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ORIGINAL ARTICLE

Exploring mathematical connections of prospective middle-grades teachers through card-sorting tasks

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Abstract Prospective teachers are expected to construct, emphasise, integrate, and make use of mathematical connections; in doing so, they acquire an understanding of mathematics that is fluid, supple, and interconnected (Evitts Dissertation Abstracts International, 65(12), 4500, 2005). Given the importance of mathematical connection making, an exploratory study was conducted to consider the ability of prospective middle-grades teachers to make mathematical connections while engaging in card-sorting activities. Twenty-eight prospective middle-grades teachers participated in both an open and closed card sort. Data were analysed using constant comparative methods to extract meta themes to describe the types of connections made. Findings indicate that these prospective teachers tended to make more procedural- and categorical-type mathematical connections and far fewer derivational or curricular mathematical connections.

Keywords Mathematical connections · Card sorting · Teacher preparation · Middle grades · Geometry

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Mathematics education literature supports the belief that mathematical understanding requires students to make connections between mathematical ideas, facts, procedures, and relationships (Hiebert and Carpenter 1992; Ma 1999; Moschkovich et al. 1993; Skemp 1978, 1989); thus, constructing, unpacking, and understanding connections are fundamental in carrying out the work of teaching mathematics. Mathematics teachers must “hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students” (Ball et al. 2008, p. 400). By constructing, decompressing, and unpacking their mathematical knowledge, teachers are better equipped to respond to students’ “why” questions, evaluate student conjectures, ask productive mathematical questions, and make connections to mathematics across the span of the curriculum. However, beginning teachers rarely make connections during instruction, or their connections are imparted in an implicit rather than explicit manner (Bartels 1995; Eisenhart et al. 1993; Hiebert 1989). Without understanding the connections among the important functional concepts in mathematics, prospective teachers cannot effectively engage K-12 students in mathematical connection making, reasoning, and problem solving.

If prospective teachers are expected to construct, emphasise, integrate, and make use of mathematical connections, then they must acquire an understanding of mathematics that is fluid, supple, and interconnected (Evitts 2005). Prospective teachers must not only be able to do the mathematics they will teach but must possess a deep conceptual understanding of the mathematics. In particular, they must learn that teaching mathematics for conceptual understanding requires that they move away from a perception of mathematics as a “set of isolated facts and procedures” (National Council of Teachers of Mathematics [NCTM] 2009, p. 3). Prospective teachers must be prepared to help K-12 students construct mathematical knowledge, establish mathematical connections, and develop mathematical habits of mind needed for problem solving (Conference Board of Mathematical Sciences [CBMS] 2001). As Hodgson (1995) points out, “…the investigation of problem situations leads naturally to the establishment and use of connections. In turn, the use of connections to solve problems brings about the need for their establishment” (p. 18). Increased attention has been given to the importance of mathematical connection making, especially as a tool for problem solving, by the NCTM (1989, 2000) standards, the *Curriculum Focal Points* (NCTM 2006), and the *Common Core State Standards for Mathematics* (CCSSO 2010). It is thought that the results of this exploratory study focused on the types of mathematical connections prospective teachers make as they engage in tasks meant to probe mathematical connections may provide insight into prospective teachers’ accessing, unpacking, and connecting mathematical knowledge.

Theoretical framework

In the last quarter century, mathematics education reform and research on the learning and teaching of mathematics have been largely influenced by constructivist theory. The emergence of constructivism in education can be attributed to “dissatisfaction with information-processing theory, concerns that students are acquiring isolated, decontextualized skills and are unable to apply them in real-world situations and an interest in Vygotsky's cultural-historical theory" (Gredler 2005, p. 89). Constructivism is grounded in the idea that all knowledge is constructed. A major tenet of constructivist theory posits that the learner constructs meaning from experiences by integrating prior knowledge with new knowledge. Through a constructivist lens, "mathematical knowledge is constructed, at least in part, through a process of reflective abstraction" (Noddings 1990, p. 10). Constructing and understanding mathematical concepts, ideas, facts, or procedures involves making connections between old and new knowledge. Hiebert and Carpenter (1992) suggest, "Many of those who study mathematics learning agree that understanding involves recognising relationships between pieces of information" (p. 67). Connections can be viewed as a natural consequence of constructivist theory in the domain of mathematics since learning for conceptual understanding involves building mental networks structured like a spider's web wherein "the junctures, or nodes, can be thought of as pieces of represented information, and threads between them as the connections or relationships" (Hiebert and Carpenter 1992, p. 67). An important concept that arises from constructivist theory to explain how making connections can aid in the learning of mathematics is *schema theory.* Mathematical connections can be described as components of a schema or connected groups of schemas within a mental network. A schema is a "memory structure that develops from an individual's experiences and guides the individual's response to the environment" (Marshall 1995, p. 15). Marshall posits that a defining feature of schema is the presence of connections. The strength and cohesiveness of a schema is dependent on connectivity of components within the schema or between groups of schemata. This model suggests that learning mathematics for understanding involves assimilating or connecting new information into mental networks, forming new connection(s) between existing knowledge components, and accommodating or reorganising schemata to address perturbations in knowledge structures and to correct misconceptions. Thus, the building and refining of such mental structures through the establishment and strengthening of connections plays an important role in the development of students' learning of mathematics. This idea is further supported by the creation of the *Connections* strand of the NCTM (2000) *Principles and Standards* emphasizing that "when students connect mathematical ideas, their understanding is deeper and more lasting, and they come to view mathematics as a coherent whole" (p. 4). From a constructivist perspective, a *mathematical connection* can be thought of as a link (or bridge) in which prior or new knowledge is used to establish or strengthen an understanding of relationship(s) between or among mathematical ideas, concepts, strands, or representations within a mental network.

A constructivist lens may provide an understanding of how prospective teachers construct, link, or bridge together relationships between mathematical concepts, ideas, and/or representations when engaged in tasks meant to probe mathematical connections. A constructivist theory of learning mathematics provided a supportive foundation for this study as the researcher attempted to understand and describe the types of mathematical connections prospective middle-grades teachers make while engaged in tasks meant to probe mathematical connections.

