

The quantitative isoperimetric inequality and related topics

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Abstract We present some recent stability results concerning the isoperimetric inequality and other related geometric and functional inequalities. The main techniques and approaches to this field are discussed.

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1 Introduction

The isoperimetric inequality is probably one of the most beautiful and at the same time one of the most powerful tools of modern mathematics. Despite the fact that the isoperimetric property of the circle among all planar figures was already well known in ancient times, the first rigorous proofs of this fact have been only obtained at the beginning of last century. In particular Hurwitz proved in 1901 that given a simple, closed curve $\gamma \subset \mathbb{R}^2$ of length L enclosing an area A , then

$$L^2 \geq 4\pi A, \tag{1.1}$$

with equality holding if and only if γ is a circle. His proof, see [86], used the Gauss–Green formula and Wirtinger inequality that is the one-dimensional Sobolev–Poincaré inequality with the sharp constant. This is not surprising since in higher dimension the connections of the isoperimetric inequality with the divergence theorem and with the Sobolev inequality are even more evident.

Passing to higher dimensions was not an easy task for the mathematicians of last century, also because it was clear that proving the isoperimetric inequality in full generality would require a deeper understanding of the concept of surface measure. In the end, three main directions were followed to this purpose. The first one, and maybe the simplest, was based on the *Brunn–Minkowski* inequality, see for instance [29, Th. 8.1.1], stating that if $H, K \subset \mathbb{R}^n$ are compact sets then

$$|H + K|^{\frac{1}{n}} \geq |H|^{\frac{1}{n}} + |K|^{\frac{1}{n}}. \tag{1.2}$$

From this inequality one gets easily that if $K \subset \mathbb{R}^n$ is compact then

$$\sigma(\partial K) \geq \sigma(\partial B_r), \tag{1.3}$$

where B_r is a ball of the same volume as K and $\sigma(\partial K)$ denotes the *outer Minkowski content* of the boundary of K , defined as

$$\sigma(\partial K) := \liminf_{\varepsilon \downarrow 0} \frac{|K + \varepsilon \bar{B}| - |K|}{\varepsilon},$$

where \bar{B} is the closed unit ball. Note that if K is the closure of a bounded smooth open set, then $\sigma(\partial K)$ coincides with the usual measure of the surface ∂K . Therefore (1.3) can be viewed as a generalized version of the classical isoperimetric inequality.

Another approach to the isoperimetric inequality comes from Calculus of Variations. Provided that an isoperimetric set exists and is a smooth bounded open set, by taking small variations of this set one immediately sees that its boundary must have

constant mean curvature. This observation led people to prove that the ball is the only smooth open set in \mathbb{R}^n whose boundary has constant mean curvature, a result that was eventually obtained by Aleksandrov in 1958, see [2].

A third approach was followed by De Giorgi in his celebrated paper [53] of 1958. He used an observation going back to Steiner: the surface area of a set E , or more generally its perimeter $P(E)$, decreases under Steiner symmetrization, see Theorem 2.7. The isoperimetric property of balls then follows, since they are the only sets that are invariant under Steiner symmetrization in every direction. The formulation given by De Giorgi in the framework of the theory of sets of finite perimeter is the most general one. It states that if $E \subset \mathbb{R}^n$ is a Lebesgue measurable set with the same volume as a ball B_r then

$$P(E) \geq P(B_r), \quad (1.4)$$

with the equality holding if and only if E is a ball.

Other proofs of the isoperimetric inequality have been proposed more recently. Among them we shall discuss the one given by Gromov in [100], see Sect. 5.1, which uses ideas related to mass transportation and ultimately reduces the isoperimetric inequality to the divergence theorem and the arithmetic-geometric mean inequality. Finally, an elegant and short proof has been given by Cabré in [30] using the properties of solutions of the Neumann problem for the Laplacian. We cannot pursue here the history of all proofs and formulations of the isoperimetric inequality, but we refer the interested reader to the survey paper of Ossermann [104] and to the book of Chavel [37].

Let us go back to the isoperimetric inequality in the plane. In the 1920's Bonnesen obtained a series of improved versions of (1.1), later on named by Osserman [105] *Bonnesen type inequalities*, of the form

$$L^2 - 4\pi A \geq d,$$

where the quantity d on the right hand side is a nonnegative geometrically significant expression vanishing if and only if γ is a circle. In particular in 1924 Bonnesen [21] proved the following sharp and elegant estimate.

Theorem 1.1 (Bonnesen) *Given a closed, simple curve $\gamma \subset \mathbb{R}^2$ enclosing a convex set C of area A , there exist two concentric circles $C_1 \subset C \subset C_2$ of radii r_1 and r_2 , respectively, such that*

$$(r_2 - r_1)^2 \leq \frac{L^2 - 4\pi A}{4\pi}.$$

This inequality has the feature of being sharp, since the constant 4π at the denominator cannot be increased, and of having an elementary proof. It provides a nice example of *quantitative isoperimetric inequality*, i.e., an inequality where a suitable distance of a set E from a ball of the same volume is controlled in terms of the difference of the perimeter of E and the perimeter of the ball.

It was only in 1989 that Fuglede extended this result to convex sets in higher dimension. Indeed, he proved in [72] that given a convex body K with the same

volume as the unit ball B one can always measure the Hausdorff distance of a translate of K from B by a suitable power of the isoperimetric deficit of K , i.e., the difference $P(K) - P(B)$, see Theorem 3.2.

His result was the starting point of modern investigations on the stability of isoperimetric inequality. In fact only a few years later Fuglede's result was extended to general sets of finite perimeter in \mathbb{R}^n in two papers by Hall et al. [84] and by Hall [82]. Precisely, in the last paper it was proved that if $E \subset \mathbb{R}^n$ is a measurable set with the same volume as the unit ball B , then

$$\min_{x \in \mathbb{R}^n} \{|(x + E) \Delta B|\}^4 \leq C(n)[P(E) - P(B)], \quad (1.5)$$

where $C(n)$ is a positive constant depending only on the dimension. Note that in dealing with general sets one has to replace the Hausdorff distance with the weaker one given by the measure of the symmetric difference between a translated of the set E and B . This is unavoidable if one thinks that by adding to B some long thin spikes the volume and the perimeter of the resulting set change very little while its Hausdorff distance from B goes to infinity. Note also that in (1.5) the power 4 on the left hand side is independent of the dimension, though Hall suggested in [82] that it should be replaced with the optimal power 2, see the discussion at the beginning of Sect. 4.

Inequality (1.5) with the power 2 was proved in [76] by Maggi, Pratelli and the author using symmetrization arguments. This result was not the end of the story. On the contrary it initiated a series of papers where other proofs of the same inequality were given or similar stability results were obtained for other inequalities such as the Sobolev, the Faber–Krahn, the Brunn–Minkowski and the isodiametric inequalities, see Sects. 5 and 6.

The aim of this paper is to present the results, the techniques and the main ideas developed in this context. Three different approaches are now available if one wants to study these stability issues: a first one using symmetrization, another one via mass transportation and a third one based on the regularity theory for minimal surfaces and other PDEs.

The paper is organized as follows. After quickly recalling some basic tools from De Giorgi's theory of sets of finite perimeter and from Geometric Measure Theory, in Sect. 3 we present Fuglede's proof of the quantitative isoperimetric inequality for convex sets and for nearly spherical sets, that are sets close to a ball in C^1 sense. In rewriting the original proofs of [72] we have tried to simplify some unnecessarily complicated parts and to highlight some ideas contained therein that have been successfully used later by various authors in different contexts.

In Sect. 4 we present the proof of the quantitative isoperimetric inequality via symmetrization given in [76], with some simplifications introduced by Maggi in [94]. Although a shorter proof has been given later by Cicalese and Leonardi [45] using the regularity theory for area minimizing sets of finite perimeter, the proof by symmetrization uses no deep results from geometric and functional analysis, is geometrically intuitive and can be adapted to other situations where a complete regularity theory is not available.

The stability proofs via mass transportation and via regularity are presented in Sect. 5. Not all the details of the proofs are given, but we have tried to explain the main ideas and the novel techniques. In this section we discuss also several generalizations of the quantitative isoperimetric inequality to higher codimension, to non local perimeters and to non Euclidean settings such as the Gauss space.

Finally, Sect. 6 gives an account on how the techniques developed for the isoperimetric inequality have been successfully applied to study the stability of other related inequalities. The quantitative versions of the Sobolev inequality and of the Faber–Krahn inequality for the first eigenvalue of the Laplacian are discussed with some details. Reference to other inequalities are given in the last section. But at that point we had to stop since for some of them new developments are foreseen in the next years.

2 Notation and preliminary results

In the following we denote by $B_r(x)$ the ball with radius $r > 0$ and center x and we use the following simplified notation

$$B_r := B_r(0), \quad B(x) := B_1(x), \quad B := B(0).$$

The measure of the unit ball B will be denoted by ω_n .

Though most of the results that we are going to present could be given first for smooth sets and then extended to sets of finite perimeters via the approximation Theorem 2.4 below, we have preferred to state them in full generality. However the reader who is not familiar with the theory of sets of finite perimeter may still read the statements and the proofs without problems by simply pretending that all sets involved are smooth. For the basic properties of sets of finite perimeter and related results we refer to [8, 59]. We start by recalling the definition. Let Ω be an open set in \mathbb{R}^n and $E \subset \mathbb{R}^n$ a measurable set. The *perimeter of E in Ω* is defined as

$$P(E; \Omega) := \sup \left\{ \int_E \operatorname{div} \varphi \, dx : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \quad \|\varphi\|_\infty \leq 1 \right\}. \tag{2.1}$$

If E is smooth we can use the classical divergence theorem to find that

$$\int_E \operatorname{div} \varphi \, dx = \int_{\partial E \cap \Omega} \varphi \cdot \nu \, d\mathcal{H}^{n-1},$$

thus, taking the supremum over all φ as in the definition, we get

$$P(E; \Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega).$$

Note that if $P(E; \Omega) < +\infty$ the map

$$\varphi \in C_c^1(\Omega, \mathbb{R}^n) \mapsto \int_E \operatorname{div} \varphi \, dx$$

is linear and continuous, so by means of Riesz’s theorem, we can find a vector valued Radon measure $\mu = (\mu^1, \dots, \mu^n)$ such that

$$\int_E \operatorname{div} \varphi \, dx = \int_{\Omega} \varphi \cdot d\mu = \sum_{i=1}^n \int_{\Omega} \varphi_i d\mu_i.$$

Thus $\mu = -D\chi_E$, where $D\chi_E$ is the distributional derivative of χ_E and the above formula can be rewritten as

$$\int_E \operatorname{div} \varphi \, dx = \int_{\Omega} \chi_E \operatorname{div} \varphi \, dx = - \int_{\Omega} \varphi \cdot dD\chi_E. \tag{2.2}$$

So, E has finite perimeter in Ω if and only if $D\chi_E$ is a Radon measure with values in \mathbb{R}^n and finite total variation. In fact, from the definition we immediately get that

$$P(E; \Omega) = |D\chi_E|(\Omega).$$

Recalling Besicovitch differentiation theorem, we also have that for $|D\chi_E|$ -a.e. $x \in \operatorname{supp}|D\chi_E|$, there exists

$$v^E(x) := - \lim_{r \rightarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \text{ and } |v^E(x)| = 1. \tag{2.3}$$

We shall denote by ∂^*E the set of all points in $\operatorname{supp}|D\chi_E|$ where (2.3) holds. The set ∂^*E is called the *reduced boundary* of E , while the vector $v^E(x)$ is the *generalized exterior normal at x* . From (2.3) we have that the measure $D\chi_E$ is obtained by integrating $-v^E$ with respect to $|D\chi_E|$, i.e.,

$$D\chi_E = -v^E |D\chi_E|.$$

Thus (2.2) can be rewritten as

$$\int_{E \cap \Omega} \operatorname{div} \varphi \, dx = \int_{\partial^*E \cap \Omega} \varphi \cdot v^E d|D\chi_E|, \quad \forall \varphi \in C_c^1(\Omega, \mathbb{R}^n). \tag{2.4}$$

Since $\partial^*E \subset \operatorname{supp}|D\chi_E| \subset \partial E$, the reduced boundary of E is a subset of the topological boundary. The next result (see e.g. [59, Sect. 5.7] or [8, Th. 3.59]) describes the structure of sets of finite perimeter. Here and in the sequel by $\mathcal{H}^k, k = 0, 1, \dots, n$, we denote the k -dimensional *Hausdorff measure* in \mathbb{R}^n . For the definition and the basic properties of *rectifiable* sets see [8, Ch. 2].

