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ORIGINAL ARTICLE

What is rate? Does context or representation matter?

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Abstract Rate is an important, but difficult, mathematical concept. Despite more than 20 years of research, especially with calculus students, difficulties are reported with this concept. This paper reports the results from analysis of data from 20 Australian Grade 10 students. Interviews targeted students' conceptions of rate, focussing on the influence of representation and context on their expression of their understanding of rate. This analysis shows that different representations of functions provide varying levels of rate-related information for individual students. Understandings of rate in one representation or context are not necessarily transferred to another representation or context. Rate is an important, but commonly misunderstood, mathematical concept with many everyday applications (Swedosh, Dowsey, Caruso, Flynn, & Tynan, 2007). It is a complicated concept comprising many interwoven ideas such as the ratio of two numeric, measurable quantities but in a context where both quantities are changing. In mathematics classes, this is commonly expressed as change in the dependent variable resulting from a unit change in the independent variable, and variously described as constant or variable rate; average or instantaneous rate. In addition, rate may be seen as a purely abstract mathematical notion or embedded in the understanding of real-world applications. This paper explores the research question: Are students' expressions of their conceptions of rate affected by either context or mathematical representation? This question was part of a larger study (Herbert, 2010) conducted with Grade 10 students from the Australian state of Victoria.

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Background

Despite considerable research in the area rate remains a troublesome concept to teach and learn even for students who go on to study calculus (see for example Orton 1983; Coe 2007; Ubuz 2007). Coe (2007) asserts that thinking about rate should start when beginning algebra early in secondary school. Ubuz (2007) reports that students experience difficulties matching the graph of a function with the graphic representation of its derivative and claims "there is a need to deepen our understanding of students' prior knowledge about calculus and pre-calculus concepts and how that knowledge can serve as an anchor for subsequent learning" (p.635). Some students become competent in symbolic manipulation and can accurately produce the symbolic representation of the derivative (delos Santos and Thomas 2005), but may not appreciate its meaning and connection to other mathematical concepts studied in earlier years.

Speed has typically been used as a foundational example for rate, on the premise that speed is familiar to all students. However research on students' understanding of speed (see for example, Groves and Doig 2003; Kaput and Schorr 2002) suggests that for many students the link between speed and rate may not be made. Groves and Doig's study with upper primary school students suggests that while students could measure distance and time they were unable to see the relationship between them. This hints that experience with speed may not easily underpin understanding of rate. In addition, Lobato and Thanheiser (1999) warn if the rate already has a label in natural language, such as speed, then students are more likely to perceive the attribute as a single entity and so not be aware of the covariance of two extensive quantities comprising the rate as Carlson et al. (2002) advise. Similarly, Lamon (1999) suggests these labels disguise the comparative nature of these rates and that students who understand rate in this way may be unable to connect the single entity with the comparison of the two extensive quantities which constitute the rate, such as distance and time, in the case of speed.

Functions and, hence, rate may be represented numerically, graphically and symbolically and these representations have acquired conventional norms with shared assumptions and agreed-upon applications. Goldin (2008) refers to conformity with these norms as the "correct" use of the representations. So teaching mathematics entails establishing awareness of and facilitating the "correct" use of these conventional representational systems and applying them to solving problems. Kaput (1999) emphasises the importance of connections between everyday experiences and these representations. Although multiple representations have the potential to enhance learning of functional relationships, some researchers have found that this potential is not always realised (Adu-gyamfi 2007). Adu-gyamfi asserts that even specific instruction, emphasising connections between representations, does not ensure that students develop flexibility in working with representations. Seufert et al. (2007) express concern about the degree of the mental integration of corresponding information involved in learning with multiple representations.

Amit and Fried (2005) warn that students may not understand that multiple representations show “different mutually reinforcing views” (p.63) and see a graph as a solution method rather than a representation.

Researchers who contributed to the SimCalc project (Kaput 1999; Bowers et al. 2002; Kaput and Schorr 2002; Schorr 2003) explored the use of computer simulation (Java MathsWorlds, Mathematics Education Researchers Group 2004) to develop students’ conceptions of distance, time and rate through linking their various mathematical representations (numeric, graphic and symbolic). They claim that this is a successful learning strategy to promote a mathematical understanding of speed (Bowers et al. 2002) but do not explore students’ understanding of rate in general. There is also little discussion of the relative contribution of different representations to students’ ability to describe the concept of rate.

This paper reports on part of a qualitative exploration of conceptions of *rate* held by Victorian Grade 10 students. The first stage of the study involved a phenomenographic analysis (Herbert and Pierce 2009) and established the four educationally critical aspects (ECA) of rate shown in Table 1.

These ECA provide a framework for a content analysis of interview data with a focus on context and representation. The second stage of the study investigated the nature of students’ connections of different representations of rate in two particular real-world contexts. This analysis forms the substance of this paper. The diagram, of a triangular dipyramid seen in Fig. 1, shows the representations and contexts employed in this exploration of conceptions of rate. The lines represent the focus of this analysis, that is, the influence of the representations and contexts on participants’ expression of their conceptions of rate. There are no lines between the representations because the focus of this analysis is not on these particular connections.

Method

The participants were 20 Grade 10 students from five Victorian secondary schools, encompassing several important dimensions of diversity, for example, two girls’ schools; two regional schools; two on the outer-suburban fringe of Melbourne; one inner-city; one in a suburb with a high level of cultural diversity; and two independent schools. The participants were selected by their teachers to represent a range of mathematical ability and a mix of gender, with 14 girls and 6 boys. They had previously experienced constant rate in the form of linear functions and at least one function where rate varies, such as quadratic or exponential functions.

Table 1 Critical aspects of rate (Herbert and Pierce 2009)

1. rate as a relationship between two changing quantities.
2. rate as a relationship between two changing quantities which may vary.
3. rate as a numerical relationship between two changing quantities which may vary.
4. rate as a numerical relationship between any two changing quantities which may vary.