Purpose of study

The purpose of the present exploratory study was to investigate the types of mathematical connections prospective middle-grades teachers make when engaged in tasks meant to probe their mathematical connections. Specifically the following question was investigated:

> What types of mathematical connections do prospective middle-grades teachers make while completing tasks meant to probe mathematical connections?

Research design and methods

The data for the present study were drawn from a larger exploratory study that utilised a concurrent mixed methods design combining both qualitative and quantitative approaches (Creswell and Plano-Clark 2007; Teddlie and Tashakkori 2009) in order to investigate prospective middle-grades teachers' mathematics knowledge for teaching geometry and the types of mathematical connections made when engaged in tasks meant to probe mathematical connections. The present study (as part of the larger mixed methods research study) is exploratory as it "generates information about unknown aspects of a phenomenon" (Teddlie and Tashakkori 2009, p. 25); in this case, the types of connections prospective middle-grades teachers make when engaged in activities meant to probe mathematical connections. Figure 1 reveals a diagram of the concurrent exploratory mixed methods design used for the larger study.

This article will focus its discussion on Card-Sort Activity (CSA) as highlighted in Figure 1. The card-sort activity was a chosen data collection tool since sorting techniques are "aligned with the constructivist approach" (Rugg and McGeorge 2005, p. 95). Furthermore, as suggested by Fincher and Tenenberg (2005), "there is evidence to suggest that the way in which participants categorize entities *externally* reflects their *internal*, mental representations of these concepts" (p. 90). Thus, the construction of the card-sorting technique was informed by a theoretical framework which assumes that learning depends upon building and refining mental structures by establishing and strengthening connections within a network.

Participants

The targeted population for this study was prospective middle-grades teachers at a large mid-south university in the United States. These future teachers were of particular interest given the recent release of the *Mathematics Teaching in the 21*st *Century* and *Breaking the Cycle* reports which found that prospective middle-grades teachers' mathematics knowledge for teaching in the areas of algebra and geometry to be weak in comparison to their international counterparts (Center for Research in Math and Science Education 2010; Schmidt et al. 2007). In particular, prospective middle-grades teachers' lack of connection making within and across mathematical domains may explain their difficulties in teaching geometric concepts.

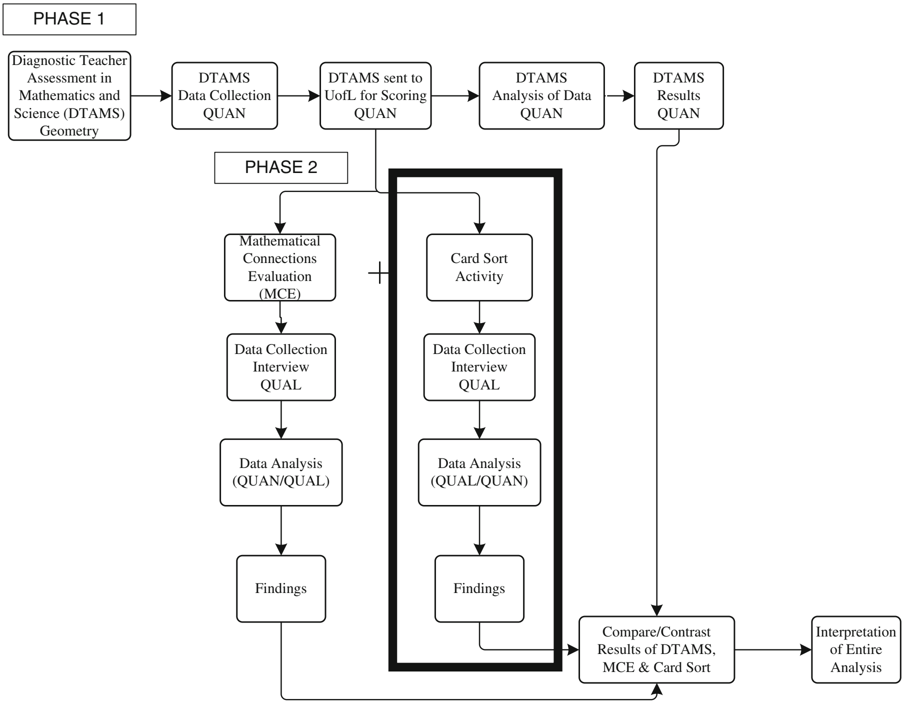

Fig. 1 Concurrent exploratory mixed methods design

A prospective middle-grades teacher is defined as an undergraduate middle-grades education student enrolled in a programme of study leading to certification with a specialisation in mathematics teaching. The sampling frame was derived from a comprehensive list of prospective middle-grades teachers meeting the following criteria: (a) declared middle-school education major, and (b) actively pursuing a middle-school certification in two content areas, one of which was mathematics. All prospective middle-school teachers meeting both criteria were contacted for voluntary participation in this study. All 58 eligible participants were contacted, of whom 28 (48.3%) elected to participate. Most participants were female ($n=22$, 78.6%). There were 14 juniors[1] (50%) and 14 seniors[2] (50%). There were six student teachers[3] (21.4%) in the study.

Instrumentation and data collection

The *Card-Sort Activity* (CSA) consisted of 20 cards in a 4×5 array labeled with various mathematical terms, concepts, definitions, and problems (see Figure 2).

[1] A *junior* is the classification for a student who has completed at least 60 credit hours of university coursework.
[2] A *senior* is the classification for a student who has completed at least 90 credit hours of university coursework.
[3] A *student teacher* is the classification for a student who is fully immersed in teaching in a classroom under the supervision of an experienced certified teacher. This classification is only given to students who have completed all university course work required for a degree in education.

