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ORIGINAL ARTICLE

Graphical construction of a local perspective on differentiation and integration

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Abstract Recent studies of the transition from school to university mathematics have identified a number of epistemological gaps, including the need to change from an emphasis on equality to that of inequality. Another crucial epistemological change during this transition involves the movement from the pointwise and global perspectives of functions usually established through the school curriculum to a view of function that includes a local, or interval, perspective. This is necessary for study of concepts such as continuity and limit that underpin calculus and analysis at university. In this study, a first-year university calculus course in Korea was constructed that integrated use of digital technology and considered the epistemic value of the associated techniques. The aim was to encourage versatile thinking about functions, especially in relation to properties arising from a graphical investigation of differentiation and integration. In this paper, the results of this approach for the learning of derivative and antiderivative, based on integrated technology use, are presented. They show the persistence of what Tall (*Mathematics Education Research Journal, 20*(2), 5–24, 2008) describes as symbolic world algebraic thinking on the part of a significant minority of students, who feel the need to introduce algebraic methods, in spite of its disadvantages, even when no explicit algebra is provided. However, the results also demonstrate the ability of many of the students to use technology mediation to build local or interval conceptual thinking about derivative and antiderivative functions.

Keywords Graphical · Derivative · Antiderivative · Pointwise · Local · Versatile

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Background

In order to do well in mathematics, students need to develop a considerable flexibility of thought. In this paper, we seek to describe the kind of versatile thinking that students need in order to build a good understanding of differentiation and indefinite integration using antiderivatives.

A number of researchers have recognised the need for flexible thinking in mathematics and sought to describe aspects of what it entails. One such attempt resulted in the concept of *adaptive expertise*, described as the flexible and creative application of meaningfully learned mathematical procedures (Baroody and Dowker 2003). Students possessing adaptive expertise have the ability to go beyond routine competencies, demonstrating flexible, innovative and creative abilities (Hatano and Oura 2003). Verschaffel et al. (2007) also identify flexible solution strategies as a key component of adaptive expertise. Another framework for flexibility in mathematics developed by Thomas (Thomas 2002, 2008; Tall and Thomas 1991; Graham and Thomas 2000) is called *versatile thinking*. He outlines the framework, which is complementary to adaptive expertise, as comprising three elements (Graham et al. 2009):

- *Process/object versatility*—the ability to switch at will in any given representational system between a perception of a mathematical entity as a process or an object
- *Visuo/analytic versatility*—the ability to exploit the power of visual schemas by linking them to relevant logico/analytic schemas
- *Representational versatility*—the ability to work seamlessly within and between representations and to engage in procedural and conceptual interactions with representations

There are many aspects to versatile thinking, but one that has been identified in the transition from school calculus to university analysis (Thomas et al. 2012) is the ability to shift the focus of attention between pointwise, local and global perspectives of function (Artigue 2009; Vandebrouck 2011). In fact, Vandebrouck (2011) claims that "working at university level on functions implies that students can adopt a local perspective on functions whereas only point-wise and global perspectives are constructed at the secondary school" (p. 2095). Often at school, students will evaluate functions, including derivatives, etc., at specific points, finding, say, $f(a)$. They may also work globally on a function representation, such as applying a translation (p,q) to a graph of a function and relating it to the algebraic formula $y-q=f(x-p)$. However, a consideration of the behaviour of a function on an interval such as $(x-\delta,\ x+\delta)$, as required for an examination of continuity, is most often not addressed. One disadvantage of a pointwise approach, demonstrated by Gray and Thomas (2001), is that it locks some students into a symbolic mode and the thinking associated with it, which affects their ability to interact with graphical representations. The important aspect of supporting students in construction of a local perspective on functions is fundamental to the ideas in this paper.

One aspect of school mathematics that can militate against construction of this versatility is an emphasis on algebraic manipulation. Hence, a number of researchers (Berry and Nyman 2003; Vandebrouck 2011; Yoon et al. 2009, 2011) have proposed that working on graphical tasks could enrich students' function perspectives. In particular, this may be one way to stimulate the local perspective needed to understand the fundamental concepts of limit, continuity, differentiability, series expansions and Riemann integration. In order to achieve this, more is required than simply adding a graphical approach to an algebraic one. True representational versatility, which is a part of versatile thinking (Thomas 2008), includes both the ability to address relationships between representations of the same concept and perform conceptual and procedural interactions with each representation of a construct.

In this study, we examine the process of student construction of versatile thinking related to graphical derivatives and indefinite integrals (using area and antiderivatives). In order to do so, we employ a theoretical framework for advanced mathematical thinking (FAMT) (Stewart and Thomas 2007; Thomas and Stewart 2011) to analyse student thinking. This framework combines the three worlds of mathematics (TWM; Tall 2004a, 2004b, 2008, 2013) and the actions, processes, objects and schemas (APOS) theory (Dubinsky and McDonald 2001).

One notion central to the theory of TWM is that a *visual* and *enactive* approach can often assist students to build important *embodied* notions before symbolic and formal ideas are introduced. Hence, in TWM, the cognitive development of mathematical thinking is postulated to take place in three worlds. In the embodied world, we make use of visual and physical attributes of concepts, combined with enactive sensual experiences to build mental conceptions, while the symbolic world is where the symbolic representations of concepts, which are often *procepts* (Gray and Tall 1994), are acted upon or manipulated. Movement between the embodied and symbolic worlds shifts the focus of learning from changes that begin in physical meaning to the properties of the symbols and relationships between them. In turn, the formal world is where properties of objects are formalised as axioms, and learning comprises the building and proving of theorems by logical deduction from these axioms.

In APOS theory, an individual uses the mechanism of reflective abstraction, to consider personal *actions*. They may examine their properties, reverse the actions, examine inputs and outputs, and hence generalise them into a mental *process*. In turn, a process becomes an *object* when the individual becomes aware of the totality of the process and is able to encapsulate it as a single entity or object. The object status may be confirmed by the ability to describe the properties of the object, understand that transformations can act on it and even construct such transformations. The mental object then becomes part of an appropriate mental *schema*.