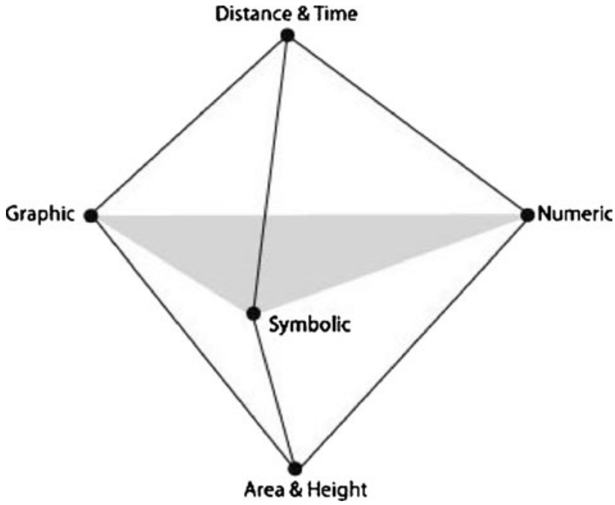


Fig. 1 Representations and contexts used to explore conceptions of rate

Two interactive computer simulations were prepared: one in Geometer's Sketchpad (GSP) (Key Curriculum Press 2006) depicting a blind partially covering a window (see Fig. 2); and the other in JMW showing a frog and clown walking (see Fig. 3). The software GSP and JMW were particularly well-suited for this purpose as they provide easy access to numeric, graphic and symbolic representations, so participants' conceptions of rate could be explored more broadly. In particular they were chosen to explore participants' expression of their understanding of a rate involving motion compared with a rate where time is not a variable.

The GSP simulation facilitated discussion of the participants' understanding of both constant rate and variable rate where area and height are the rate-related variables. The JMW simulation facilitated discussion of the participants' understanding of both constant and variable rate in a context where distance and time are the rate-related variables.

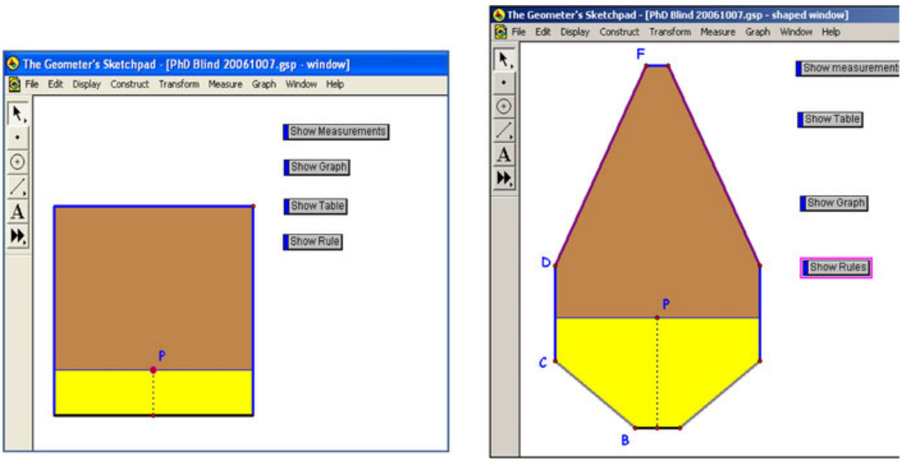
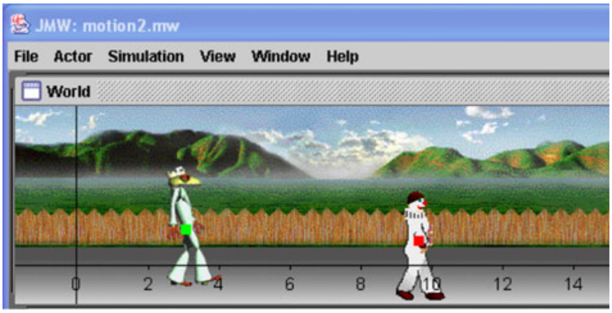


Fig. 2 GSP simulation

Fig. 3 JMW simulation



The JMW simulation involved a frog and clown walking at constant speed enabling comparison of their speeds. In addition, variable rate was explored with the frog accelerating. Like GSP, the dynamic links between the numeric, graphic and symbolic representations in JMW enabled the relationship between distance and time to be explored and provided opportunities for discussion of participants' understanding of constant speed in these representations. In video-recorded interviews participants were encouraged to explain their reasoning and think aloud as they were presented with different representational forms of rate: the simulation, table of values, graph and rule.

Each participant's responses relating to rate were coded by representation (numeric, graphic or symbolic) and context (blind or walking). All responses were examined to discern any impact of the particular representation or context on individual participants' expression of their understanding of rate. Both verbal communications and gestures provided insights into the meaning of participants' responses.

Results and commentary

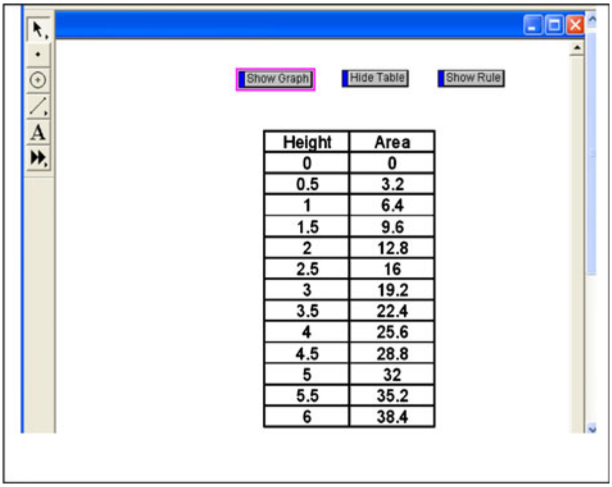
The results are illustrated by appropriate quotations from the interview data. Excerpts were chosen for inclusion within the discussion to exemplify influences identified in the participants' responses and clarify and justify inferences made by the researcher. The following sections illustrate the relative influence of representations and contexts in participants' expression of their thinking about rate.

Comparison of participants' responses to numeric representations

Area-height context and constant rate

The numeric representation (Fig. 4) of the function simulated in the area-height context provided rate-related information to most participants (13). This is illustrated by the responses given to the question "What does the table tell you about the rate that the area of the sunlight is changing?".

Twelve participants could see the pattern of constant difference in the columns of the table and use it to express the numeric relationship between the variables (for example S15). In addition, three participants could use the information in the table to provide an approximate unit rate (for example S5). Two participants provided a



Height	Area
0	0
0.5	3.2
1	6.4
1.5	9.6
2	12.8
2.5	16
3	19.2
3.5	22.4
4	25.6
4.5	28.8
5	32
5.5	35.2
6	38.4

Fig. 4 Table in constant area-height context

qualitative description (for example S16). A total of six participants made responses indicating their lack of awareness of any relationship between the variables of height and area (for example S1).