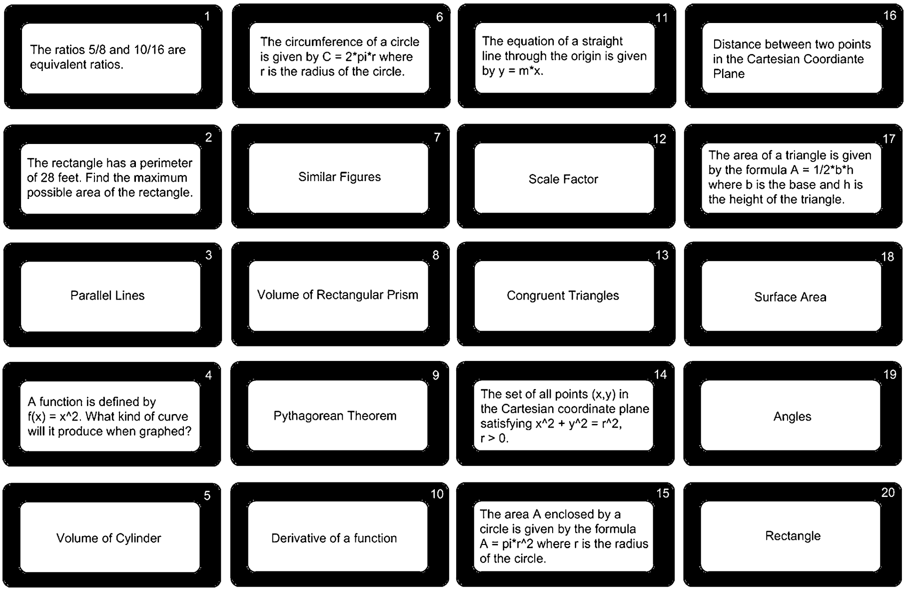

Fig. 2 Arrangement of cards for CSA open card sort

Construction of the cards was based on and aligned to national recommendations, in particular, *Recommendations for the Mathematical Education of Teachers* (CBMS 2001), *Principles and Standards for School Mathematics* (NCTM 2000), and *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics: A Quest for Coherence* (NCTM 2006). The purpose of the CSA was to examine the types of connections prospective middle-grades teachers make between various mathematical concepts, definitions, and problems. Participants were asked to complete a repeated single-criterion open card sort and closed card sort (Fincher and Tenenberg 2005; Rugg and McGeorge 2005).

In the closed card sort, five particular pairs were chosen based on national recommendations (CBMS 2001; NCTM 2000; NCTM 2006) about what middle-school teachers and students should be able to know and do. For example, prospective middle-grades teachers should know or be able to recognise that cards 6 and 11 represent linear functions and cards 4 and 15 represent quadratic functions (NCTM 2000; NCTM 2006). Cards 9 and 16 along with cards 15 and 17 were selected for the closed card sort as "prospective [middle-grades] teachers have some basic knowledge about shapes and about how to calculate area and volumes of common shapes, but many will not have explored the properties of these shapes or know why the areas and volumes are true" (CBMS 2001, p. 33). The cards chosen were also influenced by the National Science Foundation (NSF) reform middle-school curriculum textbook series *Connected Mathematics 2* (Lappan et al. 2006). This particular textbook series is currently adopted by a number of local school districts where these future teachers are likely to teach. This textbook series which emphasises a constructivist approach to learning and teaching mathematics is also used in some of the mathematics content and methods courses prospective middle-grades teachers take at the site where this study was conducted. The particular pairs of cards chosen for the closed card sort were also selected in consultation with mathematicians.

The participants sorted the cards based on a single criterion: their notion of how the statements on the cards were connected. The researcher developed a protocol of interview questions for both the open and closed card sorts that focused on students' mathematical connections (see Appendix A). The design of the protocols was influenced by the recommendations of Rugg and McGeorge (2005) for carrying out card-sorting techniques. Figure 3 illustrates the five closed sort pairings.

Participants were asked to engage in both open and closed repeated single-criterion card sort (Rugg and McGeorge 2005). This type of card-sorting technique requires participants to "sort the same entities repeatedly, categorizing in terms of a different single attribute ('criterion') each time" (Rugg and McGeorge 2005, p. 96). For the open card-sort activity, prospective teachers were shown 20 cards in a 4×5 array (see Figure 2) and asked to select a subset of 2 or more cards they felt were related or connected; they were then asked to describe this connection. The cards were returned to the array and the process was repeated until the prospective teacher indicated they could no longer make subsets. Following the open card sort, prospective teachers were asked to engage in a closed card-sort activity (see Figure 3). Five pre-selected pairs of cards were shown to the prospective teacher, one set at a time; they were then asked if there was a connection and if so, what the connection was. All card-sort activities were videotaped and interview data

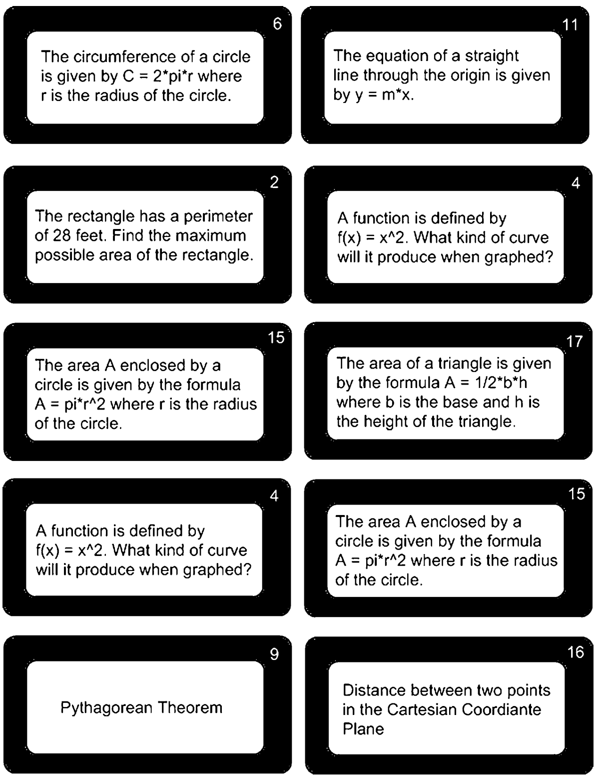

Fig. 3 CSA closed sort pairings

transcribed. On average, both the open and closed sort activities took approximately 45 min in total to complete.

Prior to the full study, pilot interviews were conducted with two student teachers (one secondary and one elementary), two prospective elementary teachers, and one in-service elementary teacher. The pilot interviews allowed the researcher to gain additional experience in conducting interviews and to become more familiar with the logistical considerations of data collection and management.