While the TWM and APOS frameworks were constructed independently, Thomas and Stewart (Stewart and Thomas 2009; Thomas and Stewart 2011) have recognised that they have complementary aspects and that combining them orthogonally could produce a gestalt effect. They argue one may consider the nature of actions, processes and objects in each of the embodied, symbolic and formal worlds of the TWM, giving rise to a matrix of cells, each of which can provide insight into mathematical thinking, with the potential to enhance teaching. The contents of these cells would include the embodied actions, embodied processes, embodied objects, symbolic actions, symbolic processes, etc. involved in any mathematical activity. In the context of this discussion, an embodied object could be the area under a graph of a function, antidifferentiation would be an example of a symbolic process, and $\int(x^2-3x+1)dx$ a symbolic object (and also a procept). In this paper, student thinking in calculus is analysed through the lens of this composite framework.

In addition, the use of digital technology was central to this research. The use of such technology in undergraduate mathematics is in some senses well established (Thomas and Holton 2003). While some recent research has demonstrated clear advantages in technology use (Pierce et al. 2010; Zbiek and Heid 2011), other researchers report opposition from students who maintain a strong belief in the superiority of by-hand work for mathematics (Stewart, Thomas, and Hannah 2005). However, there are at least two ways in which technology is less commonly used in universities. The first involves the level of integration into student learning (Oates 2011), and the second is the didactical nature of its use.

The particular way in which technology is integrated into the curriculum is one factor that influences learning outcomes (Heid et al. 2013). However, a second crucial aspect is the nature of its use. In this regard, a technology subgroup of the ICMI algebra study posits that a crucial question to ask when considering implementation of technology is how it will use influence student conceptualisation (Thomas et al. 2004, p. 166). Looking only at computer algebra system (CAS) use (Heid et al. 2013, p. 601) suggests three topics central to such practice.

> Our examination of literature across the history of CAS in mathematics education suggests three topics that are central to discussions of research, theory, or practice: the interaction of concepts and procedures; new concepts, extended procedures, and structures that can be approached with CAS; and the thinking and reasoning that CAS-use inspires or requires.

To address these kinds of issues requires recognition that introducing digital technology into learning and teaching should come with an accompanying theoretical discourse (Artigue 2002) that takes into account the mathematical system inherent in the tool. Thus, careful attention must be paid to the role of techniques (Lagrange 2003), and in particular their *pragmatic* and *epistemic* value (Artigue 2002). The former can be evaluated in terms of productive potential (efficiency, cost, field of validity) while the latter relates to their capability of producing knowledge of the mathematical object under study and new questions that promote knowledge (*ibid.*). Our aim in this project has been to be constantly aware of the need to consider the epistemic value of techniques when using technology, especially in relation to conceptual understanding.

Method

A pre-calculus course at a university in Korea was taught by the first author for 15 weeks, with one 2-h session per week. This was the first course involving calculus that the students had taken at university. These liberal arts students had taken just a single introductory calculus course at school, where the focus was on correctly performing routine procedural skills. There were 143 students across three classes, although seven withdrew after the mid-term test, leaving 136 students to sit the final test. This course, which covers linear, quadratic, cubic, exponential and logarithmic functions, differentiation, integration, probability and matrices, is required by students wanting to major in a mathematically related subject. However, it is generally unpopular, student entry grades are mixed, and the students often have little intrinsic interest in mathematics. None of the students had used any digital technology other than a scientific calculator in mathematics. Since the technology was not available, students were not able to use the CAS calculator themselves and the course was taught using lecturer demonstration with GSP, Autograph and a TI-Nspire CAS calculator. However, there were some group discussions on the assignments and exercises involving sketching different functions using Autograph where the students had an opportunity to use the graphical software.

There were two modules, the first on differentiation and the second on integration. Prior to the differentiation module, the concept of a linear function was emphasised and it was mentioned that it would be useful for the work on differentiation. Using a graphical approach with Autograph and TI-Nspire, the effect of changing the variables a and b in $y=ax+b$ was explored. This was extended to the concept of a transformation parallel to the y-axis. A similar approach, based on changes of variables in general forms, was then applied to quadratic functions, $y=ax^2+bx+c=a(x-m)^2+n=a(x-\alpha)(x-\beta)$, enabling observation of transformations parallel to the x- and y-axes. Students were then in a position to analyse for themselves the effect (including transformations) of changes of parameters for general functions such as $y=ae^{x-b}+c$ and $y=\log(x-a)+b$.

The differentiation module

For the differentiation module, the teaching was based on the concept of differentiated learning, with activities divided into the levels suggested by the National Council of Teachers of Mathematics (1989), but with the addition of a final level. Level 1 defined a rate of change function $r(h)=\frac{f(2+h)-f(2)}{h}$ for $f(x)=x^2$ and used the CAS to generate numeric approximations for r, with h from 0.1 to 0.000001, to establish the idea of the limit of r as $h\to 0$. Level 2 established the symbolic relationships:

$$f'(2)=\lim_{h\to 0}\frac{(2+h)^2-2^2}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}=\lim_{h\to 0}\frac{4h+h^2}{h}=\lim_{h\to 0}(4+h)=4.$$

In addition, using the CAS calculator the tangent line at $x=2$ on the graph was drawn to visualise the fact that $f'(2)$ is equal to the slope of tangent at $x=2$. The aim of Level 3 was to generalise the rate of change. The CAS calculator was used to introduce students to a method of obtaining the derivative at $x=a$ by defining a function slope(h)= avgRC($f(a)$, a, h), $a=\{-1, 0, 1, 2, 3\}$ as the average rate of change over an interval. In this manner, students could investigate the change of slope and, by taking the limit, see that it gets close to $\{-2, 0, 2, 4, 6\}$, and hence conjecture that the derivative is $2x$ (see Fig. 1).