Quotes from transcript	Researchers' Commentary
S15: Well, yeah three point two there's a difference between um,[pause] so for every half a metre you get three point two extra.	*RC-S15 notices the pattern of a constant difference of 3.2 in the area column and relates it to the constant difference of 0.5 in the height column.*
S5: Well, like every one centimetre the blind goes up like, the area is six point four. It's [area] always coming up three point two. It doesn't change.	*RC-S5 sees the pattern in both columns and identifies the constant unit rate.*
S16: That when the height is nothing then the area of sunlight is nothing. When the height increases, the area of sunlight increases.	*RC-S16 expresses the relationship between the variables of height and area using the word "increase" to describe the relationship.*
S1: It [table] tells you the different heights and the different area of sunlight.	*RC-S1 describes the table and does not express any awareness of a relationship between the columns.*

These responses to the numerical representation of constant rate in the area-height context demonstrate that the information in the numeric representation assisted most participants to express an understanding of rate in this context.

Distance-time context and constant rate

Figure 5 shows a screen dump of the table relating to the frog and clown walking at a constant rate. The numeric representation of the function simulated in the distance-time context (Fig. 5) provided rate-related information to all participants.

Almost every participant could express some understanding of the relationship between distance and time. Most participants (17) could even use the information in the table to calculate a numeric rate (for example S20). In this case, the values in the table encouraged the expression of the relationship as a unit rate. This is seen in participants' responses to the question "What does the table tell you about the rate

Fig. 5 Table in constant rate distance-time context

Table

Time	A	B
0.00	0.00	0.00
1.00	3.11	1.00
2.00	6.22	2.00
3.00	9.33	3.00
4.00	12.44	4.00
5.00	15.56	5.00
6.00	18.67	6.00
7.00	21.78	7.00
8.00		8.00
9.00		9.00
10.00		10.00

that the frog and clown are walking?”. Only two participants were unable to give a quantitative description of rate (for example S4).

Quotes from transcript	Researchers’ Commentary
S20: in 1 s he [clown] went 3.11 m. The frog is going um, 1 m per second.	*RC-S20 first considers the distance column for the clown and connects this pattern to the pattern in the time column, expressing the relationship between them. Then they consider the distance column for the frog and connect this pattern to the pattern in the time column, expressing the relationship between them.*
S4: both of them are increasing. I would need distance and that’s the time and [pause] see I only know because [pause] we were looking at it like we were looking how yeah no distance because …[pause] it’s not very clear.	*RC-S4 expresses the relationship between distance and time and shows an awareness of a relationship but does not express this relationship numerically.*

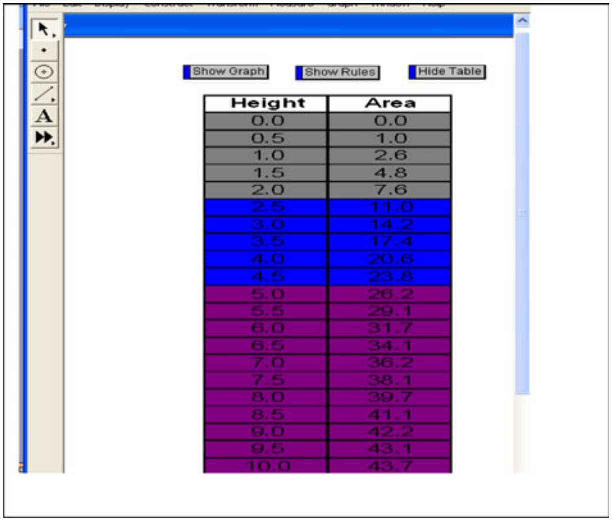
These responses to the numeric representation of constant rate in the distance-time context demonstrate that the information in the numeric representation assisted all participants to express an understanding of rate in this context.

Area-height context and variable rate

Participants were shown a table (Fig. 6) relating to variable rate in the area-height context of the blind partially covering a window with a rectangular section and two non-rectangular sections.

The numeric representation of the function simulated in the area-height context provided some information about variable rate to most participants. This can be seen in their responses to the question “What does the table tell you about the rate that the area of sunlight is changing?”. Seven participants could recognise and quantify the constant rate in the middle section of the table section. They realised that, in the other sections, the rate was changing and attempted to quantify it (for example S10). Four participants

Fig. 6 Table in variable rate area-height context



Height	Area
0.0	0.0
0.5	1.0
1.0	2.6
1.5	4.8
2.0	7.6
2.5	11.0
3.0	14.2
3.5	17.4
4.0	20.6
4.5	23.8
5.0	26.2
5.5	29.1
6.0	31.7
6.5	34.1
7.0	36.2
7.5	38.1
8.0	39.7
8.5	41.1
9.0	42.2
9.5	43.1
10.0	43.7

expressed an understanding that the rate was not the same in all sections (for example S7). A total of five participants struggled to discuss variable rate (for example S1).

Quotes from transcript	Researchers' Commentary
S10: On the wide one the bottom bit there is that about changes that one, one to two, and that one there's about one to three nearly one to four. So it gets quicker as it keeps going up. So umm it gets bigger when you go higher because it's wider down there. It's more just same until about five.	*RC-S10 points to grey section of the table and notices the differences are increasing.* *Notices differences are the same in the blue section. Points to the first height value in the purple section of the table.*
It's getting narrower, this one here's just above five whereas the first one and this one here's at about four just above four 'cause it's getting smaller.	*Points to the purple section of the table. Notices the differences in the values in the area column are getting smaller.*
S7: Um, it [area] goes up by about [pause] so as the height increases it [height] goes up by point five, it [area] goes up by one point six or something, like a higher number about two, and then it goes to the blue range, so it [area] goes up by about three, so the area increases more because it's more wider and then it goes deeper again to a triangle [purple section] and increases by about two and then it goes smaller and smaller	*RC-S7 notices the pattern of 0.5 in the values in the height column.* *Looks at the first difference in the area values in the grey section of the table.* *Looks at the first difference in the area values in the blue section of the table.*
The rate would um, that differs and everything, it's different, it wouldn't be a constant rate, it's an irregular shape, it's not a rectangle or, a square, it's got a triangle and a hexagon and it so, it would have to, it would have to have different measurements because the actual of the area of the thing goes, the way it goes up [pause] it changes the area.	*Looks at the first difference in the area values in the purple section of the table.* *Concludes the rate is different in different sections.* *Explains why the rate is different.*
S1: The higher up it [the blind] is the more light is getting in and the lower it is the less light get.	*RC-S1 is aware of the existence of a relationship between height and area, but do not express any awareness that the rate is changing.*

These responses to the numeric representation of variable rate in the area-height context demonstrate that the information in the numeric representation assisted about half the participants to express an understanding of rate in this context.

