her. She mentioned that the shape remains the same and the object and image are equidistant from the line of symmetry. The analytical properties of equidistance of the object and image from line of symmetry, and congruence of length of object and image were well-established for her. However, at no point did she make any reference to perpendicularity. The fact that she talked about 'folding' allows us to say that she carried the image of symmetry in terms of visual-mental folding. Her image of reflection as 'folding' was also apparent when we asked her to explain why she drew the requisite number of lines of symmetry for the common polygons (square, rectangle, triangle, trapezium and parallelogram including the rhombus). For instance, for the parallelogram, which is a well-known obstacle for many students (Leikin et al. 2000), she visually-mentally folded the sides horizontally, vertically and diagonally to justify that there was no line of symmetry 'because if you fold it, then they are not going to meet each other'.

She did not seem to relate the angle properties to the line of reflection. For example, she determined the two lines of symmetry in a rhombus, by visually-mentally folding the shape and considering the fact that the lengths of the sides were equal. Although she mentioned that the 'opposite angles are equal' in a rhombus, she made no apparent link between the angle properties and the lines of symmetry. Brittany showed much flexibility in visually-mentally folding a given shape or alphanumeric character. For example, in determining whether the letter 'S' has any line of symmetry, she explained that she visualised a solid object for the letter: 'Like, the shape. It's the shape of the S. So the paper was like cut out in the shape of the S and you can fold it.'

Sara's description of symmetry and analytical strategy

Sara had a well-articulated analytic conception of reflection in terms of equal distance between object and image and line of symmetry. Particularly, her concept of

Table 1 Tasks in set 1 and set 2

Task 1.1 	Task 1.2 	Task 1.3 	Task 1.4 
Task 1.5 	Task 1.6 	Task 1.7 	Task 1.8 
Task 1.9 	Task 1.10 	Task 1.11 	Task 1.12 
Task 1.13 	Task 1.14 	Task 2.1 	Task 2.2 
Task 2.3 	Task 2.4 	Task 2.5 	Task 2.6 
Task 2.7 			

perpendicularity empowered her to reliably reflect an object in an inclined line of symmetry. She focused on the points at the end of line segments (see Fig. 1) and often labelling them in the application of the equidistance and perpendicularity. For example, in Fig. 1, in order to draw the image of the given object, Sara focused on the points at the ends of the line segments in the object (which she labelled A, B and C). Points A', B' and C' are the images of the points A, B and C, respectively. She then constructed the image by connecting the points. She clearly stated that points on a mirror line have no reflection. Her conception of symmetry can be inferred from her description: 'Reflection is like … you have a line of symmetry, like a mirror or something. And you have an object in front of it. And then it reflects back. It's like flipped over and that's the image.' The image of reflection as being associated to a mirror was a useful metaphor for Sara when she could not explicitly apply equidistance and perpendicularity for the reflection of the letter S. We would like to highlight that while Sara used a ruler and exact measurements in reflecting the given objects, Brittany chose to draw free hand images.

The responses of the two students to the same set of tasks were compelling enough (in terms of the different ways in which they operated on the reflection and symmetry situations) to warrant a close scrutiny. Although our study involves only two participants, our findings align with what has been found in previous research on reflection and symmetry.

Research design

Accessing mental images and visualisation processes are methodologically challenging. As argued by Piaget and Inhelder (1956): "The chief obstacle to any developmental study of the psychology of space derives from the circumstance that the evolution of

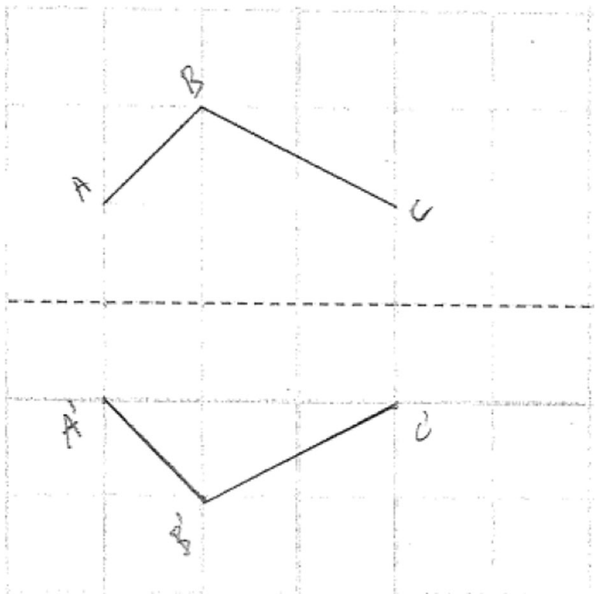


Fig. 1 Sara's focus on end points and perpendicularity

spatial relations proceeds at two different levels. It is process which takes place at the perceptual level and at the level of thought or imagination" (p. 3). In an attempt to capture the moment-by-moment responses of the participants of the study, two interviewers (authors of the paper) followed the students' reasoning in four task-based interviews. Each participant was individually interviewed twice at the university research centre after school hours. Due to the participants' availability, the time gap between the first and second interview was approximately three weeks for each participant. To assess the robustness of our claims, we followed the same students over a large number of tasks so that consistency in behaviour (in terms of the criteria that the students used for performing reflection) could be identified.

At the start of each interview, we explained clearly to the students that we were looking for the visual images or imaginative actions that they used in reflecting the given objects and that we would ask them to describe what they could see as they performed the task. We used two cameras to record the interviews to capture their inscriptions, the movements that they made with their pencils and their kinaesthetic actions (Hall 2000). We also recorded the directions in which they moved the worksheets, especially when they were asked to reflect the object in an inclined mirror line. We waited for the students to fully complete each task before asking them to describe their strategies so as not to distort their thinking processes as suggested by Gutiérrez (1996). We could observe that they made explicit attempts to describe their visual or imaginative actions.

The video records were transcribed in their entity. The analysis of the video on a moment-by-moment basis was conducted by the two interviewers independently. The responses to each of the four sets of tasks were scrutinised task-by-task. During the analysis process, we focused specifically on the criteria that the students were using in reflecting the object or in finding the line of symmetry namely, equidistance, congruence of sides and perpendicularity. We looked at the consistency with which these criteria were being used. We also paid attention to the visual strategies that students appeared to be using both in situations where they could solve the given tasks as well as when they were constrained to do so. These criteria served to assemble the data.