Linking these numeric and symbolic representations to a graph enabled an epistemic exploration of the relationship between the graph, the slope of the tangent line and the derivative. When students observed the information in Fig. 2, they were able to construct local, interval knowledge of the objects, such as if the value slope of tangent line is negative, the derivative is located under the x-axis and if it is positive it is above. Further, they could relate this to pointwise constructs, such as the turning point is at $x=0$ since the slope is 0, and the slope is negative on the interval to the left and positive to the right. Hence, a student could draw the derivative by understanding the variation of the slope of the tangent line.

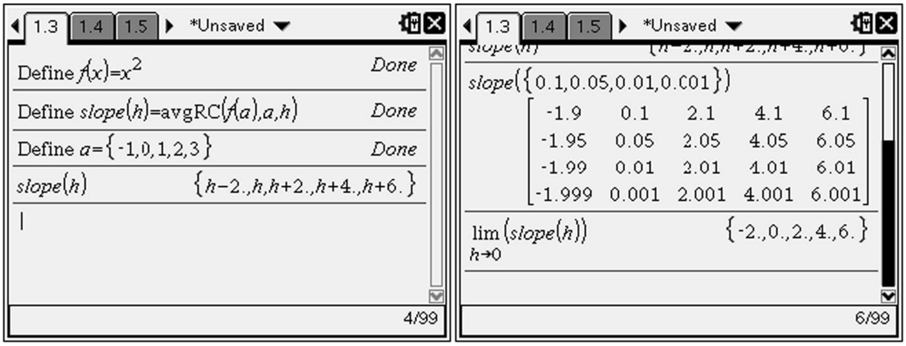


Fig. 1 The calculator screens showing the slope calculations

In this way, Level 3 sought to establish links between the graphs and the symbolic relationships:

$$f'(x) = \frac{d}{dx}f(x) = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

The role of Level 4 was to apply the results above to functions of the form $f(x)=x^n$ for $n=\{1, 2, 3, 4, 10\}$ and hence lead students to generalise and infer the derived function, $f'(x)=nx^{n-1}$. After completing level 4, students understood the derivative as the changes in the slope of the tangent line as the x values are changed and hence could sketch the derivative without being given an explicit function. This constituted the Level 5 task. For example, with the function shown in Fig. 3, students were encouraged to locate the turning points where the gradient of the tangent line is zero, at $x=0$ and approximately $x=1.5$.

Based on this, they could divide the real line into intervals whose endpoints are the critical values 0, 1 and 1.5 and produce a table of values (see Table 1) of the gradient based on the intervals $(-\infty,0)$, $(0,1)$, $(1,1.5)$ and $(1.5,+\infty)$.

The integration module

In a similar manner, the teaching for the integration module comprised five distinct levels. Level 1 considered the symbolic world calculation of Riemann sums using intervals of equal size, and a typical question is shown in Fig. 4. The approach taken to this kind of question was to use the table of values to calculate the width of a

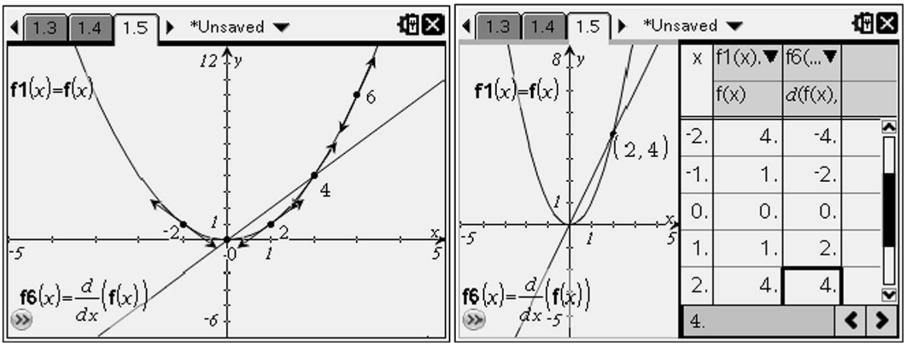


Fig. 2 The calculator screens showing the relationship between the slope function and the graphs of f and f'

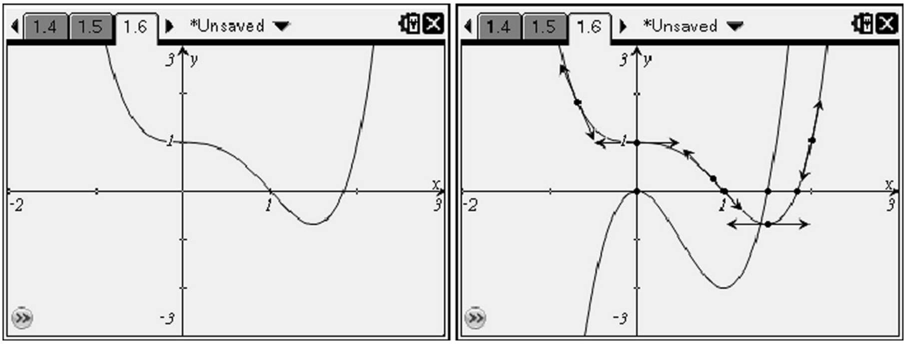


Fig. 3 The calculator screens showing construction using interval reasoning of the function f' for a function f with no explicit formula

subinterval as $\frac{(6-1)}{10}=0.5$. Thus, using leftsum, we find $x_1=1$, $x_2=1.5$, $x_3=2, \ldots\ x_{10}=5.5$ and using the values of $f(x_i)$ as the height of rectangles, the sum of the areas of the approximating rectangles with left endpoints could be found by hand as:

$$L_n = 0.5(1 + 2.25 + 4 + 6.25 + 9 + 12.25 + 16 + 20.25 + 25 + 30.25) = 63.13$$

However, students were encouraged to use the Excel spreadsheet on the graphics calculator as a pragmatic tool to find the area more easily rather than evaluating it by hand. As seen in Fig. 5, one can define the x values in column A, the values of $f(x)$ in column B, $0.5\times f(x)$ in column C and the leftsum in column D, using "=$d1+c2$", and copying this formula down the cells until $x=5.5$ produces the sum of the areas of the 10 approximating rectangles as 63.13.

