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REVIEW

Cultural-historical activity theory: Vygotsky's forgotten and suppressed legacy and its implication for mathematics education

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Abstract Cultural-historical activity theory—with historical roots in dialectical materialism and the social psychology to which it has given rise—has experienced exponential growth in its acceptance by scholars interested in understanding knowing and learning writ large. In education, this theory has constituted something like a well kept secret that is only in the process of gaining larger levels of acceptance. Mathematics educators are only beginning to realise the tremendous advantages that the theory provides over other theories. In this review essay, I articulate the theory as it may relate to the issues that concern mathematics education and educators with a particular focus on the way in which it addresses logical contradictions in existing theories.

Keywords Marxist psychology · Materialist dialectics · Praxis · Ideal · Inner contradictions · Affect · Cognition · Ethics

Introduction

For the past 30 years, the theories predominantly used in mathematics education to theorise knowing and learning have been various forms of constructivism—for example, radical and social constructivism (Cobb et al. 1992; von Glasersfeld 1987). More recently, some mathematics educators have emphasised the need of combining psychological and sociological approaches to theorising mathematical learning (e.g., Cobb 1999). In this endeavor, researchers sometimes draw on a special form of social psychology that has been developed in the Soviet Union during the early and middle parts of the 20th century (e.g., Vygotsky 1979) to suggest that there is first an inter-psychological (social) construction that precedes the intra-psychological construction of knowledge. All this revisionist work, however, does not account for the fact that there are fundamental problems with all constructivist theories (e.g., Derrida

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1990). Two attempts to address problems with the constructivist theories exist in (a) embodiment/enactivist theories, which explicitly theorise the role of bodily experiences in mathematical knowing (e.g., Núñez et al. 1999) and (b) practice theories, which shift the locus of cognition from the mind to the cultural and historical material practices that define knowledge communities (e.g., Lave 1988). However, fundamental problems remain when such frameworks are employed to understand mathematical learning in schools because the emphasis remains on the individual learner and learning theories tend not to account for the cultural-historical contingency of schooling as a particular form of human activity that reproduces not only knowledge but also societal inequities (Roth and Lee 2007). Cultural-historical activity theory, which emphasises that the locus of all (mathematical) sense is historical and political praxis (Radford et al. 2011), is a theory that overcomes the problems of other theories because it sublates any opposition between individual and collective, body and mind, or individual and its ecology (Radford 2011a). From the perspective of cultural-historical activity theory, all of these terms are but manifestations of an underlying non-self-identical unity.

Cultural-historical activity theory had been founded in the 1930s as an attempt to develop a Marxist psychology (Vygotsky 1927/1997). However, it did not begin to influence Western scholarship until the 1970s and 1980s, when the first translations of the foundational texts became available. In fact, the first translations constitute Americanised versions of the work, which do not exhibit the truly revolutionary character of the theory that derives from its Marxist underpinnings, in fact repressing this aspect, and that insiders characterise as poorly reflecting the originals (e.g., editor's introduction to Vygotskij 2002). Since then—that is, over the past 30 years—there has been an exponentially increasing interest in cultural-historical activity theory especially with respect to learning in the workplace, including mathematical practices at work (e.g., Triantafillou and Potari 2010), and human-computer interactions (Roth 2004; Roth and Lee 2007). In one of its instantiations, the theory is best known through an emblematic representation that exhibits the structural aspects of activity (Fig. 1). This representation highlights the fact that we cannot understand any action of a *subject* on the *object* of activity outside of all the relations to other aspects of the activity, which in fact mediate every other moment and relation. It is also impossible to construct this fundamental unit from any composite because of the irreducible part–whole relationship between activity and its identifiable aspects. Moreover, invisible in the representation, there are two dimensions to every activity

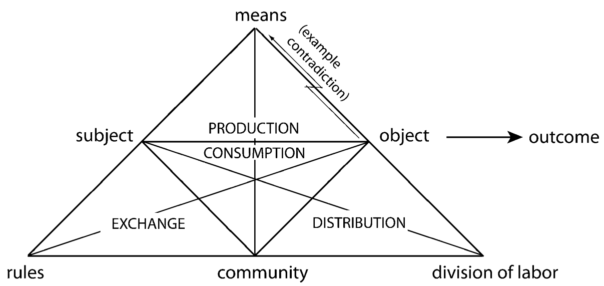

Fig. 1 This triangular representation of cultural-historical activity theory emphasizes its synchronic structure but makes invisible its diachronic dimensions

in general: the material and the ideal, which constitutes the reflection in consciousness of the former (Radford 2008). Activity, a category of analysis, is the minimal unit that allows researchers to make sense of sense making by the research participants involved in the transformation of objects into outcomes (products). To create a concrete frame of reference, consider the following three scenarios, which I subsequently use as the touchstone for explicating the theoretical concepts.