Distance-time context and variable rate

Figure 7 shows a screen dump of the table shown to the participants corresponding to this part of the interview. The numeric representation of the function simulated in the distance-time context provided some information about variable rate to most participants. This is seen in their responses to the question "What does the table tell you about the rate that the frog is walking?". Sixteen participants could recognise that the rate was changing and attempted to quantify it (for example S2). Two participants could recognise that the rate was changing but did not attempt to quantify it (for example S5). Only two participants could not give even a qualitative description of rate (for example S13).

Quotes from transcript	Researchers' Commentary
S2: He's getting faster. He's getting faster by an extra metre each time he goes a second.	*RC-S2 firstly expresses the rate qualitatively then goes on to quantify it.*
S5: He's not travelling at constant speed he is increasing his speed.	*RC-S5 identifies variable rate but does not attempt to quantify it.*
S13: It goes up [pause] It doesn't have the pattern this side [pause] although four, five and six has a pattern [time] and it just makes it more clear on how many distance between [pause]. No, only his total distance.	*RC-S13 points to the distance column and does not recognise any pattern.* *Points to the time column and notices the pattern.*

These responses to the numeric representation of variable rate in the distance-time context demonstrate that the information in the numeric representation assisted almost all participants to express an understanding of rate in the distance-time context.

Table 2 shows comparisons of responses to the numeric representation in the area-height and distance-time contexts. The table is based on the educationally critical aspects of rate developed as a result of a phenomenographic analysis (Herbert and Pierce 2009). The classifications of understanding of rate shown in the first column correspond to differences in the degree of awareness of a

Fig. 7 Table in variable rate distance-time context

Time	Distance
0	0
1	2
2	5
3	9
4	14
5	20
6	27
7	35

Table 2 Comparison of responses to the numeric representation

Understandings of rate	Constant Area & height	Variable	Constant Distance & time	Variable
numeric relationship	12	7	17	16
relationship (not numeric)	2	8	2	2
no relationship	6	4		2
no response		1	1	

relationship between the changes in variables and the nature of that relationship. The next two columns contain counts of the classifications given in the first column related to responses to both constant and variable rate in the area-height context, whilst the last two columns contain counts of the classifications given in the first column related to responses to both constant and variable rate in the distance-time context. In this way, responses relating to all the educationally critical aspects are recorded.

Scrutiny of Table 2 indicates that the numeric representation provides rate-related information for most participants in both contexts. It suggests that in the numeric representation constant rate can be seen more readily than variable rate. This is surprising as these participants were expected to have worked with both linear functions and functions where rate varies and, in particular, prepared graphs of them based on plotting points from a table of values. However, it is interesting to note that participants were better able to quantify variable rate in the distance-time context. This indicates that although walking is a familiar context which has rate-related meaning for almost all participants, this meaning does not seem to support the participant's rate-related reasoning in the area-height context even in the familiar numeric representation.

Comparison of participants' responses to graphic representations

Area-height context and constant rate

Figure 8 shows a screen dump of the graph shown to the participants corresponding to this part of the interview. It is the graphic representation of the linear function, $A=6.4$ h, associated with the rectangular window simulated in the area-height context. Participants were asked the question, "What does the graph tell you about the rate?".

The graphic representation provided some information about rate for about half (9) of the participants. This is illustrated by the following quotes, in response to the question, "What does the graph tell you about the rate?" Only two participants were able to approximately quantify constant rate in terms of a unit change in height (for example S11). Other participants demonstrated some understanding of rate, expressing awareness of the two variables, area and height, involved in the rate, and responded with a description of the

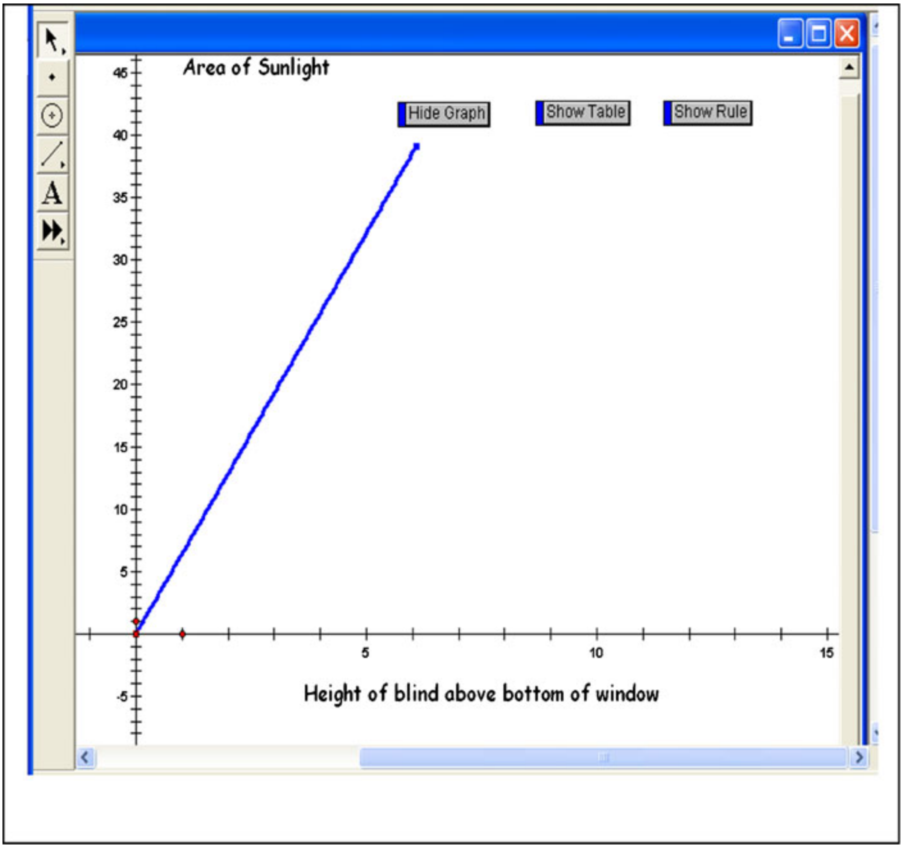


Fig. 8 Graph in constant rate area-height context

relationship between the variables in qualitative terms (for example S18). The graph provided little or no information about rate for more than half (11 out of 20) of the participants. Seven participants demonstrated a lack of awareness of the relationship between the variables involved in the rate and the actual rate (for example S1).

