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ORIGINAL RESEARCH PAPER

Catastrophe risk bonds with applications to earthquakes

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Abstract Catastrophe (CAT) risk bonds provide a solid mechanism for direct transfer of the financial consequences of extreme events (hazards) into the financial market. During the past two decades, insurance companies have been searching for more adequate liquidity funds as a consequence of increasing losses due to climate change and severe natural disasters. The aims of this study were twofold. First, we study the pricing process for CAT bonds for the structure of n financial and m catastrophe-independent risks. Second, to illustrate the applicability of our results, an application for earthquakes is considered using extreme value theory. As a numerical example, a CAT bond with historical data from California is proposed in which the magnitude, latitude, longitude, and depth are included in the model. In addition, appropriate models are constructed for the term structure of interest and inflation rate dynamics, and a stochastic process for the coupon rate. Finally, on the basis of analysis for the aforementioned catastrophe and financial market risks, we can use equilibrium pricing theory to find a certain value price for the CAT California earthquake bond.

Keywords CAT risk bonds · Extreme value theory · Equilibrium pricing · Earthquakes · California data

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1 Introduction

Losses caused by catastrophic events, such as earthquakes, tsunamis, hurricanes, and man-made disasters, are extremely large and have increased significantly during the past two decades. Insurance companies alleviate part of this risk by introducing securitization mechanics to achieve a more adequate liquidity fund. An alternative method is to issue catastrophe (CAT) bonds, which transfer the financial consequences of catastrophe events from issuers to investors in a contract to cover huge liabilities through traditional reinsurance providers or governmental budgets.

CAT bonds are inherently risky, non-indemnity-based multi-period deals that pay a coupon to investors at the end of each period and a final principal payment at the maturity date if no catastrophic events have occurred. If a major catastrophic event hits the secured region before the expiry date, investors will receive no coupon payment or only part of their principal. In general, CAT bonds carry a 3- to 5-year maturity at issuance, and a floating coupon of the LIBOR rate plus a premium at a rate between 2 and 20 % [14, 48]. Moreover, we call such a catastrophe a trigger event. In the literature, there are five types of trigger variable: indemnity triggers, industry index triggers, modelled loss index triggers, parametric index triggers, and hybrid triggers [9, 28].

The first such experimental transactions were completed in the mid-1990s after Hurricane Andrew (with losses of \$19.6 bn) and the Northridge earthquake (with losses of \$14.9 bn) by a number of specialized catastrophe-oriented insurance and reinsurance companies in the USA, including AIG, Hannover Re, St. Paul Re, and USAA [39]. The market grew rapidly from approximately \$0.6 bn to over \$2 bn following the 9/11 terrorist attacks in 2001, and losses significantly increased to \$116 bn in 2011 and down to \$45 bn in 2013 [4]. Low penetration rates for insurance leave individuals, companies, and governments to shoulder the financial losses arising from catastrophic events. In the next sections, we focus our attention on earthquakes. According to historical information from the National Earthquake Information Center,[1] 12,000–14,000 earthquakes are recorded annually throughout the world.[2] In California, two or three earthquakes of magnitude 5.5 and higher occur annually, and these are large enough to cause moderate damage [47]. Although infrequent, earthquakes and their side effects, including landslides, surface fault ruptures, liquefaction, aftershock fires, and tsunamis, have huge potential to cause injury, loss of life, and property damage. The California Geological Survey[3] has reported that more than 70 % of California residents live within the area where significant earthquakes could occur in the next 50 years according to slip rates in geological time [47]. Therefore, the potential enormous financial demands on insurance and reinsurance businesses make it realistic to introduce a mechanism for individuals against catastrophic losses caused by earthquakes. In 2007, Swiss Re launched a set of CAT bond performance indices[4] that increased the transparency of CAT bond returns.

[1] http://earthquake.usgs.gov/regional/neic/.

[2] http://earthquake.usgs.gov/earthquakes/world/historical.php/.

[3] http://www.consrv.ca.gov/CGS/Pages/Index.aspx.

[4] http://www.swissre.com/media/news_releases/swiss_re_launches_the_first_catastrophe_bond_indices.html.

In theory, the pricing of CAT bonds requires an incomplete market framework because catastrophe risks cannot be replicated by a portfolio of primitive securities [11, 12, 29, 49]. In the case of an incomplete market, there is no universal pricing theory that successfully addresses issues such as specification of hedging strategies and price robustness [51]. For example, Wang [50] addressed market incompleteness using the Wang transform, an approach adopted by Lin and Cox [32, 33], Pelsser [41], and Galeotti et al. [25]. An alternative technique is based on the principle of equivalent utility to obtain indifferent pricing. In this direction, Young [51] calculated the price of a contingent claim under a stochastic interest rate for an exponential utility function. An extension was proposed by Egami and Young [17], who introduced a more complex payment structure. Cox and Pedersen [12] used a time-repeatable representative agent utility. Their approach is based on a model of the term structure of interest rates and a probability structure for catastrophe risks assuming that the agent uses the utility function to make choices about consumption streams. They applied their theoretical results to Morgan Stanley, Winterthur, USAA and Winterthur-style bonds. Reshetar [43] used a similar setting for multiple-event CAT bonds for the first time. Zimbidis et al. [53] also adopted the Cox and Pedersen [12] framework for pricing a Greek bond using equilibrium pricing theory with dynamic interest rates. Several other important alternative pricing mechanisms have been developed for catastrophe-linked securitization pricing models in different markets. Froot and Posner [22, 23] derived an equilibrium pricing model for uncertain parameters of multi-event risks. Dieckmann [16] applied a CAT bond model based on consumption, while Zhu [52] detailed the premium spread using an intertemporal equilibrium framework. Braun [6] analysed the premium using OLS regression with robust standard errors. Föllmer and Schweizer [20] introduced a minimal martingale measure for option pricing, whereas Schweizer [45] used a variance optimal martingale measure. Other possible equivalent martingale measures are the Esscher martingale measure [7, 26] and the minimal entropy martingale measure [21, 36, 37]. Lin et al. [31] applied a Markov-modulated Poisson process for catastrophe occurrences using a similar approach to that of Vaugirard [49]. Nowak and Romaniuk [38] priced CAT bonds by focusing on the dynamics of the spot interest rate. It is important to note that Vaugirard [49] was the first to develop a simple arbitrage approach to evaluate catastrophe risk insurance-linked securities, notwithstanding the non-traded underlying framework. Pérez-Fructuoso [42] developed a CAT bond with index triggers. Loubergé et al. [34] used a compound Poisson process for the binomial interest rate. Extensions involving a compound doubly stochastic Poisson process were investigated by Burnecki and Kukla [8] and Albrecher et al. [1].

Here we develop a model with multiple catastrophes and financial risks in a discrete-time period as an extension of the approach of Cox and Pedersen [12]. We apply our theoretical results to construct a multivariable CAT bond for earthquakes, and then use data for California to derive the price density function for a 5-year bond. We apply an incomplete and no-arbitrage framework and assume that all risks are mutually independent, and that aggregate consumptions depend on financial and catastrophic risks. Section 2 describes the probability structure for the model and a price model for earthquake CAT bonds using equilibrium pricing theory. The fact that catastrophic risks are uncorrelated with financial risk movements makes the problem much simpler. In Sect. 3, we specify one-period and multi-period price formulas for CAT bonds, and analyse the term structures or the distributions of the risk variables relative to the bond. The distribution of the annual maximum earthquake magnitude in California is estimated using extreme value theory. We assume that the dynamics of the LIBOR rate is a CIR model and that the interest and inflation rates follow autoregressive integrated moving average (ARIMA) processes. Section 4 presents numerical examples for 1- and 5-year CAT bonds. The density plot for the price is derived to illustrate the applicability of our results. Finally, Sect. 5 discusses the results and suggests future research directions.

2 Modeling CAT bonds

2.1 Model description and preliminaries

In this section, a preliminary presentation for the CAT bond structure is given. Generalizing the ideas of Cox and Pedersen [12], we design a CAT bond that combines n financial market variables and m catastrophic risk variables. The model set-up requires a probabilistic structure which is given as follows.

Assume that we are trading CAT bonds in an investment market that is arbitrage-free. The time of the catastrophe(s) is independent of the term structure(s) under the relevant probability measure. We assume that we have n financial risk variables, each modelled on the probability triples $(\Omega_{1,i}, \mathcal{F}^{(1,i)}, \mathbb{P}_{1,i})$ for $i = 1, 2, \ldots, n$. Let $T < \infty$ be the maturity time of the trading interval. Let $\mathcal{F}_k^{(1,i)}$ be the σ-algebras of $\Omega_{1,i}$ representing the investment information available to the market at time k ($k = 0, 1, \ldots, T$), where $\mathcal{F}^{(1,i)}$ ($i = 1, 2, \ldots, n$) are corresponding filtrations. Thus, each probability measure $\mathbb{P}_{1,i}$ is defined for all events belonging to the $\mathcal{F}_k^{(1,i)}$ σ-algebra, $k \leq T$. Note that the measures $\mathbb{P}_{1,i}$ do not necessarily have the same distributions.