Quality review The CSA instrument underwent a *quality review* (Halff 1993; Tessmer 1993) to further strengthen the validity of each instrument. An expert quality review is an evaluation of a product (in this case the CSA instrument and protocol) on the basis of appropriateness, content accuracy, and design quality. Expert reviews consist of an expert or experts (in this case mathematicians and mathematics educators) reviewing a rough draft of the instrument along with interview protocols to determine strengths and weaknesses. The feedback and comments provided by the expert reviewers and pilot-study participants were analysed and subsequently modifications were made to the CSA instrument in order to improve the quality of the instrument and interview protocols. For instance, experts recommended that no more than 20 cards be used for the card-sort activity. This recommendation was implemented as it was consistent with findings from card-sort literature (Rugg and McGeorge 2005; Fincher and Tenenberg 2005).

Analysis

Qualitative data in the form of videotaped semi-structured interviews were collected from the Card-Sort Activity (CSA) instrument. The videotapes and transcribed interview data were analysed qualitatively. The qualitative data were then "quantitized" (Teddlie and Tashakkori 2009, p. 27) a process by which qualitative data is converted to quantitative data for further analysis.

CSA open sort

Participant responses for each open card sort were analysed using an inductive approach to the method of constant comparison (Denzin and Lincoln 2000). This method of constant comparison involved reviewing videotapes and subsequent transcribed videotape data of participants' explanations for each card sort they had constructed. In accordance with the definition of *mathematical connection* described earlier in this paper, participants' explanations were "chunked" so that each meaningful phrase or sentence could be categorised with a descriptive code. Each new chunk of data was compared with previously generated descriptive codes, so that similar chunks could be labeled with the same descriptive code (Leech and Onwuegbuzie 2007). After all the data had been coded, the codes were grouped by similarity which represented a unique emergent theme, that is, mathematical connection type. There were five types of mathematical connection themes that emerged from the data: *categorical*, *procedural*, *characteristic/property*, *derivation*, and *curricular* (see Appendix B).

The researcher and an outside consultant coded the open card sorts using a coding guide (see Appendix B). The coding guide provided a description of each of the five emergent mathematical connection types along with examples for each type. The second coder was a mathematician at the site where the study was being conducted and who has taught mathematics content courses for prospective middle-grades teachers. The researcher and consultant together categorised 12 open card sorts (with each mathematical connection type represented at least twice) in order to become more familiar with the descriptions for each mathematical connection type and to help establish consistency in the coding. The second coder independently coded a randomly selected sample of approximately 53% of the open card sorts ($n=137$). Inter-rater reliability analysis using a kappa statistic (Cohen 1960) was performed to determine consistency among coders. The level of agreement among coders was found to be "substantially strong" (Landis and Koch 1977, p. 165) with kappa=.74. The CSA open sort data were "quantitized" (Teddlie and Tashakkori 2009, p. 27) by tallying the number of open sorts that fell into each mathematical connection category.

CSA closed sort

In the closed card sort, the researcher in consultation with mathematicians and mathematics educators selected five pairs of cards and asked if each pair of cards were related or connected and, if so, why. Participants' responses for each pair of cards were analysed using an inductive approach to the method of constant comparison (Denzin and Lincoln 2000) for extracting themes. The method of constant comparison carried out for the open card sort was the same for each pair of cards in the closed sort. The CSA closed-sort data were *quantitized* by tallying the number of responses that fell within each theme. The mathematical connection categories used to interpret the open card sorts was also used to interpret the closed-sort pairings.

Results and discussion

The purpose of this exploratory mixed methods study was to investigate the types of mathematical connections prospective middle-grades teachers make when engaged in card-sorting tasks meant to probe their mathematical connections.

Types of connections made during the CSA open sort

There were a total of 258 open card sorts. On average each participant made nine open card sorts. The unique emergent themes (i.e., the types of mathematical connections made by prospective middle-grades teachers during the open cards sort) resulting from an inductive analysis of participants' responses using the method of constant comparison were as follows: *categorical*, *procedural*, *characteristic/property*, *derivation*, and *curricular* (see Appendix B).

Although there were 258 open card sorts, there were 287 mathematical connections since a participant's response for grouping particular cards together could fall into one or more of the five types of mathematical connections categories. Table 1 lists the number of connections that fell into each mathematical connection category.

As a group, the prospective middle-grades teachers made more categorical and procedural connections and far fewer derivational and curricular connections (see Table 1). Since the card-sorting technique is "an advanced level sorting task that can be used to identify how concepts in a content area are organized in a learner's knowledge structures" (Jonassen et al. 1993, p. 45) the number of sorts under each connection type provides a glimpse into how these prospective middle-grades teachers tend to unpack, relate, and connect the concepts presented in the open card sort. The majority of card sorts made by prospective middle-grades teachers were categorical and procedural in nature is not surprising for three potential reasons:

- The majority of participants had never engaged in a card-sort activity and, thus, may have related or connected the cards based on the most "obvious" relationships or links between the mathematical concepts, ideas, and terms presented on the cards.
- The majority of participants' experiences with learning mathematics had been dominated by traditional curriculum focused on instrumental rather than relational understanding of mathematics (Skemp 1978).
- The majority of participants had not yet taken mathematics methods courses so perhaps they did not think about creating subsets from the perspective of what a future middle-school teacher should know and be able to do.

Another potential reason why the majority of participants made fewer curricular and derivational connections may reside in the order in which the Mathematical Connections Evaluation (MCE) (other instrument as part of larger exploratory study) and CSA were conducted. All participants engaged in the CSA immediately following the MCE. The MCE was focused more on mathematical content connections and less on pedagogical connections and, thus, participants may not have been in the frame of mind to create subsets from the perspective of what a future middle-school teacher should know and be able to do.

However, the fact that nearly 25% of the subsets were curricular and/or derivational in nature (see Table 1) is an encouraging result. Faculty at the site

Table 1 CSA open sort counts by connection category (n=28)

Mathematical connection type	Count	Frequency
Categorical	97	34%
Procedural	68	23%
Characteristic/Property	51	18%
Curricular	36	13%
Derivational	35	12%
Totals	287	100%

where the study was conducted currently use and draw upon NSF reform curriculum emphasising a constructivist approach to learning and teaching mathematics in the prospective middle-grades teacher content and methods courses. The development, improvement, and refinement of these prospective teacher courses include a focus on how to make "visible the connection to the kinds of mathematical thinking, judgment, [and] reasoning one has to do in teaching" (Ball et al. 2009, p. 29).