Quotes from transcript	Researchers' Commentary
S11: Well the rate would be five to one, [because] well we've got a red mark here which is on one and we have got a red mark here which is also on one so that tells us there is five up here to every one across here.	*RC-S11 points at (1,5) on the line and explains that it means when the height is 1, the area is 5. Expresses the rate as an approximate ratio. Expresses the rate-related connection with the words "across" and "up."*
S18: The height of the blind makes more area for the sunlight.	*RC-S18 expresses relationship between the height and area in qualitative terms.*
S1: Um, that the height is 7.	*RC-S1: expresses rate in terms of only one of the variables involved in the rate* i.e., *height*

These responses to the graphic representation of constant rate in the area-height context demonstrate that the information in the graphic representation assisted about half participants to express an understanding of rate in this context.

Distance-time context and constant rate

Participants were shown a graph (see Fig. 9) relating to the frog and clown walking at a constant rate. There was often spontaneous use of the term "speed" and other speed-related terms, such as "fast," when discussing the graphic representation of the linear functions associated with the frog and clown walking at constant rate (for example S12). However, it is not clear whether participants are considering the steepness of the gradient or the absolute height when making these comments.

The graphic representation in this context provided some rate-related information to every participant. Five participants could read values off the graph and express the rate as a change in distance per unit time (for example S13). Most participants (19) attempted to quantify rate as a numerical relationship between the variables of distance and time (for example S21). S18 could describe rate only in qualitative terms. Information in the graphic representation assisted all participants to express an understanding of rate in the distance-time context.

Quotes from transcript	Researchers' Commentary
S12: The clown's going faster than the frog.	*RC-S12 notices the differences in the graphs and concludes this means something about the speeds.*
S13: Well he [clown] goes 3 m per second. Divide twenty-two by seven [pause] just divide the ten, this number here which is just ten, wait that'd be one yeah, yep it'd [speed of frog] be one.	*RC-S13 uses a rule to calculate the unit rate of the clown.* *Points to the end of the green line on the graph and calculates the unit rate of the frog.*
S21: it takes him [frog] 2 m, 2 s to do 2 m where it takes the clown roughly to do two it takes him six.	*RC-S21 reads values from the graph but is a little unclear about the relationship between distance and time.*
S18: It looks like the clown is walking faster um, because he's covered more metres.	*RC-S18 determines the faster walker, but could not express the rate numerically.*
R: Can you work out how fast the clown is going?	
S18: No, I have no idea.	

Area- height context and variable rate

Figure 10 shows a screen dump of the graph relating to variable rate in the area-height context. The graphic representation of the three-part piece-wise function ($A(h)=1.2h^2+1.4h$; $6.4h$-5;-$0.5h^2+11h$-16.3) associated with the non-rectangular window provided some information about rate to more than half (12) of the participants. This is illustrated by the following quotes, in response to the