We consider m catastrophic risk variables, which are modelled on probability triples $(\Omega_{2,j}, \mathcal{F}^{(2,j)}, \mathbb{P}_{2,j})$, where $\mathcal{F}_k^{(2,j)}$ are the σ-algebras of $\Omega_{2,j}$ representing the risk information available at time k ($k = 0, 1, \ldots, T$) and $\mathbb{P}_{2,j}$ ($j = 1, 2, \ldots, m$) are the probability measures governing the catastrophe structure (not necessarily with the same distribution). The filtrations $\mathcal{F}^{(2,j)}$ are indexed by the same times $k = 0, 1, \ldots, T$ as previously. We can now construct the sample space of the full model such that

$$\Omega = \left(\Omega_{1,1} \times \Omega_{1,2} \times \cdots \times \Omega_{1,n}\right) \times \left(\Omega_{2,1} \times \Omega_{2,2} \times \cdots \times \Omega_{2,m}\right).$$

A typical event of the full model sample space is of the form $\omega = (\widetilde{\omega}_{1,n}, \widetilde{\omega}_{2,m})$, where $\widetilde{\omega}_{\kappa,\ell} = (w_{\kappa,1}, w_{\kappa,2}, \ldots, w_{\kappa,\ell})$, $\kappa = 1,2$, $\ell = n,m$, such that $w_{1,i} \in \Omega_{1,i}$ $(i = 1,2,\ldots,n)$ and $w_{2,j} \in \Omega_{2,j}$ $(j = 1,2,\ldots,m)$.

Assuming that the events $w_{\kappa,1}, w_{\kappa,2}, \ldots, w_{\kappa,\ell}$ $(\kappa = 1,2,\ \ell = n,m)$ are pairwise independent, then the probability measure on the sample space Ω is given by

$$\mathbb{P}(\omega) = \prod_{i=1}^{n} \mathbb{P}_{1,i}(\omega_{1,i}) \cdot \prod_{j=1}^{m} \mathbb{P}_{2,j}(\omega_{2,j}), \quad i = 1,2,\ldots,n, \quad j = 1,2,\ldots,m.$$

In addition, the natural filtration produced by the σ-algebras of Ω is denoted by $\mathcal{F}$ and given by

$$\mathcal{F}_k = \left(\mathcal{F}_k^{(1,1)} \times \mathcal{F}_k^{(1,2)} \times \cdots \times \mathcal{F}_k^{(1,n)}\right) \times \left(\mathcal{F}_k^{(2,1)} \times \mathcal{F}_k^{(2,2)} \times \cdots \times \mathcal{F}_k^{(2,m)}\right)$$

for $k = 0,1,\ldots,T$. Thus, with all the elements defined as above, $(\Omega, \mathcal{F}, \mathbb{P})$ constitutes a probability triple for the full model. For the full model that depends on either financial variables or catastrophic variables, we introduce the increasing filtrations $\mathcal{A}_k^{(1)} \subset \mathcal{A}^{(1)}$ and $\mathcal{A}_k^{(1,i)} \subset \mathcal{A}^{(1,i)}$ $(i = 1,\ldots,n)$, and similarly $\mathcal{A}_k^{(2)} \subset \mathcal{A}^{(2)}$ and $\mathcal{A}_k^{(2,j)} \subset \mathcal{A}^{(2,j)}$ $(j = 1,\ldots,m)$ generated from the following σ-algebras:

$$\begin{aligned}
\mathcal{A}_k^{(1)} &= \mathcal{F}_k^{(1,1)} \times \cdots \times \mathcal{F}_k^{(1,n)} \times \{\emptyset, \Omega_{2,1}, \ldots, \Omega_{2,m}\}, \\
\mathcal{A}_k^{(1,i)} &= \mathcal{F}_k^{(1,i)} \times \{\emptyset, \Omega_{2,1}, \ldots, \Omega_{2,m}\}, \quad i = 1,\ldots,n, \\
\mathcal{A}_k^{(2)} &= \{\emptyset, \Omega_{1,1}, \ldots, \Omega_{1,n}\} \times \mathcal{F}_k^{(2,1)} \times \cdots \times \mathcal{F}_k^{(2,m)}, \\
\mathcal{A}_k^{(2,j)} &= \{\emptyset, \Omega_{1,1}, \ldots, \Omega_{1,n}\} \times \mathcal{F}_k^{(2,j)}, \quad j = 1,\ldots,m,
\end{aligned}$$

for $k = 1,\ldots,T$. An $\mathcal{A}_T^{(\kappa)}$ measurable random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ (or an $\mathcal{A}^{(\kappa)}$ adapted stochastic process Y) depends on financial risk variables $(\kappa = 1)$ or catastrophic risk variables $(\kappa = 2)$. Let financial events be $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and catastrophic events be $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Therefore, $\alpha_{1,i} = A_{1,i} \times \Omega_{2,1} \times \cdots \times \Omega_{2,m}$ and $\alpha_{2,j} = \Omega_{1,1} \times \cdots \times \Omega_{1,n} \times A_{2,j}$, for some $A_{\kappa,\ell} \in \mathcal{F}_T^{(\kappa,\ell)}$, $\kappa = 1,2$, $\ell = i,j$. We need the independent notion of $\mathcal{A}_T^{(\kappa,\ell)}$ because we cannot refer to $\mathcal{F}_T^{(\kappa,\ell)}$ as being independent under $\mathbb{P}$, since each of $\mathcal{F}_T^{(\kappa,\ell)}$ does not contain events defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 1 *For $i = 1,\ldots,n$ and $j = 1,\ldots,m$, the σ-algebras $\mathcal{A}_T^{(1,i)}$ and $\mathcal{A}_T^{(2,j)}$ are independent under the probability measure $\mathbb{P}$.*

Proof For $i = 1,\ldots,n$ and $j = 1,\ldots,m$, let $a_{1,i} \in \mathcal{A}_T^{(1,i)}$ and $a_{2,j} \in \mathcal{A}_T^{(2,j)}$. Then $a_{1,i} = A_{1,i} \times \Omega_{2,1} \times \cdots \times \Omega_{2,m}$ for some $A_{1,i} \in \mathcal{F}_k^{(1,i)}$, and $a_{2,j} = \Omega_{1,1} \times \cdots \times \Omega_{1,n} \times A_{2,j}$ for some $A_{2,j} \in \mathcal{F}_k^{(2,j)}$, and we have that

$$\begin{aligned}
\mathbb{P}\left[\left(\bigcap_{i=1}^{n}\alpha_{1,i}\right)\bigcap\left(\bigcap_{j=1}^{m}\alpha_{2,j}\right)\right] &= \mathbb{P}\Big(A_{1,1}\times\cdots\times A_{1,n}\times A_{2,1}\times\cdots\times A_{2,m}\Big)\\
&= \prod_{i=1}^{n}\mathbb{P}_{1,i}(A_{1,i})\cdot\prod_{j=1}^{m}\mathbb{P}_{2,j}(A_{2,j})\\
&= \prod_{i=1}^{n}\mathbb{P}_{1,i}(A_{1,i})\prod_{j=1}^{m}\mathbb{P}_{2,j}(\Omega_{2,j})\prod_{i=1}^{n}\mathbb{P}_{1,i}(\Omega_{1,i})\prod_{j=1}^{m}\mathbb{P}_{2,j}(A_{2,j})\\
&= \prod_{i=1}^{n}\mathbb{P}(\alpha_{1,i})\cdot\prod_{j=1}^{m}\mathbb{P}(\alpha_{2,j}).
\end{aligned}$$

□

2.2 The valuation framework

In this section, we show how to implement valuation under the full model by choosing the equivalent measure. Similar to Cox and Pedersen [12] and Magill and Quinzii [35], the setting of a representative agent is adopted to price uncertain cash flow streams.

Assume that we are in a T-period economy in which agents can make choices and consume during each period. An agent makes choices about his future consumption, represented by the stochastic process $\{c(k); k = 0, 1, \ldots, T\}$. The aggregate consumption stochastic process is denoted by $\{C^*(k); k = 0, 1, \ldots, T\}$. Both these processes are adapted to filtration of the full model. Only the first choice is known with certainty at time $k = 0$. For $i = 1, 2, \ldots, n$, let $\{r_i(k); k = 0, 1, 2, \ldots, T-1\}$ be the one-period financial market rates. Then these one-period financial market rates can be defined through the conditional expectation

$$\prod_{i=1}^{n}\frac{1}{1+r_i(k)} = \frac{1}{u'_k(C^*(k))}\mathbb{E}^{\mathbb{P}}\left[u'_{k+1}(C^*(k+1))|\mathcal{F}_p\right], \quad k = 0, 1, 2, \ldots, T-1,$$

where $u_0, u_1, \ldots, u_T$ represent the utility functions corresponding to $\{c(k); k = 0, 1, \ldots, T\}$. The Randon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is defined in the same vein as [12]

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \prod_{i=1}^{n}\prod_{s=0}^{T-1}[1+r_i(s)]\left[\frac{u'_T(C^*(T))}{u'_0(C^*(0))}\right]. \tag{1}$$

Note that this new random variable is measurable with respect to $\mathcal{F}_T$. In addition, we clearly need to ensure that $\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$ (Lemma 2). First, for notation simplicity we denote the one-period financial market discount rates by $B(k) = \prod_{i=1}^{n}\prod_{s=0}^{k-1}[1+r_i(s)]$, for $k = 1, 2, \ldots, T$, and $B(0) = 1$. Then we can define the stochastic processes $\{\xi(k); k = 0, 1\ldots, T\}$ and $\{\zeta(k); k = 0, 1\ldots, T\}$ as

$$\xi(k) = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_k\right] = \frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_k} \quad \text{and} \quad \zeta(k) = B(k) \cdot \frac{u_k'(C^*(k))}{u_0'(C^*(0))},$$

with $k = 1, \ldots, T$ and $B(0) = 1$, which leads to $\zeta(0) = 1$. By Eq. (1) it holds that $\zeta(T) = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{F}_T$. Similar to Lemma B.1 and Theorem B.1 of Cox and Pedersen [12], we have the following lemma and theorem.