In the following the perimeter of E in \mathbb{R}^n will be denoted by $P(E)$. If $P(E) < \infty$ we say that E is a *set of finite perimeter*.

Theorem 2.1 (De Giorgi) *Let $E \subset \mathbb{R}^n$ be a measurable set of finite perimeter, then the following hold:*

- (i) $\partial^* E$ is countably $(n-1)$ -rectifiable, i.e., $\partial^* E = \bigcup_i K_i \cup N_0$, where $\mathcal{H}^{n-1}(N_0) = 0$ and K_i are compact subsets of C^1 manifolds M_i of dimension $n-1$;
- (ii) $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$, in particular $P(E; \Omega) = \mathcal{H}^{n-1}(\partial E^* \cap \Omega)$ for any open set $\Omega \subset \mathbb{R}^n$;
- (iii) for \mathcal{H}^{n-1} -a.e. $x \in K_i$, the generalized exterior normal $\nu^E(x)$ is orthogonal to the tangent hyperplane to the manifold M_i at x ;
- (iv) for all $x \in \partial^* E$, $\frac{|E \cap B_r(x)|}{|B_r(x)|} \rightarrow \frac{1}{2}$ as $r \rightarrow 0$;
- (v) for all $x \in \partial^* E$, $\lim_{r \rightarrow 0} \frac{\mathcal{H}^{n-1}(\partial^* E \cap B_r(x))}{\omega_{n-1} r^{n-1}} = 1$.

From (ii) above we have that (2.4) can be rewritten as

$$\int_E \operatorname{div} \varphi \, dx = \int_{\partial^* E \cap \Omega} \varphi \cdot \nu^E \, d\mathcal{H}^{n-1} \quad \text{for all } \varphi \in C_c^1(\Omega, \mathbb{R}^n).$$

Another characterization of the reduced boundary is contained in the following result. To this aim we recall that if $E \subset \mathbb{R}^n$ is a measurable set E has density $t \in [0, 1]$ at a point $x \in \mathbb{R}^n$ if

$$D(E; x) := \lim_{r \rightarrow 0^+} \frac{|E \cap B_r(x)|}{|B_r|} = t.$$

We shall denote by $E^{(t)} := \{x \in \mathbb{R}^n : D(E; x) = t\}$. Then the *measure theoretic boundary* of E is defined as $\partial^M E := \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)})$. For the proof of the next theorem see [8, Th. 3.61].

Theorem 2.2 (Federer) *Let E be a set of finite perimeter in \mathbb{R}^n . Then*

$$\partial^* E \subset E^{(1/2)} \subset \partial^M E \quad \text{and} \quad \mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(0)} \cup \partial^* E \cup E^{(1)})) = 0.$$

Let E_h be a sequence of measurable subsets of \mathbb{R}^n . We say that the sets E_h converge in measure to a set $E \subset \mathbb{R}^n$ in an open set $\Omega \subset \mathbb{R}^n$ if $\chi_{E_h} \rightarrow \chi_E$ in $L^1(\Omega)$. The *local convergence in measure* is defined accordingly. Note that from the definition (2.1) of perimeter it follows immediately that the perimeter is lower semicontinuous with respect to the local convergence in measure, i.e. if the sets E_h converge locally in measure in Ω to a set E , then $P(E; \Omega) \leq \liminf_h P(E_h; \Omega)$. Note also, as a consequence of Rellich–Kondrachov theorem, that sets of finite perimeter are compact with respect to the convergence in measure, see [8, Th. 3.39].

Theorem 2.3 *Let E_h be a sequence of measurable subsets of \mathbb{R}^n such that $\sup_h P(E_h; \Omega) < \infty$. Then, there exists a subsequence E_{h_k} locally converging in measure in Ω to a set E of finite perimeter in Ω . Moreover,*

$$P(E; \Omega) \leq \lim_{k \rightarrow \infty} P(E_{h_k}; \Omega).$$

Another useful property of sets of finite perimeter is stated in the next approximation result.

Theorem 2.4 *Let E be a set of finite perimeter in Ω . Then there exists a sequence of smooth, bounded open sets $E_h \subset \mathbb{R}^n$ such that $E_h \rightarrow E$ in measure in Ω and $P(E_h; \Omega) \rightarrow P(E; \Omega)$.*

Note also that in Theorem 2.4 one may replace the smooth sets E_h with polyhedra, i.e. bounded open sets obtained as the intersection of finitely many half-spaces.

The next result is a special case of the *coarea formula* for rectifiable sets, see [8, Th. 2.93]. To this aim, given $k = 1, \dots, n - 1$, we set $x = (x', y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$. Similarly, given a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ we set $v_{x'} = (v_1, \dots, v_{n-k})$, $v_y = (v_{n-k+1}, \dots, v_n)$, and given a measurable set $E \subset \mathbb{R}^n$ we denote the section of E over $x' \in \mathbb{R}^{n-k}$ by $E_{x'} := \{y \in \mathbb{R}^k : (x', y) \in E\}$.

Theorem 2.5 *Let E be a set of finite perimeter in \mathbb{R}^n . Given $k \in \{1, \dots, n - 1\}$, for \mathcal{H}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$ the slice $E_{x'}$ is a set of finite perimeter in \mathbb{R}^k and $\mathcal{H}^{k-1}(\partial^*(E_{x'}) \Delta (\partial^* E)_{x'}) = 0$. Moreover if $g : \mathbb{R}^n \rightarrow [0, +\infty]$ is a Borel function we have*

$$\int_{\partial^* E} g(x) v_y^E(x) d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-k}} dx' \int_{(\partial^* E)_{x'}} g(x', y) d\mathcal{H}^{k-1}(y).$$

Fix k and a measurable set $E \subset \mathbb{R}^n$. Let $v_E : \mathbb{R}^{n-k} \rightarrow [0, \infty)$ be the function measuring the k -dimensional slices of E , i.e.,

$$v_E(x') := \mathcal{H}^k(E_{x'}) \quad \text{for } \mathcal{H}^{n-k}\text{-a.e. } x' \in \mathbb{R}^{n-k}. \tag{2.5}$$

Note that v_E is \mathcal{H}^{n-k} -measurable. Moreover, if E is a set of finite perimeter, v_E is a function of bounded variation (BV -function) and even a Sobolev function if the reduced boundary of E has *no vertical parts*. For the definition and the main properties of BV functions we refer to [8, Ch. 3]. The following result is a consequence of Lemma 3.1 and Propositions 3.4 and 3.5 of [11].

Theorem 2.6 *Let E be a set of finite perimeter with finite measure and $k \in \{1, \dots, n - 1\}$. Then the function v_E defined in (2.5) belongs to $BV(\mathbb{R}^{n-k})$. Moreover if $\mathcal{H}^{n-1}(\{(x \in \partial^* E : v_y^E(x) = 0\}) = 0$, then $v_E \in W^{1,1}(\mathbb{R}^{n-k})$. In addition, if $U \subset \mathbb{R}^{n-k}$ is a Borel set*

$$P(E; U \times \mathbb{R}^k) \geq \int_U \sqrt{p_E(x')^2 + |\nabla v_E(x')|^2} dx' + |D^s v_E|(U),$$

where ∇v_E and $D^s v_E$ are the absolutely continuous part and the singular part, respectively, of the gradient measure Dv_E with respect to Lebesgue measure in \mathbb{R}^{n-k} and for \mathcal{H}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$ we have set $p_E(x') := \mathcal{H}^{k-1}(\partial^*(E_{x'}))$.

Given a measurable set $E \subset \mathbb{R}^n$ and $k \in \{1, \dots, n - 1\}$ we define the *essential projection of E* over the first $n - k$ coordinate hyperplanes by setting

$$\pi(E)^+ := \{x' \in \mathbb{R}^{n-k} : v_E(x') > 0\}.$$

If $x' \in \pi(E)^+$ we denote by $r_E(x')$ the radius of a k -dimensional ball with \mathcal{H}^k -measure equal to the one of $E_{x'}$. Then the *Steiner symmetral of E* of codimension k with respect to the subspace $\{y = 0\}$ is defined by setting

$$E^S := \{(x', y) \in \mathbb{R}^{n-k} \times \mathbb{R} : x' \in \pi(E)^+, |y| < r_E(x')\}.$$

Note that by definition $v_E(x') = v_{E^S}(x')$ for \mathcal{H}^{n-k} -a.e. $x' \in \mathbb{R}^{n-k}$, hence $|E| = |E^S|$. On the other hand perimeter decreases under Steiner symmetrization. Precisely, we have the following result whose proof can be found for instance in [11, Section 3].

Theorem 2.7 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and E^S its Steiner symmetral with respect to first $n - k$ coordinate hyperplanes. Then E^S is a set of finite perimeter and for any Borel set $U \subset \mathbb{R}^{n-k}$ we have*

$$P(E^S; U \times \mathbb{R}^k) \leq P(E; U \times \mathbb{R}^k). \tag{2.6}$$

Moreover,

$$P(E^S; U \times \mathbb{R}^k) = \int_U \sqrt{p_{E^S}(x')^2 + |\nabla v_{E^S}(x')|^2} dx' + |D^S v_{E^S}|(U).$$

Let us apply the above formula to a set of finite perimeter E , axially symmetric with respect to the x_n axis and let us set $v(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (x', t) \in E\})$. For any open set $U \subset \mathbb{R}$ we have

$$P(E; \mathbb{R}^{n-1} \times U) = \int_U \sqrt{(n - 1)^2 \omega_{n-1}^{\frac{2}{n-1}} v(t)^{\frac{2(n-2)}{n-1}} + v'(t)^2} + |D^S v|, \tag{2.7}$$

provided $n \geq 3$ or $n = 2$ and $v > 0$ in U .