Three cases of concrete mathematical practice

To set up this review of cultural-historical activity theory and its potential for mathematics education research, consider the following concrete scenarios, which are of the type described in the literature on mathematical knowing and learning.

> A fish culturist, who has nothing but a high school certificate uses graphs to record, predict, and adjust the average growth rate of the approximately 1,000,000 coho salmon in her care. When the average weight of a monthly sample lies below the target, she uses food-to-weight conversion rates, total estimated fish population, and targeted above-normal monthly growth to calculate how much feed there is to be thrown into the fish pond each day. (Roth 2005)

> A group of theoretical physicists, using the Internet to interact, attempts to calculate the cohomology for a given operator on a given space. In their computation, they draw on "a lot of abstract techniques," among them the "spectral sequence technique." They attempt to understand the behavior of the equations involved so that they can publish these prior to other groups working on the same mathematical problem. (Merz and Knorr-Cetina 1997)

> Seventh-grade students are given a sheet containing several columns of numbers from which they may choose any two. The teacher explains that they are to find relationships using one of the techniques he has taught them before, including pie charts, bar graphs, and scatterplots. He distributes graphing paper and reminds students to use pencil so that they can easily erase and redo a graph should they have made an error. (Roth and Barton 2004)

One of the boys in the seventh-grade class, Davie, has been defined as learning disabled. The video of the mathematics class could easily be used to confirm this assessment, because the boy spent only 90 s on the 28-minute task. He does not arrive at doing anything other than drawing the axis in the way he had seen his desk mates do. However, other evidence collected during the science unit, which allowed Davie to frame his own research questions the results of which he presented at an open-house event of a local environmentalist group, shows that he used a graph to display relationships he had identified between multiple variables. Such differences are astonishing and not easily theorised by constructivist theories, which hold that knowledge exists in the form of mental structures that should be employable in any setting of interest. When individuals employ mathematical knowledge in one setting but not in another, analytical concepts such as "transfer" or the knowing/application contrast are employed to explain performance differences (e.g., Lave 1988).

Whereas most theoretical approaches would be concerned with identifying what this or that individual really knows, cultural-historical activity theory begins by identifying the overall frame: What is the culturally and historically contingent *activity* that is realised by the actions of the participants in the three scenarios? Those who use cultural-historical activity theory in the way it was intended point out that there are three distinct activities, each of which is identified by what it produces to meet some generalized need of society. The fish culturist produces a healthy brood of 1,000,000 coho smolt, which she releases after having taken care of it for 18 months since she fertilised the eggs by mixing them with the milt (sperm) taken from male coho. She does employ mathematics, models the coho population, and uses statistics; but, in her own words, she does not do mathematics but raises fish (Roth 2005). When the adult coho return, they will contribute to several local industries, including tourism, commercial fishery, and basic food supply. The theoretical physicists, however, investigate the mathematical properties of the operators and spaces so that they can calculate the cohomology; their chosen product is a scientific paper, which contributes to the collectively available knowledge in the field of theoretical physics and applied mathematics. The paper will be circulated and read within a community of scientists. Finally, the students produce graphs that end up in the garbage can once the particular mathematics unit is completed and student achievement has been assessed. This mathematical task is part of the overall schooling activity that produces a stratified population with differential access to further schooling and workplace; and it also produces individual characteristics such as learning disability (McDermott 1993; Roth and Barton 2004). Even if all three scenarios were to involve exactly the same kind of mathematical representations, the actions we observe would differ, because the intended outcomes differ, as do the means of production (tools), the community that defines the legitimate practices, and the division of labour. That is, actions are different because the entire unit of analysis has changed, not because of the individuals—who, in any event, tend to exhibit different mathematical practices as soon as the situation is changed (Lave 1988; Saxe 1991). To understand individual aptitudes, we are therefore held to follow the same persons through multiple settings and contexts rather than giving one test to multiple persons (Corno et al. 2002).

Defining activity and its structure

Activity as the fundamental unit and category

Activity is the basic category of the theory; it is the smallest analytic unit for understanding human performances: their practices, the sense they make, or the actions they perform. In English, there is a confusion arising from the fact that the term “activity” translates into two distinctly different concepts in the languages in which activity theory was formulated initially. Thus, the German/Russian term *Tätigkeit/deyatel’nost’* (“activity”) denotes a structural moment of society that produces something for a generalised, common need: the fish culturist produces, and thereby maintains the stock of, salmon, the theoretical physicists produce mathematical and physical knowledge, and the seventh-grade students, in and through their participation, reproduce in not only schooling but also society and its culturally specific (cognitive, material) practices. That is, from a cultural-historical activity theoretic perspective, for the latter going to school is the activity rather than doing a graph or doing a problem-solving task. *Schooling* activity is the precursor of and mediates any learning that follows (Seeger 2011). The satisfaction of the generalised need is the motive of the activity. The German/Russian terms *Aktivität/aktivnost'* ("activity") refers to doing something without a collective motive and without being oriented toward a generalised need. For example, the seventh-grade students are doing something: they produce graphs, but these end up, at the time of the research, in the garbage pail. What they have produced or failed to produce, in this instance, the graph, does not satisfy a generalised need—though Davie's science graphs, exhibited at the open house, do contribute to increasing visitors' understandings of the creek that they represent (Roth and Barton 2004). In (Anglo-Saxon) mathematics education, the two different concepts come to be conflated when doing some task is named "activity."