Lemma 2 *The process $\{\zeta(k); k = 0, 1, 2, \ldots, T\}$ is a $\mathbb{P}$-martingale on the filtration $\mathcal{F}$, and $\zeta(k) = \xi(k)$ for $k = 0, 1, 2, \ldots, T$.*

Remark 1 An immediate consequence of Lemma 2 is that

$$1 = \mathbb{E}^{\mathbb{P}}\Big[\zeta(0)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\zeta(T)\Big] = \mathbb{E}^{\mathbb{P}}\Big[\xi(T)\Big] = \mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right],$$

which ensures that the Radon-Nikodym derivative in Eq. (1) indeed defines a new probability measure.

Intuitively, the probability measure $\mathbb{Q}(\cdot)$ is equivalent to knowledge of the representative investor's utility function and the aggregate consumption process.

Theorem 1 *Under the assumptions of the representative agent pricing model, the price of a generic future cash flow process $\{d(k); k = 1, 2, \ldots, T\}$ at time 0 is given by*

$$V(d) = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{T} \frac{1}{\prod_{i=1}^{n}\prod_{s=0}^{k-1}[1 + r_i(s)]} d(k)\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{T} \frac{1}{B(k)} d(k)\right]. \tag{2}$$

Remark 2 When in incomplete markets, there is no unique interpretation for the prices that we assign to CAT bonds unless we introduce the probability distribution of the catastrophe risk [12].

Using similar arguments to those in Theorem B.2 of Cox and Pedersen [12], the general intertemporal valuation of a future cash flow can be expressed in terms of the equivalent measure $\mathbb{Q}(\cdot)$.

Theorem 2 *Under the assumptions of the representative agent pricing model, the price of a generic future cash flow process $\{d(k); k = p+1, p+2, \ldots, T\}$ is given by*

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{k=p+1}^{T} \frac{u_k'(C^*(k))}{u_p'(C^*(p))} d(k)\middle|\mathcal{F}_p\right] = \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=p+1}^{T} \frac{B(p)}{B(k)} d(k)\middle|\mathcal{F}_p\right],$$

where $p = 0, 1, \ldots, T$, with the convection $\sum_{\gamma}^{\lambda} = 0$ for $\lambda < \gamma, \lambda, \gamma \in \mathbb{N}$.

For analysis of CAT bonds, hereafter we assume that the aggregate consumption depends only on financial risks, given as $C^*(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m};\ k) = C^*(\tilde{\omega}_{1,n}; k)$ for $\omega \equiv (\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}) \in \Omega$. Then C^* is $\mathcal{A}^{(1)}$ adapted. This is quite a natural approximation since global economic conditions in terms of exchange and production are not strongly related to localized catastrophes [12], assuming that the aggregate consumption process depends only on financial risk information available at time k, and that the structure at time 0 is known.

Lemma 3 *Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, for any random variable X that is $\mathcal{A}_T^{(2)}$ measurable we have*

$$E^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X].$$

In particular, for any catastrophic events $\alpha_{2,j}$ ($j = 1, 2, \ldots, m$) that are $\mathcal{A}_T^{(2,j)}$ measurable, it holds that

$$\mathbb{Q}\left(\bigcap_{j=1}^{m}(\alpha_{2,j})\right) = \mathbb{P}\left(\bigcap_{j=1}^{m}(\alpha_{2,j})\right) = \prod_{j=1}^{m}\mathbb{P}_{2,j}(A_{2,j}), \tag{3}$$

where $A_{2,j} \in \mathcal{F}_T^{(2,j)}$.

Proof Note that $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (1) is $\mathcal{A}_T^{(1)}$ measurable because C^* and $B(T)$ are $\mathcal{A}^{(1)}$ adapted. Since the r.h.s. X and $\frac{d\mathbb{Q}}{d\mathbb{P}}$ are $\mathcal{A}_T^{(2)}$ measurable and independent under $\mathbb{P}$, together with Lemma 3.2.5 of Shreve [46], we have

$$\mathbb{E}^{\mathbb{Q}}[X] = \mathbb{E}^{\mathbb{P}}\left[X\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}[X]\mathbb{E}^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}[X]\cdot 1 = \mathbb{E}^{\mathbb{P}}[X].$$

Moreover, we define

$$X = \prod_{j=1}^{m}\mathbb{1}_{\alpha_{2,j}} = \mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}},$$

where $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}, j = 1, 2, \ldots, m$. Substituting into Eq. (3), we have

$$\begin{aligned}
\mathbb{Q}\left(\bigcap_{j=1}^{m}(\alpha_{2,j})\right) &= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right] = \mathbb{E}^{\mathbb{Q}}[X] = E^{\mathbb{P}}[X] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right] \\
&= \mathbb{P}\left(\bigcap_{j=1}^{m}(\alpha_{2,j})\right) = \mathbb{P}\left[\bigcap_{j=1}^{m}\left\{\Omega_{1,1}\times\cdots\times\Omega_{1,m}\times A_{2,j}\right\}\right] \\
&= \prod_{j=1}^{m}\left[\left(\prod_{i=1}^{n}\mathbb{P}(\Omega_{1,j})\right)\mathbb{P}(A_{2,j})\right] = \prod_{j=1}^{m}\mathbb{P}_{2,j}(A_{2,j}).
\end{aligned}$$

□

Remark 3 Under the measure $\mathbb{P}(\cdot)$ and the assumption that C^* depends only on financial risk variables, the catastrophic events $\alpha_{2,j}$ that depend on the jth catastrophic risk ($j = 1, \ldots, m$) are independent.

To implement Theorems 1 and 2, we need to assume that the events are mutually independent, that is, they depend only on financial risks and only on catastrophic risks, under the measure $\mathbb{Q}$.

Lemma 4 *Under the assumption that C^* is $\mathcal{A}^{(1)}$ adapted, the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent under $\mathbb{Q}$.*

Proof Let $\alpha_{1,i} \in \mathcal{A}_T^{(1,i)}$ and $\alpha_{2,j} \in \mathcal{A}_T^{(2,j)}$. Applying Lemma 3.2.5 of Shreve [46], we have

$$\mathbb{Q}\left(\left(\bigcap_{i=1}^{n}\alpha_{1,i}\right)\bigcap\left(\bigcap_{j=1}^{m}\alpha_{2,j}\right)\right) = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right]$$
$$= \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\frac{d\mathbb{Q}}{d\mathbb{P}}\right].$$

Since $\frac{d\mathbb{Q}}{d\mathbb{P}}$ in Eq. (1) is $\mathcal{A}_T^{(1)}$ measurable,

$$\mathbb{1}_{\bigcap_{i=1}^{m}\alpha_{1,i}}\frac{d\mathbb{Q}}{d\mathbb{P}} \quad \text{and} \quad \mathbb{1}_{\bigcap_{\mathrm{j}=1}^{\mathrm{m}}\alpha_{2,\mathrm{j}}}$$

are independent under $\mathbb{P}$. Consequently,

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\right]$$
$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\right]\mathbb{P}\left[\bigcap_{j=1}^{m}\alpha_{2,j}\right]$$
$$= \mathbb{Q}\left[\bigcap_{i=1}^{n}\alpha_{1,i}\right]\prod_{j=1}^{m}\mathbb{P}_{2,j}[\alpha_{2,j}].$$

Referring back to Lemma 3, we have

$$\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\bigcap_{i=1}^{n}\alpha_{1,i}}\mathbb{1}_{\bigcap_{j=1}^{m}\alpha_{2,j}}\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = \mathbb{Q}\left[\bigcap_{i=1}^{n}\alpha_{1,i}\right]\prod_{j=1}^{m}\mathbb{P}_{2,j}[\alpha_{2,j}]$$
$$= \mathbb{Q}\left[\bigcap_{i=1}^{n}\alpha_{1,i}\right]\mathbb{Q}\left[\bigcap_{j=1}^{m}\alpha_{2,j}\right].$$

Therefore, we conclude that under $\mathbb{Q}$ the σ-algebras $\mathcal{A}_T^{(1)}$ and $\mathcal{A}_T^{(2)}$ are independent. □

As a direct implication of Lemmas 3 and 4, the current value of cash flows X depending on catastrophic risks has the simple form as below. For notation simplicity, we denote the current value of non-defaultable zero-coupon bond maturing at time k with face amount 1 as $P(k) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(k)}\right]$.