Level 2 employed an embodied world perspective, assisting students visually with the GC to see the difference between leftsum, middlesum and rightsum. It also used interval thinking, focussing student attention on what was occurring on intervals such as $[x_{i-1},x_i]$ and $[x_i,x_{i+1}]$. By using an increasing number of subintervals, $n=5$, 10, 20, and 50, the aim was also to conjecture the exact area under the curve. In Fig. 6, representative screens showing the results for $n=10$ and 50 are presented. In these, left-endpoint rectangle approximation method (LRAM) is the leftsum (which increased, LRAM=55, 63.1, 67.3, 69.9), midpoint rectangle approximation method (MRAM) is the middlesum (with values MRAM=71.3, 71.6, 71.6, 71.7), and right-hand rectangle approximation method (RRAM) is the rightsum (which decreased, RRAM=90, 80.6, 76.1, 73.4) with $n=5$, 10, 20, and 50 subintervals.

It was anticipated that the students would see that as the number of rectangles increases, the interval size gets smaller and the difference between leftsum and rightsum gets closer to 0, and since these are also lower and upper sums, respectively, for the given function, this implies that they are squeezing the exact area (which is $71\frac{2}{3}$)

Table 1 The table employing interval reasoning to construct the function f' for a function f

x	$-\infty$	…	0	…	1	…	1.5	…	$+\infty$
$f(x)$			1		0				
$f'(x)$		−	0	−		−	0	+	
	$-\infty$	↗		↘		↗		↗	$+\infty$

The table gives the values of a function obtained from $f(x) = x^2$. Find the area of the region that lies under the graph of $f(x)$ between x=1 and x=6 using 10 equal subintervals with left endpoints.

x	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6
$f(x) = x^2$	1	2.25	4	6.25	9	12.25	16	20.25	25	30.25	36

Fig. 4 A question on Riemann sums from the integration module

between them. These epistemic techniques lead to knowledge of the Riemann integral through a linking of the embodied approximation and the symbolic world through the limits of left and right sums equal the integral $\int_1^6 x^2 dx$, as see in Fig. 7.

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(f\left(x_i\right)\cdot\Delta x\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(f\left(x_i\right)\cdot\Delta x\right) = \int_1^6 x^2 dx = 71\frac{2}{3}$$

Thus, the aim was that after completing Level 2, the students might be in a position to understand the definite integral as the limit of Riemann sum.

Level 3 again took an embodied function stance, using an Excel spreadsheet in the GC to plot graphs of leftsum and rightsum as functions of x for a given n (see Fig. 8). Thus, students were helped to appreciate that if the function f, with $f(x)=x^2$, is increasing on the interval [1, 6], then as the number of rectangles increases, the leftsum and rightsum approximate the integral $\int_1^6 x^2 dx$.

Level 4 used the concept of antiderivative as the inverse of the differentiation to deduce a sketch of the definite integral for a graph without an explicit function. The embodied world method employed used interval thinking, considering the sign of the function and the antiderivative on crucial intervals. For example, in the question given in Fig. 9, $x=-1$ and $x=3$ are critical values, giving rise to intervals $(-\infty,-1)$, $(-1,3)$ and $(3,+\infty)$.

Once again, a table of values (see Table 2) was used to consider the sign of the function and its antiderivative on these intervals.

Finally, level 5 allowed students to relate the embodied and symbolic worlds in order to investigate Cavalieri's principle on an interval [a, b] (that $\int_a^b f(x)-(f(x)-g(x))dx$

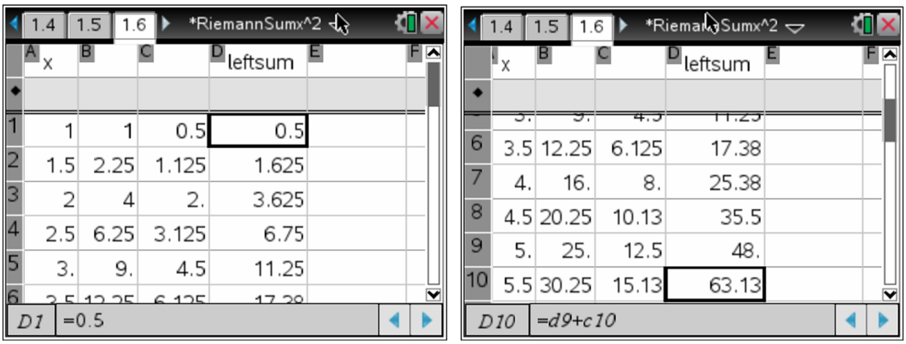


Fig. 5 Pragmatic use of the graphic calculator to find a Riemann leftsum

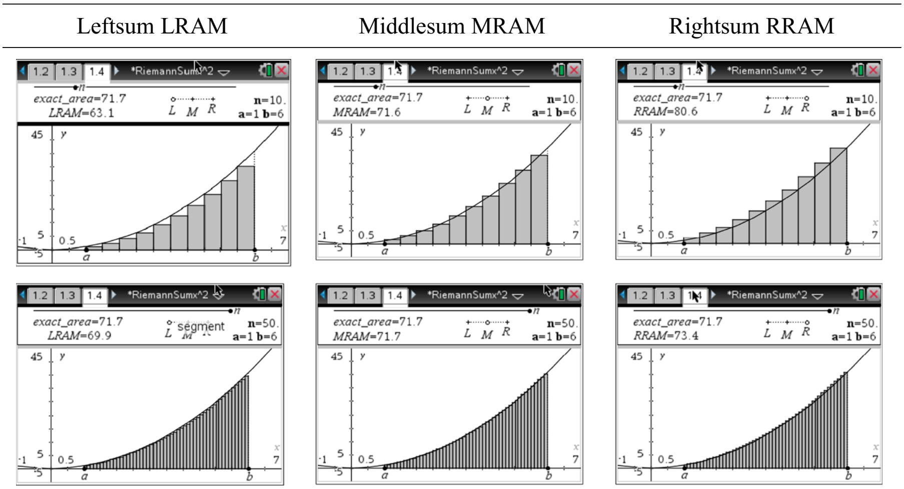


Fig. 6 Leftsum, middlesum and rightsum for $n=10$ and 50 subintervals

$= \int_a^b g(x)dx)$ and form generalisations about the concept of area under the graph of a function.