Activities and actions

Activities are concretely realised by goal-directed actions, but actions are performed only because they realise an activity. The relationship between these two levels therefore is mutually constitutive: One cannot exist or thought without the other because they are bound in a whole–part relation. In fact, the sense of an action derives from the activity that it realises, which means that the same action will have a different sense if it is performed to realise a different activity (Leontjew 1982). Thus, in mathematics class, the graph *is the transitive object* at which Davie and his peers direct their actions; during the open house, however, Davie's concern is the local creek and the graphs he uses are but representational *tools* that assist him to communicate what there is to know and what might be done to improve environmental and human health in the community. As Fig. 1 shows, the actions are different because the graphs are differentially embedded in the respective activities.

Actions and operations

Actions do not just exist but they need to be actually performed: They are realised by series of unconscious operations. But these operations are executed only because there is conscious goal-directed (conscious) action that calls for realisation. Thus, Davie does not have to think about holding his pen, drawing the axes, or writing numbers to produce a scale. These operations are executed without him attending to it while he is focussed on producing a graph that will convince the visitors of the open house. In the course of development, however, conscious actions may become unconscious operations, for example, when students who at first have to learn how to scale and prepare axes subsequently draw them without having to think about these explicitly.

Integrating across levels

The important contribution cultural-historical activity theory makes is to stipulate that not only actions but also operations are contingent upon the activity. We know that people who appear competent on certain tasks may fail and commit simple errors in examination conditions. That is, although the object is the same, the very fact that one is an examination activity changes the execution of basic, unconscious operations. Whereas most mathematics educators at present think about and theorise the fundamental processes involved in the realisation of mathematical praxis as context independent and person specific, the theory presented here stipulates otherwise. Thus, for example, speaking and thinking are theorised as changing processes that constitute each other in their mutual evolution; however, any act of speaking cannot be understood by considering the speaker alone but needs to take into account the speaker–listener relation (Bakhtine [Volochinov] 1977; Vygotskij 2002). That is, even the specific word that a research participant utters—the production of which is an operation, like a tone in a melody (Luria 1973)—is a function of the activity as a whole. Thus, the nature and sense of individual words specifically and local discourses more generally are functions of the activity such that the talk of any one of the participants in the three scenarios needs to be understood differently if it arises during an interview, think-aloud protocol, or teaching experiment, on the one hand (research activity) or the natural settings.

The mental and the material

An important aspect of cultural-historical activity theory is that it does not privilege the mental over the practical material dimensions of mathematical activity. In fact, there are two dimensions to any activity: the material and its (ideal) reflection in consciousness (Leontjew 1982). The relation between the material and ideal frequently is expressed in a confusing way: as the unity of opposites (e.g., Il'enkov 1982). This dual nature is made most salient in the concept of the object/motive. Some mathematics educators suggest that the "object" is something like the material embodiment of the collective purpose (Williams et al. 2007), but other mathematics educators find this concept difficult to work with and simply conflate the object and outcome as an operationalisation of the dual nature of the object (Beswick et al. 2010). However, to make advances in the theorising of mathematical learning, such as abstraction, we need to find the single source of the different manifestations of activity (e.g., Ozmantar and Monaghan 2007; Roth and Hwang 2006). In the theory, the object refers to the materials at hand that are transformed in the productive process and turned into the outcome; this outcome exists in the consciousness—that is, ideally—from the beginning and serves as the activity-driving motive. Thus, at the beginning of the work process, there are concrete entities on the material plane, reflected on the ideal plane, where the intended outcomes also already exist.

Sublating difference

The dual (ideal, material) nature of activity generally and the object/motive specifically leads us, in the context of classical Western scholarship, to think activity as somehow composed by some combination of the opposite—much in the way the dual nature of light became a hindrance to the progress of physics until quantum mechanics showed that the wave and particle natures are but manifestations. A better way of approaching this unity is by thinking it in terms of a basic category that is not identical with itself, and, therefore, can manifest itself in different ways across people and within people across time. That is, some mathematical entity such as a graph is not thought about as "having different meaning" because individuals "construct" it differently based on their subjectivities; rather, the ways in which graphs are used have to be understood within activity as a whole under their material and ideal aspects simultaneously. Those mathematics educators who have taken up theoretical concepts from post-structuralist approaches already emphasise the substitutability and supplemental nature of any sign (Brown 2011).