Corollary 1 *The current value of an $\mathcal{A}_k^{(2)}$ measurable cash flow X paid at time k is given by*

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B(k)}X\right] = P(k)\mathbb{E}^{\mathbb{P}}[X].$$

Under the discrete time framework, we can express the valuation measure as a product measure of the probability measures $\mathbb{Q}_1$ and $\mathbb{P}_{2,j}$,

$$\begin{aligned}
\mathbb{Q}(\omega) &= \frac{d\mathbb{Q}}{d\mathbb{P}}(\omega)\mathbb{P}(\omega) \\
&= B(\omega;T)\frac{u_T'(C^*(\omega;T))}{u_0'(C^*(\omega;0))}\mathbb{P}(\omega) \\
&= \prod_{s=0}^{T-1}\left[\prod_{i=1}^{n}[1+r_i(\omega_{1,i};s)]\right]\frac{u_T'(C^*(\tilde{\omega}_{1,n};T))}{u_0'(C^*(\tilde{\omega}_{1,n};0))}\prod_{i=1}^{n}\mathbb{P}_{1,i}(\omega_{1,i})\prod_{j=1}^{m}\mathbb{P}_{2,j}(\omega_{2,j}) \\
&= \mathbb{Q}_1(\tilde{\omega}_{1,n})\prod_{j=1}^{m}\mathbb{P}_{2,j}(\omega_{2,j}), \qquad (4)
\end{aligned}$$

where

$$\mathbb{Q}_1(\tilde{\omega}_{1,n}) = \prod_{s=0}^{T-1}\left[\prod_{i=1}^{n}[1+r_i(\omega_{1,i};s)]\right]\frac{u_T'(C^*(\tilde{\omega}_{1,n};T))}{u_0'(C^*(\tilde{\omega}_{1,n};0))}\prod_{i=1}^{n}\mathbb{P}_{1,i}(\omega_{1,i}). \qquad (5)$$

The probability measure in Eq. (5) is generated in terms of the term structure of financial risks term [40]. It is practical to have Eq. (4) since the empirical probabilities of catastrophic events can be used for the probability measures $\mathbb{P}_{2,j}$, where $j = 1, \ldots, m$.

2.3 Implication for valuation

In this subsection we present a concrete form for pricing certain CAT bonds under the discrete time framework. The valuation structure of CAT bonds can be further simplified because the discount factors $B(k)$ are $\mathcal{A}_k^{(1)}$ measurable and depend only on financial risks. Consider a generic future cash flow process $d(\omega;k) = d(\tilde{\omega}_{1,n}, \tilde{\omega}_{2,m}; k)$ depending on financial and catastrophic risks. In addition, we define an associated process of future cash flow as

$$\bar{d}(k) = \mathbb{E}^{\mathbb{Q}}\left[d(k)|\mathcal{A}_k^{(1)}\right],$$

which is the conditional expectation over the loss distribution of catastrophic risks given fixed financial risk variables. The value of $\bar{d}$ reflects the financial events by filtration $\mathcal{A}^{(1)}$; thus, $\bar{d}(k)$ is $\mathcal{A}_k^{(1)}$ measurable. We now reformulate Eq. (2) using the process $\bar{d}$, with $B(k)$ and $\bar{d}(k)$ $\mathcal{A}_k^{(1)}$ measurable. We have

$$V(d)=\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{T}\frac{1}{B(k)}\bar{d}(k)\right]=\mathbb{E}^{\mathbb{Q}_1}\left[\sum_{k=1}^{T}\frac{1}{B(k)}\bar{d}(k)\right],\tag{6}$$

where $\mathbb{Q}_1$ is the valuation measure in terms of n financial risk variables given in Eq. (5). This is practical since we can use Eq. (6) to price the CAT bond by choosing a term structure for arbitrage-free financial risks and calculating the expected cash flow conditionally on the financial risk process.

However, to complete the valuation, we also need to verify the structure of the cash flow process. A direct deduction from Corollary 1 is the case in which the CAT bond cash flows depend only on the catastrophic risk variables.

Theorem 3 *For CAT bond cash flows that are $\mathcal{A}^{(2)}$ adapted,*

$$\bar{d}(k)=\mathbb{E}^{\mathbb{Q}}\left[d(k)|\mathcal{A}_k^{(1)}\right]=\mathbb{E}^{\mathbb{P}}[d(k)]$$

and the value of the CAT bond can be given as

$$V(d)=\sum_{k=1}^{T}P(k)\mathbb{E}^{\mathbb{P}}[d(k)].\tag{7}$$

The pricing formulas for CAT bonds given in Eqs. (6) and (7), which are an extension of work by Cox and Pedersen [12], are the core results of this paper.

3 Application of the results for earthquakes

In this section, we introduce a model with three financial risks (LIBOR, real interest, and inflation rates) and two catastrophe risks (earthquake magnitude and depth). We use the maximum earthquake magnitude in one region as the parametric index trigger for this CAT bond. It becomes clearer later that the region in which the earthquake occurs is included in the payoff function.

We model a financial market risk, the real interest rate, via a discrete process $\{r_k; k=1,2,\ldots,T\}$ within $(\Omega_{1,1},\mathcal{F}^{(1,1)},\mathbb{P}_{1,1})$, that is equipped with the filtration $\mathcal{F}^{(1,1)}$. Similarly, the inflation rate process $\{\pi_k; k=1,2,\ldots,T\}$ is modelled on another complete probability triple $(\Omega_{1,2},\mathcal{F}^{(1,2)},\mathbb{P}_{1,2})$ equipped with the filtration $\mathcal{F}^{(1,2)}$. The final financial risk US LIBOR rate $\{R_k; k\in[0,T]\}$ is modelled within $(\Omega_{1,3},\mathcal{F}^{(1,3)},\mathbb{P}_{1,3})$, which is equipped with the filtration $\mathcal{F}^{(1,3)}$.

Catastrophic risks are modelled via three random variables. We model the annual maximum-magnitude earthquake using the random variable $\{M_k; k=1,2,\ldots,T\}$ within the probability space $(\Omega_{2,1},\mathcal{F}^{(2,1)},\mathbb{P}_{2,1})$, which is equipped with the filtration $\mathcal{F}^{(2,1)}$, and the depth $\{D_k; k=1,2,\ldots,T\}$ within $(\Omega_{2,2},\mathcal{F}^{(2,2)},\mathbb{P}_{2,2})$, which is equipped with the filtration $\mathcal{F}^{(2,2)}$.

One-period and multi-period models are developed and the CAT bond valuation is performed in three stages. In the first stage we specify cash flows to the bondholder, which are dependent on the above risk variables. In the next stage we analyse the dynamics of financial risks and catastrophic risks by assuming a suitable distribution function and estimating parameters from historical data. In the final stage we generate sequences of a discrete-time process for future risks and obtain the price of CAT bonds in an arbitrage-free framework.

3.1 One-period (basic) model

In this subsection, we formulate a simple one-period model in which the dynamics of financial risks (real interest rate, inflation rate, and LIBOR rate) are constant. Under the discrete-time framework of the analysis, we first define the following symbols and notations:

K	Face amount of the CAT bond.
r	One-period risk-free real interest rate (e.g. 1-year US Treasury securities rate).
π	One-period inflation rate [e.g. represented by the consumer price index (CPI)].
R	Deterministic coupon payment rate for the 1-year period given that a specified catastrophic event does not occur (e.g. 12-month US LIBOR rate on the bond issuance date).
e	Extra premium loading for the earthquake risk (normally positive considering risk-averse investors).
M	Maximum earthquake magnitude within all selected regions. If we have two regions, $M = \max\{M^1, M^2\}$, where M^1 and M^2 represent the maximum-magnitude earthquake in each of the two regions.
D	Depth (km) of the earthquake.
$V(d)$	Price of the CAT bond at time of issuance.
$d(R; M, D)$	Piecewise cash value of the CAT bond on maturity. Zimbidis et al. [53] gave a similar expression for CAT bond cash flows that depend on M and D. As an illustration, the structure of the cash value is given by

$$d(R; M, D) = \begin{cases} K \cdot (1 + f(R)), & M \in [0, \mu_1], \text{with } \{\mathrm{D} \leq \delta_1\} \text{ or } \{\mathrm{D} > \delta_1\} \\ K \cdot (1 + g(R)), & M \in (\mu_1, \mu_2], \text{with } \{\mathrm{D} \leq \delta_2\} \text{ or } \{\mathrm{D} > \delta_2\} \\ K \cdot (1 + h(R)), & M \in (\mu_2, \mu_3], \text{with } \{\mathrm{D} \leq \delta_3\} \text{ or } \{\mathrm{D} > \delta_3\} \\ K, & M \in (\mu_3, \mu_4] \\ \phi(K), & M \in (\mu_4, \mu_5], \text{with } \{\mathrm{D} \leq \delta_4\} \text{ or } \{\mathrm{D} > \delta_4\} \\ \gamma(K), & M \in (\mu_5, \mu_6], \text{with } \{\mathrm{D} \leq \delta_5\} \text{ or } \{\mathrm{D} > \delta_5\} \\ \eta(K), & M \in (\mu_6, \infty), \end{cases},$$

where the trigger points $\mu_1, \mu_2, \ldots, \mu_6$ and $\delta_1, \delta_2, \ldots, \delta_5 \in \mathbb{R}_+$ are the prespecified levels for magnitude and depth, respectively, and $0 < \mu_1 < \mu_2 < \cdots < \mu_6$,