Tasks design, selection and characteristics

The participants were presented with four sets of tasks (set 1, set 2, set 3 and set 4) from the school mathematics curriculum. In the first set, they were required to find the line(s) of symmetry of a polygon (square, rectangle, equilateral triangle, cross, parallelogram and rhombus) and alphanumeric characters (S, X and Z). In the second set, they were asked to find the image of a given line segment (tasks 1.1–1.14) and polygon (tasks 2.1–2.7) reflected on a line of symmetry on grid paper (see Table 1). In the third set, they had to complete the object from the partial object and the given number of lines of symmetry (see Table 2). In the fourth set, they had to reflect objects on inclined lines of symmetry, without the support of a grid (see Table 3). Some of the tasks were chosen and modified from Küchemann (1981), Grenier (1985), Hoyles and Healy (1997) and Leikin et al. (2000).

We present only sample tasks from set 4 in Table 3 and describe the remaining ones as the tasks have the same structure, except that the object of reflection changes. This set of tasks requires the reflection of a given object embedded in a grid-free square (see

Table 2 Tasks in set 3

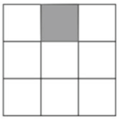
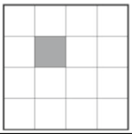
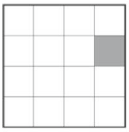
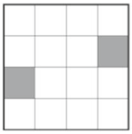
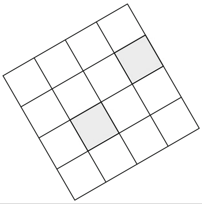
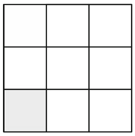
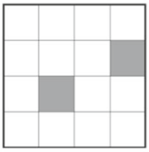
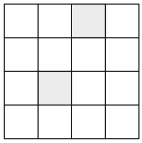
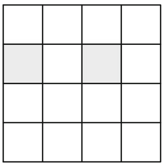
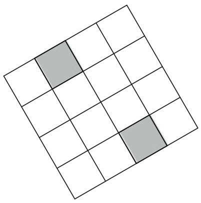
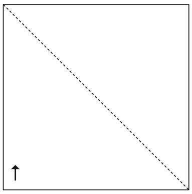
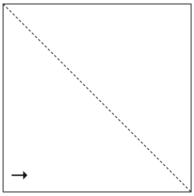
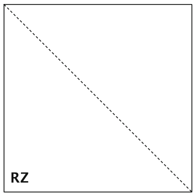
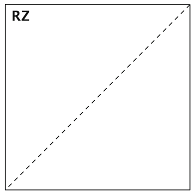
Task 3.1: Shade one more small square on the diagram given so that it has two lines of symmetry. 	**Task 3.2:** Shade three more small squares on the given diagram so that it has four lines of symmetry. 
Task 3.3: Shade two small squares on the given diagram so that it has one line of symmetry. 	**Task 3.4:** Shade two small squares on the given diagram so that it has two lines of symmetry. 
Task 3.5: Shade two more squares so that it has one line of symmetry. 	**Task 3.6:** Shade one small square on the given diagram so that it has two lines of symmetry. 
Task 3.7: Shade two small squares on the given diagram so that it has one line of symmetry. 	**Task 3.8:** Two small squares are shaded in the diagram. Shade one more square so it has one line of symmetry. 
Task 3.9: Two small squares are shaded in the diagram. Shade one more small square, so that it has one line of symmetry. 	**Task 3.10:** Shade two more squares on the given diagram so that it has two lines of symmetry. 

Table 3). Tasks 4.1, 4.2, 4.3 and 4.4 involve the reflection of an arrow in a vertical, horizontal and inclined line of symmetry, respectively. Tasks 4.5–4.9 require the reflection of the alphanumeric characters L, S, Z, RZ and CAP in an inclined line of symmetry. Unlike task 4.8, task 4.10 involves the reflection of RZ when the line of reflection is inclined in the opposite direction (see Table 3).

Table 3 Tasks in set 4

Task 4.3	Task 4.4	Task 4.8	Task 4.10
			

We use the following notations in the transcript, namely I1: interviewer 1, I2: interviewer 2, B: Brittany and S: Sara.

Findings

We analysed the data on a task-by-task basis to identify patterns in strategy use in an attempt to understand the underlying visual and analytical mechanisms. Similarly, we scrutinised the patterns of errors in line with research question 1. The initial analysis led us to identify the different criteria that the two students used in reflecting an object on a line of symmetry. Sara consistently used two criteria, namely (1) equidistance between object, image and line of symmetry (referred to as *equidistance criterion* in this paper) and (2) perpendicularity between a point on the object and the line of symmetry (referred to as *perpendicularity criterion* in this paper). Mathematically, two points are equidistant from a line when the perpendicular distances between each point and the line are equal. Thus, perpendicularity is an important element when the distance from a point to a line is considered.

For some of the tasks, Brittany maintained the perpendicularity between points on the object, image and line of symmetry. However, in the majority of the tasks involving an inclined line of symmetry, she did not maintain the perpendicularity between points on the object, image and line of symmetry, indicating that she did not have a robust notion of perpendicularity. She intuitively knew that on reflection the image should be 'exactly opposite' (the term that she used). Thus, she could visually position the image on the opposite side of the mirror line but she was constrained by her lack of a formal notion of perpendicularity to determine the precise location of the image. When she experienced such a type of constraint, she would use congruence of sides (i.e. the lengths of the sides of the object and image are the same) as the alternative criterion for reflection. In summary, Brittany focused on three criteria, although not consistently: (1) equidistance, (2) congruence of sides (i.e. the lengths of the sides of the object and image are the same) and (3) 'exactly opposite' as the intuitive counterpart of perpendicularity. We confirmed the robustness of the above criteria in the second interview, conducted approximately 3 weeks later.

We present critical events from each of the four sets of tasks. Our explanation of the students' responses is based on what they said and appeared to be doing. We acknowledge that the interpretations provided in the data analysis for the selected tasks are not necessarily the only possible explanation of how the students might have solved the problem.

Analysis of set 1 tasks

Set 1 tasks involve determining the number of lines of symmetry of a polygon (square, rectangle, equilateral triangle, cross, parallelogram and rhombus) and alphanumeric characters (S, X and Z). We started to see the expression of the mental operation ‘visual-mental folding’ in this set of tasks as the students used the phrase ‘if I imagine folding into half’. We asked Brittany to justify why there are four lines of symmetry in a square. Her response showed how she visually-mentally folded the square:

I2: How do you know that there are four (lines of symmetry)?