A characterization of the equality cases in the perimeter inequality (2.6), under suitable assumptions on the set E , has been given in [38] for the case $k = 1$ and in [11] for the higher codimension case. The complete characterization of the equality cases for the codimension 1 case has been recently proven in [33].

We conclude this section by recalling the *coarea formula* for functions. Though it holds more generally for functions of bounded variations, we are going to apply it only to Sobolev functions. In this case we have, see for instance [8, Th. 3.40].

Theorem 2.8 *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W_{loc}^{1,1}(\Omega)$. Then, for every Borel function $g : \Omega \rightarrow [0, \infty]$ we have*

$$\int_{\Omega} g(x) |\nabla f| dx = \int_{-\infty}^{\infty} dt \int_{\{f=t\}} g(x) d\mathcal{H}^{n-1}(x). \tag{2.8}$$

3 The Fuglede’s approach

3.1 Nearly spherical sets

In this section, following Fuglede [72], we consider a *nearly spherical set*, that is a set $E \subset \mathbb{R}^n$ whose boundary can be written as a graph over the boundary of the ball with the same volume and the same barycenter as E . Thus, up to a translation and a dilation, we shall assume that there exists a Lipschitz function $u : \mathbb{S}^{n-1} \rightarrow (-1, 1)$ such that

$$E = \{y \in \mathbb{R}^n : y = tx(1 + u(x)), \text{ with } x \in \mathbb{S}^{n-1}, 0 \leq t < 1\}, \tag{3.1}$$

that $|E| = |B|$ and that E has barycenter at the origin.

Fuglede’s result states that if E is sufficiently close to B , then the isoperimetric deficit, that is the difference between $P(E)$ and $P(B)$, controls the L^2 norm of the tangential gradient $\nabla_\tau u$.

Theorem 3.1 *There exists $\varepsilon(n) > 0$ such that if E is as in (3.1), $|E| = |B|$, the barycenter of E is the origin and $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon$, then*

$$P(E) - P(B) \geq \frac{1}{4} \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{8\omega_n} |E \Delta B|^2. \tag{3.2}$$

Proof Step 1 Let us prove that if E is as in (3.1) then

$$\mathcal{H}^{n-1}(\partial E) = \int_{\mathbb{S}^{n-1}} \sqrt{(1 + u)^{2(n-1)} + (1 + u)^{2(n-2)} |\nabla_\tau u|^2} d\mathcal{H}^{n-1}. \tag{3.3}$$

In fact, from the area formula we have that

$$\mathcal{H}^{n-1}(\partial E) = \int_{\mathbb{S}^{n-1}} J_{n-1}\varphi d\mathcal{H}^{n-1}, \tag{3.4}$$

where $J_{n-1}\varphi$ is the $(n - 1)$ -dimensional Jacobian of the map $\varphi(x) = x(1 + u(x))$, $x \in \mathbb{S}^{n-1}$. Recall that, if $T_x\mathbb{S}^{n-1}$ is the tangential plane to \mathbb{S}^{n-1} at x , then $J_{n-1}\varphi = \sqrt{\det((d_x\varphi)^* \circ d_x\varphi)}$, where the linear map $d_x\varphi : T_x\mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is the tangential differential of φ at x and $(d_x\varphi)^* : \mathbb{R}^n \mapsto T_x\mathbb{S}^{n-1}$ denotes the adjoint of the differential. Since for any $\tau \in T_x\mathbb{S}^{n-1}$ we have $d_x\varphi(\tau) = \tau(1 + u(x)) + xD_\tau u(x)$, the coefficients of the matrix $d_x\varphi$ relative to an orthonormal base $\{\tau_1, \dots, \tau_{n-1}\}$ of $T_x\mathbb{S}^{n-1}$ and to the standard base $\{e_1, \dots, e_n\}$ are $\tau_i \cdot e_h(1 + u(x)) + x_h D_{\tau_i} u$, for $i = 1, \dots, n - 1, h = 1, \dots, n$. Thus, for all $i, j \in \{1, \dots, n - 1\}$ the coefficients a_{ij} of the matrix $(d_x\varphi)^* \circ d_x\varphi$ are given by

$$\begin{aligned} a_{ij} &= \sum_{h=1}^n (\tau_i \cdot e_h(1 + u) + x_h D_{\tau_i} u)(\tau_j \cdot e_h(1 + u) + x_h D_{\tau_j} u) \\ &= \delta_{ij}(1 + u)^2 + D_{\tau_i} u D_{\tau_j} u, \end{aligned}$$

where in the last equality we have used the fact that $\tau_i \cdot \tau_j = \delta_{ij}$ and $\tau_i \cdot x = 0$ for all $i, j = 1, \dots, n-1$. Hence, recalling that for $a, b \in \mathbb{R}^k$ one has $\det(I+a \otimes b) = 1+a \cdot b$, we have that

$$J_{n-1}\varphi = \sqrt{\det(a_{ij})} = \sqrt{(1+u)^{2(n-1)} + (1+u)^{2(n-2)}|\nabla_\tau u|^2}$$

and thus (3.3) follows immediately from (3.4). Let us extend u by setting $u(x) : u(x/|x|)$ for all $x \neq 0$. Then $E = \Phi(B)$, where $\Phi : B \rightarrow E$ is the map defined by setting $\Phi(x) := x(1+u(x))$. Note that $D\Phi(x) = (1+u(x))I + x \otimes Du$ and that $x \cdot Du(x) = 0$ for all $x \neq 0$. Thus, we conclude that the n -dimensional Jacobian of Φ is given by $(1+u(x))^n$. Hence, the assumptions $|E| = |B|$ and $\int_E y \, dy = 0$ can be rewritten as follows

$$\frac{1}{n|B|} \int_{\mathbb{S}^{n-1}} (1+u(x))^n \, d\mathcal{H}^{n-1} = 1, \quad \int_{\mathbb{S}^{n-1}} x(1+u(x))^{n+1} \, d\mathcal{H}^{n-1} = 0. \tag{3.5}$$

Step 2 Using (3.3) we have

$$\begin{aligned} P(E) - P(B) &= \int_{\mathbb{S}^{n-1}} \left[(1+u)^{n-1} \sqrt{1 + \frac{|\nabla_\tau u|^2}{(1+u)^2}} - 1 \right] d\mathcal{H}^{n-1} \\ &= \int_{\mathbb{S}^{n-1}} [(1+u)^{n-1} - 1] d\mathcal{H}^{n-1} \\ &\quad + \int_{\mathbb{S}^{n-1}} (1+u)^{n-1} \left[\sqrt{1 + \frac{|\nabla_\tau u|^2}{(1+u)^2}} - 1 \right] d\mathcal{H}^{n-1}. \end{aligned}$$

From the Taylor expansion of the square root it follows that for $t > 0$ sufficiently small $\sqrt{1+t} \geq 1 + \frac{t}{2} - \frac{t^2}{7}$. Hence from the smallness assumption $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$ we get

$$\begin{aligned} P(E) - P(B) &\geq \int_{\mathbb{S}^{n-1}} [(1+u)^{n-1} - 1] d\mathcal{H}^{n-1} \\ &\quad + \int_{\mathbb{S}^{n-1}} (1+u)^{n-1} \left[\frac{1}{2} \frac{|\nabla_\tau u|^2}{(1+u)^2} - \frac{1}{7} \frac{|\nabla_\tau u|^4}{(1+u)^4} \right] d\mathcal{H}^{n-1} \\ &\geq \int_{\mathbb{S}^{n-1}} [(1+u)^{n-1} - 1] d\mathcal{H}^{n-1} + \left(\frac{1}{2} - C\varepsilon \right) \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1}. \end{aligned} \tag{3.6}$$

Note that the first equation in (3.5), that is the assumption $|E| = |B|$, implies that

$$\int_{\mathbb{S}^{n-1}} [(1+u)^n - 1] d\mathcal{H}^{n-1} = 0,$$

that is

$$\int_{\mathbb{S}^{n-1}} \left(nu + \sum_{h=2}^n \binom{n}{h} u^h \right) d\mathcal{H}^{n-1} = 0. \tag{3.7}$$

From this identity, recalling again that $\|u\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon$, we have

$$\int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} \geq -\frac{n-1}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}.$$

Therefore, using this last inequality and the smallness assumption, we may estimate

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} [(1+u)^{n-1}-1]d\mathcal{H}^{n-1} &= (n-1) \int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} + \sum_{h=2}^{n-1} \binom{n-1}{h} \int_{\mathbb{S}^{n-1}} u^h d\mathcal{H}^{n-1} \\ &\geq (n-1) \int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} + \frac{(n-1)(n-2)}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} \\ &\geq -\frac{n-1}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}. \end{aligned}$$

In conclusion, recalling (3.6), we have proved that if $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon$, then

$$P(E) - P(B) \geq \left(\frac{1}{2} - C\varepsilon\right) \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} - \left(\frac{n-1}{2} + C\varepsilon\right) \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}. \tag{3.8}$$

Step 3 Now, for any integer $k \geq 0$, let us denote by $y_{k,i}$, $i = 1, \dots, G(n, k)$, the spherical harmonics of order k , i.e., the restriction to \mathbb{S}^{n-1} of the homogeneous harmonic polynomials of degree k , normalized so that $\|y_{k,i}\|_{L^2(\mathbb{S}^{n-1})} = 1$, for all k and for $i \in \{1, \dots, G(n, k)\}$. For instance, since $\int_{\mathbb{S}^{n-1}} 1 d\mathcal{H}^{n-1} = n\omega_n$ and, for $i = 1, \dots, n$, $\int_{\mathbb{S}^{n-1}} x_i^2 d\mathcal{H}^{n-1} = n^{-1} \int_{\mathbb{S}^{n-1}} |x|^2 d\mathcal{H}^{n-1} = \omega_n$, we have $y_0 = 1/\sqrt{n\omega_n}$ and $y_{1,i} = x_i/\sqrt{\omega_n}$, respectively. The functions $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on \mathbb{S}^{n-1} and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2)y_{k,i}.$$

Therefore if we write

$$u = \sum_{k=0}^\infty \sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where } a_{k,i} = \int_{\mathbb{S}^{n-1}} u y_{k,i} d\mathcal{H}^{n-1},$$

we have

$$\|u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=0}^\infty \sum_{i=1}^{G(n,k)} a_{k,i}^2, \quad \|\nabla_\tau u\|_{L^2(\mathbb{S}^{n-1})}^2 = \sum_{k=1}^\infty k(k+n-2) \sum_{i=1}^{G(n,k)} a_{k,i}^2. \tag{3.9}$$