$0 < \delta_1 < \delta_2 < \cdots < \delta_5$. Selection of $\mu_1, \mu_2, \ldots, \mu_6$ affects the securitization level of the bond, which an individual company should balance between profit and marketability by analysing historical earthquake loss data. Finally, coupon payment functions $f(R)$, $g(R)$, $h(R)$, $\phi(K)$, $\gamma(K)$, and $\eta(K)$ are normally designed according to company policy. Here we illustrate a possible example:

$$f(R) = \begin{cases} 2.6R \cdot \mathbb{1}_{\{D \le \delta_1\}} + 2.8R \cdot \mathbb{1}_{\{D > \delta_1\}}, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ 2.9R \cdot \mathbb{1}_{\{D \le \delta_1\}} + 3R \cdot \mathbb{1}_{\{D > \delta_1\}}, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases},$$

$$g(R) = \begin{cases} 1.6R \cdot \mathbb{1}_{\{D \le \delta_2\}} + 1.9R \cdot \mathbb{1}_{\{D > \delta_2\}}, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ 1.8R \cdot \mathbb{1}_{\{D \le \delta_2\}} + 2R \cdot \mathbb{1}_{\{D > \delta_2\}}, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases},$$

$$h(R) = \begin{cases} 0.5R \cdot \mathbb{1}_{\{D \le \delta_3\}} + 0.6R \cdot \mathbb{1}_{\{D > \delta_3\}}, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ R \cdot \mathbb{1}_{\{D \le \delta_3\}} + 1.1R \cdot \mathbb{1}_{\{D > \delta_3\}}, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases},$$

$$\phi(K) = \begin{cases} 0.8K \cdot \mathbb{1}_{\{D \le \delta_4\}} + 0.85K \cdot \mathbb{1}_{\{D > \delta_4\}}, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ 0.95K \cdot \mathbb{1}_{\{D \le \delta_4\}} + 0.98K \cdot \mathbb{1}_{\{D > \delta_4\}}, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases},$$

$$\gamma(K) = \begin{cases} 0.55K \cdot \mathbb{1}_{\{D \le \delta_5\}} + 0.6K \cdot \mathbb{1}_{\{D > \delta_5\}}, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ 0.7K \cdot \mathbb{1}_{\{D \le \delta_5\}} + 0.75K \cdot \mathbb{1}_{\{D > \delta_5\}}, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases},$$

$$\text{and} \quad \eta(\mathrm{K}) = \begin{cases} 0.2K, & \text{for } \mathrm{M} = \mathrm{M}^1 \\ 0.5K, & \text{for } \mathrm{M} = \mathrm{M}^2 \end{cases}.$$

In the one-period case, we assume that K, r, π, R, and e are constant. Therefore, cash flow is independent of financial risks, and we can apply Eq. (7) and obtain the price of the CAT bond:

$$V(d) = \frac{1}{1 + (r + e)} \cdot \frac{1}{1 + \pi} \mathbb{E}^{\mathbb{P}}[d(R; M, D)], \tag{8}$$

where $\mathbb{P}$ is the probability measure corresponding to the distribution of M^1, M^2, D. It is important to note that one of our financial market rates $(r + e)$ is a shift of the interest rate, which makes CAT bonds more attractive than normal return bonds.

Assuming that expectation exists in Eq. (8), we can approximate the CAT bond price using the same logic as Zimbidis et al. [53] according to equilibrium pricing theory:

$$V(d) = \lim_{h \to \infty} V(d)^{(h)},$$

where

$$V(d)^{(h)} = \frac{1}{1 + (r + e)} \frac{1}{1 + \pi} \frac{1}{h} \sum_{l=1}^{h} d(R^{(l)}; M^{(l)}, D^{(l)}). \tag{9}$$

Therefore, we approximate the value of $V(d)$ based on Eq. (9) by generating h events [5, 44].

3.2 Multi-period (advanced) model

Under the discrete-time framework, we now introduce the notation for multi-period models. K, e, and the coupon payment functions $f(R)$, $g(R)$, $h(R)$, $\phi(K)$, $\gamma(K)$, and $\eta(K)$ have the same form as in the one-period model.

T Maturity date for the CAT bond.

r_k Market yield at the 1-year US Treasury securities rate at time k. More precisely, r_k gives the annual compounded interest discount rate of a typical cash flow for the period $k+1$. We assume that r_k follows an ARIMA $(1, 1, 1)$ model [2] with parameters θ_1 and α_1 for any $k = 1, 2, \ldots, T$, which simulates the interest rate well [15]. For $r_k > 0$,

$$\Delta r_k = C_1 + \theta_1 \Delta r_{k-1} + \varepsilon_k + \alpha_1 \varepsilon_{k-1},$$

where $\Delta r_k = r_k - r_{k-1}$, C_1 is constant, and the error terms ε_k are assumed to be independent, identically distributed variables sampled from a normal distribution with zero mean.

π_k 1-year inflation rate at time k. In a similar setting as for the treasury rate, we assume that $\pi_k (\pi_k > 0)$ follows an ARIMA $(1, 0, 0)$ model with parameters α_2 for any $k = 1, 2, \ldots, T$. For $\pi_k > 0$,

$$\pi_k = C_2 + \tilde{\varepsilon}_k + \alpha_2 \tilde{\varepsilon}_{k-1},$$

where $\tilde{\varepsilon}_k$ are i.i.d. normal distributed random variables representing the white noise of the model.

R_k The 12-month LIBOR rate at time k. Here, we assume that the fundamental process for the instantaneous LIBOR rate $\{R_k; k \in [0, T]\}$ is the CIR process [13] given by the following stochastic differential equation:

$$dR_k = \alpha_3(\beta_3 - R_k)dk + \sqrt{R_k}\sigma_3 dW_k, \tag{10}$$

where $\theta_3 = (\alpha_3, \beta_3, \sigma_3)$ are the model parameters and W_k is standard Brownian motion. The LIBOR rate process R_k stays in a positive domain guaranteed by the diffusion function $R_k\sigma_3^2$.

M_k Annual maximum earthquake magnitude in the kth year, $M_k = \max\{M_k^1, M_k^2\}$, for $k = 1, 2, \ldots, T$, where M_k^1 and M_k^2 have the common distributions described in Sect. 3.3.1.

D_k Depth of the earthquake in the kth year, $k = 1, 2, \ldots, T$.

$d(R_k; M_k, D_k)$ Cash value received by the CAT bondholder at time $k = 1, 2, \ldots, T$, constructed in the following form:

$$d(R_k; M_k, D_k) = \begin{cases} Kf(R_k)1\!\!1_{\{0 \le M_k \le \mu_1\}} + Kg(R_k)1\!\!1_{\{\mu_1 < M_k \le \mu_2\}} + Kh(R_k)1\!\!1_{\{\mu_2 < M_k \le \mu_3\}}, \\ \text{for}\, k = 1, 2, \ldots, T-1 \\ K(1 + f(R_k))1\!\!1_{\{0 \le M_k \le \mu_1\}} + K(1 + g(R_k))1\!\!1_{\{\mu_1 < M_k \le \mu_2\}} \\ + K(1 + h(R_k))1\!\!1_{\{\mu_2 < M_k \le \mu_3\}} + K1\!\!1_{\{\mu_3 < M_k \le \mu_4\}} + \phi(K)1\!\!1_{\{\mu_4 < M_k \le \mu_5\}} \\ + \gamma(K)1\!\!1_{\{\mu_5 < M_k \le \mu_6\}} + \eta(K)1\!\!1_{\{M_k > \mu_6\}}, \quad \text{for}\, k = T. \end{cases} \tag{11}$$

Therefore,

$$\begin{aligned} \bar{d}(R_k; M_k, D_k) &= \mathbb{E}^{\mathbb{Q}}\left[d(k)|\mathcal{A}_k^{(1)}\right] \\ &= \begin{cases} \mathbb{E}^{\mathbb{Q}}\left[Kf(R_k)1\!\!1_{\{0 \le M_k \le \mu_1\}} + Kg(R_k)1\!\!1_{\{\mu_1 < M_k \le \mu_2\}} + Kh(R_k)1\!\!1_{\{\mu_2 < M_k \le \mu_3\}}\right], \\ \quad \text{for}\, k = 1, 2, \ldots, T-1 \\ \mathbb{E}^{\mathbb{Q}}[K(1 + f(R_k))1\!\!1_{\{0 \le M_k \le \mu_1\}} + K(1 + g(R_k))1\!\!1_{\{\mu_1 < M_k \le \mu_2\}} \\ \quad + K(1 + h(R_k))1\!\!1_{\{\mu_2 < M_k \le \mu_3\}} + K1\!\!1_{\{\mu_3 < M_k \le \mu_4\}} + \phi(K)1\!\!1_{\{\mu_4 < M_k \le \mu_5\}} \\ \quad + \gamma(K)1\!\!1_{\{\mu_5 < M_k \le \mu_6\}} + \eta(K)1\!\!1_{\{M_k > \mu_6\}}], \quad \text{for}\, k = T. \end{cases} \end{aligned}$$

We assume that the random variables $\{M_k; k = 1, 2, \ldots, T\}$, $\{D_k; k = 1, 2, \ldots, T\}$, $\{\varepsilon_k; k = 1, 2, \ldots, T\}$, and $\{\tilde{\varepsilon}_k; k = 1, 2, \ldots, T\}$ and stochastic processes $\{r_k; k = 1, 2, \ldots, T\}$, $\{\pi_k; k = 1, 2, \ldots, T\}$, $\{R_k; k = 1, 2, \ldots, T\}$, and $\{W_k; k = 1, 2, \ldots, T\}$ are mutually independent.