B: Uh, because, if like you imagine, like folding the shape in half and they would be the same.

I1: When you were drawing did you fold it in your mind or you just drew it?

B: Yeah. I did I fold it in my mind.

Similarly, Sara’s flexibility to visually-mentally fold the given objects to determine the number of lines of symmetry became evident as she went through the first set of tasks. The only constraint that she momentarily experienced was in deciding whether a parallelogram has a line of symmetry. She drew one line of symmetry midway parallel to the sides (Fig. 2) initially.

The following transcript shows the justification she gave for her answer:

I1: And why do you say it is a line of symmetry?

S: Because if I fold it. Actually not, that’s not going to be a line of symmetry. I don’t think there is any.

I2: Do you think there is any? Or are you not sure?

S: I am not sure.

I1: But why did you draw the line?

S: I was just trying to imagine whether if I fold it based on this line (the line of symmetry), it would fit onto this half.

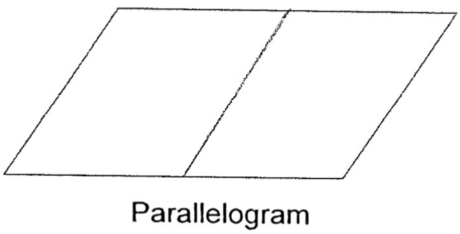


Fig. 2 Sara’s drawing with one line of symmetry for the parallelogram

I1: Will it fit?

S: Don't think so. May be. Not sure.

What could account for Sara's uncertainty with regard to whether the parallelogram has a line of symmetry or not? It is more likely that she focused on the congruence of the two parallelograms (by drawing the line of symmetry) in Fig. 2 to infer that there was one line of symmetry. However, in the second interview, she justified that a parallelogram does not have a line of symmetry.

Analysis of set 2 tasks

In this subsection, we present extended analyses of Brittany's responses given the constraints she encountered in reflecting the given objects. Sara used the equidistance criterion and perpendicularity criterion to correctly reflect all the given objects.

Brittany's reliance on congruence of sides to reflect an object

As we observed Brittany's strategy to reflect a line segment on an inclined line of symmetry in tasks 1.3–1.10 (except task 1.5 which has a horizontal line of symmetry but the object has a slanting side) in the second set of problems (set 2, Table 1), we immediately realised that she was focusing on congruence of lengths of the object and image as the main criterion for reflection. In other words, she did not pay explicit attention to the perpendicularity property of reflection. Figure 3 shows the outcome of her focus on equidistance and congruence of lengths. When the given object (line

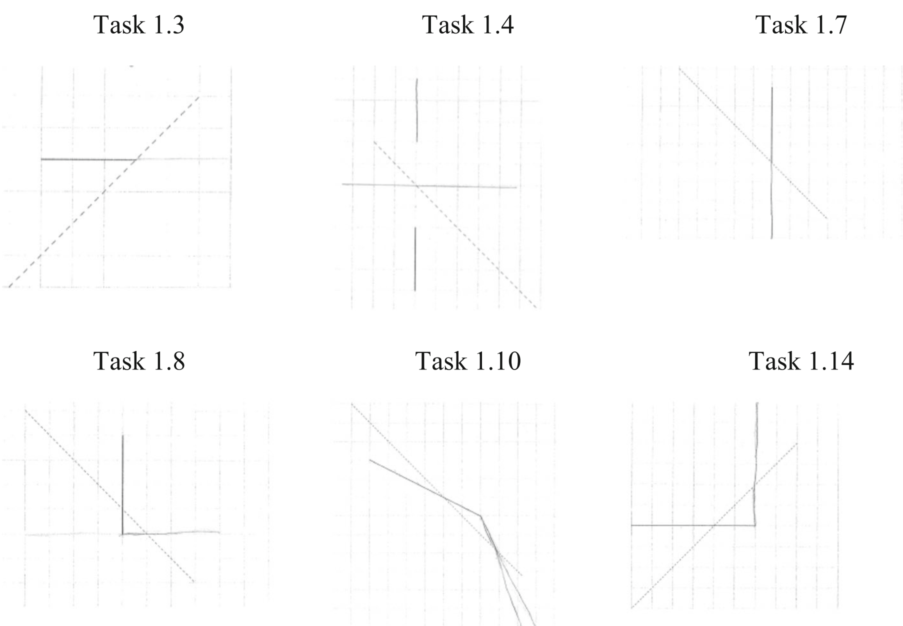


Fig. 3 Brittany's responses to set 2

segment) crossed the line of symmetry, she interpreted the object as consisting of two parts in tasks 1.8, 1.10 and 1.14.

We give the detailed response to task 1.8 to show the outcome of Brittany's focus on congruence of lengths. We have redrawn the intermediate sketches she made in Fig. 4a, b from her workings for explanation purposes. As the given object (i.e. the vertical line segment) crossed the line of symmetry, she interpreted the object as consisting of two parts. She started by counting the length of the object. She seemed to be not very sure how to proceed and asked: 'So that's the object, like full?' The motion of her pencil suggested that she was thinking about moving either to the right or left, at right angle to the object. She first decided to draw the image to the left of the object (see Fig. 4a). She joined the two end points (labelled X and Y for explanation purposes) to find a means of getting equal distance between X and Y. Since the length on either side of the line segment XY was different, this led her to realise that this step is incorrect: 'That's not really…' She then changed her solution by drawing the image on the right of the object (see Fig. 4b) and mentioned: 'Because well this is one square like pass it (line of symmetry). So if we do this it's got one part on this side too, the same amount on the other side.' Because the length of the object and image, above and below the line of symmetry was the same, she felt confident that she performed the reflection correctly.

We tested the robustness of her criteria for reflection in the second interview, 3 weeks later. Further evidence of her reliance on equidistance and congruence of lengths of object and image is illustrated in Fig. 5.

These types of misconceptions have been reported in earlier studies (e.g., Grenier 1985; Küchemann 1981). The congruence of length between the object and image logically convinced her of the correctness of the reflection of the given line segments and may have suppressed the need for questioning the soundness of the answer.