Affect and cognition as reflections of praxis

Affect—which is theorised in much of psychology as a separate human factor that mediates cognition, generally diminishing it (e.g., because of fear of mathematics)—is an integral aspect of activity. Cognition therefore cannot ever be thought independently of affect, because the two constituted themselves in the evolution of the human psyche as mutually constitutive (Leontyev 1981). Without affect, there is no intentional movement, and, therefore, no activity or action. Affect and cognition constitute different manifestations of how the material situation is reflected in consciousness. The presence of affect is apparent in the fact that activity is oriented to the generalised human needs at the level of society. As long as individuals contribute to the generalised needs, division of labour and exchange processes allow them to meet their individual needs. The fish culturist contributes as part of the generalised need for food and sustaining a local economy. She receives an income that allows her to purchase clothing, food, and shelter to meet the needs of her family. Because cognition and affect are different manifestations of the same unit, they can, in contrast to factors that are external on to the other, directly influence each other. Thus, a surprising lay-off notice literally *affected* her and the worries about the future came to affect the way in which she worked and used mathematics (Roth 2007). That is, some event at the level of the fish hatchery directly influenced her bodily states and these, providing the context for the unconscious operations, therefore also shaped how mathematics was enacted. Thus, cultural-historical activity theory models the interrelation not only of material and ideal dimensions but also between cognition and affect. In fact, cognition and affect are reproduced and transformed with every action, which allows us to understand why some action in mathematics class takes a student from elation that he knows what he is doing to utter frustration about not knowing what he is doing or is supposed to do (Roth and Radford 2011). Because these dimensions and constructs generally are thought about in terms of opposites that cannot be combined, we need to consider how cultural-historical activity theory draws on the category of inner contradictions.

Ethics

In cultural-historical activity theory, each action, in transforming the natural world, consciousness, and activity as a whole has an indelible effect on all other human beings that are part of the activity system and the society-constituting network of activities as a whole. It therefore does not come as a surprise that mathematics educators working from this paradigm have come to theorise the ethical dimensions that comes from the participation in collective endeavours (Radford and Roth 2011).

Society as a network of activity

Activities do not exist in and for themselves but are subtended in a network of activities; and it is this network that is constitutive of society, reflecting the division of labour necessary for a distributed, generalised provision for meeting basic and extended human needs. Because the products of an activity are exchanged with other activities where they find specific use, the production process is shaped not only by the inner relations within the activity but also by the relations that connect the different activities; and any production may in turn create further needs.

Inner contradictions

A widely misunderstood category of dialectical thinking generally and cultural-historical activity specifically is that of *inner contradictions*. In general use, which also dominates research, contradictions are exemplified by logical contradictions, perturbations, innovations, or breakdowns that arise from mathematical activity (e.g., David and Tomaz 2011). As such, contradictions may be removed; and, because the actions produce movement and change in the activity, contradictions are thought of as the fundamental engine underlying change in and of activity. But within dialectical approaches, inner contradictions are understood differently. Most fundamentally, these cannot be removed because they are characteristic (constitutive) of the thing itself. In the preceding section, we encounter one such contradiction in the opposition of the material and ideal dimensions of activity specifically and the human life form generally.

The developers of cultural-historical activity theory (e.g., Leontjew 1982; Vygotsky 1927/1997) explicitly ground themselves in dialectical materialism, which is an epistemology interested in understanding and modelling the apparently ever-changing world. This change, however, was not thought about in terms of some external force that influences a system; rather, change was to be the effect of the activity itself. Thus, linguists and language philosophers working on the same theoretical ground understand that a language changes *because* it is used, not because there is some external factor bring about change; and a language is inherently dead when no longer used (Bakhtin 1981). Every time a word is uttered, its sense changes together with language as a whole (Bakhtine [Volochinov] 1977; Vygotskij 2002).

The categories that are required to model change *itself* have to embody an inner contradiction. To concretise the discussion, take the following example. A mathematics educator interested in understanding the impact some intervention has on students gives a test or uses (clinical) interviews to establish what students know (their mental framework or discourses) before the intervention (K_1). They then teach a unit or do a teaching experiment and then interview/test the students again, this time identifying knowledge after the unit (K_2). Learning is conceived as the difference Δ between the two states $\Delta = K_2 - K_1$. Here, knowledge is taken as something unproblematic, assessable using this or that method; and learning how students get from one to the other state is problematic. The category of learning is reduced to the difference between two states. But this category does not model *change in itself*. There is some teaching or learning that makes (“constructs”) a change in knowledge. What is required is a category that embodies change itself. That is, acting *itself* has to produce the change in the way speaking inherently changes language rather than language change being the result of some constructive effort (Bakhtine [Volochinov] 1977; Vygotskij 2002). The fundamental unit of learning, therefore, has to be something like (K_1, K_2) which is irreducible to K_1 and K_2 or a mixture thereof. Any time we are conducting an observation (by giving a test, doing an interview), this unit will manifest itself in one or another way precisely because it is not identical with itself. We can use the same reasoning for evaluations of knowledge across different settings (activities); again, a particular student or person may exhibit different mathematical knowing. From the present perspective, this is not surprising but is in fact central to the theory. Theoretical physicists would not be surprised reading this, because phenomena such as light manifest themselves in different ways not because of the subjectivity of the researcher but because the manifestation is itself a function of the particular situation. By using such a category, learning, change, and performance differences between settings are unproblematic, but what knowledge is and how it could be assessed as something independent of context and specific to the person does become a problem (Lave 1993).