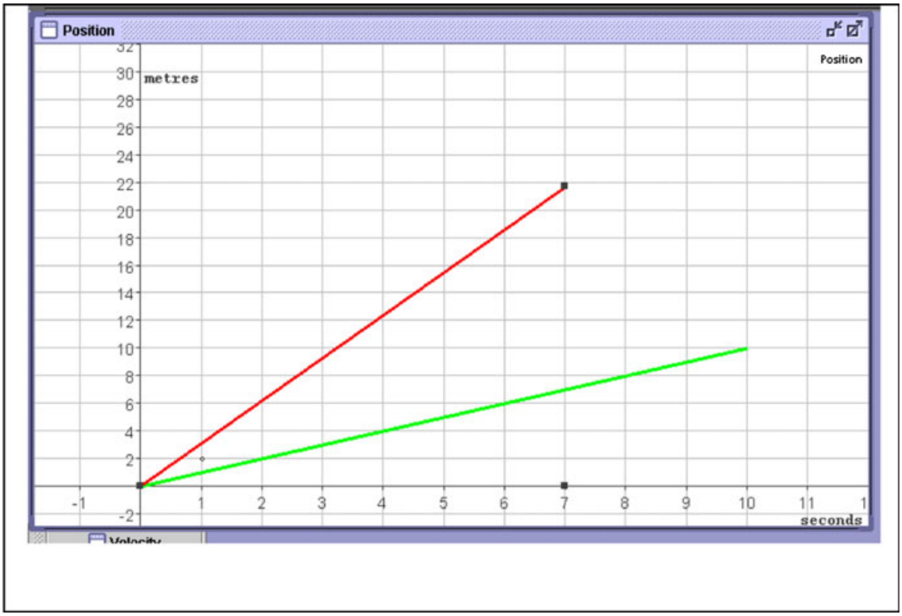


Fig. 9 Graph in constant rate distance-time context

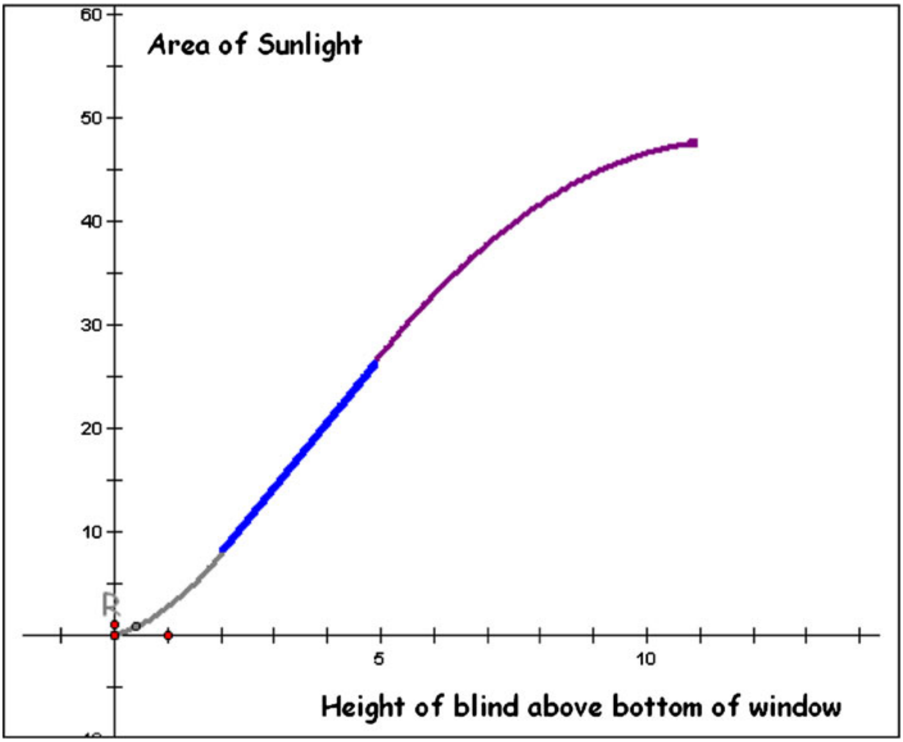


Fig. 10 Graph in variable rate area-height context

question, “What does the graph tell you about the rate?” Five participants attempted to quantify the variable rate (for example S7), whilst two were able to give correct average rates. Five participants could give a qualitative description of the rate for the different sections (for example S11). The graph provided little or no information about rate to almost half (8) of the participants (for example S12).

Quotes from transcript	Researchers' Commentary
S7: actually it [area] would keep going up but not as by as steady amount as it was before, like three point two by three point two, it might go up by one or two, so it would be a little bit more. Oh, well just say this was the same formula for that and instead of going up by three point two. For every point five it might go up by two or one or one point five because, you can't keep going [the same] because it's not square. One point five for every point five, well just the area because it doesn't decrease, it just um, it doesn't increase as much.	*RC-S7 notices that this graph is not straight like the one seen for constant rate and interprets that to mean not a "steady amount."* *Attempts to quantify rate.* *Makes a distinction between increasing area and decreasing rate.*
S11: Where the line is the steepest it would mean that's where there's most sunlight entering the window [pause] then the rate is decreasing as the blind, [pause] increasing up to a certain point and then decreasing from there.	*RC-S11thinks aloud to make sense of the graph.*
If we start here then the rate's going to increase until we reach here and then from this dot to this dot [rate] would be fairly much the same, there's no bend in the line, and then as we go from here to here there's less area so the rate would decrease.	*Begins to express thinking more clearly.* *Points to grey section of the graph.* *Points to blue section of the graph.* *Points to beginning and end of the blue section of the graph and explains that this straight section infers constant rate.* *Points to beginning and end of the purple section of the graph and concludes this curved section means decreasing rate.*
S12: Because the window's a different shape, the area is different in the sunlight.	*RC-S12 Makes no rate-related comment.*

These responses to the graphic representation of variable rate in the area-height context demonstrate that the information in the graphic representation assisted about half the participants to express some understanding of rate in this context.

Distance-time context and variable rate

Figure 11 shows a screen dump of the graph related to the frog walking with variable rate. The graphic representation of the quadratic function, associated with the frog accelerating, provided some rate-related information to every participant.

Five participants successfully expressed the variable rate with a quantitative description in language appropriate to their year level (for example S4). Eight participants attempted to quantify the variable rate with varying degrees of success.

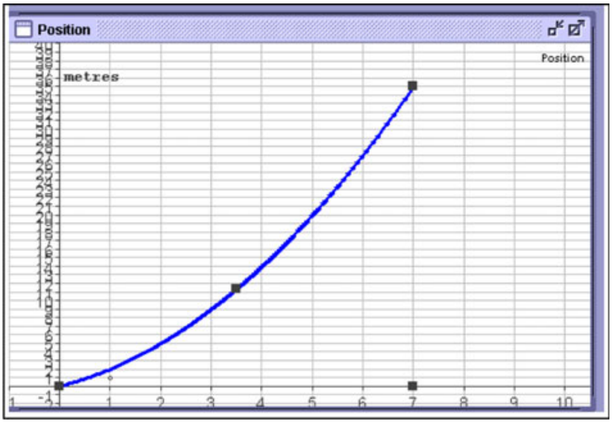


Fig. 11 Graph of variable rate distance-time context

This is most clearly illustrated by example S1. Seven participants could give only a qualitative description of the rate (for example S15).

Quotes from transcript	Researchers' Commentary
S4: So at the start in 1 s it's done 2 m and 2 s it's done five, so it's going up. It's not going at the same. [pause] First its done two, in the next second it has done three, in 3 s it has done [pause] ten or nine, so its going up, so the speed is increasing because it is covering more distance in each second [pause] first second two, second about five so 3 m it's done, it's done 4 m [pause] in four, so four, fourteen so it's gone up five, so it's gone up a metre every time [pause] yeah that's six [pause] yeah the speed is increasing each second it's not going at the same.	*RC-S4 notices that the change in distance for each second is not the same.*
He's walking faster. It's covering an extra metre every time, every second, first it's 2 m, next one it's three then it's 4 m and then five and 6 m. He's going faster.	
	Attempts to quantify the rate.
	Thinks aloud, working out the average rate over each second.
	Uses the word "speed" when discussing rate.
	Summarises thinking and concludes that the rate is increasing by 1 m each second.
S1: In eleven and a half metres he is going three and a half seconds and then 35 m he's going 7 s.	*RC-S1 Selects two points on the graph and compares distance with time for each.*
S15: He's walking fast. He's accelerating. He's going faster as he's walking along.	*RC-S15 Interprets the curve to mean acceleration.*

These responses to the graphic representation of variable rate in the distance-time context demonstrate that the information in the graphic representation assisted all participants to express an understanding of rate in the distance-time context.

Table 3 shows comparisons of responses to the numeric representation in the area-height and distance-time contexts. Scrutiny of Table 3 indicates that the graph

Table 3 Comparison of responses to the graphic representation

Understandings of rate	Constant Area & height	Variable	Constant Distance & time	Variable
numeric relationship				
unit	2	2	5	5
incorrect/incomplete	5	5	14	8
relationship (not numeric)	2	5	1	7
no relationship	11	7		
no response		1		

provides rate-related information for all participants in the distance-time context. The distribution and expressions of understandings of constant rate, from the graph, in the distance-time context were entirely different from the area-height context where about half of the participants did not express an awareness of any relationship between height and area.