In the second interview, in probing further, her conception of reflection when the object (line segment) crosses the line of symmetry, we asked Brittany whether a point on the line of symmetry has a reflection, and she mentioned 'It does not have a reflection; I am not sure; I don't think so'. Apparently, she did not realise the necessity to consider the point of intersection of the object and the line of symmetry as in tasks 1.11, 1.12 and 1.13 (Fig. 5).

To further explore the ways in which the participants solved tasks involving inclined line of symmetry, we presented six more tasks (tasks 2.1–2.6), involving polygons (Table 1). Brittany's focus on congruence of sides (between object and image) served her well in tasks 2.1, 2.2, 2.4 and 2.5. For example, in task 2.2, Brittany first reoriented

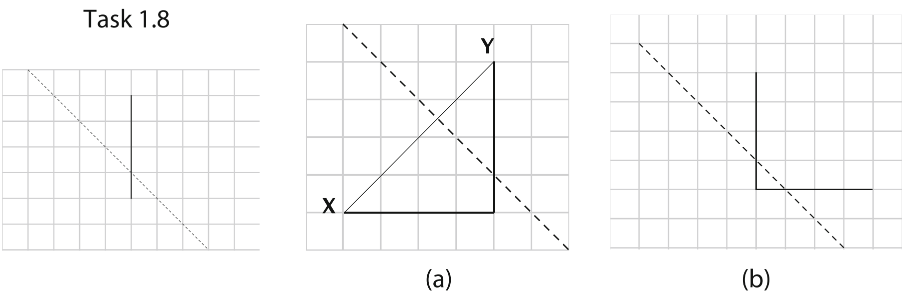


Fig. 4 Brittany's response to task 1.8

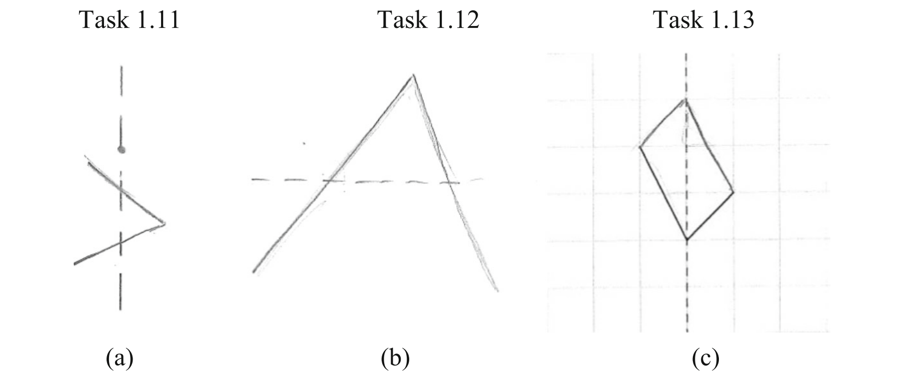


Fig. 5 Brittany's response to tasks 1.11, 1.12 and 1.13

the inclined line of symmetry vertically by turning the worksheet. Then she reflected the horizontal segment (touching the slanting line of symmetry) in the L-shaped polygon. Next, she reflected the vertical segment (touching the slanting line of symmetry). These two initial constructions apparently served as a trigger for her to spontaneously identify the next part of the image and she quickly proceeded to construct the image, counting the length of the different parts of the object (via the grid) as she did so. Brittany mentioned she could visualise what the object would look like. There must be an underlying mental operation which allowed Brittany to see the image of the object on reflection although she may not be able to find the exact location of the image straightaway. Such observations where the participants could anticipate the image of the object before actually counting the grids prompted us to define the mental operation termed as visual-mental reflection (as described in the conceptual framework).

We would also like to highlight that Brittany's response in this problem shows how her analytical strategy (maintaining equal length) combined with the visual anticipation of the image, supported her construction of the image of the L-shaped polygon in the inclined line of symmetry. She turned the slanting line of symmetry vertically after the construction to match the object and the image at a global level. She explained why she reoriented the slanting line of symmetry: 'I guess this angle (showing the vertical direction) is easier to visualise'. Since the lengths of the segments in the object were equal to those of the image, this validated her response. We would like to underline that in task 2.2 the reflection of one part of the object served to open the space for reflecting the whole object. We could observe a similar trigger in task 2.4, where the reflection of the segment at 45° to the line of reflection led her to spontaneously and promptly construct the image.

In the second interview, we asked Brittany to reflect an L-shaped polygon at a distance from the inclined line of symmetry (task 2.3). She started with the vertex closest to the line of symmetry and identified the corresponding point on the image diagonally opposite. Then she reflected the adjacent vertex (located vertically above) to obtain a line segment. After this point, she attempted to visually complete the image of the L shape as could be inferred from the different L that she drew and erased. She also tried to draw another diagonal line from the top vertex (see Fig. 6). When asked to explain why she deleted the different images, she mentioned: 'Because it does not look like a reflection.' Finally, she settled with the image shown in Fig. 6 which has the right

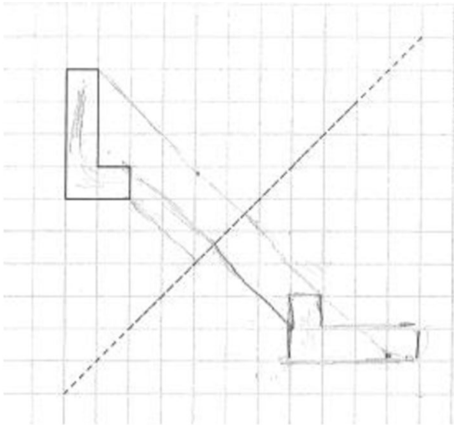


Fig. 6 Brittany's reflection of L-shape polygon on inclined line of symmetry

orientation and size but not the correct location. When asked if the image she drew is correct, she mentioned: 'It looks right but I am not sure'. This points to the possibility that the visual-mental reflection operation may have been deployed here, although the lack of perpendicularity criterion prevented the precise location of the image.

For the other tasks in set 2, she seems to have used a similar strategy of visualising, counting and reflecting (although the order in which these processes take place is difficult to disentangle) in attempting to maintain equal length and equal distance. For instance, in task 2.4, she said she visualised a shape in the form of a V: 'Just a V, the full shape of it.' After making the image, she turned the paper to make the line of symmetry vertical to globally verify the correspondence between the object and the image.