In the preceding section, I suggest that cognition and affect are reproduced and transformed in and because of acting. This assertion expresses precisely the same point made about activity as a category for modelling *the process of* learning. It does so because the category describes *change itself* rather than being the difference between two (self-identical) states. The activity changes *with* an action rather than *because of* an action. The category of activity is useful precisely because it models the change of a continuously changing world rather than focussing on the states that this world goes through, animated by some external process.

This inner unit of learning cannot be composed, somehow, by construing some form of assemblage of the different stages or forms of knowledge. From a cultural-historical activity theoretic perspective, we cannot compose learning by bringing together (the difference between) K_1 and K_2. This is so because the required unity "between *nature* and *culture*, intellect and affect, and the higher and lower forms of behavior cannot be phenomena rooted in mutual determination of the 'different aspects' of that unity" (Mikhailov 2001, p. 19). The inner unity and difference precede any manifestations of differences.

Constructivist theory is based on the knowledge–application distinction typical of all Kantian and neo-Kantian approaches and the correlative distinction between mind and the world. Radical forms of constructivism in mathematics emphasise that knowledge is in the mind and its usefulness arises from its viability. Social constructivists frequently ground their work in Vygotsky (1978) to suggest that constructions first occur in the "social" sphere and between individuals (inter-psychologically) before they occur within the individuals (intra-psychologically). A clear separation is maintained between internal and external processes. Most importantly, this idea has found its way into the notion of the *zone of proximal development*, which mathematics educators use to understand that a student can perform at a developmentally more advanced stage in the presence of a teacher or more advanced peer (Meira and Lerman 2001). Apart from the fact that this asymmetrical application of the concept fails to theorise teacher learning that occurs simultaneously (Roth and Radford 2010), activity theorists who read Vygotsky in the original suggest that this distinction between the internal and the external processes is not consistent with Vygotsky's other work, a logical contradiction that the scholar would have clearly addressed had it not been for his early death at the age of 34 (Mikhailov 2001). It is evident that a fourth-grade student who produces an algebraic generalisation of the kind $y = \mathrm{a} \cdot x + \mathrm{b}$ in the presence of his teacher also is active mentally doing precisely the generalisation (Roth and Radford 2011). In fact, it does not matter whether the teacher is present or not: the higher psychological function that expresses itself as the relation with the teacher *is* the same higher psychological function that expresses itself sometime later when the student is alone. Because the relation is external (Vygotskij 2005), it is an objectified and objectively available expression of the higher psychological function. This fact has led, in mathematics education, to the theory of objectification (Radford 2011b). We are therefore held to understand that "*the very existence of mind is possible only at the borderline [between inside and outside] where there is a continual coming and going of one into the other*, at their dynamic interface, as it were" (Mikhailov 2001, p. 20). Thus, there is a "single process of their mutual generation and mutual determination" (p. 21). This does not mean, as some might fear (e.g., Fried 2011), that the distinction between inner and outer, private and public is made to disappear but rather that the distinction receives its proper place in an encompassing theory where the two aspects are irreducibly intertwined.

In educational applications, the theory is often associated with the triangular representation (Fig. 1), which mathematics educators, too, have found useful (e.g., Zevenbergen and Lerman 2008). One of the main problems with the triangular representation of the theory is that it does not make (sufficiently) salient the *inner* contradictions; in fact, the representation reifies the static perspective on activity rather than emphasising its dynamic nature and the inner contradictions that explain the dynamic. This corresponds to the distinction made in linguistics between the synchronic and diachronic dimensions of (and perspectives on) language. The contradiction between the two perspectives is not overcome by somehow collating them into a common theory but rather by making the dimensions (the grammatical, the historical) two manifestations of the same, non-self-identical linguistic activity (Bakhtine [Volochinov] 1977). Thus, a living language is one that changes in use and while being used, exhibits specific (stable) structural relations only when it is considered fixed for the moment. But such a synchronic (structural) representation of a living phenomenon inherently is a fiction.