Cash flows from this multi-period CAT bond depend on both financial and catastrophic risk variables. Therefore, according to Eq. (6), the T-period price of the CAT bond is

$$V(d) = \mathbb{E}^{\mathbb{Q}_1}\left[\sum_{k=1}^{T} \frac{1}{\prod_{s=0}^{k-1}[1 + r_s + e][1 + \pi_s]} \bar{d}(R_k; M_k, D_k)\right], \tag{12}$$

which can be calculated using the same method as for Eq. (9). Assuming that expectation exists in (12), similar to the one-period model, the CAT bond price can be approximated by the strong law of large numbers:

$$V(d) = \lim_{h \to \infty} V(d)^{(h)},$$

where

$$V(d)^{(h)} = \frac{1}{h} \sum_{l=1}^{h} \sum_{k=1}^{T} \frac{1}{\prod_{s=0}^{k-1}[1 + r_s^{(l)} + e][1 + \pi_s^{(l)}]} \bar{d}(R_k^{(l)}; M_k^{(l)}, D_k^{(l)}). \tag{13}$$

Therefore, similar to the one-period model, we approximate the value of $V(d)$ based on Eq. (13) by generating h events. For future convenience, we used the magnitude and depth trigger points $\mu_1 = 5.4, \mu_2 = 5.8, \mu_3 = 6.2, \mu_4 = 6.6, \mu_5 = 7.0, \mu_6 = 7.4$, and $\delta_1 = 20, \delta_2 = 15.\delta_3 = 10, \delta_4 = 10, \delta_5 = 10$. A catastrophe might or might not occur before the maturity date T. According to the cash flow stream given in Eq. (11), a CAT bond with face amount \$$K$ will pay coupons $f(R)$, $g(R)$, and $h(R)$ to bondholders at the end of each period if an earthquake of maximum magnitude in the intervals $(0, 5.4]$, $(5.4, 5.8]$, and $(5.8, 6.2]$, respectively, occurred in this period, or no coupon payment if the magnitude is greater than 6.2. On the maturity date, the CAT bond is scheduled to repay the full principal payment plus a coupon $\phi(K)$, $\gamma(K)$, or $\eta(K)$, if the magnitude is in the interval $(0, 6.6]$, $(6.6, 7.0]$, $(7.0, 7.4]$, or $(7.4, \infty)$, respectively.

3.3 California earthquake data for catastrophic risk variables

We use earthquake data from California as an example and estimate distributions of earthquake magnitude and depth for the future time period. Figure 1 shows recent significant earthquakes in California, with a darker colour representing more severe earthquakes. The two circles denote locations where the most significant earthquakes occurred. We analyse the earthquakes that hit these circled areas, San Francisco (region 1) and Los Angeles (region 2), over the period 1968–2011. Table 1 (data from Southern California Earthquake Data Center, http://www.data.scec.org/) lists the annual maximum-magnitude (M) earthquakes in each region, and the latitude (La), longitude (Lo), and depth (D). These two regions include the biggest cities in California and claim the majority of the economic losses. We present elements of the data set according to Coles [10] and analyse the data using standard numerical algorithms (e.g. the Newton–Raphson method).

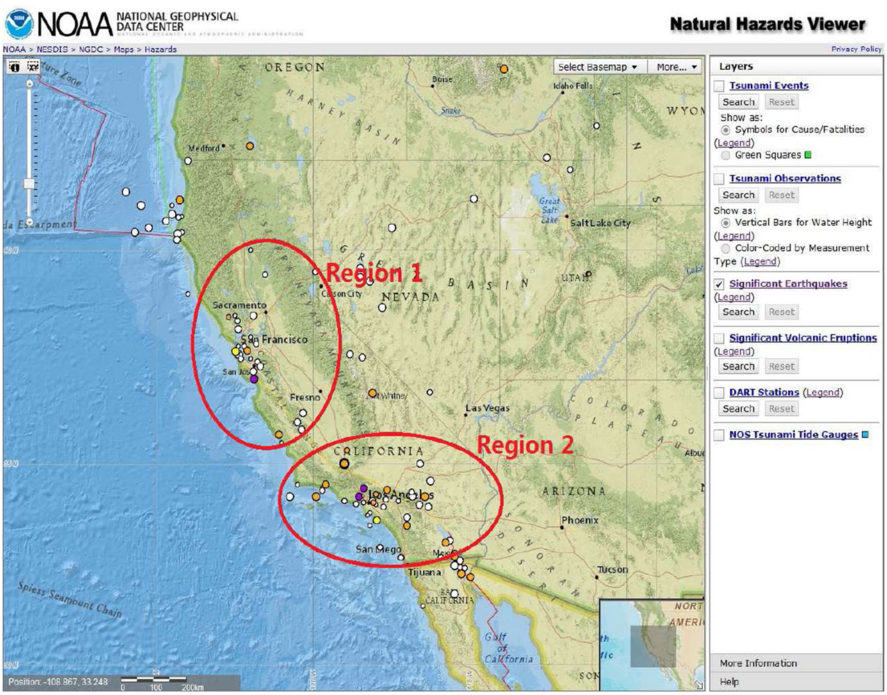

Fig. 1 Recent significant earthquakes in California. *Source* NOAA National Geophysical Data Center

Table 1 Annual maximum-magnitude earthquakes in two regions in California

Years	Region 1				Region 2				Years	Region 1				Region 2			
	La	Lo	D	M	La	Lo	D	M		La	Lo	D	M	La	Lo	D	M
1968	37.0	−121.7	12	4.3	33.2	−116.1	10	6.6	1990	36.9	−121.7	6	5.4	34.1	−117.7	4	5.5
1969	38.5	−122.8	5	5.7	33.3	−116.4	6	5.5	1991	35.8	−121.3	10	5.2	34.3	−118.0	9	5.8
1970	36.8	−121.4	10	4.7	34.3	−117.5	11	5.2	1992	38.9	−122.8	4	4.5	34.2	−116.4	1	7.3
1971	36.6	−121.2	4	4.7	34.4	−118.4	9	5.8	1993	37.0	−121.5	8	4.8	35.1	−119.1	21	5.2
1972	36.6	−121.2	4	5.1	34.8	−118.9	4	4.3	1994	38.8	−122.4	12	4.2	34.2	−118.5	18	6.7
1973	39.3	−123.4	18	4.8	34.1	−119.0	8	5.3	1995	36.6	−121.2	7	4.9	34.4	−118.7	13	5.0
1974	36.9	−121.5	5	5.2	34.4	−118.5	24	4.7	1996	36.9	−121.6	5	4.9	34.0	−116.3	6	4.5
1975	39.4	−121.5	5	5.7	34.5	−116.5	0	5.3	1997	39.9	−120.8	5	4.3	35.0	−116.8	2	5.3
1976	36.5	−120.5	12	4.6	33.1	−115.6	5	5.1	1998	36.8	−121.5	9	5.1	34.3	−116.8	6	4.8
1977	39.4	−123.3	15	4.8	34.4	−118.5	11	4.4	1999	37.9	−122.7	8	4.6	34.6	−116.3	0	7.1
1978	39.2	−123.2	12	4.5	34.3	−119.7	18	5.1	2000	38.4	−122.4	10	4.9	32.9	−115.5	10	4.5
1979	37.1	−121.5	9	5.8	32.7	−115.4	15	5.8	2001	39.8	−120.6	18	5.3	33.5	−116.5	15	5.0
1980	37.8	−121.8	15	5.8	33.5	−116.5	19	5.3	2002	37.0	−121.6	7	4.9	34.8	−116.3	5	4.8
1981	37.4	−121.7	9	4.8	33.1	−115.6	19	5.8	2003	35.7	−121.1	9	6.5	34.3	−116.8	1	5.0
1982	36.6	−121.3	11	4.6	33.5	−116.7	13	4.8	2004	36.0	−120.5	11	4.9	35.4	−118.6	4	5.0
1983	35.8	−121.4	10	5.2	32.5	−117.4	10	4.7	2005	38.8	−122.8	0	4.3	33.5	−116.6	14	5.2
1984	37.3	−121.7	9	6.2	34.0	−119.0	13	4.6	2006	38.8	−122.8	5	4.7	32.7	−116.0	14	4.6
1985	39.5	−120.6	7	4.6	35.5	−118.9	6	4.6	2007	37.4	−121.8	10	5.5	33.7	−117.5	13	4.7
1986	37.5	−121.7	9	5.7	34.0	−116.6	10	5.7	2008	40.0	−120.9	0	4.5	34.0	−117.8	15	5.4
1987	36.2	−120.7	28	4.8	33.0	−115.9	11	6.6	2009	35.6	−121.0	6	4.4	33.3	−115.7	6	4.8
1988	37.4	−121.7	10	5.3	34.9	−118.7	7	5.4	2010	37.5	−121.8	11	4.1	32.7	−115.9	5	5.7
1989	37.0	−121.9	17	7.0	33.9	−118.6	12	5.0	2011	36.6	−121.2	8	4.6	32.7	−115.7	15	4.4

3.3.1 Magnitude

The traditional approach for defining extremes is to focus on the statistical behavior of

$$M_k^q = \max\{X_{1k}^q, X_{2k}^q, \ldots, X_{ok}^q\},$$

where $q = 1, 2$ and $X_{1k}^q, X_{2k}^q, \ldots, X_{ok}^q$ is a sequence of $o = 365$ independent random variables with a universal distribution function F that measures the daily maximum-magnitude earthquake in each region for the period $[k, k+1)$. $X_{ok}^q = 0$ if no earthquake occurs in region q on that day. Thus, the sequence $M_k = \max\{M_k^1, M_k^2\}$ corresponds to the kth annual maximum-magnitude earthquake over the observation period. The distribution of M_k^q can be derived for each year k using the generalized extreme value (GEV) distribution. The rescaled sample maxima $(M_k^q)^* = (M_k^q - b_k)/a_k$ is a heavy-tailed distribution and the possible distribution is provided by the well-known Fisher–Tippet–Gnedenko theorem [10, 18, 19, 27].