In task 2.6, the line of symmetry is not inclined at 45° and this means that the analytical property of perpendicularity is necessary to accurately draw the image, a knowledge element that was not available to Brittany. We could infer the constraints that Brittany experienced as she turned the diagram in different orientations and started from either end of the object, drawing and erasing segments. She explained that she was trying to visualise the image: 'like the full shape as it would be if I drew there.' As in task 2.4, she attempted to imagine the 'full shape' (i.e. object and image together) to verify whether the object and image corresponded. Finally she drew the image shown in Fig. 7 and was convinced that it was correct as the lengths of corresponding segments were equal.

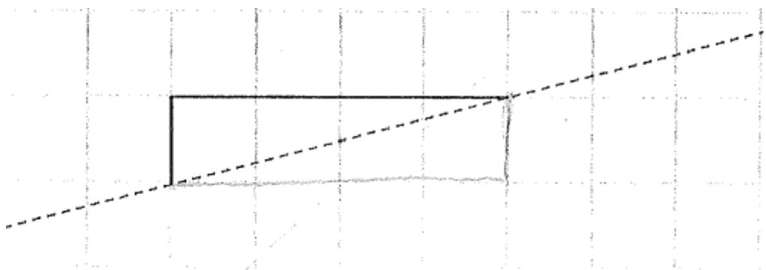


Fig. 7 Brittany's attempt to maintain equal length

She could visually anticipate that she needed to have a triangle as an image, an inference we made when we asked her to explain what the image should look like. Since she did not have the analytical property of perpendicularity, she experienced a dissonance which she resolved by maintaining congruence of sides.

Sara's consistent application of the equidistance and perpendicularity criteria

In contrast, Sara's consistent approach involving equidistance and perpendicularity criteria showed that she had a well-established scheme (in terms of a consistent way of acting) for lines of symmetry. For example, in task 1.3, she mentioned: 'I used the boxes. I made a perpendicular line with my ruler. And then I saw how many boxes I needed.' Her succinct notion of perpendicularity (perpendicularity criterion) is clearly illustrated in Fig. 8 for task 2.6, where she explicitly drew a segment (see dotted segment) perpendicular to the line of symmetry.

Analysis of set 3 tasks

We present the data from set 3 to give more evidence of the visual-mental reflection operation in symmetry tasks. In contrast to sets 1 and 2 where the full object is presented, in set 3 the object has to be constructed on the basis of the given symmetry. Thus, it involves imagining different shaded configurations and verifying if the resulting grid has the required number of lines of symmetry.

Brittany's approach to set 3 tasks

In these situations, Brittany positioned an imaginary line of symmetry (as could be inferred by the motion of her pencil) generally in the sequence, vertical, horizontal and slanting to then visually-mentally reflect the shaded cells given in the problem. Brittany's actions give evidence of the possible involvement of kinaesthetic imagery. She focused not only on the shaded parts but equally looked at the continuous shape made by the unshaded parts. For example, in task 3.1 (see Table 2), she focused on the letter H formed on shading the required cell to confirm that there were two lines of symmetry (see Fig. 9a). Similarly in task 3.6 she focused on the unshaded part to verify if her solution works: 'I checked the shape of the white (unshaded part) to see if it would fold.' She imagined folding along the lines of symmetry (see Fig. 9b). In task 3.2, she shaded three more cells to make a square using the properties of square: 'It's in

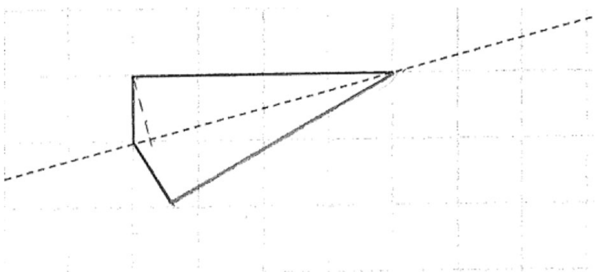


Fig. 8 Sara's robust perpendicularity criterion

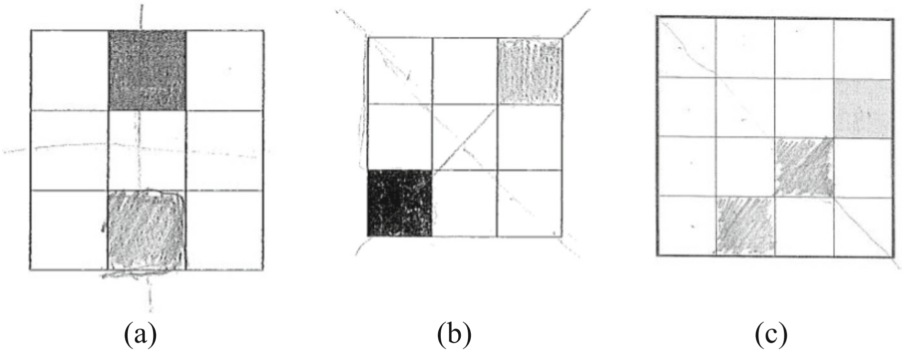


Fig. 9 Set 3 sample responses from Brittany and Sara

the shape of a square and a square has 4 lines of symmetry.' In the case of the slanting line of symmetry (e.g. task 3.3), she equally counted the equal distance by working across the diagonal of the grid.

Sara's approach to set 3 tasks

In set 3, Sara positioned the line of symmetry (vertically, horizontally and diagonally) and then attempted to determine which cells had to be shaded. For instance, in task 3.3 (see Fig. 9c), she gave all the possible solutions along the two diagonals after drawing them. She equally gave the following explanation to show the strategy that she used: 'I imagine, in my head, of a paper with small grids and three shaded ones (the given shaded square and the two squares that she shaded in her worksheet) and folded it half diagonally here and this would match with this (showing corresponding cells with her pencil)… It's like how I did it with my other method. I use this point (referring to one point on the given shaded cell) and drew a perpendicular line to the line of symmetry.' It appears that she used the visual as well as the analytical properties (equidistance and perpendicularity) in conjunction.