Observe that from formula (3.7) we have

$$a_0 = \frac{1}{\sqrt{n\omega_n}} \int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} = -\frac{1}{n\sqrt{n\omega_n}} \sum_{h=2}^n \binom{n}{h} \int_{\mathbb{S}^{n-1}} u^h d\mathcal{H}^{n-1},$$

hence

$$|a_0| \leq C \|u\|_2^2 \leq C\varepsilon \|u\|_2.$$

Similarly, from the second equality in (3.5), that is the assumption that the barycenter of E is at the origin, using the equality $\int_{\mathbb{S}^{n-1}} x_i = 0$, we immediately get that for all $i = 1, \dots, n$,

$$|a_{1,i}| = \left| \frac{1}{\sqrt{\omega_n}} \int_{\mathbb{S}^{n-1}} u x_i d\mathcal{H}^{n-1} \right| \leq C\varepsilon \|u\|_2.$$

Therefore, from (3.9) we get

$$\|u\|_2^2 \leq C\varepsilon^2 \|u\|_2^2 + \sum_{k=2}^{\infty} \sum_{i=1}^{G(n,k)} |a_{k,i}|^2 \implies \|u\|_2^2 \leq \frac{1}{1 - C\varepsilon} \sum_{k=2}^{\infty} \sum_{i=1}^{G(n,k)} |a_{k,i}|^2.$$

But since for $k \geq 2$, $k(k + n - 2) \geq 2n$, from (3.9) we have

$$\|u\|_2^2 \leq \frac{1}{2n(1 - C\varepsilon)} \|\nabla_{\tau} u\|_2^2$$

and thus, recalling (3.8) and choosing ε sufficiently small, in dependence on n , we get

$$\begin{aligned} P(E) - P(B) &\geq \left(\frac{1}{2} - C\varepsilon\right) \int_{\mathbb{S}^{n-1}} |\nabla_{\tau} u|^2 d\mathcal{H}^{n-1} - \left(\frac{n-1}{2} + C\varepsilon\right) \frac{1}{2n(1 - C\varepsilon)} \|\nabla_{\tau} u\|_2^2 \\ &\geq \frac{1}{4} \int_{\mathbb{S}^{n-1}} |\nabla_{\tau} u|^2 d\mathcal{H}^{n-1} \geq \frac{n}{3} \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \geq \frac{1}{3\omega_n} \|u\|_{L^1(\mathbb{S}^{n-1})}^2. \end{aligned} \tag{3.10}$$

This proves the first inequality in (3.2). To get the second inequality we observe that, choosing again ε sufficiently small

$$|E \Delta B| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} |(1 + u(x))^n - 1| d\mathcal{H}^{n-1} \leq \frac{n+1}{n} \int_{\mathbb{S}^{n-1}} |u| d\mathcal{H}^{n-1}.$$

Therefore, from the last inequality of (3.10) we conclude that

$$P(E) - P(B) \geq \frac{1}{3\omega_n} \|u\|_{L^1(\mathbb{S}^{n-1})}^2 \geq \frac{n^2}{3(n+1)^2\omega_n} |E \Delta B|^2 \geq \frac{1}{8\omega_n} |E \Delta B|^2.$$

□

3.2 Convex sets

We shall now discuss the case of convex sets for which Fuglede in [72] proved a quantitative estimate where the distance from a ball is measured in terms of the Hausdorff distance. To this aim we recall that if E, F are any two sets in \mathbb{R}^n the Hausdorff distance between E, F is defined as

$$d_H(E, F) := \inf\{\varepsilon > 0 : E \subset F + B_\varepsilon, F \subset E + B_\varepsilon\}.$$

In this section we shall always assume that K is a closed convex set with non empty interior and $|K| = |B|$. For such a set we define the *isoperimetric deficit* $\mathcal{D}(K)$ and the *asymmetry index* $\mathcal{A}(E)$ as

$$\mathcal{D}(K) := P(K) - P(B), \quad \mathcal{A}(K) := \min_{x \in \mathbb{R}^n} d_H(K, B(x)),$$

respectively. With these definitions in hand the result proved in [72, Th. 2.3] reads as follows.

Theorem 3.2 (Fuglede) *Let $n \geq 2$. There exist δ, C , depending only on n , such that if K is convex, $|K| = |B|$, and $\mathcal{D}(K) \leq \delta$, then:*

$$\mathcal{A}(K) \leq \begin{cases} C\sqrt{\mathcal{D}(K)}, & n = 2 \\ \sqrt{\mathcal{D}(K) \log\left(\frac{1}{\mathcal{D}(K)}\right)}, & n = 3 \\ C\mathcal{D}(K)^{\frac{2}{n+1}}, & n \geq 4. \end{cases} \quad (3.11)$$

To be precise, in his paper [72] Fuglede only deals with the case $n \geq 3$, since when $n = 2$ the result was already known with a better estimate and a more elementary proof. Indeed, the case of planar convex sets was already studied by Bernstein [13] in 1905 and by Bonnesen [21] in 1924, see Theorem 1.1 in Sect. 1. Note that the estimates (3.11) are sharp with respect to the order of magnitude as $\mathcal{D}(K) \rightarrow 0$, see [72, Sect. 3].

The proof of Theorem 3.2 is based on the quantitative estimate proved in the previous section. The key point is the observation that a convex sets K with the same volume as the unit ball and small isoperimetric deficit is nearly spherical. Indeed, the boundary of K can be written as the graph of a Lipschitz function u over the boundary of the sphere centered at the barycenter of K with $\|u\|_{W^{1,\infty}}$ small. This is precisely the content of the next lemma.

Lemma 3.3 *For all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if K is a closed convex, $|K| = |B|$, the barycenter of K is the origin and $\mathcal{D}(K) < \delta_\varepsilon$, then there exists a Lipschitz function $u : \mathbb{S}^{n-1} \rightarrow (0, \infty)$, with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon$, such that*

$$K = \{tx(1 + u(x)) : x \in \mathbb{S}^{n-1}, 0 \leq t \leq 1\}. \quad (3.12)$$

Proof Note that if K is any closed convex set with barycenter at the origin and non-empty interior, then K can be always represented as in (3.12) for some Lipschitz function $u : \mathbb{S}^{n-1} \rightarrow (0, \infty)$. The point here is to show that the $W^{1,\infty}$ norm of u is small when $\mathcal{D}(K)$ is small.

Step 1 We start by proving a weaker statement, namely that for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if K is a closed convex set with barycenter at 0, $|K| = |B|$ and $\mathcal{D}(K) < \delta_\varepsilon$, then K can be represented as in (3.12) for some Lipschitz function u with $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq \varepsilon$.

To prove this we argue by contradiction assuming that there exist $\varepsilon > 0$ and a sequence of closed convex sets K_j such that $|K_j| = |B|$, the barycenter of K_j is the origin, $\mathcal{D}(K_j) \rightarrow 0$, but $\|u_j\|_{L^\infty(\mathbb{S}^{n-1})} \geq \varepsilon_0$, where u_j is the function representing K_j as in (3.12).

The following fact is well known, see for instance [58, Lemma 4.1]. If $n \geq 2$ there exists $C(n)$ such that for any convex set K with non empty interior

$$\text{diam}(K) \leq C(n) \frac{P(K)^{n-1}}{|K|^{n-2}}.$$

Using this inequality we deduce that the sets K_j are equibounded and so, up to a not relabeled subsequence, we may assume that they converge in the Hausdorff distance to a closed set K . Note that K is convex and that $|K_j \Delta K| \rightarrow 0$ as $j \rightarrow \infty$. Therefore, by the lower semicontinuity of the perimeter we have that $P(K) \leq \liminf_{j \rightarrow \infty} P(K_j)$. Therefore, since $\mathcal{D}(K_j) \rightarrow 0$, we conclude that $P(K) = P(B)$ and thus that K is the closed unit ball centered at the origin. This gives a contradiction, since it is not possible that the sets K_j converge in the Hausdorff sense to \bar{B} , while $\|u_j\|_{L^\infty(\mathbb{S}^{n-1})} \geq \varepsilon_0$ for all j .

Step 2 Let us now assume that $\|u\|_{L^\infty(\mathbb{S}^{n-1})} \leq \varepsilon$ for some $\varepsilon < 1/2$. To conclude the proof of the lemma we show that if this is the case, then the following stronger inequality holds

$$\|\nabla_\tau u\|_\infty \leq 2\sqrt{\|u\|_\infty} \frac{1 + \|u\|_\infty}{1 - \|u\|_\infty}. \tag{3.13}$$

To prove this estimate we observe that if K is represented as in (3.12), then it is not too hard to show that for \mathcal{H}^{n-1} -a.e. $x \in \mathbb{S}^{n-1}$ the exterior normal $\nu^K(y(x))$ to ∂K at the point $y(x) = x(1 + u(x))$ is given by

$$\nu^K(y(x)) = \frac{x(1 + u(x)) - \nabla_\tau u(x)}{\sqrt{(1 + u(x))^2 + |\nabla_\tau u(x)|^2}}. \tag{3.14}$$

Thus, if $x \in \mathbb{S}^{n-1}$ is a point where u is differentiable from the previous equality we have that

$$x \cdot \nu^K(y(x)) = \frac{1 + u(x)}{\sqrt{(1 + u(x))^2 + |\nabla_\tau u(x)|^2}},$$

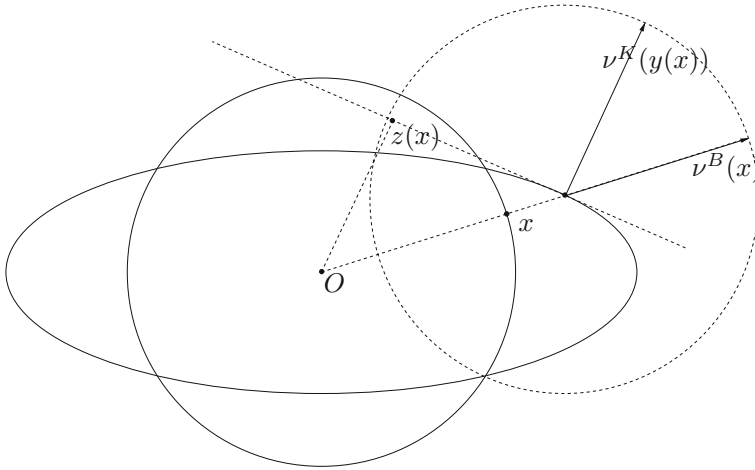


Fig. 1 The construction in Lemma 3.3

where we used the fact that $x \cdot \nabla_\tau u(x) = 0$. Then, denoting by $z(x)$ the projection of the origin on the tangent plane to K at $y(x)$, the triangle with vertices $O, z(x), y(x)$ is rectangle at $z(x)$ and the angle at the origin is equal to the angle formed by the vectors x and $\nu^K(y(x))$, see Fig. 1. Therefore $|z(x)| = |y(x)|(x \cdot \nu^K(y(x)))$. Observe also that

$$|y(x)| \leq 1 + \|u\|_\infty, \quad |z(x)| \geq 1 - \|u\|_\infty,$$

where the last inequality follows from the convexity of K , and thus

$$\frac{1 - \|u\|_\infty}{1 + \|u\|_\infty} \leq x \cdot \nu^K(y(x)) = \frac{1 + u(x)}{\sqrt{(1 + u(x))^2 + |\nabla_\tau u(x)|^2}},$$

from which we get

$$\frac{|\nabla_\tau u(x)|^2}{(1 + u(x))^2} \leq \left(\frac{1 + \|u\|_\infty}{1 - \|u\|_\infty} \right)^2 - 1 = \frac{4\|u\|_\infty}{(1 - \|u\|_\infty)^2},$$

thus concluding

$$|\nabla_\tau u(z)|^2 \leq 4\|u\|_\infty \left(\frac{1 + \|u\|_\infty}{1 - \|u\|_\infty} \right)^2,$$

whence (3.13) follows. □

The proof of Theorem 3.2 now follows quite easily from the lemma we have just proved and from the next interpolation result, whose proof is given in [72, Lemma 1.4].