Constant rate can be seen in the graphic representation more readily than variable rate in the distance-time context. Given the participants' prior experience with the graphs of both linear and non-linear functions, it is remarkable that the graphic representation provided so little rate-related information in the area-height context. Again it appears that understandings of rate evident in a distance-time context were not available to participants when considering the graph in the area-height context. In addition it is intriguing to see participants attempting to quantify variable rate in the distance-time context but not in the area-height context. Speed, hence constant rate, was more accessible to the participants than acceleration. This contrasts with the participants' willingness to use the table to attempt to quantify variable rate.

Comparison of participants' responses to symbolic representations

Area-height context and constant rate

Figure 12 shows a screen dump of the rule shown to the participants corresponding constant rate in the area-height context.

The participants expressed almost no rate-related comments in response to the question, "What does the rule tell you about the rate?", even for the linear function, A=6.4 h, associated with the rectangular window (for example S22). Other participants responded to this same question by translating the symbolic representation into words. Two participants went further by describing the process of substitution of values for height into the rule to calculate the corresponding values of area (for example S2). These participants did not isolate the rate from the

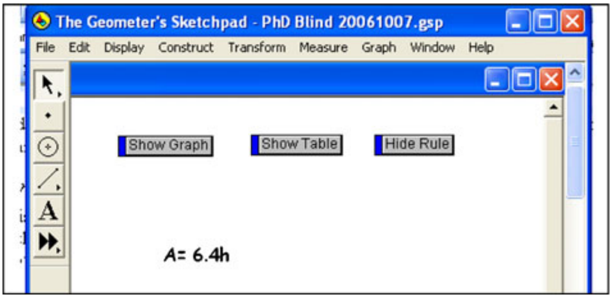


Fig. 12 Rule in constant rate area-height context

symbolic representation. It is surprising that no participant commented on the connection of the symbolic representation to gradient, hence rate, as it was expected that their prior experience with the symbolic representation of linear functions (for example see Bull et al. 2004) would have emphasised the meaning of the coefficient of the variable as gradient. Possibly they were trying to tell the researcher all they knew about the symbolic representation and did not focus their response on rate.

Quotes from transcript	Researchers' Commentary
S22: That might even be [pause] to be quite honest, I have no idea.	*RC-S22 makes no rate-related comment.*
S2: It would be six point four times the height from the bottom.	*RC-S2 demonstrates understanding of symbolic representation by translating it into words without any rate-related meaning.*

These responses demonstrate that the information in the symbolic representation assisted no participant in expressing their understanding of rate in the area-height context.

Distance-time context and constant rate

Figure 13 shows a symbolic representation including constant rate from the distance-time context of frog and clown walking.

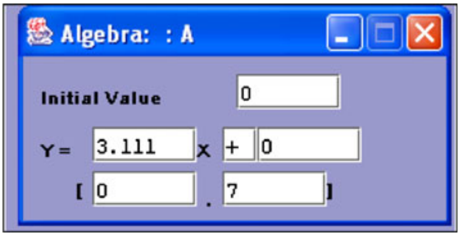


Fig. 13 Rule in constant rate distance-time context

The symbolic representations of the linear functions associated with the frog and clown walking at constant rate provided some rate-related information to most participants. Seven participants could correctly identify the walker from the symbolic representation suggesting that they could connect the rate of the walker with the symbolic representation (for example S10) when responding to the question "This is the rule for one of my walkers, who is it for, the frog or the clown?". Seven participants' responses (for example, the response of S20) demonstrated little rate-related reasoning. The discussion with S5 demonstrates that this participant could isolate the rate from the symbolic representation.

Quotes from transcript	Researchers' Commentary
S10: Clown because of 3 m. The frog [would be] one. It'd be [one] there.	*RC-S10 points to the number in the white box in front of "X."*
S20: It's for the um, for the frog because um, hmm.	*RC-S20 demonstrates no rate-related thinking in response to the symbolic representation even for constant rate.*
R to S5: This is the rule for one of the walkers, Y equals two point eight six X.	*RC-S5 is given a different rule and takes it to mean that after 1 s the distance is 2.86, thus expresses the relationship between the variables.*
S5: Well you'd think it was in seconds, two point eight six when he's like 1 s.	

These responses to the symbolic representation of constant rate in the distance-time context demonstrate that the information in the symbolic representation assisted most participants to express an understanding of rate in this context.

Area-height context and variable rate

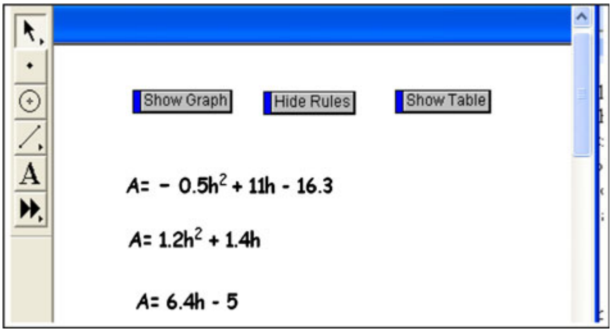
Figure 14 shows a screen dump of the rules shown to the participants corresponding to this part of the interview.

When considering the non-rectangular window participants demonstrated even less connection between the rule and their understanding of rate. No participant expressed any understanding of the connection between the rate and the rule in their response to the question "What do the rules tell you about the rate that the area is changing?" (for example S11).

Quotes from transcript	Researchers' Commentary
S11: It's umm nine point five height squared plus eleven height minus sixteen point three. But it, well for every zero point 5 m squared you add eleven and minus sixteen point three.	*RC-S11 gives a translation of the symbols into words, but attempts to express it in rate-related terms with the use of "for every."*

These responses demonstrate that no participant was assisted to express an understanding of rate in the area-height context by the information in the symbolic representation.

Fig. 14 Rules in variable rate area-height context



Distance-time context and variable rate

Figure 15 shows a screen dump of the rule related to the frog walking and accelerating. For four participants the symbolic representation conveyed the information that the speed was increasing (for example S19). However most (16) participants demonstrated little or no understanding of the connection between the symbolic representation and the speed (for example S1). These participants either had no response or responded negatively to the question "What does the rule tell about the rate the frog is walking?"(for example S12).