Analysis of set 4 tasks

In set 4, the objects to be reflected are not embedded in a grid but they are confined in a square boundary. Like the previous tasks, our aim was to create situations that would allow us to capture the ways in which the participants interacted with the reflection tasks. We obtained further opportunities to observe how the analytical strategy was used in conjunction with the visual strategy. In task 4.3 (see Table 3), Brittany first tilted the worksheet to make the inclined line of symmetry vertically oriented, mentioning 'it's kind of easy, to like, to see'. She explained that she first considered the distance from the tip of the arrow to the line of symmetry (as indicated by the vertical and horizontal marks that she drew (Fig. 10) only when we asked her to explain her strategy). Her equidistance criterion guided her to locate the position of the image of the arrow. While explaining her strategy, she realised that the opposite diagonal distance on either side of the line of symmetry should also be equal: 'and probably there as well (referring to the diagonal distances) on either side'. Similarly, in task 4.5 (involving the reflection of the letter L in the diagonal line of symmetry), she tilted the

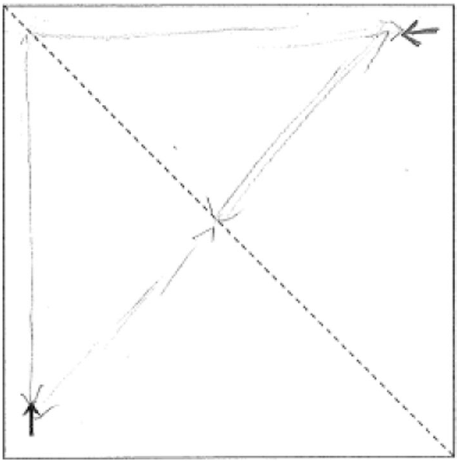


Fig. 10 Brittany's strategy for task 4.3

worksheet to make the line of symmetry vertically oriented. When asked if she could visualise the image, she said 'I see like the shape of it'. To decide where the shorter and longer segments of the letter L should be positioned, she focused on the directions towards which they pointed with respect to the line of symmetry.

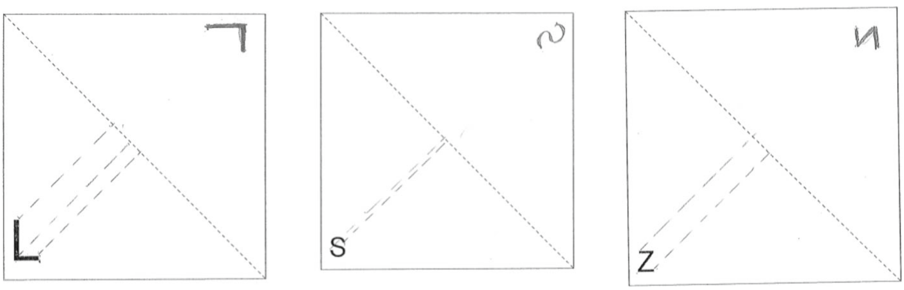
In reflecting the alphanumeric characters, S, Z, RZ and CAP in tasks 4.6–4.10, we identified another criterion that Brittany used to reflect the letters. She took into consideration the fact that the orientation of the letters was opposite on reflection: 'because it's like reflecting, it's got to be like opposite, like the opposite way'. Further, she mentioned that unlike the L shape, for the S shape, it is not easy to draw diagonal lines from object to line of symmetry to find the image 'it's like a shape and so it is hard to get to see where this part was (referring to one point on the S).' She first deduced that the image of S should be on the opposite corner and use the fact that the image should be in opposite orientation to draw it. With regard to task 4.8 which involved two adjacent letters (R and Z), she mentioned 'well the Z is like closer to the line of symmetry. So on this side (i.e. on the other side of the line of symmetry) it has to be closer to the line of symmetry and then the opposite way' and for the reflection of the letter R: 'it's just next to the Z so I just draw it the opposite way to it'. When we asked her how she deduced the orientation of the image of the letter R, she mentioned: 'I just know that it is opposite to how it normally looks like; the opposite way we normally write it'. In task 4.10, where the diagonal line of symmetry is inclined differently from the previous tasks, we could identify three distinct steps from her explanation: First, she focused on the lower end of the letter Z and located its image point (by drawing a diagonal across the line of symmetry) on the other side of the line of symmetry. Secondly she drew the letter Z in the opposite orientation. Thirdly, she drew the letter R in the opposite orientation next to the Z. It appears that Brittany used kinaesthetic imagery in starting her solution to set 4 problems as she pulled the page up on the corner in imitating a folding action.

Sara's approach for set 4 tasks

Sara used her ruler to measure the exact distance between the object and the line of symmetry as well as the length of the arrow in tasks 4.1 to 4.4 (see Table 3). In task 4.3, like Brittany, she also reoriented the line of symmetry vertically (which she apparently did more for drawing convenience than visualisation). She applied her perpendicularity criterion as could be explicitly inferred from her explanation: 'so I tried to make this perpendicular (meaning the distance from the tip of the arrow to the inclined line of symmetry), so I use the lines in the ruler here (meaning using the subdivisions in the ruler to get exact perpendicularity).' She equally indicated that she visualised the result of the reflection by imagining folding it on an origami paper: 'I imagine there is a piece of square paper, like origami paper and there is a dot like on this side (meaning in the corner) and another dot like there (meaning on the opposite side) and I tried folding it and it fits perfectly.' Thus, like Brittany, it seemed that first she localised the spot where the image should be and then worked for its exact positioning in the second place. In task 4.5, she used her ruler to find the perpendicular distance on three points on the letter L (see Fig. 11a).

She applied the perpendicularity criterion again in the reflection of the letter S in task 4.6. She mapped the starting and end points of the letter S perpendicularly on the other side of the line of reflection (Fig. 11b). However, she felt unsure about the orientation of the image: 'but then after I knew where it would end and start, I was not very sure, how to draw it and then I just draw something like that; it's not very good. So, then I thought doing that (tracing the upper part of the image of S with her right hand) following that way (tracing the corresponding part on the object with her left hand) like a mirror.' She also added later: 'I knew the rough position but I do not know like where, like how it's going to face.'

In task 4.7 (see Fig. 11c), besides using the perpendicularity and equidistance criteria, she used the side of the embedding square parallel to the horizontal part of the Z as an indicator to decide the orientation of the image of Z. Similarly, in task 4.8 (requiring the reflection of RZ), she used the sides of the embedding square as an indicator. In task 4.9 (requiring the reflection of CAP), she started with the letter C which is closer to the corner of the embedding square and inductively constructed the image of A and P. She used a similar strategy in task 4.10, first spotting the location of the image by visually-mentally folding it as could be explicitly inferred from her response: 'If I fold it', illustrating the action with her hand and after reorienting the line of symmetry vertically in her direction. Sara found task 4.10 to be more demanding than task 4.8, although both of them involve the reflection of the same letters. This may suggest that the orientation of the inclined line of symmetry (left or right) may also be an influential element in performing a reflection.