Lemma 3.4 *If $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}} v = 0$, then*

$$\|v\|_{L^\infty(\mathbb{S}^{n-1})}^{n-1} \leq \begin{cases} \pi \|\nabla_\tau v\|_2, & n = 2 \\ 4 \|\nabla_\tau v\|_2^2 \log \frac{8e \|\nabla_\tau v\|_\infty}{\|\nabla_\tau v\|_2^2}, & n = 3 \\ C(n) \|\nabla_\tau v\|_2^2 \|\nabla_\tau v\|_\infty^{n-3}, & n \geq 4, \end{cases}$$

where the constant $C(n)$ depends only on the dimension and e is the Neper number.

Proof of Theorem 3.2 Fix $\varepsilon > 0$. Thanks to Lemma 3.3 we may assume that K is a convex set represented as in (3.12) by means of a function u such that $\|u\|_{W^{1,\infty}} < \varepsilon$. Set $v := \frac{(1+u)^n - 1}{n}$. From the volume constraint $|K| = |B|$ we have, see the first equality in (3.5),

$$\int_{\mathbb{S}^{n-1}} v d\mathcal{H}^{n-1} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} [(1+u)^n - 1] d\mathcal{H}^{n-1} = 0.$$

Moreover, since

$$v = u + \frac{1}{n} \sum_{h=2}^n \binom{n}{h} u^h,$$

if $\varepsilon > 0$ is small enough we have

$$\frac{1}{2}|u| \leq |v| \leq 2|u|, \quad \frac{1}{2}|\nabla_\tau u| \leq |\nabla_\tau v| \leq 2|\nabla_\tau u|.$$

Assume now, to fix the ideas, that $n \geq 4$. If ε is smaller than or equal to the one provided by Theorem 3.1, from the interpolation Lemma 3.4 and (3.13) we have

$$\begin{aligned} \|u\|_\infty &\leq 2\|v\|_\infty \leq C(n) \|\nabla_\tau v\|_2^{\frac{2}{n-1}} \|\nabla v_\tau\|_\infty^{\frac{n-3}{n-1}} \\ &\leq C(n) \|\nabla u_\tau\|_2^{\frac{2}{n-1}} \|\nabla_\tau u\|_\infty^{\frac{n-3}{n-1}} \leq C(n) \|\nabla_\tau u\|_2^{\frac{2}{n-1}} \|u\|_\infty^{\frac{n-3}{2(n-1)}}, \end{aligned}$$

hence $\|u\|_\infty \leq C(n) \|\nabla_\tau u\|_2^{\frac{4}{n+1}}$. Thus, recalling the first inequality in (3.2), we may conclude that

$$A(K) \leq d_H(K, B) = \|u\|_\infty \leq CD(E)^{\frac{2}{n+1}}.$$

The cases $n = 2, 3$ are proved in the same way. □

When passing from a convex set to a general set E of finite perimeter one cannot expect to estimate the isoperimetric deficit with the Hausdorff distance from a ball, see the discussion at the beginning of the next section. However, this is still possible if some additional structure is imposed on the set E . At this regard we mention two

results that have been recently obtained in this direction. The first one, proved in [73], deals with sets satisfying a uniform interior cone condition at the boundary. To simplify the statement we shall assume as before that E is a set with the same volume as the unit ball. The general case can be recovered by suitably rescaling all the relevant quantities, see [73, Theorem 1.1].

Given $x \in \mathbb{R}^n$, $r > 0$ and $v \in \mathbb{S}^{n-1}$ the *spherical sector* with vertex at x , axis of symmetry parallel to v , radius r and aperture $\pi/2$ is defined as

$$S_{v,r}(x) = \{y \in \mathbb{R}^n : |y - x| < r, \sqrt{2}\langle y - x, v \rangle > |y - x|\}.$$

Then, we say that a closed set E satisfies the *interior cone condition at the boundary* with radius r and aperture $\pi/2$ if for any $x \in \partial E$, there exists $v_x \in \mathbb{S}^{n-1}$ such that $S_{v_x,r}(x) \subset E$ and we denote by \mathcal{C}_r the family of all closed sets E , with $|E| = |B|$, satisfying the interior cone condition at the boundary with radius r . It can be easily checked [73, Lemma 2.9], that if $E \in \mathcal{C}_r$ and $\mathcal{D}(E)$ is sufficiently small then E is compact.

The interior cone condition at the boundary is a rather mild regularity condition. In fact, given any $\varrho > 0$ and any $\theta \in (0, \pi/2)$ one may construct a set $E \in \mathcal{C}_r$ which does not satisfy the standard interior cone condition with height ϱ and aperture θ , see [73, Example 2.3]. On the other hand, a set in \mathcal{C}_r has always finite perimeter [73, Prop. 2.4], but the \mathcal{H}^{n-1} -measure of its topological boundary may be strictly larger than its perimeter, [73, Example 2.6]. The main result proved in [73] then reads as follows.

Theorem 3.5 *For any $r > 0$ there exist δ, C , depending only on n and r , such that if $E \in \mathcal{C}_r$ and $\mathcal{D}(E) \leq \delta$, then:*

$$A(E) \leq \begin{cases} C\sqrt{\mathcal{D}(E)}, & n = 2 \\ \sqrt{\mathcal{D}(E) \log\left(\frac{1}{\mathcal{D}(E)}\right)}, & n = 3 \\ C\mathcal{D}(E)^{\frac{2}{n+1}}, & n \geq 4. \end{cases} \tag{3.15}$$

We observe that the powers appearing in (3.15) are the same as in Theorem 3.2. Another estimate with the same powers has been also obtained by Rajala and Zhong in [106, Th. 1.5] for John domains whose complement with respect to a suitable ball is also a John domain. Note that though the sets considered in [106] do not necessarily belong to \mathcal{C}_r , they cannot have singularities such as inward cusps, which are instead admissible for sets in \mathcal{C}_r .

4 The quantitative isoperimetric inequality: the approach via symmetrization

We now discuss the quantitative isoperimetric inequality for general sets of finite perimeter. In this case it is clear that we cannot use the Hausdorff distance to measure the asymmetry of a set. Think for instance of a set which is the union of a large ball

and a far away tiny one. For this reason we replace the asymmetry index considered in the previous section with an L^1 -type distance, the so-called *Fraenkel asymmetry*, which is defined for a measurable set E , as the minimum of the symmetric difference between E and any ball with the same measure

$$\alpha(E) := \min_{x \in \mathbb{R}^n} \left\{ \frac{|E \Delta B_r(x)|}{|B_r|} : |E| = |B_r| \right\}. \tag{4.1}$$

We shall refer to a ball minimizing $\alpha(E)$ as to an *optimal ball*. Note that an optimal ball needs not to be unique. Note also that $\alpha(E)$ is scaling invariant. It is convenient to define the *isoperimetric deficit* of a set E also in a scaling invariant way by setting

$$D(E) := \frac{P(E) - P(B_r)}{P(B_r)},$$

where again r is the radius of a ball with the same measure of E .

The first quantitative estimate for sets of finite perimeter was obtained in 1992 by Hall [82]. Using some previous results proved in collaboration with Hayman and Weitsman [84], he showed that there exists a constant $C(n)$ such that for all measurable sets of finite measure

$$\alpha(E)^4 \leq C(n)D(E). \tag{4.2}$$

It is interesting to observe that the power 4 on the left hand side of this estimate does not depend on the dimension, while in Fuglede’s Theorem 3.2 it does. However, in his paper Hall conjectured that the right power should be 2 and actually proved that (4.2) holds with the exponent 2 if E is an axially symmetric set, see [82, Th. 2]. Moreover he observed that one cannot expect a power smaller than 2. To see this, take $\varepsilon > 0$ and consider in any dimension $n \geq 2$ the ellipsoid

$$E_\varepsilon = \left\{ \frac{x_1^2}{1 + \varepsilon} + x_2^2(1 + \varepsilon) + x_3^2 + \dots + x_n^2 \leq 1 \right\}.$$

It can be proved, see Proposition 4.3, that $\alpha(E_\varepsilon) = |E_\varepsilon \Delta B|$. Then one can show that

$$\frac{D(E_\varepsilon)}{\alpha^2(E_\varepsilon)} \rightarrow \gamma > 0, \quad \text{as } \varepsilon \rightarrow 0^+.$$

In 2008 Maggi, Pratelli and the author proved in [76] the following quantitative isoperimetric inequality with the sharp exponent.

Theorem 4.1 *There exists a constant $\gamma(n)$ such that for any measurable set E of finite measure*

$$\alpha(E)^2 \leq \gamma(n)D(E). \tag{4.3}$$

Note that inequality (4.3) can be rewritten in the following equivalent way: if $|E| = |B_r|$,

$$P(E) \geq P(B_r) \left(1 + \frac{\alpha(E)^2}{\gamma(n)} \right).$$

Thus the asymmetry index $\alpha(E)$ can be viewed as a quantity estimating the second order term in the Taylor expansion of $P(E)$ around $P(B_r)$. This point of view has been exploited in the proof of Theorem 3.1 where the idea is precisely to view the perimeter of E as an integral functional over the sphere depending on the function u and then to write the second order Taylor expansion of that functional.

In this section we will present the proof of the quantitative isoperimetric inequality given in [76] which is mostly based on symmetrization arguments. Though different and even shorter proofs can be given, based on completely different strategies, the one we present here has the advantage of not using any sophisticated technical tool, it is geometrically intuitive and can be adapted to other contexts where symmetry plays a fundamental role. Other proofs of the quantitative isoperimetric inequality will be discussed in the Sect. 5.

Let us now give a short description of how the proof goes. The main idea is to reduce the problem, by means of suitable geometric constructions, to the case of more and more symmetric sets, namely to axially symmetric sets that are also symmetric with respect to n orthogonal hyperplanes. For these sets one may either use the already mentioned result by Hall [82, Th. 2], or simpler ad hoc arguments.

Observe that both the asymmetry and the isoperimetric deficit are scaling invariant. Therefore, throughout all this section we will only deal with sets with the same measure of the unit ball B and with small isometric deficit. In fact, since $\alpha(E)$ is always smaller than 2 inequality (4.3) becomes trivial when $D(E)$ is large.