Results and analysis

Graphical derivatives

One of the questions on the mid-term test required students to sketch the graph of a derived function from a graphical representation, with no explicit algebraic function given. This derivative question is given in Fig. 10. There were a number of different approaches taken to these questions, and these form the content of the two scenarios described below.

Scenario 1: symbolic process algebraic thinking

Students whose thinking is dominated by symbolic world thinking initially may find such a question difficult since there is no algebra to work with. The modelling method employed by

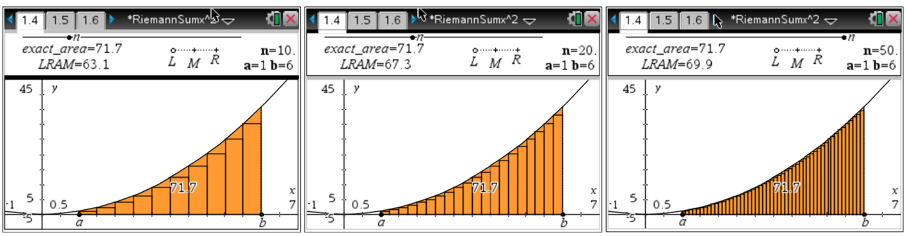


Fig. 7 Embodied and symbolic relationships between leftsum, rightsum and the exact area as the number of rectangles increases to ∞

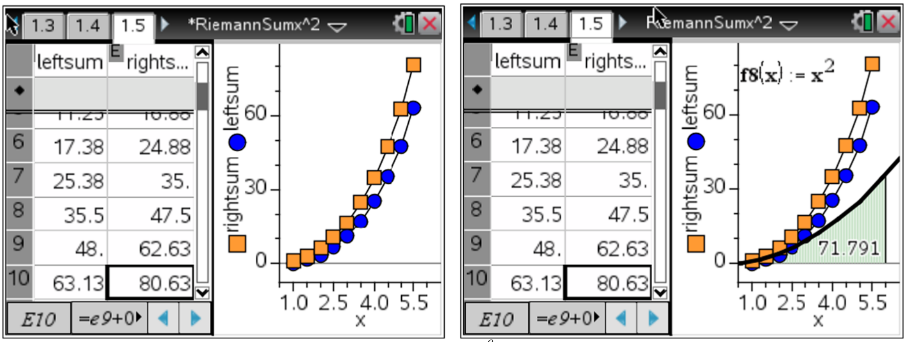


Fig. 8 Graphing function approximations to the integral $\int_1^6 x^2 dx$

these symbolic process-oriented students in the second example was (1) to assume the graph is a polynomial and determine its order, (2) try to fit it to the general formula for such a polynomial function using $y=a(x-b)(x-c)(x-d)$ ($y=a(x-b)^2+c$ in the first case), (3) employ information from the given graph to find the parameters and model the function (for example, the point (0, 2) lies on the graph, so $a = \frac{1}{3}$), (4) expand the trinomial, (5) differentiate the polynomial function obtained, (6) complete the square on the resulting quadratic function and then (7) draw the derived function.

In this research, 51 students, or 35.7 %, attempted the derivative questions this way. This is typified in the working of students A and B, seen in Fig. 11.

In both cases, they have solved problem a) by starting with a function of the form $y=a(x+2)^2+2$ and using the vertex (−2, 2) to find the value of a from the y-intercept (0, −2). Once found, the function $y=-(x+2)^2+2$ is then differentiated to give $f'(x)=-2x-4$ or $f'(x)=-2(x+2)$. In the second problem b), both defined a function $y=a(x+1)(x-2)(x-3)$, then substituted (0, 2) for x and y to find $a = \frac{1}{3}$. The function $y = \frac{1}{3}(x+1)(x-2)(x-3)$ was then differentiated, giving $y'(x) = x^2 - \frac{8}{3}x + \frac{1}{3} = \left(x-\frac{4}{3}\right)^2 - \frac{13}{9}$. Both students not only gave $f'(x) = x^2 - \frac{8}{3}x + \frac{1}{3}$ but also completed the square to get $f'(x) = \left(x-\frac{4}{3}\right)^2 - \frac{13}{9}$, enabling them to find the vertex of their parabola.

No doubt these students were influenced by the procedural teaching they had received in school. However, there is some considerable merit and versatility in the

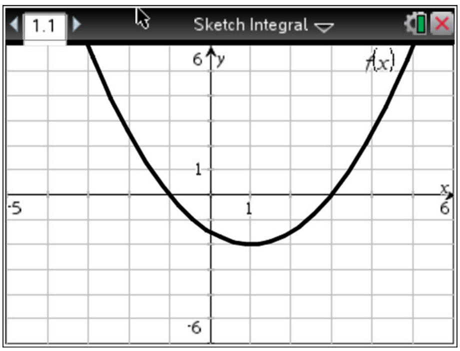


Fig. 9 Graphing an antiderivative of a function. The graph of a function is shown in the figure. Make a rough sketch of an antiderivative function F, $F(0)=0$

Table 2 The table employing interval thinking to establish the antiderivative

x	$-\infty$	…	-1	…	3	…	$+\infty$
$f(x)$		+	0	−	0	+	
$F(x)$	$-\infty$	↗		↘		↗	$+\infty$

accomplishments of such students; not least a measure of commendable persistence, with all the 43 students successful in their approach. In this method, they have to be engaged in a conversion, or translation between representations (or registers), from the graphical to the algebraic in this case, something that Duval (2006) has shown to be difficult for many students. However, we can see that while this symbolic process route requires some pointwise thinking, it circumvents the need for local or interval thinking involved in an embodied process when working within the graphical representation.