(a) (b) (c)

Fig. 11 Evidence of Sara's perpendicularity criterion

Discussion and conclusion

This study examined the intermediate manoeuvers that were apparent in the production of the image from the object under a reflection, focusing both on the visual and analytical components. The moment-by-moment responses of the two contrasting case studies provide a fine-grained perspective on how students' actions in mathematical tasks (symmetry and reflection) are regulated by their knowledge elements. We now address the research questions that guided the study.

1. What are the sources of conceptual difficulties associated with inclined lines of symmetry?

While the situations involving horizontal and vertical lines of symmetry did not pose any constraint for Brittany, the inclined lines of symmetry revealed the inadequacy of her conception of symmetry. She focused on equidistance, congruence of length and 'exactly opposite' as the criteria for reflection and was not formally aware of the perpendicularity criterion. The second interview also confirmed that Brittany was not aware that the reflection of a point on a line of symmetry is invariant under such a transformation. Brittany's erroneous conception of symmetry associated with inclined lines is attributed to her intuitive notion of perpendicularity as being opposite. For vertical and horizontal lines of symmetry, perpendicularity is readily ensured, especially if a grid is available. However, for inclined lines of symmetry such is not the case, although the visual appearance of the task in a grid may help in establishing perpendicularity. On the other hand, Sara had a formal conception of perpendicularity as could be observed by the way she positioned her ruler consistently to ensure right angles.

Our results compare with that of Küchemann (1981) in terms of the influence of the slope of the inclined line of symmetry. For example, Brittany had no difficulty with task 2.7 (where the slope of the inclined mirror line was 45°) compared to task 2.6 (where the mirror line was not inclined at 45°) that caused much challenge for her. Similar to the observations made by Grenier (1985), objects that intersect the inclined line of symmetry were problematic for Brittany. We explain the challenges that Brittany obtained in these circumstances as a result of lack of a formal conception of perpendicularity.

Our findings led us to conjecture that students' responses to inclined line of symmetry tasks are also dependent on whether the objects have a shape or are merely line segments. A line segment (e.g. tasks 1.3 and 1.7 in Table 1) was more challenging to reflect as compared to when it was part of a figure (e.g. task 2.2 in Table 1), where the L-shaped polygon contained vertical and horizontal line segments. A second example could be observed by comparing Brittany's response to task 1.7 to that in task 2.4 (which she did correctly). In task 1.7, involving a vertical line segment touching the inclined line of symmetry, Brittany just extended the line segment on reflection (as illustrated Fig. 3) but in task 2.4, involving a shape with vertical line segments and still touching the inclined line of symmetry, she drew the correct figure. What could account for such differential behaviour? It appears that it may be easier to imagine or perceive the reflection of an object as one whole as it has more defining characteristics rather than an individual line segment. It seems that the object in task 2.4

may have been perceived globally in the first instance as one whole rather than consisting of three different segments. Furthermore, it appears that the orientation of the object relative to the inclined line of symmetry tends to suppress the global perception necessary to visually check the soundness of the image produced. In task 1.7, Brittany merely extended the object vertically down by 4 units, while in task 1.6, she produced the correct image. The above observations point to the fact that spatial visualisation is dependent on the perceived features of the object and the relation between the object and the line of symmetry.

2. How are visual strategies enacted in bilateral symmetry and reflection tasks?

When the participants used terms such as 'I visualised', 'I imagine' or 'I think of (a particular object)', we inferred that they were using imagery in reflecting or finding the line of symmetry of the given objects. Similarly, when they made a particular movement (with their finger or pencil) showing the possibilities that they were considering, these were taken as evidence of the use of imagery (more specifically kinaesthetic imagery). As highlighted earlier, we refer to a *visual strategy* when attention is given to the use of imagery as related to shape, location, position, orientation, local and global perception. We now comment on the distinct ways in which visual strategies directed the participants' actions in performing the symmetry and reflection tasks.

Visual-mental folding and visual-mental reflection

We provided two apparent mental operations namely, visual-mental folding and visual-mental reflection which seemed to be at the basis of the students' action in their visual strategies for some of the symmetry and reflection tasks. Visual-mental folding was apparent in set 1 where the students had to find the number of lines of symmetry of given polygons and alphanumeric characters. Visual-mental reflection was particularly evident in Brittany and Sara's solution procedure in set 2 and set 3 tasks.

Reorientation of inclined line of symmetry to the vertical

To work with situations involving inclined line of symmetry, particularly when a grid was not available, Brittany and Sara would turn the line of symmetry in a vertical orientation. This reorientation was particularly apparent in set 4. The visual or perceptual facility afforded by the vertical reorientation of the line of symmetry was explicitly highlighted by Brittany: 'If you tilt it (to the vertical) this way, I could just have to work out where it will be'. Psychologists in the area of perception (Giannouli 2013) made similar observations, claiming that the vertical orientation is favoured by human beings.

Reflection of part of object as a visual trigger

In some cases, the reflection of one part of the given object served to open the space for reflection of the whole object in the inclined line of symmetry. We could observe such a visual trigger in tasks 2.2, 2.4, 2.5 and 2.7. Küchemann (1981) described this strategy as semi-analytic. Edwards and Zazkis (1993) also observed how the salient parts of figures tend to draw the attention of students in geometrical transformations. However, such a visual trigger was not always cued. For example, when we compare Brittany's response to tasks 2.2 and 2.3 involving the same object and same line of symmetry but at different positions, she experienced much constraint in the second task.

The visual strategy as a visual check

In a number of cases, we could observe how the students inspected their solution as a whole (global perception) to verify whether the image was correctly drawn. For instance, in task 2.6, where the line of symmetry was not inclined at 45°, both Brittany and Sara could observe that their initial solution was incorrect. Another example is in set 4 (tasks 4.5 and 4.6) where Brittany compared the orientation of the alphanumeric characters and their image after reflection in the inclined line of symmetry, justifying the image by mentioning: 'It looks correct'. Although Sara used such a visual check, she tended to rely more on her local perception emanating from her primarily analytical approach.

Imagining lines of symmetry

The responses of the two students in set 3 explicitly illustrate how they used imagery in terms of imagining different lines of symmetry and 'mentally shading' different cells to satisfy the conditions in the given tasks. For instance, the motion of Brittany's pencil in the vertical, horizontal and slanting positions gave clear evidence how she was imagining different configurations.