The first step in the proof is to reduce to sets that are confined in a sufficiently large cube. This step is needed in order to take advantage of the compactness properties of bounded sets with equibounded perimeter. This first reduction is achieved by using the concavity property of the isoperimetric function which is the function that associates to every $t > 0$ the perimeter $n\omega^{1/n} t^{(n-1)/n}$ of the ball of measure t , see Lemma 4.2.

The second step in the proof is to reduce to the case of n -symmetric sets. We say that E is n -symmetric if E is symmetric with respect to all coordinate hyperplanes. This reduction is the most delicate part the proof. On the other hand, when restricted to n -symmetric sets, the proof of (4.3) will be relatively easy. The reason is that while in general it can be quite hard to find an optimal ball, when E is n -symmetric the ball centered at the center of symmetry of E plays the same role of the optimal ball, even if it is not optimal. Indeed, if E is n -symmetric and $|E| = |B|$, Lemma 4.4 states that if $B(x_0)$ is an optimal ball then

$$|E \Delta B(x_0)| \leq |E \Delta B| \leq 3|E \Delta B(x_0)|,$$

a simple, but very useful property.

The next step is to pass from an n -symmetric to an axially symmetric set. This further reduction is achieved by proving, see Proposition 4.9, that if E is n -symmetric

and E^* denotes its Steiner symmetral with respect to an axis passing through its center of symmetry, then we have

$$\alpha(E) \leq \alpha(E^*) + C\sqrt{D(E)} \tag{4.4}$$

for some positive constant C depending only on n , provided that $n = 2$ or $n \geq 3$ and the quantitative isoperimetric inequality holds true in dimension $n - 1$. To conclude the proof one has only to show that (4.3) holds true for the axially symmetric and n -symmetric set E^* , i.e.

$$\alpha(E^*)^2 \leq C(n)D(E^*).$$

Then, (4.3) for E follows at once by combining this inequality with (4.4) and recalling that $D(E^*) \leq D(E)$, see Theorem 2.7.

4.1 Reduction to n -symmetric sets

As we said before the first step is to reduce to the case where the set E is contained in a cube of fixed edge length. To this aim, given $l > 0$ we denote by Q_l the open cube $(-l, l)^n$. Thus, we may state the following result, proved in [76, Lemma 5.1].

Lemma 4.2 *There exist two constants, $l, C > 0$ depending only on n with the following property: given a measurable set $E \subset \mathbb{R}^n$, with $|E| = |B|$, it is possible to find a new set $F \subset Q_l$, such that $|F| = |B|$ and*

$$D(F) \leq CD(E), \quad \alpha(E) \leq \alpha(F) + CD(E). \tag{4.5}$$

Proof By rotating E a little if necessary, we may assume that

$$\mathcal{H}^{n-1} \left(\left\{ x \in \partial^* E : \nu^E(x) = \pm e_n \right\} \right) = 0. \tag{4.6}$$

Thus, Theorem 2.6 applied with $k = n - 1$ yields that the function

$$v_E(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (x', t) \in E\}) \quad \text{for } t \in \mathbb{R},$$

belongs to $W^{1,1}(\mathbb{R})$. Hence, v_E is continuous. Set $E_t^- = \{x \in E : x_1 < t\}$ for all $t \in \mathbb{R}$ and recall that if F is a set of finite perimeter, then by Theorem 2.2 we have $\mathcal{H}^{n-1}(F^{(1/2)} \setminus \partial^* F) = 0$. Therefore from (4.6) we have that for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$

$$P(E_t^-) \leq P(E; \{x_n < t\}) + v_E(t), \quad P(E \setminus E_t^-) \leq P(E; \{x_n > t\}) + v_E(t). \tag{4.7}$$

Let us now define a function $g : \mathbb{R} \rightarrow \mathbb{R}^+$ as

$$g(t) := \frac{|E_t^-|}{\omega_n}.$$

From Fubini’s theorem it follows that g is a non-decreasing absolutely continuous function such that $g'(t) = v_E(t)/\omega_n$ for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$. Therefore there exist $-\infty \leq a < b \leq +\infty$ such that $\{t : 0 < g(t) < 1\} = (a, b)$. Fix any $t \in (a, b)$ and note that by definition

$$|g(t)^{-\frac{1}{n}} E_t^-| = |B|.$$

Therefore, by the isoperimetric inequality (1.4) $P(g(t)^{-1/n} E_t^-) \geq P(B)$, hence

$$P(E_t^-) = g(t)^{\frac{n-1}{n}} P\left(g(t)^{-\frac{1}{n}} E_t^-\right) \geq g(t)^{\frac{n-1}{n}} P(B).$$

Similarly,

$$P(E \setminus E_t^-) \geq (1 - g(t))^{\frac{n-1}{n}} P(B).$$

Therefore, from (4.7) we get that

$$P(E) + 2v_E(t) \geq P(B) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} \right)$$

for all $t \in (a, b)$. On the other hand, from the definition of isoperimetric deficit we have $P(E) = P(B)(1 + D(E))$, and thus

$$v_E(t) \geq \frac{1}{2} P(B) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} - 1 - D(E) \right). \tag{4.8}$$

Let us now define a function $\psi : [0, 1] \rightarrow \mathbb{R}^+$ as

$$\psi(t) := t^{\frac{n-1}{n}} + (1 - t)^{\frac{n-1}{n}} - 1.$$

Note that $\psi(0) = \psi(1) = 0$ and that ψ is a concave function attaining its maximum at $1/2$, $\psi(1/2) = 2^{1/n} - 1$. Therefore,

$$\psi(t) \geq 2(2^{1/n} - 1)t \quad \text{for all } 0 \leq t \leq \frac{1}{2}. \tag{4.9}$$

We may assume, without loss of generality, that $2D(E) < \psi(1/2)$. Otherwise, since $\alpha(E) \leq 2$, the assertion immediately follows by choosing $F = B$ and $C = 4/\psi(1/2)$. Then, let $a < t_1 < t_2 < b$ be two numbers such that $g(t_1) = 1 - g(t_2)$ and $\psi(g(t_1)) = \psi(g(t_2)) = 2D(E)$; therefore,

$$\psi(g(t)) \geq 2D(E) \quad \text{for all } t \in (t_1, t_2) \tag{4.10}$$

and by (4.9)

$$g(t_1) = 1 - g(t_2) \leq \frac{D(E)}{2^{1/n} - 1}. \tag{4.11}$$

Thanks to (4.8) and (4.10), for \mathcal{H}^1 -a.e. $t_1 \leq t \leq t_2$ we have

$$\begin{aligned}
 v_E(t) &\geq \frac{1}{2} P(B)(\psi(g(t)) - D(E)) \\
 &\geq \frac{1}{4} P(B)\psi(g(t)) + \frac{1}{4} P(B)(\psi(g(t)) - 2D(E)) \geq \frac{n\omega_n}{4} \psi(g(t)).
 \end{aligned}
 \tag{4.12}$$

Therefore, recalling that $v_E(t) = \omega_n g'(t)$,

$$t_2 - t_1 \leq \frac{4}{n} \int_{t_1}^{t_2} \frac{g'(t)}{\psi(g(t))} dt = \frac{4}{n} \int_{g(t_1)}^{g(t_2)} \frac{1}{\psi(s)} ds \leq \frac{4}{n} \int_0^1 \frac{1}{\psi(s)} ds = \alpha \tag{4.13}$$

for some constant $\alpha = \alpha(n)$. Let us now set

$$\begin{aligned}
 \tau_1 &= \max \left\{ t \in (a, t_1] : v_E(t) \leq \frac{n\omega_n D(E)}{2} \right\}, \\
 \tau_2 &= \min \left\{ t \in [t_2, b) : v_E(t) \leq \frac{n\omega_n D(E)}{2} \right\}.
 \end{aligned}$$

Note that τ_1 and τ_2 are well defined since v_E is continuous and $v_E(a) = v_E(b) = 0$ and that by (4.10) and (4.12) $v_E(\tau_1) = v_E(\tau_2) = (n\omega_n D(E))/2$. Moreover, from (4.11), we have

$$t_1 - \tau_1 \leq \frac{2}{n\omega_n D(E)} \int_{\tau_1}^{t_1} v_E(t) dt = \frac{2}{nD(E)} \int_{\tau_1}^{t_1} g'(t) dt \leq \frac{2g(t_1)}{nD(E)} \leq \frac{2}{n(2^{1/n} - 1)}$$

and a similar estimate holds for $\tau_2 - t_2$.

Let us now set $\tilde{E} = E \cap \{x : \tau_1 < x_1 < \tau_2\}$. From the above estimate and (4.13), we have that $\tau_2 - \tau_1 < \beta(n)$. Moreover, (4.11), the definition of τ_1, τ_2 and (4.7) immediately yield

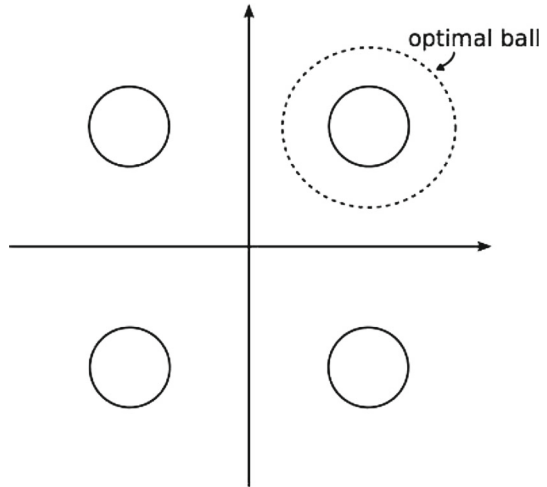
$$|\tilde{E}| \geq |B| \left(1 - 2 \frac{D(E)}{2^{1/n} - 1} \right), \quad P(\tilde{E}) \leq P(E) + n\omega_n D(E). \tag{4.14}$$

Let us now assume that $D(E) < (2^{1/n} - 1)/4$ and set

$$\sigma := \left(\frac{|B|}{|\tilde{E}|} \right)^{1/n}, \quad F := \sigma \tilde{E}.$$

Clearly, $|F| = |B|$ and F is contained in a strip $\{\tau'_1 < x_1 < \tau'_2\}$, with $\tau'_2 - \tau'_1 \leq \beta'$, where β' is a constant depending only on n . Let us now show that F satisfies (4.5) for a suitable constant C depending only on n .