Quotes from transcript	Researchers Commentary
S19: Ok, time, he's increasing, like um. I dunno whether I would try to, try and get a figure here by itself, by moving them to the other side, I can't explain, I'm not sure. It is increasing by one point five?	*RC-S19 claims the speed is increasing, but had difficulty explaining why they thought that.*
S1: Distance equals time squared over two plus one point five.	*RC-S1 translates the symbols into words in response to the question "What does the rule tell you about how fast he's walking?"*
S12: that [pause] not sure.	*RC-S12 acknowledges their lack of understanding.*

Table 4 shows comparisons of responses to the symbolic representation in the area-height and distance-time contexts. Scrutiny of Table 4 indicates that the symbolic representations of rate provide little or no rate-related information for most participants in both contexts.

Fig. 15 Rule in variable rate distance-time context

$$D = \frac{t^2}{2} + 1.5t$$

Table 4 Comparison of responses to the symbolic representation

Understandings of rate	Constant Area & height	Variable	Constant Distance & time	Variable
numeric relationship				
unit			7	
incorrect/incomplete			6	
relationship (not numeric)				4
no relationship	20	20	7	15
no response				1

Group summary

Tables 2, 3 and 4 recorded all responses relating to the educationally critical aspect of rate and provide a numerical summary of all participants' understanding of rate in the numeric, graphic and symbolic representations and their understanding of rate in the distance-time and area-height contexts.

This information in Tables 2, 3 and 4 is summarised in Fig. 16 in order to show the strength of connections for the group as a whole: A dark thick line indicates a strong connection; a thin solid line indicates a medium strength connection; a dotted line indicates a weak connection; and no line indicates no connection. For the group, rate was most clearly expressed as a numeric relationship between the variables of distance and time.

Discussion

The educationally critical aspects of the concept of rate provided the framework for content analysis of the data. Tables based on the educationally critical aspects of rate

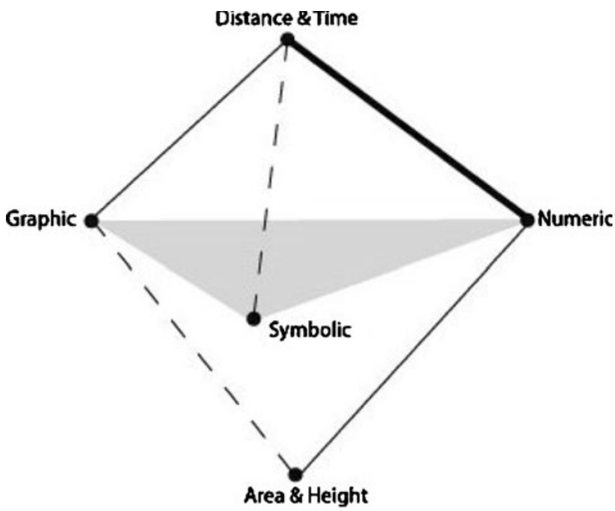


Fig. 16 Strength of connections between context and representations of rate

similar to those seen in Tables 2, 3 and 4 appear to be useful in summarising students' understanding of rate. Analysis of data showed that participants were most confident to extract information about rate from the numeric representation. In fact, the graphic representation was often used to determine coordinate points from which to calculate constant rate. Only one participant actually referred to the gradient of the line and its relationship to rate and no-one linked the symbolic representation to rate in the area-height context. This was unexpected since it was likely that the prior experience with the symbolic representation of linear functions (for example see VCAA 2007) would have introduced them to the meaning of the coefficient of the independent variable as gradient. It appears that the participants of this study did not transfer their understanding of rate demonstrated in tables and graphs to the corresponding symbolic representation. These students did not move seamlessly between representations, and understandings demonstrated in one representation did not necessarily transfer to other representations. This is of particular concern to the teachers of introductory calculus, especially if such an introduction relies heavily on the symbolic form of functions.

Findings of this study suggest that understandings of rate are inconsistent across representations. This aligns with Amit and Fried's (2005) report that questions whether the potential of multiple representations to enhance students' understanding of functions is realised in the classroom. It appears that the participants of this study did not transfer their understanding of rate demonstrated in tables and graphs to the corresponding symbolic representation. The numeric, graphic and symbolic representations of the functions associated with the simulations were all presented to participants to provide them with several opportunities to express their understanding of rate. It was anticipated that individual participants would have different preferences and differing abilities in working with the different representations as reported by Adu-gyamfi (2007) and Pape and Tschoshanov (2001). It was clear that participants did vary in their facility to express their understanding of rate in the different representations. The results of this study suggest that an understanding of rate may be dependent on the representation chosen since participants did not naturally make explicit or even implicit connections between tables, graphs and rules.

It is interesting that the distance-time scenario provided a context that at least some students could discuss and in which they could make some connection between the symbolic representation and rate. Possibly this is because of the understanding of speed participants brought to their discussions of rate. Findings of this study indicate that whilst rate appeared to be quite well understood in the context of walking, this understanding did not automatically transfer to the blind context. So, specific instruction connecting rate in a motion context to other rates would be necessary to capitalise on the understanding of rate in a motion context, which many participants seemed to bring from their prior experiences in science classrooms, or experiences outside of the classroom. Alternatively, initial treatment of rate emphasising the co-variational approach suggested by Carlson et al. (2002) utilising speed, density and other rates as examples, rather than substitutions in formulae, may support the development of rate as a numeric relationship. Although the researcher's interview questions used the word "rate," some participants spontaneously used the word "speed" when discussing rate in the distance-time context, so for these participants, rate in this context seems to be synonymous with speed. So, as Lamon (1999) suggests, there may be learning difficulties associated with this particular rate having a label in natural language. It appears that speed may be seen as a single entity with little emphasis on the covariance of the variables of distance and time.

Conclusions

This study confirms the notion that rate is a complicated concept and it informs teachers of the different ways in which students may read meaning into the concepts they are learning. It shows that alternative representations of functions provide different rate-related information for different students and that understandings of rate expressed in one representation or context are not necessarily transferred to other representations or contexts. Specifically, the numeric representation appeared to provide the most information about rate for most participants, whilst the graphic representation appeared to provide some information about rate for many participants. However, the findings show that symbolic representations held almost no rate-related meaning for the participants of this study. Results also indicate that while speed, as a phenomenon, was quite well understood this understanding was not necessarily helpful to these participants in understanding rate in a context not involving speed.

Specific instruction emphasising a co-variational approach and connecting rate in a motion context to other rates is necessary to capitalise on prior understandings of rate as speed. Students' learning is strengthened by building conceptual links between school mathematics topics. Approaches to the teaching and learning of functions need to include frequent and strong connections between alternative representations of functions with explicit emphasis on rate.

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