Scenario 2: embodied process interval thinking

Of the 143 students, 80 (55.9 %) were able to draw correctly the derived function graphs by a consideration of local or interval thinking (in addition, one student used both algebra and interval methods), without requiring an algebraic function. Examples of the working of two students who typify this approach are given in Fig. 12. Some of the students made comments such as "if $f(x)$ is increasing, $f'(x)>0$, if $f(x)$ is decreasing, $f'(x)<0$, if the gradient of the tangent line is zero on $f(x)$, $f'(x)=0$", "if the slope values change from positive to negative, then the values of the derivative change from positive to negative" and "if the slope values change from negative to positive, then the values of the derivative change from negative to positive".

These students are thinking in a versatile manner, constructing a method based on principles they have learned. They are not constrained to work in the world of symbolic

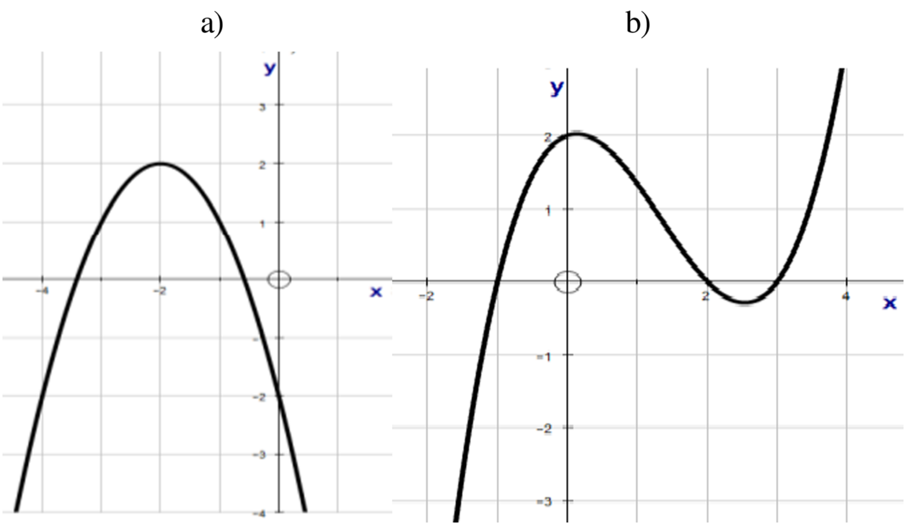


Fig. 10 The test questions on graphical derivative. Sketch the derivative for the given graphs

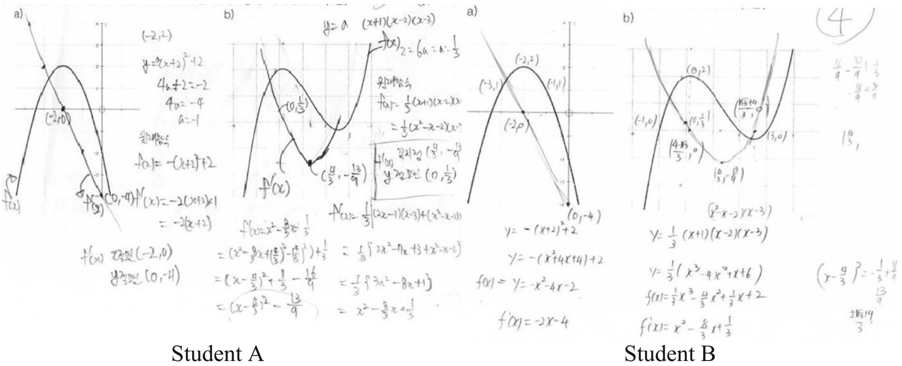


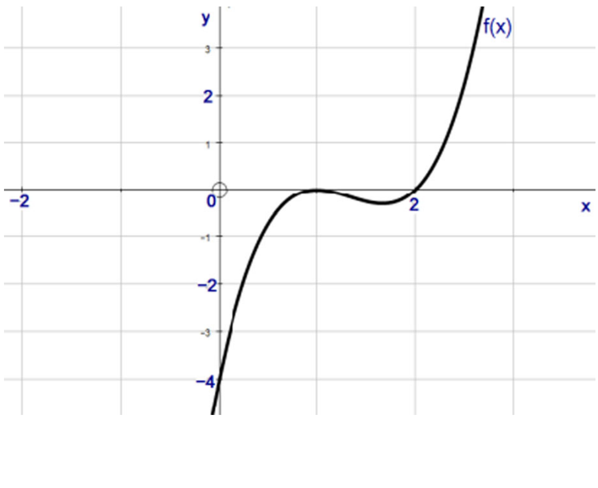
Fig. 11 Algebraic solutions of two students to the questions on graphical derivative

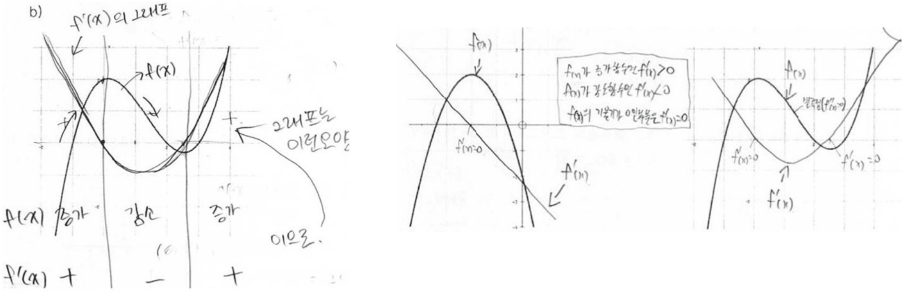
algebra thinking but can use embodied process reasoning in their interactions with the graphs of the functions. In addition, they do not need to construct the derivative graph in a pointwise fashion, as many students might, but are able to employ an epistemic technique of working with intervals. For graph a), having established the pointwise relationship of the stationary point on the parabola, where the gradient is zero, to the point where $f'(x)=0$ on the derived function, they can then reason on the intervals $(-\infty,-k)$, $(-k,+\infty)$ to the left and right of this point. Similar reasoning is applied to graph b), where there are two stationary points and hence three intervals to consider. Student D was one of only two students who also realised that the point of inflection (identified in Fig. 12) corresponded to the greatest negative gradient and hence the local minimum on the derived function graph.