Fig. 2 The optimal ball of a n -symmetric set is not always centered at the origin



To this aim, note that since we are assuming $D(E)$ small, from (4.14) we get that $1 \leq \sigma \leq 1 + C_0D(E)$, with $C_0 = C_0(n)$. Thus, from (4.14), we get

$$\begin{aligned} P(F) &= \sigma^{n-1} P(\tilde{E}) \leq \sigma^{n-1} (P(E) + P(B)D(E)) \\ &= \sigma^{n-1} P(B)(1 + 2D(E)) \leq P(B)(1 + C(n)D(E)). \end{aligned}$$

Hence, the first inequality in (4.5) follows. To prove the second inequality, let us denote by $B(x)$ an optimal ball for F . From the first inequality in (4.14), we then get

$$\begin{aligned} |B|\alpha(E) &\leq |E \Delta B(x/\sigma)| \leq |E \Delta \tilde{E}| + |\tilde{E} \Delta B_{1/\sigma}(x/\sigma)| + |B_{1/\sigma}(x/\sigma) \Delta B(x/\sigma)| \\ &= |E \setminus \tilde{E}| + \frac{|B|\alpha(F)}{\sigma^n} + |B \setminus B_{1/\sigma}| \\ &\leq C(n)D(E) + |B|\alpha(F) + C(n)(\sigma - 1) \leq |B|\alpha(F) + CD(E). \end{aligned}$$

Thus, the set F satisfies (4.5). Starting from this set, we may repeat the same construction with respect to the x_{n-1} axis, thus getting a new set, still denoted by F , uniformly bounded with respect to the first two coordinate directions and satisfying (4.5) with a new constant, still depending only on n . Thus, the assertion follows by repeating this argument with respect to all remaining coordinate directions. \square

The next step is to show that we may reduce to a n -symmetric set. To this aim, following the terminology introduced in [76] we say that $E \subset \mathbb{R}^n$ is an n -symmetric set if it is symmetric with respect to the n coordinate hyperplanes. Note that for an n -symmetric set it is not true in general that the optimal ball for the Fraenkel asymmetry is the one centered at the center of symmetry, see Fig. 2. However, this is true if the set E is convex as shown in the next result that was proved in [11, Lemma 5.9].

Proposition 4.3 *Let $E \subset \mathbb{R}^n$ be an n -symmetric bounded convex set with $|E| = |B_r|$. Then*

$$\min_{x \in \mathbb{R}^n} |E \Delta B_r(x)| = |E \Delta B_r|. \tag{4.15}$$

Proof Given $F, G \subset \mathbb{R}^n$, consider their Steiner symmetral F^S, G^S with respect to any coordinate hyperplane. Then, it is easily checked that $|F^S \cap G^S| \geq |F \cap G|$. Since E is n -symmetric and convex, E is stable under the Steiner symmetrization with respect to any coordinate hyperplane. Hence for any $x \in \mathbb{R}^n$ we have

$$|E \cap B_r(x)| \leq |E \cap B_r(x_1, \dots, x_{n-1}, 0)| \leq \dots \leq |E \cap B_r|.$$

from which (4.15) immediately follows. □

On the other hand, though for n symmetric sets the ball centered at the origin is not always a minimizer for the Fraenkel asymmetry, yet it is ‘optimal’ up to a constant.

Lemma 4.4 *Let E be n -symmetric, $|E| = |B_r|$. Then*

$$\min_{x \in \mathbb{R}^n} |E \Delta B_r(x)| \leq |E \Delta B_r| \leq 3 \min_{x \in \mathbb{R}^n} |E \Delta B_r(x)|$$

Proof Let $B_r(x_0)$ be an optimal ball for E . Since E is n -symmetric, also $B_r(-x_0)$ is optimal. Thus, using twice the triangular inequality and the inequality $|B_r(x_0) \Delta B_r| \leq |B_r(x_0) \Delta B_r(-x_0)|$, we have

$$\begin{aligned} |E \Delta B_r| &\leq |E \Delta B_r(x_0)| + |B_r(x_0) \Delta B_r| \leq |E \Delta B_r(x_0)| + |B_r(x_0) \Delta B_r(-x_0)| \\ &\leq |E \Delta B_r(x_0)| + |B_r(x_0) \Delta E| + |E \Delta B_r(-x_0)| = 3|E \Delta B_r(x_0)|. \end{aligned}$$

□

Remark 4.5 The same argument used to prove the above lemma, shows that if E is symmetric with respect to k orthogonal hyperplanes $H_1, \dots, H_k, k = 1, \dots, n$, and $H = \cap_{i=1}^k H_i$, then

$$\min_{x \in \mathbb{R}^n} |E \Delta B_r(x)| \leq \min_{y \in H} |E \Delta B_r(y)| \leq 3 \min_{x \in \mathbb{R}^n} |E \Delta B_r(x)|. \tag{4.16}$$

The next theorem is the key point of the whole proof of the quantitative isoperimetric inequality via symmetrization.

Theorem 4.6 *There exist δ and C , depending only on n , such that if $E \subset Q_l, |E| = |B|, D(E) \leq \delta$, then there exists an n -symmetric set F such that $F \subset Q_{2l}, |F| = |E|$ and*

$$\alpha(E) \leq C\alpha(F), \quad D(F) \leq 2^n D(E). \tag{4.17}$$

The proof of Theorem 4.6 is quite tricky. In order to explain the strategy we start with some definitions. Given a hyperplane H , we consider the two half spaces H^+, H^- in which \mathbb{R}^n is divided by H and denote by r_H the reflection about H . Let E be

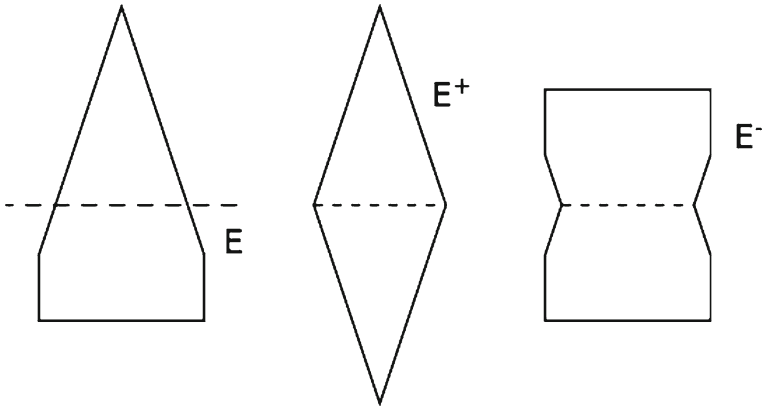


Fig. 3 Construction of E^+ and E^-

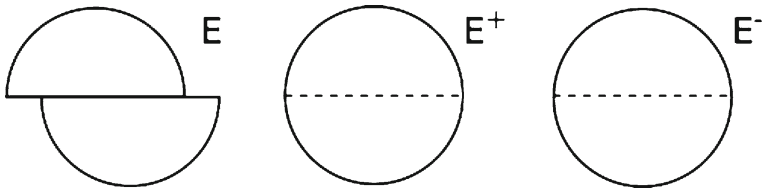


Fig. 4 Counterexample to (4.20)

a measurable set divided by the hyperplane H in two parts of equal volume. Then consider the two halves in which E is divided by H : $E \cap H^+$ and $E \cap H^-$ and the sets obtained by adding to each half its symmetral with respect to H , i.e.,

$$E^+ := (E \cap H^+) \cup r_H(E \cap H^+), \quad E^- := (E \cap H^-) \cup r_H(E \cap H^-). \quad (4.18)$$

The construction is illustrated in Fig. 3. By construction, $|E^+| = |E^-| = |E|$. Moreover, using Theorem 2.2 it is not too hard to show that

$$P(E^+) + P(E^-) \leq 2P(E), \quad \text{hence} \quad D(E^\pm) \leq 2D(E), \quad (4.19)$$

with the first inequality possibly being strict. Thus, if for some universal constant $C(n)$ one had

$$\alpha(E) \leq C(n)\alpha(E^+) \quad \text{or} \quad \alpha(E) \leq C(n)\alpha(E^-), \quad (4.20)$$

iterating this estimate we would immediately get (4.17). Unfortunately, (4.20) is false, as shown by the example in Fig. 4, where we have $\alpha(E) > 0$, but $\alpha(E^+) = \alpha(E^-) = 0$.

The following proposition provides the right strategy in order to deal with such unpleasant situation.

Proposition 4.7 *There exist $\delta_0, C_0 > 0$ such that if E is a measurable set with finite measure and $D(E) \leq \delta_0$, given any two orthogonal hyperplanes H_1, H_2 dividing E*

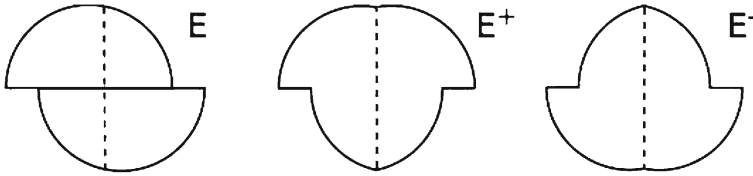


Fig. 5 A different symmetrization may give a bigger asymmetry

in four parts of equal measure and the four sets E_1^\pm, E_2^\pm defined as in (4.18) with H replaced by H_1 and H_2 , respectively, we have that at least one of them, call it \tilde{E} , satisfies the estimate

$$\alpha(E) \leq C_0 \alpha(\tilde{E}). \tag{4.21}$$

In other words, even if some of the sets E_i^\pm have a small asymmetry (or even zero asymmetry as in Fig. 4), at least one of them must have a sufficiently large asymmetry, compared to the one of E . To convince oneself that indeed this is the case, observe that if we symmetrize the set of the previous example in the horizontal direction we get a larger asymmetry, see Fig. 5.

Before proving Proposition 4.7 let us show the continuity of the asymmetry index with respect to the isoperimetric deficit which is a simple consequence of the compactness properties of sets of finite perimeter.

Lemma 4.8 *Let $l > 0$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset Q_l$, $|E| = |B|$, and $D(E) \leq \delta$ then $\alpha(E) \leq \varepsilon$.*

Proof We argue by contradiction. If the assertion is not true, there exist $\varepsilon > 0$ and a sequence of sets $E_j \subset Q_l$, with $|E_j| = |B|$, $D(E_j) \rightarrow 0$ and $\alpha(E_j) \geq \varepsilon > 0$ for all $j \in \mathbb{N}$. Since the sets E_j are all contained in Q_l , thanks to Theorem 2.3 we may assume that up to a subsequence the E_j converge in measure to some set E_∞ of finite perimeter. Thus $|E_\infty| = |B|$, and by the lower semicontinuity of the perimeters $P(E_\infty) \leq P(B)$, so E_∞ is a ball. However the convergence in measure of E_j to E_∞ immediately implies that $|E_j \Delta E_\infty| \rightarrow 0$, against the assumption $\alpha(E_j) \geq \varepsilon$. The contradiction concludes the proof. \square

Following [94, Th. 6.1] we may now give the proof of Proposition 4.7.