Types of connections made during the CSA closed sort

In the closed card sort, five particular pairs of cards were selected: cards 6 and 11; cards 2 and 4; cards 15 and 17; cards 4 and 15; and cards 9 and 16 (see Figure 3). Participant explanations were qualitatively analysed using an inductive approach to the method of constant comparison for each closed-sort pairing. For the closed-sort pairing of cards 6 and 11 the following themes emerged: *yarn explanation*; *radius as a "line"*; *both are formulas*; *both are equations*; *both are linear functions; none*. These themes, exemplars, frequencies with which each occurred, and the open-sort criteria apparent in this closed sort are shown in Table 2.

Table 2 Themes and exemplars for closed-sort pair 6 and 11 (n=28)

Themes	Exemplars of participant responses	Count	Frequency
Yarn explanation	If you take a piece of yarn at a certain point around the circle and brought it all the way around, then straightened it out, it would make a straight line that you could lay against a ruler.	6	21%
Radius as a "line"	If you were to graph the circle on the coordinate plane, the line [$y=mx$] could be the radius of that circle.	7	25%
Both are formulas	Right off the bat, I think they are both formulas. It's kind of one of the second-nature formulas that you just know. Hopefully, your teachers help you derive it and you know what they are. I think this is another case like with the last two, I wouldn't teach together. From a teacher's perspective they are kind of unrelated in terms of how I would teach it.	3	11%
Both are equations	They are both equations. I don't really know if finding the slope of a straight line would help you find the circumference of a circle, but they are both equations. They are both equations. This $y=mx$ gives you a line and the other gives you a circle.	2	7%
Both are linear functions	I think they can be related because they are both functions, really. Well, the x I would just think of it relating C the circumference can be a function of the radius. If you change the radius, it will change the circumference. Whenever you change the x value it's going to change the y, the output. They are both input/output. They are both lines.	1	4%
None	I don't think they are related because that [card 6] has to do with a shape [a circle] and this [card 11] has to do with a line.	9	32%
Totals		28	100%

Card 6 read "The circumference of a circle is given by $C=2\pi r$ where r is the radius of the circle." Card 11 read "The equation of a straight line through the origin is given by $y=mx$"

For card pairing 6 and 11 (see table 2), only one participant (4%) was able to identify the expression in both cards as linear functions represented algebraically. Eighteen percent of participants used the surface features of the cards as a basis for their connection. In particular, these participants focused on the equal sign on both cards and said the cards were related because both represented equations or formulas. Nearly a third of the participants said that the two cards were not related or connected. The remainder of the participants (46%) tried to make a connection between the two cards by focusing on a visual or graphical representation for the statement on each card. When talking about circumference of a circle, participants tended to draw a picture of a circle, labeling the distance from the center of the circle to a point on the circle, *r*, for radius. When looking at card 11 they tended to focus on the visual representation of a line, rather than the equation given on the card. They would use the pictorial representation of a circle to build a connection to a pictorial representation of a line. The participants who gave the "yarn explanation" indicated that you could take a piece of yarn, wrap it around the circle and then you could straighten out the piece of yarn and it would be a "line." The participants who gave the "radius as a line" explanation indicated that the radius could be thought of as a straight line. For this closed-sort pairing, the majority of participants either did not make a connection, made a connection based purely on the surface features of the card, or made an algebraic/geometric misconnection. This is consistent with the research literature as these participants failed to make a connection between "a particular feature of a function in one representation to the same feature in another representation" (Leinhardt et al. 1990, p. 24).

For the closed-sort pairing of cards 2 and 4 the following themes emerged: *max area most square like*; *calculus problem*; *derivative to find max*; *graphing possibilities*; *none* (see Table 3).

For card pairing 2 and 4, more than half the participants said there was no connection between the two cards (see Table 3). These participants tended to focus on a geometric representation for the statement on each card. For card 2 they focused on a geometric representation of a rectangle and for card 4 they focused on a geometric representation of a parabola. These participants said there was no connection because they could not see how the graph of a parabola had any relationship to finding the maximum area of a rectangle.

For the closed-sort pairing of cards 15 and 17 the following themes emerged: *both area formulas*; *geometric/relational*; *volume of cone*; *none* (see Table 4).

For card pairing 15 and 17, the majority of participants indicated that the two cards were related, the most popular response being that the two cards were both area formulas for two different objects. These participants focused on the surface features of the statements on the card to make a connection. However, nearly a third of participants were able to go beyond the surface in making a connection between the two cards. These participants tried to make more of a derivational connection in relating the two cards. In particular, they focused on how to use the area of a triangle to investigate the area of a circle. The following illustrates how participants made a derivational connection by making connections between the algebraic and geometric representations of the area of a triangle and the area of a circle.

> They're both area, just of different shapes. I'm trying to figure out how much more I can relate them than that. I guess if you have your circle and you make

Table 3 Themes and exemplars for closed-sort pair 2 and 4 (n=28)

Themes	Exemplars of participant responses	Count	Frequency
Max area most square like	I'm trying to find the max possible area of the rectangle. I think it relates because the max possible area of rectangle is going to be given by length times width which is 7 times 7 so you could say 7 squared so the is some kind of connection to x squared.	3	11%
Calculus problem	Here I think about, there is some calculus interwoven in this, when trying to find the maximum area with a given perimeter. When you do the arithmetic, the math is going to create a parabola and that maximum value….I would need to flush this one out, but they are related.	3	11%
Derivative to find max	I think these are related. I think you have to take the derivative to find the maximum. We did problems like this last semester where sometimes it was undefined and sometimes a maximum. I need my notes for this one.	1	4%
Graphing possibilities	To find the maximum area of a rectangle you can graph it which is usually going to be a parabola and this is the equation that gives you a parabola. You could graph every possibility and the graph would look like this [participant uses hands to indicate a downward opening parabola] which is a parabola.	5	17%
None	I don't see how finding the max area of a rectangle has to do with a parabola…nope…nothing.	16	57%
Totals		28	100%

Card 2 read, "A rectangle has perimeter 28 ft. Find the maximum possible area of the rectangle." Card 4 read, "A function is defined by *f(x)=x^2*. What kind of curve will it produce when graphed?"