Graphical antiderivatives

In the final test, the students were given the following question, which asked them to find an antiderivative function graphically. As in the differentiation question above, only the graph was given, with no explicit algebraic function.

> The graph of a function is shown in the figure. Make a rough sketch of an antiderivative F, given that $F(0)=0$.





Student C Student D

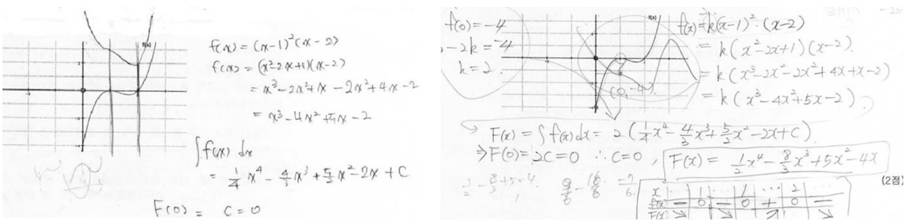
Fig. 12 Graphical, interval solutions to the questions on graphical derivative

Scenario 1: symbolic process algebraic thinking

As with the differentiation question, there was a minority of students who employed a symbolic world modelling approach to the problem. This usually involved a number of steps: (1) assuming that the graph is a cubic polynomial; (2) modelling the given graph with a version of $f(x)=k(x–1)^2(x–2)$, based on reading the function zeros from the graph; (3) using data from the graph that $f(0)=-4$ to find the value of k; (4) expanding the trinomial; (5) finding symbolically the antiderivative, F, of this cubic; (6) using an assumed fact that $F(0)=0$ to find the constant of integration and (7) sketching the graph of this quartic function.

Of the 136 students who sat the final test, 39 (28.7 %) attempted a symbolic process method of this kind using algebra to solve the problem, and 16 (11.8 %) of these did so successfully. In Fig. 13, we see typical working from students H and I, both of whom have employed this algebraic method.

We can see that student H has modelled the function using $f(x)=(x–1)^2(x–2)$, taking k as 1. Although he finds $F(0)=0$ when he sketches the antiderivative graph, it does not pass through the origin. In contrast, student I does find $k=2$ from $f(0)=-4$, as well as $C=0$. Interestingly, at that point, presumably unable to sketch the quartic graph, he resorts to an interval approach, drawing a table with critical values and considering the sign of f and F in the intervals. Thus, he is able to make a reasonably good sketch of the graph. While this versatile thinking is commendable, we note that he did not attempt to

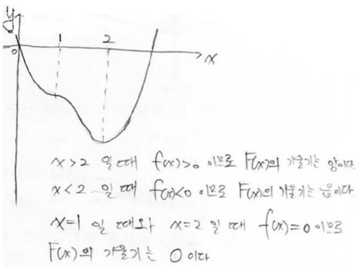


Student H Student I

Fig. 13 Algebraic solutions of two students to the questions on graphical antiderivative extract the information in the table from the original graph before entering the symbolic world.

Scenario 2: embodied process interval thinking

In contrast with the symbolic thinking exemplified above, 97 students or 71.3 % were able to tackle the question using an epistemic technique based on embodied interval thinking. Of these, 79 could draw the antiderivative graph reasonably correctly. Figure 14 exemplifies this in the working of four of these students, J, K, L and M. All four of these were able to identify the critical points at $x=1$ and $x=2$, and thus, most considered some or all of the intervals $(-\infty, 1)$, $(1,2)$ and $(2,+\infty)$. However, we can see a qualitative difference in thinking by these students. Student M uses mostly embodied thinking in a pointwise manner, considering the behaviour at $x=1$ and $x=2$ and talking about the "tangent line". However, he does consider what happens in an interval $(2-k,$

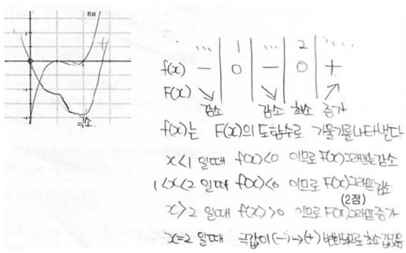


"When $x > 2, f(x) > 0$, the gradient of the tangent line on $F(x)$ is positive.

When $x < 2, f(x) < 0$, the gradient of the tangent line on $F(x)$ is negative.

When $x = 1$ and $x = 2, f(x) = 0$, the gradient of the tangent line on $F(x)$ is 0."

Student J



"$f(x)$ represents the derivative of the function $F(x)$.

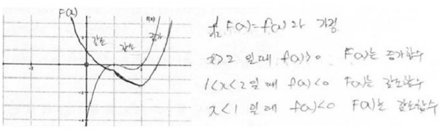
$x < 1, f(x) < 0$, the graph of $F(x)$ is decreasing

$1 < x < 2, f(x) < 0$, the graph of $F(x)$ is decreasing

$x > 2, f(x) > 0$, the graph of $F(x)$ is increasing

$x = 1$, there is a local minimum value $f(x) < 0$, the graph of $F(x)$ is decreasing."

Student K



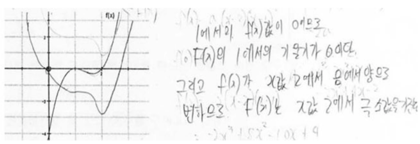
"Let $\frac{d}{dx}F\ x\ = f(x)$

When $x > 2,\ f(x) > 0, F(x)$ is an increasing function.

When $1 < x < 2, f(x) < 0, F(x)$ is a decreasing function.

When $x < 1, f(x) < 0, F(x)$ is a decreasing function."

Student L



"The value of $f(x)$ is 0 at $x = 1$, the gradient of the tangent line is 0 in at $x = 1$ for $F(x)$. And $F(x)$ has minimum value at $x = 2$ since the value of $f(x)$ changes from negative to positive at $x = 2$."

Student M

Fig. 14 Graphical, interval solutions to the questions on graphical antiderivative

$2+k$), saying "the value of $f(x)$ changes from negative to positive at $x=2$", in order to deduce a local minimum at $x=2$.