3. How do visual and analytical strategies interact in the production of the image from the object?

We refer to the interaction between visual and analytical strategies when they are used conjointly in the production of the image from the given object. The visual strategy may serve to orient the problem solver in terms of the approximate location of the image (i.e. its position), its orientation or its shape. The visual strategy acts as a cue, trigger or prompt to start the solution (such as providing an insight or a global perception of the solution). The analytical strategy then builds on the visual strategy in the application of its constituent rules (i.e. equidistance and perpendicularity criteria), whether correct or incomplete.

The visual strategy as a scaffold for the analytical strategy

In some of the tasks, the students asserted that they made a global picture of how the image would look like before actually applying the analytical properties of reflection and symmetry. For example, in task 4.5, in the reflection of the letter L on an inclined line of symmetry, Sara first visually-mentally folded the enclosing square along the diagonal line of symmetry to locate the position of the image before applying the equidistance and perpendicularity criteria. In cases where she experienced constraints, Brittany depended on the visual to scaffold her thinking to maintain equal length as in task 1.8 (see Fig. 4). She attempted to construct the image visually by redrawing the shape so that the length of the object and image is maintained. Here, it appeared that the visual strategy took over as a more intuitive fallback measure.

Brittany's responses to tasks 3.2 and 3.8 illustrate further the interaction between visual and analytic strategies in the reflection and symmetry tasks. In task 3.2, she shaded three more cells to make a square using the properties of square: 'It's in the shape of a square and a square has 4 lines of symmetry.' For task 3.8, as she turned the page anticlockwise with the diagonal facing her vertically, she immediately deduced the cell she had to shade and gave the following justification: 'I knew that for it to have a line of symmetry it had to be on a diagonal because there are only 3 squares. So one of them has to be in half'. When drawing the other diagonal to verify if there is another possibility, she inferred that: 'that would not work; the diagonal does not go through any of the (shaded) boxes.'

Global perception serving the visual strategy and local perception prompting the analytical strategy

The concepts of global and local perception provide some insight in understanding how the visual strategy may support the analytical strategy. In task 2.3, involving the reflection of the L-shaped polygon, Sara explained how she made a global picture of the image and then applied the analytical property:

> I actually don't always do the points (meaning focusing on the vertices), I just. I can imagine this (the given object in task 2.3) folding into; like that's a piece of paper, that's shape like an L. So I just imagine if I fold it into two like diagonally here (pointing to the inclined line of symmetry), imagine where it would go. I can imagine that (the leftmost vertical line segment in the object) would go horizontally here…. Then after that I checked, doing the points thing (meaning focusing on the equidistance and perpendicularity criteria). Then, like measuring how many squares from the line of symmetry is the object and the image.

Thus, the initial global perception of the image gave her a visual feel of the solution. Then she focused on the individual vertices of the object (local perception) in the application of the analytical properties (equidistance and perpendicularity). Similarly, in task 2.6, in her explanation of how she constructed the image, she mentioned:

> I just had a rough idea of how it would be if I folded it. After that I tried. Because if I folded it diagonally and this line is horizontal (pointing to the upward horizontal segment in task 2.6), then this means the other line would definitely be vertical.

Summary and implications

By focusing on spatial visualisation, this study enhances our understanding of the visual and analytical strategies deployed in the solution of symmetry and reflection tasks. It explains the constraints associated with the inclined line of symmetry identified by the seminal work conducted by Küchemann (1981) and Grenier (1985). As noted by Hoyles and Healy (1997), it explains how the meaning of symmetry is negotiated via visual and analytical strategies. More importantly, it attempts to make explicit the layers of complexity inherent in specific tasks requiring spatial visualisation.

Limitation and future work

As argued by Piaget and Inhelder (1956), it is methodologically challenging to track processes that involve both mental and visual elements, especially if they are almost occurring simultaneously and swiftly, as is the case of reflection and symmetry in the current study. The inferences we made about the visual and mental strategies are subjective to what we could observe and deduce from the students' responses. We attempted to minimise these inferential limitations in a number of ways. We used a relatively diverse set of reflection and symmetry tasks in two interviews. Purposively, two interviewers and two cameras were used to track the moment-by-moment actions of the students. The data were analysed independently by the authors.

Moreover, most of the tasks that we used were set on grids. This may have facilitated the reflection of the objects. It may be an insightful endeavour to investigate the problems in this study without a grid. Another research enterprise may involve a fine-grained understanding of the ways in which students articulate other transformations involving spatial skills such as rotations. This study equally brings forth the importance of local and global perception as influential elements in students' reasoning, an aspect that requires further exploration. Furthermore, with the increasing use of technology in the teaching and learning of mathematics, students may be expected to perform reflection and symmetry tasks on a digital platform. The spatial visualisation demand of such transformations on the digital media is an element of future research and may be an extension to our work.

Curricular and instructional implications

In the Australian mathematics curriculum (Australian Curriculum Assessment and Reporting Authority 2014), the concept of reflection and symmetry starts as from year 2, beginning with the notion of flips and symmetry in the environment, till the end of secondary schooling where students encounter more advanced reflection and symmetry problems. These concepts may be regarded as relatively simple in the school mathematics curriculum. Yet, our case study shows that this may not be necessarily the case. By bringing to the forth students' strategies and constraints, it advances research knowledge in the field of reflection and symmetry by making explicit how visual and analytical strategies may interact as students engage in such tasks.

It is acknowledged that this two-participant contrasting case study is bound to be limited in scope. However, the ways in which it portrays the explicit students' actions with the symmetry and reflection tasks is informative for teachers. The constraints that Brittany encountered with inclined lines of symmetry are not uncommon among students as the review of literature reveals (Grenier 1985; Küchemann 1981; Leikin et al. 2000). Our findings prompt us to highlight the necessity to give more explicit attention to the perpendicularity criterion in reflection tasks. Ignorance of this criterion may be carried over to adulthood. It is very likely from their responses that Brittany and Sara received different quality of instructions. The more formal perpendicularity criterion that Sara used is most likely to be the result of instruction compared to the intuitive meaning of 'opposite' that Brittany held. It is important to develop the notion of perpendicularity as a foundational concept for the further learning of mathematics.

Carefully designed instruction is necessary for students to go beyond the intuitive notion of reflection and symmetry.

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