Subjectification and personality

Cultural-historical activity theory allows us to think in new and productive ways about individual development generally and about mathematical development specifically. There are two relevant concepts: *subjectification* and *personality* (Roth and Radford 2011). The first concept can be understood in terms of the changing relations within activity (Fig. 1). Because activity is a category that models change, any of its moments (observables) also are modelled in terms of change itself. *Subjectification* refers us to the change process relating to the (individual, collective) subject, which I understand in terms of a related concept used in political theory (Rancière 1999). It names a process by means of which new capacities for actions of a body and previously not identifiable enunciations within a particular field of experience are produced in the course of acting and discoursing; the identification of actions, discourse, and subjects "are part of the reconfiguration of the field of experience" (p. 35). That is, any action changes the activity system and, with it, the subject—both in its bodily material and ideal (cognitive, emotional) dimensions (Marx/Engels 1962). A person, therefore, can be identified by its actions that arise from particular subject positions (due to division of labour) within systems of activity; these actions, though a function of the activity as a whole, are ascribed to the individual, who identifies with these and its position. Thus, the people in the three scenarios *are*—a form of the verb "to be," Lat. *esse*, pointing us to what is *essential*—fish culturist, theoretical physicist, and student because of the activity in which they (have chosen to) participate, because of what they do (their role in the division of labour), and because of how they do it (expert, novice, beginner). That is, the individuals are identified on the basis of the object/motive that is constitutive of each activity.

Subjectification does not explain the development of the individual person as a whole. From a cultural-historical activity theoretic perspective, it does not make sense to think about "mathematics identity" in the way some mathematics educators do (e.g., Nyamekye 2010), including those subscribing to the theory (e.g., Black et al. 2010). This leads these scholars to make assertions such as those postulating relations between "mathematical identity" and "racial identity" (Martin 2007), as an "amalgam of partial identifications" (Brown 2011, p. 82), or as a "leading identity" among other forms of identity (e.g., Black et al. 2010). The problem with this approach arises from (a) attempting to theorise the person as a whole (consisting of the sum total and relation between different identities), (b) assuming that identities are the result of individual constructions, and (c) using a concept such as "(self-) identity" for theorising an inherently non-self-identical phenomenon.

In *personality*, cultural-historical activity theory offers a concept that allows us to understand the person as a singularity and as collective phenomenon simultaneously without reducing it to one of its observable moments (Leontjew 1982; Roth and Radford 2011), including "mathematics identity," "racial identity," and "leading identity." Each and every day, a person participates in multiple systems of activity: the fish culturist is a shopper, mother, worker, gardener, wife, moviegoer, and so on. In each activity, her development would be understood in terms of the *collective* object/motives that characterises the particular system and by the associated process of subjectification. The person, therefore, participates in realising multiple object/motives in the course of her day. However, these collective object/motives are integrated into an individual, hierarchical "knotwork" of object/motives. That is, although the latter are characteristic of society and the generalised needs that they meet, each knotwork is highly individual in the specific place where a particular object/motive appears and the strength it entertains with all the others (Roth 2011). Personality is a process that stands in a whole–part relation to all processes of subjectification that are part of the personality constituting knotwork as a whole. We cannot theorise subjectification with respect to a particular person in any activity without also considering the personality as a whole. There *cannot* therefore by something like "mathematical identity," because the process of subjectification in the relevant activity (e.g., school mathematics) cannot be understood independent of the entire network of object/motives that is constitutive of personality of the person. If there were something like "mathematical identity," then it would have to be theorised, given its status as a part in a whole, as a singular plural, that is, as an inherently plural unity (Nancy 1996) that is inherently heterogeneous and non-self-identical (Nancy 1993).

Possibilities for research and practice in mathematics education

Cultural-historical activity theory opens up new ways of thinking about teaching and learning in mathematics and to go about doing research in the field. In the following, I sketch some of the areas/topics that mathematics educators may want to pursue.

Cognition and the societal-political dimensions

Mathematics educators tend to analyse learning irrespective of the societal dimensions of schooling (Roth 2009). In this way, such research makes it look as if cognition and student participation were independent of the structure of society and the inequities embedded therein. Yet careful ethnographic studies have shown that children from working-class families tend to become working-class citizens rather than moving up the ladder (Eckert 1989; Willis 1977). Merely analysing what happens in mathematics classrooms without taking into account *schooling* in the way many scholars do (e.g., Jurdak 2006) is a logical contradiction in the context of cultural-historical activity theory. The processes that reproduce and transform society in this manner, although they occur on a continuing basis, do not become salient within most current theoretical frameworks and research methods. This is astonishing once we accept that personality is the ensemble of relations in which a human being participates. That is, the kinds of relations in which students with working-class biographies engage during their school time are those that produce their working-class status in life. Cultural-historical activity theory, because its minimum analytic unit is *activity*, which in the present case is *schooling* ("going to school"), offers precisely the analytic tools to understand how societal relations as an ensemble mediate every act within the activity system. Despite all its shortcomings, the mediational triangle (Fig. 1) can orient researchers toward recognising how society shapes the means in educational production, the rules (e.g., state- or district-ordered curriculum), division of labour (hierarchy of educational institutions and their exchange relations with the society at large), and the communities of practice.