Student J's approach involves embodied process thinking applied to intervals using statements such as "when $x>2$, $f(x)>0$, the gradient of the tangent line on $F(x)$ is positive". Here, we see that the interval $(2,+\infty)$ is used, along with the embodied notion of a tangent line to a graph. The graph is then constructed through a process employing gradients rather than the use of an antiderivative function object. The piece of information that "when $x=1$ and $x=2$, $f(x)=0$, the gradient of the tangent line on $F(x)$ is 0" is sufficient, in this case, to avoid problems with concavity of the graph on the intervals $(-\infty,1)$ and $(1,2)$, but it may be that he would have experienced more difficulty with concavity if the point of inflection had not had a zero gradient.

Students K and L both present an explicit functional relationship between the function, f, whose graph is given as the derivative of a function F, that they are trying to find, indicating an object perspective on the function F. They also exhibit interval thinking, considering all three intervals $(-\infty,1)$, $(1,2)$ and $(2,+\infty)$. In addition, student K expresses embodied thinking, stating "the graph of $F(x)$ is decreasing", while student L says that "$F(x)$ is an increasing function", more removed from the embodied perspective. Further, his use of $\frac{d}{dx}F(x) = f(x)$ suggests, unsurprisingly, an influence from symbolic world thinking.

There was virtually no use of pointwise thinking to recognise the point of inflection on the antiderivative graph, although one student, N, who also used interval thinking to sketch the antiderivative, used this same approach to explain why there is a point of inflection at $x=1$. His reasoning was "the inflection point exists because there is no change of decreasing-increasing at $x=1$ in the region of a and b", where the regions a and b refer to the intervals $(-\infty,1)$ and $(1,2)$. He first claimed that "the region of a is negative, it is a decreasing function on this part. The region of b is negative, it is a decreasing function on this part" but then did not expand on how he arrived at his conclusion.

Other question results

In the discussion above, we have identified two distinct approaches to the derivative and antiderivative questions: one restricted to symbolic process algebraic thinking and a second, more versatile approach, primarily based on embodied process interval thinking. Some questions arise about the performance on the rest of the test questions of the groups of students exhibiting these kinds of thinking. For example, did those who expended much time and effort on the algebraic approach manage to answer as many questions as the others? How did the overall performance compare between the groups?

A statistical analysis of the number of questions attempted by those students in each group and their level of success in them is given in Table 3. While it is not straightforward to interpret these figures, they do reveal that in both the mid and final tests, the group using algebraic thinking attempted significantly fewer questions than the versatile group and performed significantly worse overall. One possible explanation is that they comprised a weaker group of students. However, it is worth noting that membership of the algebra group was not stable and, by the final test, 26 students had moved from that group to the versatile one. This may suggest that, as well as innate ability

Table 3 A comparison of group performance

	Algebraic thinking (*N*)	Versatile interval thinking (*N*)	*t*	*p*
Mid-term test number of questions attempted	5.6 (65)	7.9 (78)	13.05	<0.000001
Final term test number of questions attempted	6.5 (39)	9.0 (97)	6.18	<0.000001
Mid-term test mean mark	26.5 (65)	34.1 (78)	6.22	<0.000001
Final term test mean mark	16.3 (39)	30.9 (97)	10.81	<0.000001

being a variable in their performance, the algebraic method is both slowing them down so that they have less time and so attempt fewer questions and they are less likely to succeed in the questions they do attempt. Reversing this, we might say that there is some evidence versatile thinking has a positive effect on performance in the questions set at this level.

Conclusion

The aim in this study was to investigate whether curriculum materials integrating digital technology and designed to give an improved cognitive base for a flexible, proceptual understanding of limit, derivative, integration and other concepts develop a more balanced dual view of concepts as process and concept. This accords with Heid et al. (2013) who maintain that technology use can help calibrate the balance and interplay of procedural and conceptual knowledge if different concepts are emphasised, concepts are studied more deeply, investigations of procedures are extended and increased attention is placed on structure.

There are some observations emerging from this research. One is that structuring the teaching materials using the level method appears advantageous in that it engages students in appropriate, staged activities that require them to think through each embodied or symbolic process involved. Further, at each level, the technology focus promotes active thinking about what is happening, encouraging students to develop epistemic mathematical techniques. Thus, rather than a focus on just routine, pragmatic algebraic computations, the learner is assisted to know and to understand.

Second, the results confirm that an emphasis on a pointwise and symbolic process algebraic thinking in schools tends to produce students with a reliance on this form of working (Gray and Thomas 2001). This type of thinking strongly persisted for around 30 % of the students, even though the lectures did not teach this method and the questions did not provide an explicit algebraic function. However, we note that during the course, the number of students whose thinking was restricted to algebraic methods did decrease. While strong algebraic manipulation is advantageous in mathematics, students do need to be versatile and especially to build their ability to use interval thinking. This is especially the case if, as here, the students solely employing algebraic thinking are disadvantaged in terms of available time and perform less well in examinations.

The outcomes of this research project suggest that a pedagogical approach, as developed here, integrating digital technology in teaching, may have positive effects, encouraging versatile, embodied and inter-representational thinking. Further, a key contribution of this study is to provide evidence that developing engagement with the numeric and graphical representations (or registers) can support students in the development of epistemic techniques involving local or interval thinking. In this case, around 56 % of the students were able to demonstrate versatile thinking, applying this kind of strategy to find, correctly, the graphs of derived functions and 58 % to graphs of antiderivative functions. We have also provided a clear indication of the kind of activities that can be successful in encouraging epistemic techniques leading to versatility. Since this local or interval thinking is vital in the progression from calculus to the formal world thinking of analysis at university (Artigue 2009; Vandebrouck 2011), then modules such as the one described here, which support a graphical approach to derivative and antiderivative, and others that allow an even more active engagement with the technology could usefully be included as part of a multi-faceted strategy to develop such thinking, at school, in bridging courses and in first-year university service courses. Since these issues are a matter of some concern in the transition from school to university mathematics (Thomas et al. 2012), it suggests the need to consider the construction of didactic situations aimed at addressing a reliance on symbolic algebra process thinking.

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