Table 4 Themes and exemplars for closed-sort pair 15 and 17 (n=28)

Theme	Exemplars of participant responses	Count	Frequency
Both area formulas	That's just going back to area because you are trying to find area in each. If you want to find the area of a triangle you use this formula and if you want to find area of circle you use this one and that's how they are related. They are formulas for area but just different objects.	17	60%
Geometric/ Relational	They're both area, just of different shapes. I'm trying to figure out how much more I can relate them than that. I guess if you have your circle and you make it into a bunch of different pie pieces which is kind of similar to a triangle you could end up using this formula [card 17] to roughly get to this one [card 15]. The more triangles you put into the circle, the closer it will get to the area of a circle.	9	32%
Volume of cone	If you go by what I said earlier about multiplying the area of a triangle times the area of a circle, then it might be volume of a cone.	1	4%
None	There is something there but I can't remember what it is, I can't put my finger on it. It is something I've done and I don't remember when and where.	1	4%
Totals		28	100%

Card 15 read, "The area *A* enclosed by a circle is given by the formula *A=πr^2* where *r* is the radius of the circle." Card 17 read, "The area of a triangle is given by the formula *A=1/2bh* where *b* is the base and *h* is the height of the triangle"

> it into a bunch of different pieces which is kind of similar to a triangle you could end up using this formula [card 17] to roughly get to this one [card 15]. The more triangles you put into the circle, the closer it will get to the area of a circle. (Participant 137, interview transcript, April 15, 2008)

For the closed-sort pairing of cards 4 and 15 the following themes emerged: *both have "squares"*; *both are quadratic functions*; *invalid geometric*; *none* (see Table 5).

For card pairing 4 and 15, over a third of the participants indicated that the cards were connected because they "both have squares," referring to the exponent of the variable for each equation on each card. Similar to previous closed-sort pairings, participants focused solely on the surface features of the card resulting in a superficial rather than mathematical connection. Nearly a third of the participants tried to make a connection between the two cards by relating what they indicated to be geometric representation for each equation. For card 4, participants would describe the graph of the function *f(x)* as a parabola or "U" shape. For card 15, participants associated the equation for the area of a circle with the geometric representation of a circle by saying that the "curve" for card 15 was a circle. In other words, participants indicated that in card 4 the "curve" is a U shape and in card 15 the "curve" is a circle. They would then try to establish a connection between the two cards by comparing the geometric representations, that is, the "curves." The following illustrates how participants tried to establish a connection between the two cards by comparing the "curves."

> Again, I'm going to go with they are connected because area squared and this [function] is squared. This one says what kind of curve will it produce when graphed and we know what kind of curve a circle is going to produce. I guess

Table 5 Themes and Exemplars for Closed-Sort Pair 4 and 15 (n=28)

Theme	Exemplars of participant responses	Count	Frequency
Both have "Squares"	The variable in both formulas is squared. They both have "squares" in them.	10	35%
Both are quadratic functions	You have two functions squared. You could substitute *pi x* for *r*. They are both even quadratic functions.	2	7%
Invalid geometric	Again, I'm going to go with they are connected because area squared and this [function] is squared. This one says what kind of curve will it produce when graphed and we know what kind of curve a circle is going to produce. I guess half of it is going to be a parabola. The function is going upward like a U shape. If it continued or if you flip it, rotate it, then you could find the area of a circle.	8	29%
None	I'm not sure I can think of a relationship between 4 and 15. This [card 4] could be the area of a wedge of a circle, but that is pretty obscure.	8	29%
Totals		28	100%

Card 4 read, "A function is defined by *f(x)=x^2*. What kind of curve will it produce when graphed?" Card 15 read, "The area *A* enclosed by a circle is given by the formula *A=πr^2* where r is the radius of the circle"

> half of it is going to be a parabola. (Participant 876, interview transcript, April 21, 2008)

There were only two participants (7%) who recognised the equations on both cards were algebraic representations of particular quadratic functions, that is, when graphed in the Cartesian coordinate plane each equation would produce the graph of a parabola. The remaining eight participants (29%) could not make a connection between the two cards. In some of these cases, the participants indicated that they could not see a connection between the two cards because one card was describing the area of a circle, while the other card was focused on the graph of a particular curve. “I don’t see how they are related because this number 4 is talking about curves on the graph and number 15 is the area of a circle” (Participant 421, interview transcript, May 7, 2008). Other participants indicated they were not related because of where the topics typically fall within K-12 curriculum.

> I would say they’re not related. Again, they are far apart. I feel like area is such a basic math that you really have to understand that before you can move on to understand the *x-y* coordinate plane. Before you ever got to sketching curves you have to understanding what this was [participant points to card 15]. The area of a circle has nothing to do with knowing how to sketch a curve. But I feel like this [participant points to card 15] is something you have to understand before you every get to understand this [participant points to card 4]. This one [participant points to card 15] is something you learn in middle school whereas this one [participant points to card 4] is something you learn to do in high school. (Participant 190, interview transcript, April 18, 2008)

The previous statement is of particular interest when thinking about prospective middle-grades teacher preparation, mathematical connection making, K-12 curriculum, and “horizon knowledge” (Ball 1993). Horizon knowledge is an “awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al. 2008, p. 403). Knowledge at the mathematical horizon is “useful in seeing connections to much later mathematical ideas” (p. 403). Prospective middle-grades teachers’ ability to unpack mathematics and make insightful connections between mathematics learned in college courses to the mathematics they will teach may be related to the extent to which their knowledge of mathematics is connected. With respect to the preparation of prospective middle-grades teachers in this study, perhaps greater care must be taken toward explicitly demonstrating how certain geometric concepts, themes, or topics their future middle-school students will encounter will again reappear and be examined in greater depth and complexity as they move into high school and beyond.

For the closed-sort pairing of cards 9 and 16 the following themes emerged: *given triangle*; *create triangle*; *distance formula looks like Pythagorean Theorem*; *Pythagorean theorem is the distance formula*; *none* (see Table 6).