Theorem 4 (Fisher–Tippet–Gnedenko) *If there exist sequences of constants* $\{a_k : k > 0\}$ *and* $\{b_k : k > 0\}$ *such that*

$$\mathbb{P}\left\{\frac{M_k^q - b_k}{a_k} \le z\right\} \to G(z) \quad \text{as } k \to \infty$$

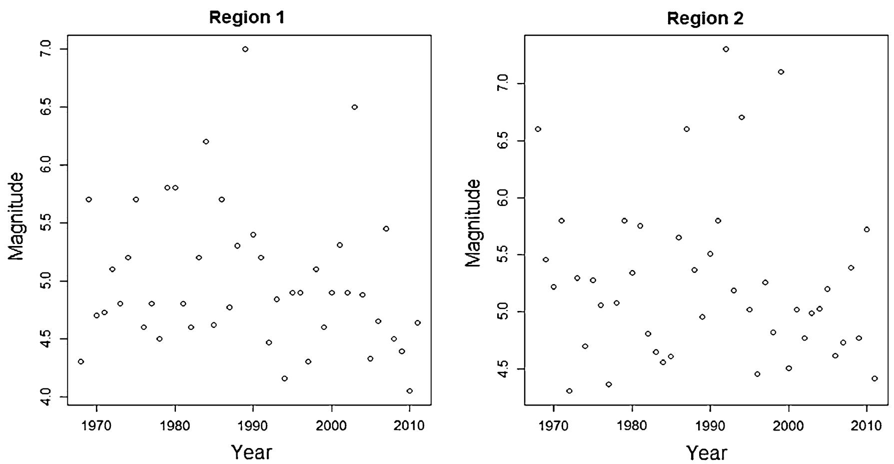

Fig. 2 Scatter plot of the annual maximum-magnitude earthquakes M_k^1 in region 1 and M_k^2 in region 2 in California

for a non-degenerate distribution function G, then G is a member of the GEV family

$$G(z) = \exp\left\{ -\left[1 + \xi\left(\frac{z - \beta_4}{\sigma_4}\right)\right]^{-1/\xi} \right\} \tag{14}$$

defined on $\{z : 1 + \xi(z - \beta_4)/\sigma_4 > 0\}$, *where* $-\infty < \beta_4 < \infty$, $\sigma_4 > 0$, $-\infty < \xi < \infty$, *and* $\beta_4 = \mathbb{E}(M_k^q)$, $\beta_4 = \sqrt{\mathrm{Var}(\mathrm{M_k^q})}$.

The model has three parameters: location parameter β_4, scale parameter σ_4, and shape parameter ξ. When $\xi = 0$ is the limit of Eq. (14) as $\xi \to 0$, the model corresponds to the Gumbel family. For the cases $\xi > 0$ and $\xi < 0$, Eq. (14) leads to Frechét and Weibull family distributions, respectively. Then the GEV parameters can be estimated by maximizing the log-likelihood function, as carried out by Zimbidis et al. [53].

According to time series plots of the maxima for both regions (Fig. 2), it is reasonable to assume that the patterns of variation have stayed constant over the observed period, which suggests that the data are independent observations from the GEV distribution [10, 53].

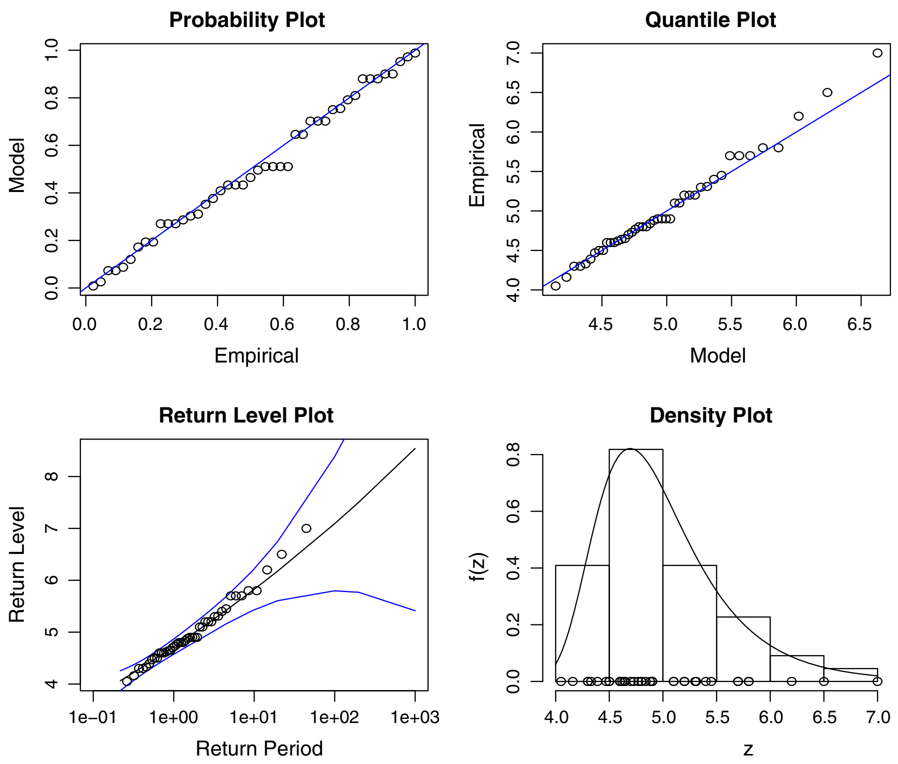

Fig. 3 Diagnostic plots for GEV fitting to the annual maximum-magnitude earthquakes in region 1 in California

We take region 1 as an example for analysis. We maximize the GEV log-likelihood for these data and achieve the estimate

$$\left(\hat{\beta_4}, \hat{\sigma_4}, \hat{\xi}\right) = (4.71946946, 0.44861472, 0.05866229),$$

for which the log-likelihood is 36.01543. The approximate variance–covariance matrix of the parameter estimates is

$$V = \begin{bmatrix} 0.005854675 & 0.001935385 & -0.003127097 \\ 0.001935385 & 0.003228341 & -0.001542433 \\ -0.003127097 & -0.001542433 & 0.013764031 \end{bmatrix}.$$

Therefore, we can easily obtain standard errors 0.07651585, 0.05681849, and 0.11732021 for β_4, σ_4, and ξ, respectively, with approximate 95 % confidence intervals of $\beta_4 \in [4.64, 4.80]$, $\sigma_4 \in [0.39, 0.51]$, and $\xi \in [-0.06, 0.18]$.

To assess the accuracy of the GEV model fitted to the California earthquake data, we show various diagnostic plots of M_k^1 in Fig. 3. The probability and quantile plots are close to linear, which confirms the validity of the fitted model. The estimate of ξ is close to zero, and the estimated curve in the return level plot is nearly linear. According to the histogram density plot of the data, the density estimate is consistent. Consequently, the analysis provides strong evidence that the GEV model provides a good fit.

Furthermore, the tail behaviour [3, 18] of the distribution displayed in Fig. 4 reflects the sample mean excess, and the downward trend suggests a very short tail behaviour for the annual maximum-magnitude earthquakes in region 1 in California.

Similar analysis can be conducted for region 2, and the exceeding probabilities intervals M_k^1 and M_k^2 for the GEV distributions are listed in Table 2. The possibility of an earthquake of magnitude greater than 6.6 occurring in the target regions is less than 8 %, so we can introduce a bond with 92 % capital guarantee.

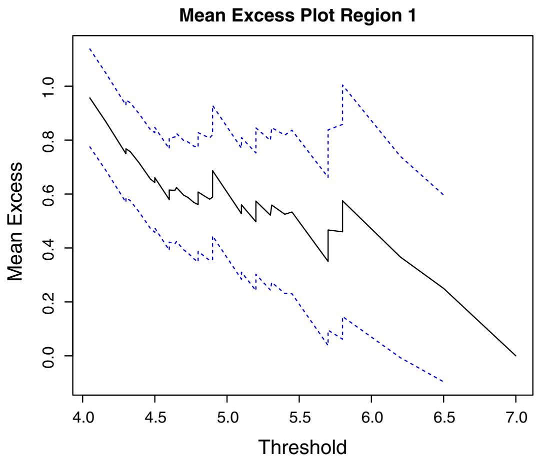

Fig. 4 Sample mean excess for annual maximum-magnitude earthquakes in region 1 in California with 95 % confidence interval

Table 2 Exceeding probabilities for the model in regions 1 and 2 California

	Region 1	Region 2
$\mathbb{P}(5.0 < M \leq 5.4)$	0.205599827	0.250286661
$\mathbb{P}(5.4 < M \leq 5.8)$	0.105005438	0.150399888
$\mathbb{P}(5.8 < M \leq 6.2)$	0.049947001	0.080828401
$\mathbb{P}(6.2 < M \leq 6.6)$	0.023619037	0.042623696
$\mathbb{P}(6.6 < M \leq 7.0)$	0.011371076	0.022816466
$\mathbb{P}(7.0 < M \leq 7.4)$	0.005618113	0.012543028
$\mathbb{P}(M > 7.4)$	0.006178647	0.01813135