Cultural-historical activity theory has very practical dimensions, which have arisen from the critical psychology expansion of Leont'ev's work conducted in Germany (e.g., Holzkamp 1993). In this expansion, the focus is on the individual subject, who acts based on how the world appears to its consciousness. The structure of activity is determined in ways that generally are not apparent to the subject. Because the individual person learns when it can expand its agency and control over life conditions, cultural-historical activity theory in this version is aligned with the goals of those researchers and practitioners, who take mathematics education as the possibility to educate students for citizenship, increasing agency, entertaining critical relations to power, and engagement in political matters (Brown 2011; Radford 2011a; Valero and Stentoft 2010). I am convinced that cultural-historical activity theory is a suitable framework for bringing together often-disparate forms of investigation in mathematical cognition and the often-differently oriented concerns of ethnomathematics.

Histories of institutions and artifacts (tools)

A fundamental presupposition of cultural-historical activity theory is that activity as a whole and every one of its constitutive moments is culturally and historically contingent. We cannot therefore just take the triangular representation and attempt to place it, like a cookie cutter, on some school mathematics class trying to identify the various moments. To understand any action within the system (Fig. 1), we need to know the history (biography) of each moment (i.e., subject, object, tools, rules, community, and division of labour); that is, we need to know the personal, institutional, cultural, and theoretical histories that embed every instance and moment we might observe while studying mathematical teaching and learning. This happens, as my analysis of a body of text has shown, all too infrequently (if ever) in the study of mathematical cognition and learning (Roth 2009). For example, it is well known that artifacts and practices crystallise cultural knowledge (Radford and Puig 2007; Williams and Wake 2007) such that what once were necessary skills come to be sedimented and no longer are needed—though they may be re-awakened at any time, as a study of the history of geometry shows (Husserl 1939). Cultural-historical activity theory, because it explicitly takes into account the tools in theorising cultural productions (see Fig. 1) captures how the system as a whole changes and how it requires *different operations* (*i.e., skills*) for successfully producing the actions that realise the activity. Such changes are well known and pervasive in society. Whereas many adults—including teachers—took a long time to familiarise themselves with new information technologies, children who grew up with these tools use them in the same facile ways that they use their mother tongue; when the Rubik's cube was invented (in 1974), adults attempting to solve the puzzle struggled and most adults never achieved success even when they worked at it for hours, whereas some of those growing up with the cube solve the puzzle in a few seconds. Pertaining to mathematics education, there are calls for "Back to the basics," which are antithetical to this common knowledge. A cultural-historical analysis of the skills required in doing division—from paper-and-pencil, long-hand division via the use logarithmic tables to slide rules and calculators—shows that the "basic" operations (skills) required change (Roth 2008). From a cognitive perspective, there are no reasons why doing a division with a calculator requires competency in doing a long-hand division; and the estimation required for doing a ball-park check of the results of long-hand-, log-table-, slide-rule-, or calculator-mediated division requires different kinds of "basic" operations (skills). It is precisely because of its recognition of the role of tools (signs) and the changes these undergo that cultural-historical activity theory has shown itself to be useful to mathematics educators interested in how tools shape mathematical cognition (e.g., Carlsen 2009, 2010; Falcade et al. 2007; Lagrange and Erdogan 2009).

Personal histories

Persons who take up the subject positions in an activity come with their own histories (biographies), having been shaped in very different ways depending on social class or gender, for example. These histories are sedimented in the current knotwork of object/motives that constitute the specific person and in their structured structuring dispositions (e.g., Lave 1988). Opening any mathematics education journal shows that most studies of mathematical learning neglect carefully studying the histories of those who take up subject positions (teachers, students) even though there are studies that show how the very relations girls have had with their mothers is formative of the mathematical discourses and understandings that they come to employ and embody in schools (Walkerdine 1988). In contrast, one of our own studies of the mathematics in a fish hatchery investigated (a) the history of fish hatching in British Columbia, (b) the history of one specific "indicator" hatchery that contributes to constitution of the collective history, and (c) the histories of the persons as a basis for understanding fish hatchery mathematics and its development over the course of a five-year ethnographic study (Roth et al. 2008). In this study, therefore, mathematical competencies observed were understood from the dialectical relations of personal history and the cultural-historical field of the activity (salmon enhancement). Moreover, we studied the changing nature of the means of production available, finding out why some tools were introduced and abandoned again, to understand what people were doing, how they were doing them, and why.