For card pairing 9 and 16, all but one participant (96%) indicated the two cards were connected which is not surprising given that the Pythagorean Theorem is arguably the most popular and remembered mathematical statement from high-school geometry. As seen in the majority of responses here, the Pythagorean Theorem is often remembered as “*a* squared plus *b* squared equals *c* squared,” and

Table 6 Themes and exemplars for closed-sort pair 9 and 16 (n=28)

Theme	Exemplars of participant responses	Count	Frequency
Given triangle	These are connected because if you have a right triangle on the coordinate plane you can figure out, easily figure out, the base and the height and then you could use the Pythagorean theorem to figure out the hypotenuse.	15	54%
Create triangle	Like, I'm picturing if I want to find this line and I wanted to find the distance between these two points, I could make a triangle out of that. I would put two points in the plane, I was picturing a line between the two points, and then so I was picturing to draw a triangle. Then finding the distance between these two points would be like finding this line. If this was my triangle and this was my right angle then using the Pythagorean theorem to find the line.	5	17%
DF looks like PT	Yeah [indicating the statement on the two cards are related], because the Pythagorean theorem is pretty much the distance formula. Because *a* squared plus *b* squared equals *c* squared and square root all that to find *c* by itself which is the distance equal to the square root of *a* squared plus *b* squared. The *a*'s could be the *x*'s, the *b*'s could be the *y*'s and so square root of *a* squared plus *b* squared is square root of (*x2* minus *x1*) squared plus (*y2* minus *y1*) squared which equals the distance which equals *c*.	3	11%
PT is DF	The Pythagorean theorem is the distance formula in the coordinate plane. Here I thought about the Pythagorean theorem, actually…. because I have never been able to remember the distance formula and I've learned in two classes this year that you can use the Pythagorean theorem to find the distance between two points instead of having to memorize the distance formula which I found to be really helpful.	4	14%
None	I'm not sure if they are related. I can't remember right now.	1	4%
Totals		28	100%

Card 9 read, "Pythagorean Theorem." Card 16 read, "Distance between two points in the Cartesian Coordinate Plane"

when prompted participants usually recalled that *a*, *b*, and *c* represent the lengths of the legs and hypotenuse, respectively, of a right triangle. More than half the participants' responses for relating the two cards fell under the "given triangle" theme. That is, given a right triangle in the Cartesian coordinate plane, the Pythagorean Theorem could be applied to find the distance between the two endpoints of the hypotenuse. During their explanations, participants would sketch a right triangle oriented in the coordinate plane with one leg of the right triangle parallel to the *x*-axis and the other leg parallel to the *y*-axis. Given this orientation, participants indicated that finding the length of the legs of the right triangle was a matter of counting grid marks and once these lengths had been found, the Pythagorean Theorem could be applied.

> These are connected because if you have a right triangle on the coordinate plane you can figure out, easily figure out, the base and height and then you could use the Pythagorean Theorem to figure out the hypotenuse. Because you can't just count the points like you did on the base and height because they are not exact. Like on a grid it would go through just a corner of a box or half of a

> box or three-quarters of a box, it wouldn't be accurate. (Participant 876, interview transcript, April 21, 2008)

These findings have interesting mathematical and pedagogical implications. What if the right triangle was not oriented in the way described above but was rotated 30°? How would these participants have responded to a situation in which simply counting grid marks would not yield a precise solution?

In contrast to those participants who indicated the need to be given a triangle in the coordinate plane in order to apply the Pythagorean Theorem, there were only three participants (11%) who made a connection to finding the distance between two points in the coordinate plane by creating a triangle and then applying the Pythagorean Theorem.

There were approximately 25% of participants who made a procedural connection to card 16 by stating there was a formula for the distance between two points in the Cartesian coordinate plane. These participants then used this procedural connection to make a connection to the Pythagorean Theorem. In some cases, participants described how the distance formula looks like the Pythagorean Theorem while others made the connection that the distance formula is just an application of the Pythagorean Theorem in the coordinate plane.

> Yeah [indicating the statement on the two cards are related], because the Pythagorean Theorem is pretty much the distance formula. Because *a* squared plus *b* squared equals *c* squared and square root all that to find *c* by itself which is the distance equal to the square root of *a* squared plus *b* squared. The *a*'s could be the *x*'s, the *b*'s could be the *y*'s and so the square root of *a* squared plus *b* squared is square root of (*x2* minus *x1*) squared plus (*y2* minus *y1*) squared which equals the distance which equals *c*. (Participant 137, interview transcript, April 15, 2008)
>
> The Pythagorean Theorem is the distance formula in the coordinate plane. Here I thought about the Pythagorean Theorem, actually…because I have never been able to remember the distance formula and I've learned in two classes this year that you can use the Pythagorean Theorem to find the distance between two points instead of having to memorise the distance formula which I found to be really helpful. Because you already have to know the Pythagorean Theorem anyway so well just use it for that [participants points to card 16] too. (Participant 914, interview transcript, April 24, 2008)

The findings exhibited in the statements above are encouraging because we are beginning to see how prospective middle-grades teachers are able to move away from a rote memorisation of formulas to making potential derivational connections between algebraic and geometric representations of distance in order to reason out and explain why the distance formula is just an application of the Pythagorean Theorem in the coordinate plane.

Conclusions and implications

This exploratory study describes the types of mathematical connections prospective middle-grades teachers made while engaged in tasks meant to probe mathematical connections. In the open card-sorting task, five types of mathematical connections were identified: *categorical*, *procedural*, *characteristic/property*, *curricular*, and *derivational*. The majority of the open card sorts were categorical and procedural. The majority of responses to the closed card sort were also predominantly categorical in nature as prospective middle-grades teachers tended to focus mainly on the surface features of the cards when relating each preselected pairing. Perhaps these participants' mathematical experiences have been dominated by traditional curriculum placing focus on procedural fluency rather than conceptual understanding of mathematics (Boaler and Humphreys 2005; Battista 2007). There were very few (13%) subsets made in the open card sort that were curricular. The majority of participants (79%) had yet to take methods courses and, thus, may have not considered making subsets from the perspective of what a future teachers need to know and be able to do in the context of teaching. Given that the majority of participants had yet to take methods courses, perhaps this lack of curricular connection making could be improved by integrating more pedagogy into all mathematics content courses for teachers. By infusing pedagogy in content courses, mathematicians and mathematics educators could help to make "visible the connections to the kinds of mathematical thinking, judgment, [and] reasoning one has to do in teaching" (Ball et al. 2009, p. 29).