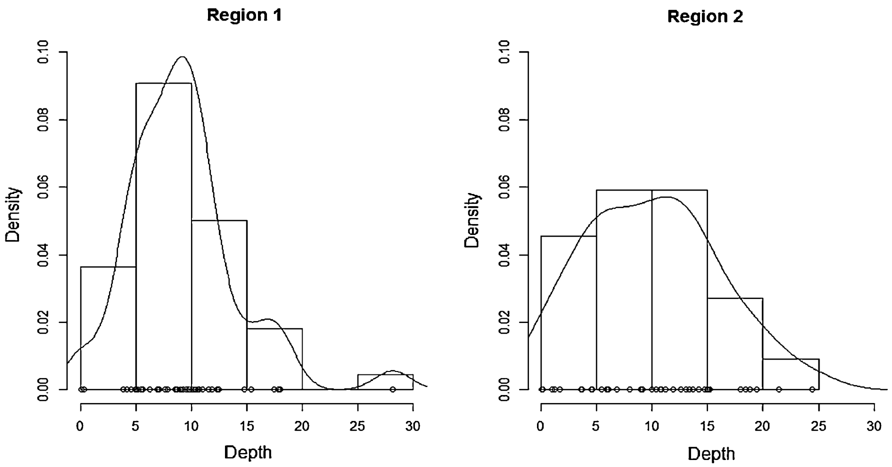

Fig. 5 Density depth plot for the annual maximum-magnitude earthquakes in regions 1 and 2

3.3.2 Depth

The next stage is to analyse the earthquake depth distribution. According to the density plot in Fig. 5, earthquake depth follows a right-skewed heavy-tailed distribution and we fit it as a gamma distribution

$$f(x; \alpha_5, \beta_5) = \beta_5^{\alpha_5} \frac{1}{\Gamma(\alpha_5)} x^{\alpha_5 - 1} e^{-\beta_5 x}$$

for which the estimated parameters are $(\hat{\alpha}_5, \hat{\beta}_5) = (2.35378504, 0.25460951)$ and $(\hat{\alpha}_5, \hat{\beta}_5) = (1.44878306, 0.14585340)$ for regions 1 and 2, respectively. This model is realistic since earthquakes that occur near the surface tend to be of higher magnitude compared with deeper earthquakes [24].

4 Numerical example

4.1 Numerical example for the one-period model

We consider the one-period model with face value $\$K = 1{,}000$, interest rate $r = 0.12\,\%$, and inflation rate $\pi = 3.16\,\%$ (Board of Governors of the Federal Reserve System[5]). Given risk premium $e = 3\,\%$ and LIBOR rate[6] $R = 1.13\,\%$, Eq. (9) yields the price of a one-period CAT bond as $V = \$940$.

4.2 Pricing for the multi-period model

We consider a 5-year period CAT bond with payments depending on earthquake magnitude in selected areas. Because the probability of large-magnitude earthquakes is low, large number of events need to be considered to estimate the price of CAT bonds with a relatively small error [44]. We build the simulation using the following five steps.

Step 1

First, the maximum-magnitude earthquakes in each region can generate 100,000 sequence values via GEV distributions representing the 5-year period up to the maturity date. Moreover, we can generate 100,000 depth sequences for both regions using a gamma distribution. Then we select depth in the larger-magnitude branch for future simulation.

Step 2

Second, we obtain 100,000 paths for the LIBOR rate R_k for $k \in [0, 5]$ using Monte Carlo simulations. Following Romaniuk [44], we use an iterative stochastic equation with the concept of local characterizations for the Levy process.

In our simulation, we let $[0, T]$ be the lifetime interval for the CAT bond and discretize this into v different steps. The time moments are $\tau = \{\tau_0 = 0, \tau_1, \ldots, \tau_\delta = T\}$, where δ is the number of steps. The steps are constant at 1 day (250 business days a year), with $\Delta\tau = \tau_{\upsilon+1} - \tau_\upsilon$, where $\upsilon = 1, 2, \ldots, v-1$. The discrete version of Eq. (10) given by Kladivko [30] takes the form

$$R_{\tau+\Delta\tau} - R_\tau = \alpha_3(\beta_3 - R_\tau)\Delta\tau + \sigma_3\sqrt{R_\tau}\sqrt{\Delta\tau}\bar{\varepsilon}_\tau, \tag{15}$$

where $\bar{\varepsilon}_\tau$ follow $N(0, \Delta\tau)$ as a white noise process for $\tau = 1, 2, \ldots$.

The MATLAB implementation of the estimation processes provided by Kladivko [30] suggests use of the ordinary least square of Eq. (15) to find the starting point for the parameters. Then the log-likelihood function of the CIR process is maximized. Then statistical analysis of 12-month LIBOR[7] historical data for 2000–2011 yields the parameter estimates $\hat{\theta} = (\hat{\alpha_3}, \hat{\beta_3}, \hat{\sigma_3}) = (0.212421, 1.084655, 0.420791)$. For the initial value in Eq. (15) we set $R_0 = 1.13\,\%$, which was the actual LIBOR rate in December 2011.

[5] http://www.federalreserve.gov/.

[6] On 30/12/2011; http://www.bba.org.uk/.

[7] http://www.bba.org.uk/.

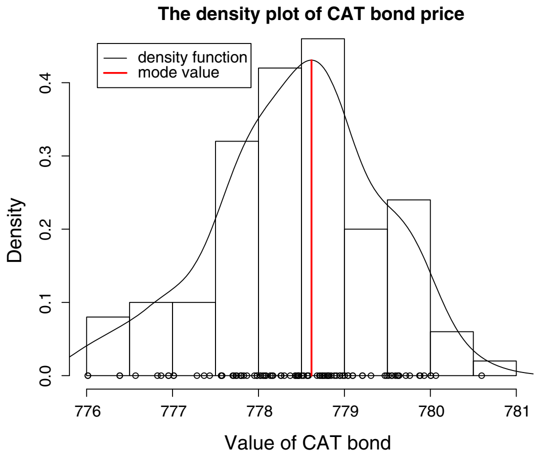

Fig. 6 Density plot and cumulative density plot of the CAT bond price after running the algorithm 100 times with $h = 100{,}000$

Step 3

The next step is the generation of sequences for the annual interest and inflation rates (data from Board of Governors of Federal Reserve System[8] for the period 1968–2011). Recall from Sect. 3.1 that r_k follows an ARIMA $(1, 1, 1)$ model with parameters $(\hat{C_1}, \hat{\theta_1}, \hat{\alpha_1}) = (-0.0976, -0.2833, 1)$, and π_k follows an ARIMA $(1, 0, 0)$ model with parameters $(\hat{C_2}, \hat{\alpha_2}) = (0.7867, 0.7867)$, for any $k = 1, 2, 3, \ldots$, $r_k \geq 0$, and $\pi_k \geq 0$, according to the maximum log-likelihood estimate of the 1-year US Treasury securities rate and inflation rate for 1968–2011.

Step 4

The next step is to calculate the coupon payments [cash flows $d(k)$] of the CAT bond for the 5-year period. It should be mentioned that this procedure is quite complex and involves logical functions and many subroutines. According to the cash flow stream in Eq. (11), the capital of our CAT bond may decrease if and only if an earthquake of magnitude greater than 6.6 hits California before the maturity date. Moreover, we assume a face amount of $\$K = 1{,}000$ and a risk premium of $e = 3\,\%$.

Step 5

The final step is to calculate the present value of cash flows for every year, and then average over all the discounted values based on r_k, π_k for each period. According to Eq. (13), the price of the $T = 5$ CAT bond is approximately \$779.73.

Now we test the validity of the results. In the above process, we use the equilibrium pricing theory given in Eq. (13) for $h = 100{,}000$ and run the algorithm 100 times to generate 100 possible value of the CAT bond, for which the variance equals 0.91. It can easily be derived that the price variance dramatically decreases as h increases, and is asymptotically equal to zero after 10,000. Figure 6 is a density plot of price values in which the density reaches the mode at $V = \$778.62$ at a

8 http://www.federalreserve.gov/.

density of 0.43. This is quite a promising result since the low volatility level suggests that our pricing model is both consistent and computationally efficient. Compared to a zero-coupon bond with price $V = \$935$, which depends only on financial risks, our CAT bond with a 92 % capital guarantee is very attractive to investors.

5 Concluding remarks

We built a valuation framework for earthquake CAT bonds with n financial and m catastrophic independent risks. These securitization products can play a vital role in the financial sustainability of insurers and re-insurers, as well as for governmental authorities. The high return of the CAT bond identified here can generate sufficient funds to pay claims and post-disaster reconstruction costs if a significant catastrophic event occurs in an area. Furthermore, the assumptions made in this paper are quite standard and realistic, so the valuation model is easy to modify further and apply in industry. To simplify the model, all the risks are mutually independent. It is quite natural that earthquakes occur only in certain regions, and such event generally do not affect exchange and production levels and the economic environment on a global scale.

We also demonstrated how to construct a practical pricing model for earthquakes in California from 1968 to 2011. We used extreme value theory for the maximum-magnitude earthquakes in each year and concludes that they follow a Frechét distribution in our case. In addition, earthquake depth fitted a gamma distribution. For financial risks, we chose the classical ARIMA model for interest and inflation rates, and a CIR model for the stochastic process of coupon payment as a predetermined function of the annual LIBOR rate. Consequently, we were able to identify an equilibrium price for an earthquake CAT bond that depends on the risk variables above. This model, as an extension of the Cox and Pedersen approach [12], provides a more accurate approximation of price by considering multiple variables cross financial and catastrophic risks. However, because of the catastrophic risks, CAT bonds cannot be perfectly hedged in an incomplete market and the high yields received may not be sufficient to balance investor risk.

The dependence between the CAT market and the financial market cannot be used within our methodology and framework for bond pricing. Consequently, it should be characterized as a separate problem. In general, to the best of our knowledge, the problem of dependence within CAT risks or/and the financial market is still open and is very challenging with respect to bond pricing. This issue will be considered in future research.

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