A cultural-historical approach orients us to simultaneous and constitutive processes of subjectification within the system and its role within the larger process of the development of personality, the shifting and developing relations between the ensemble of activities (societal relations) in which a person takes part. This allows us to understand, for example, how the different, often-contradictory mathematical requirements in school and at the workplace become constitutive discursive resources within activities (school, workplace), between activities, and in the life of a person as a whole. Electricians, for example, have to use trigonometry in their vocational courses for calculating how and where to bend conducting pipe but use a specifically marked tool for doing the bending on the job (Roth 2011). But talking about the contradiction between what they do on the job site and at college is part of being a competent electrician, who, even while using the tool, needs to be able to justify his/her decisions according to the codified (academic) knowledge.

Capturing learning within the system

Studies in mathematics education tend to focus either on teachers or on students. In a typical teaching experiment, for example, only the children studied are investigated without consideration that the person conducting the experiment also changes. In educational praxis, teachers are made responsible for learning outcomes, which may come, depending on the jurisdiction, with repercussions when the students of a teacher consistently underperform. This responsibility of the teacher is further salient in some of the inappropriate uses of the concept of the zone of proximal development, where learning opportunities arise from the presence and skills of the teacher (or more experienced peer). Cultural-historical activity theory, however, forces us to look at the entire system and study the changes therein. This leads us to a more symmetric approach. Thus, for example, it is common knowledge that teachers learn to teach mathematics while teaching mathematics (and often come to better understand mathematics in so doing); and yet the analyses of student learning or failure to learn do not simultaneously exhibit the concomitant teacher learning. Thus, it has been shown how in the interactions of teachers with their students all participants have to seek out what the problems are and, often by means of trial and error, evolve appropriate teaching strategies (Roth and Radford 2010, 2011). That is, while second-grade students learn to name objects using the proper geometric discourse or while fourth-grade students make an abstraction of the kind $y = \mathrm{m} \cdot x + \mathrm{b}$, the respective teachers learn to teach (i.e., expand their agency). Both teachers and students participate in *societal* relations, which, as is presupposed in cultural-historical activity theory, are the concrete forms in which the higher psychological functions exist in the world (Vygotskij 2005). In fact, it can be shown that students *assist* or provide the resources for the teachers' learning (e.g., Roth and Radford 2010, 2011). This led us to assert that teaching is learning and learning is teaching. It is precisely the changing nature of activity as a whole that captures the changes that occur throughout the system, irrespective of the nature of the moment and irrespective of the operant division of labour.

Emotion, motive, motivation

About 80 years ago, Vygotsky complained that (standard) psychology separates cognition and affect, which makes it appear as if thought constituted an autonomous process that thinks itself (Vygotskij 2002). The psychologist suggested that the relation can be understood only when we consider the "fullness of life," the "personal needs and interests," "the inclinations and impulses of the thinker" (p. 54). This approach bars a true understanding of cognition, making it an epiphenomenon detached from affecting the world (e.g., the separation of knowledge and its application). Cultural-historical activity theory does not conceive of cognition and emotion as separate factors in mathematical performance. Rather, cognition (consciousness) and affect are two manifestations of the same phenomenon. Both are reflections of the material state of the activity and the distance between currently anticipated and intended outcome. Mathematical cognition and affect are but two sides of the same coin (Roth 2007). A recent analysis shows how the two mediate and transform each other in and as a result of activity, which allows us to understand the entire rollercoaster of advances and regressions, frustrations and elations, or needs and tendencies in mathematical teaching-learning processes (Roth and Radford 2011).

Coda

Cultural-historical activity theory has much to offer to mathematics educators, as it opens new areas for research and new ways of theorising phenomena that emphasise relations and histories. It is by no means a complete theory, which not only has been expanded in recent years but also requires further development. In fact, the theoretical development rather than eternal reification of theory is at the heart of the dialectical materialist agenda that underlies cultural-historical activity theory (e.g., Il'enkov 1982). Much work needs to be done because, as I show here, contradictions have arisen when a theoretical framework based on dialectical materialism and a very different form of thought (logic of inner contradictions) is absorbed into a way of thinking based on classic logic (the excluded third).

In this brief review article, I merely sketch some of the possible openings that this theory prepares for those interested in mathematics education research and practice. The theory orients us to praxis, which, inherently, is concretely available to participants in societal relations. Even the most "abstract" forms of mathematical thought are concretely available in societal relations—for example, the proof of Gödel's theorem (Livingston 1986). Just as implied in pragmatic approaches to knowledge (e.g., Wittgenstein 1953/1997), cultural-historical activity theory does not require us to make hypotheses about the contents of peoples' minds but asks us to study societal relations that are the origin of anything that might be attributable to the individuals and their minds. Drawing on the title a Vygotsky (1978) book, I offer this concluding aphorism that might shape how mathematics educators want to think about research and classroom practice: "Mind is in society to the extent that society is in the mind." What we, mathematics educators, have to endeavor is to look for the right places (societal relations) so that we can find it (mind). The mathematical mind then no longer is something ephemeral, an inaccessible structure underneath the skull, but something concrete and alive among all of us and in the societal relations that we perform with others.

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