The results of this study also have implications for K-12 and prospective middle-grades teachers' methods preparation. In methods courses, prospective middle-grades mathematics teachers focus on lesson planning, instructional strategies, and assessment. However, prospective middle-grades mathematics teachers are rarely afforded the opportunity in their methods courses to reflect on the role mathematical connections play in lesson planning, instructional strategies, and assessments. The CSA activities could serve as one model for both formative and summative assessment techniques for mathematical connection making that could be implemented during K-12 classroom instruction and lesson planning. In understanding the role mathematical connections play in carrying out the work of teaching, prospective middle-grades teachers will also be better prepared to carry out best mathematical practices addressed in the recently released draft of *College and Career Readiness Standards for Mathematics* (CCSSO 2009). According to this document,

> Proficient students expect mathematics to make sense. They take an active stance in solving mathematical problems. When faced with a non-routine problem, they have the courage to plunge in and try something, and they have the procedural and conceptual tools to carry through. They are experimenters and inventors, and can adapt known strategies to new problems. (p. 5)

By strengthening prospective middle-grades teachers' mathematical connection making and its role in carrying out the work of teaching, mathematics educators will be helping these future teachers implement and carry out college and career readiness standards.

The identification of mathematical connections categories in this study may aid those wishing to construct mathematics tasks for explicit connection making with the intention of strengthening prospective teachers' conceptual understanding of underlying mathematical concepts and mathematics knowledge for teaching. The card-sorting task presented here provides a "snapshot" of how prospective middle-grades teachers externalise their internal representations of mathematical connections within their mental framework. Overall, the findings of this study may be particularly useful to mathematics educators, curriculum developers, and researchers seeking further understanding behind effective and ineffective teacher preparation.

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Appendix A

Card-sort activity interview protocol

Researcher: "I have a series of cards which contain mathematical ideas, concepts, terms, definitions and problems. I would like you to go through and read each of these cards. When you have finished reading each card, please hand them back to me."

(*Hand cards over to participant.*)

(*Ask for participant's initial thoughts upon reading each card.*)

Researcher: "What were some of your first thoughts after reading card number ___?"

(*When the participant has finished giving initial thoughts about each card, the researcher will lay the card on the table in columns by number on the card. The arrangement is four columns with five cards in each column.*)

Researcher: I would like you to select a group of more than 1 card that you believe are related. Do not assume that any of these topics are connected or that they are all connected—just select your subsets as you see fit.

1. Participant will select a subset of cards he/she feels are connected.
2. ***Researcher:*** "How are these concepts or ideas connected? What were you thinking when you selected these cards?"
3. Participant will give an explanation.
4. ***Researcher:*** "Any other cards you would add to this group?"
5. ***Researcher:*** "Please return the cards. Now select another subset of cards that you feel are related."
6. Repeat steps 1–5 for approximately eight sorts (the average number of sorts from pilot study).
7. ***Researcher:*** "Can you make any more subsets?"

Researcher: "If you could make up your own cards, what kinds of cards would you make to help create additional subsets of related cards? Or what kinds of cards would you create to add to the subsets you already selected?"

Researcher: "Are there any cards here that you believe are connected to or related to what is on card 14? Are there any cards here that you believe are connected or related to what is on card 10?" (Card 10 and 14 were selected as these particular cards were less frequently selected during the pilot study.)

1. If response is NO, then researcher will ask "What kind of cards would you create that would show a connection or relation to this particular card?"

Researcher: "I'm going to select a couple of cards and would like to know if you think these cards are connected or related in some way. Do not assume that the cards I select are related or connected. I just want to hear your thoughts." (Researcher selects two cards, paired as follows: cards 6 and 11; cards 15 and 4; cards 16 and 9; cards 17 and 15; cards 2 and 4. These particular pairs were chosen in consultation with expert mathematicians.)

(*Once participants have completed the entire card sort activity, the researcher will ask the following reflective questions below*)

Researcher: "What did you think of the card-sort activity?"

Researcher: "What are some advantages or disadvantages to doing a card-sort activity?"

Researcher: "What do you think is the purpose of this particular activity?"

Researcher: "Why would I do an activity like this with a prospective middle-grades teacher?"

Appendix B

Description of mathematical connections for coding open card sort

- **Categorical**: *use of surface features primarily as a basis for defining a group or category.*
  - *Example*: Cards 9 and 14

"The formulas look similar. The *a* would be the *x* and *b* would be your *y* so *c* would be your *r*."

- **Procedural**: *relating ideas based on a mathematical procedure or algorithm possibly through construction of an example; may include description of the mechanics involved in carrying out procedure rather than the mathematical ideas embedded in the procedure.*
  - *Example*: Cards 4 and 10

"The derivative is move the exponent in front and subtract exponent by 1. So the derivative of *f(x)=x^2* is *2x*. Whenever I've seen derivative they always use *f(x)=x^2* or whatever and *f* prime of *x* is the derivative. I've had experience taking the derivative of things that look like this."

Characteristic/Property: *defining characteristics or describing the properties of concepts in terms of other concepts.*

○ *Example*: Cards 19, 20, and 3

"A rectangle has two sets of parallel sides and four 90-degree angles."

- **Derivation**: *knowledge of one concept to build upon or explain another concept; including but not limited to the recognition of the existence of a derivation.*

  ○ *Example*: Cards 5, 15, 18, 8, and 6

"I can derive the formula for the volume and surface area of a cylinder using the area of a circle and circumference of a circle… [Participant gives detailed explanation/justification]"

- **Curricular**: *relating ideas or concepts in terms of impact to the curriculum, including the order in which one would teach concepts/topics.*

  ○ *Example*: Cards 15 and 6

"If you were going to teach a lesson on circles you would have to teach them area and circumference rules. They would fall in the same lesson you would teach them. They would have to understand pi and radius for both of them. The circumference of a circle its perimeter; think like triangle and rectangle so my students would understand what circumference is."

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