Proof of Proposition 4.7 Step 1 Without loss of generality we may assume $|E| = |B|$. Let $E_1^+, E_1^-, E_2^+, E_2^-$, be the four sets obtained by reflecting E around the two orthogonal hyperplanes H_1, H_2 dividing E in parts of equal measure. For $i = 1, 2$ let B_i^\pm four balls of radius one such that

$$|E_i^\pm \Delta B_i^\pm| = \min_{x \in H_i} |E_i^\pm \Delta B(x)|.$$

Observe that for $i = 1, 2$, from the definition of B_i^\pm and the triangular inequality we have

$$\begin{aligned}
 \min_{x \in \mathbb{R}^n} |E \Delta B(x)| &\leq |E \Delta B_i^+| = |(E \Delta B_i^+) \cap H_i^+| + |(E \Delta B_i^+) \cap H_i^-| \\
 &\leq |(E \Delta B_i^+) \cap H_i^+| + |(E \Delta B_i^-) \cap H_i^-| + |(B_i^+ \Delta B_i^-) \cap H_i^-| \\
 &= \frac{1}{2} |E_i^+ \Delta B_i^+| + \frac{1}{2} |E_i^- \Delta B_i^-| + \frac{1}{2} |B_i^+ \Delta B_i^-|. \tag{4.22}
 \end{aligned}$$

To conclude the proof it is enough to show that if $D(E)$ is sufficiently small at least one of the following two inequalities

$$\begin{aligned}
 |B_1^+ \Delta B_1^-| &\leq 16[|E_1^+ \Delta B_1^+| + |E_1^- \Delta B_1^-|] \\
 \text{or } |B_2^+ \Delta B_2^-| &\leq 16[|E_2^+ \Delta B_2^+| + |E_2^- \Delta B_2^-|] \tag{4.23}
 \end{aligned}$$

holds. Indeed, if for instance the first inequality is true, from (4.22) applied with $i = 1$, recalling the definition of B_1^\pm and (4.16), we get

$$\begin{aligned}
 \min_{x \in \mathbb{R}^n} |E \Delta B(x)| &\leq 9[|E_1^+ \Delta B_1^+| + |E_1^- \Delta B_1^-|] \\
 &\leq 27 \left[\min_{x \in \mathbb{R}^n} |E_1^+ \Delta B(x)| + \min_{x \in \mathbb{R}^n} |E_1^- \Delta B(x)| \right],
 \end{aligned}$$

thus proving (4.21) with $C_0 = 54$ and \tilde{E} equal to E_1^+ or E_1^- .

Step 2 To prove (4.23) we argue by contradiction assuming that

$$|E_1^+ \Delta B_1^+| + |E_1^- \Delta B_1^-| < \frac{|B_1^+ \Delta B_1^-|}{16} \quad \text{and} \quad |E_2^+ \Delta B_2^+| + |E_2^- \Delta B_2^-| < \frac{|B_2^+ \Delta B_2^-|}{16} \tag{4.24}$$

and introduce the following unions of half balls

$$S_1 = (B_1^+ \cap H_1^+) \cup (B_1^- \cap H_1^-), \quad S_2 = (B_2^+ \cap H_2^+) \cup (B_2^- \cap H_2^-).$$

Then, by (4.24) we get that

$$|S_1 \Delta S_2| \leq |S_1 \Delta E| + |E \Delta S_2| = \frac{1}{2} \sum_{i=1}^2 (|E_i^+ \Delta B_i^+| + |E_i^- \Delta B_i^-|) < \frac{1}{32} \sum_{i=1}^2 |B_i^+ \Delta B_i^-|.$$

The contradiction will be achieved if we show that

$$\max\{|B_1^+ \Delta B_1^-|, |B_2^+ \Delta B_2^-|\} < 16|S_1 \Delta S_2|, \tag{4.25}$$

provided $D(E)$ is sufficiently small. To this aim, recall that given $\varepsilon > 0$, Lemma 4.8, (4.16) and the second inequality in (4.19) yield that there exists $\delta_0 > 0$ such that if $D(E) < \delta_0$, then

$$\max\{\alpha(E), |E_i^\pm \Delta B_i^\pm| : i = 1, 2\} < \varepsilon.$$

Note that this inequality tells us that E is close in the sense of convergence in measure to its optimal balls. Similarly, also the balls B_i^\pm are close to the sets E_i^\pm in the same sense. Hence the centers of these balls are all close to each other and to the center of an optimal ball for E which in turn is close to the ball of radius 1 centered at the intersection of H^1 and H^2 . Therefore all the four regions $H_1^\pm \cap H_2^\pm$ contain almost one quarter of the balls B_i^\pm . Therefore we may choose ε so small, and δ_0 accordingly, that for all possible choices of the $\sigma, \tau \in \{+, -\}$ we have

$$|(B_1^\sigma \Delta B_2^\tau) \cap (H_1^\sigma \cap H_2^\tau)| > \frac{|B_1^\sigma \Delta B_2^\tau|}{8}.$$

Thus, recalling the definition of S_1 and S_2 , we have

$$|S_1 \Delta S_2| \geq |(S_1 \Delta S_2) \cap (H_1^\sigma \cap H_2^\tau)| = |(B_1^\sigma \Delta B_2^\tau) \cap (H_1^\sigma \cap H_2^\tau)| > \frac{|B_1^\sigma \Delta B_2^\tau|}{8}.$$

From this inequality we have in particular that

$$|B_1^+ \cap B_1^-| \leq |B_1^+ \cap B_2^+| + |B_2^+ \cap B_1^-| < 16|S_1 \Delta S_2|$$

and in a similar way that $|B_2^+ \cap B_2^-| < 16|S_1 \Delta S_2|$, thus proving the desired contradiction (4.25). □

At this point we have all the ingredients for the proof of Theorem 4.6.

Proof of Theorem 4.6 Take $\delta_1 = \delta_0 2^{-(n-1)}$, where δ_0 is chosen as in Proposition 4.7. By applying this proposition $n - 1$ times to different pairs of orthogonal directions and recalling (4.19) we find a set \tilde{E} , with $|\tilde{E}| = |B|$ such that

$$\alpha(E) \leq C_0^{n-1} \alpha(\tilde{E}), \quad D(\tilde{E}) \leq 2^{n-1} D(E).$$

Moreover, by translating \tilde{E} and relabeling the coordinate axes, if needed, we may assume without loss of generality that \tilde{E} is symmetric about all the coordinate hyperplanes $\{x_1 = 0\}, \dots, \{x_{n-1} = 0\}$. In order to get the last symmetry we take a hyperplane H orthogonal to e_n and dividing \tilde{E} into two parts of equal measure and consider the corresponding sets \tilde{E}^+, \tilde{E}^- . Again, by translating \tilde{E} in the direction of e_n , if necessary, we may assume that $H = \{x_n = 0\}$. As before we have

$$D(\tilde{E}^\pm) \leq 2D(\tilde{E}) \leq 2^n D(E).$$

To control the asymmetry of \tilde{E}^\pm observe that since \tilde{E} is symmetric with respect to the first $n - 1$ coordinate hyperplanes, \tilde{E}^+ and \tilde{E}^- are both n -symmetric so we can use Lemma 4.4 to get

$$\begin{aligned} |B|\alpha(\tilde{E}) &\leq |\tilde{E} \Delta B| = |(\tilde{E} \Delta B) \cap \{x_n > 0\}| + |(\tilde{E} \Delta B) \cap \{x_n < 0\}| \\ &= \frac{1}{2}[|\tilde{E}^+ \Delta B| + |\tilde{E}^- \Delta B|] \leq \frac{3|B|}{2}[\alpha(\tilde{E}^+) + \alpha(\tilde{E}^-)]. \end{aligned}$$

Thus, at least one of the sets \tilde{E}^\pm has asymmetry index greater than $\frac{1}{3}\alpha(\tilde{E})$. Therefore, denoting by F this set, we have

$$D(F) \leq 2D(\tilde{E}) \leq 2^n D(E) \quad \text{and} \quad \alpha(E) \leq C_0^{n-1}\alpha(\tilde{E}) \leq 3C_0^{n-1}\alpha(F).$$

Finally, the inclusion $F \subset Q_{2l}$ follows immediately from the construction performed in the proof of Proposition 4.7 and the one performed here. \square

4.2 Reduction to axially symmetric sets

From the results obtained in the previous section it is clear that in order to prove the quantitative isoperimetric inequality (4.3) we may assume without loss of generality that there exist $\delta_0 \in (0, 1)$ and $l > 0$ such that

$$|E| = |B|, \quad E \subset Q_l, \quad D(E) \leq \delta_0, \quad E \text{ is } n\text{-symmetric.} \tag{4.26}$$

In fact, since $\alpha(E) \leq 2$ it is clear that if $D(E) \geq \delta_0$ (4.3) follows immediately with $\gamma(n) = 4/\delta_0$. Thus, if δ_0 is sufficiently small Lemma 4.2 and Theorem 4.6 tell us that we may assume without loss of generality that E is contained in some cube of fixed size and that it is n -symmetric. Therefore, throughout this section we shall always assume that E satisfies the above assumptions (4.26).

The next step consists in reducing the general case to the case of an *axially symmetric set*, i.e., a set E having an axis of symmetry such that every non-empty cross section of E perpendicular to this axis is an $(n - 1)$ -ball. To this aim we recall that the *Schwarz symmetral* of a measurable set E with respect to the x_n axis is defined as

$$E^* = \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \in \mathbb{R}, |x| < r_E(t)\},$$

where $r_E(t)$ is the radius of the $(n - 1)$ -dimensional ball having the same measure of the section E_t , that is $\mathcal{H}^{n-1}(E_t) = \omega_{n-1}r_E(t)^{n-1}$. Note that E^* coincides with the Steiner symmetral of E with respect to the line $\{x' = 0\}$. Therefore, from Theorem 2.7 we have that $P(E^*) \leq P(E)$. Moreover the properties stated in Theorems 2.6 and 2.7 apply to the functions defined for \mathcal{H}^1 -a.e. $t \in \mathbb{R}$ as $v_E(t) := \mathcal{H}^{n-1}(E_t)$ and $p_E(t) := \mathcal{H}^{n-2}(\partial^* E_t)$.

Proposition 4.9 *Let E satisfy assumptions (4.26). If $n = 2$ or if $n \geq 3$ and the quantitative isoperimetric inequality (4.3) holds true in \mathbb{R}^{n-1} , then there exist a constant $C(n)$ such that*

$$|E \Delta E^*| \leq C\sqrt{D(E)} \quad \text{and} \quad D(E^*) \leq D(E). \tag{4.27}$$

Proof The second inequality in (4.27) follows immediately from the fact that $|E^*| = |E|$ and $P(E^*) \leq P(E)$. In order to prove the first inequality in (4.27) we start by assuming that

$$\mathcal{H}^{n-1}(\{x = (x', t) \in \partial^* E : \nu_{x'}^E(x) = 0\}) = 0. \tag{4.28}$$

Thus, from Theorem 2.6 we have that v_E is an absolutely continuous function in \mathbb{R} . We use Theorem 2.6 again and Theorem 2.7 to estimate

$$\begin{aligned} P(E) - P(B) &\geq P(E) - P(E^*) \geq \int_{\mathbb{R}} \left(\sqrt{p_E^2 + v_E'^2} - \sqrt{p_{E^*}^2 + v_E'^2} \right) dt \\ &= \int_{\mathbb{R}} \frac{p_E^2 - p_{E^*}^2}{\sqrt{p_E^2 + v_E'^2} + \sqrt{p_{E^*}^2 + v_E'^2}} dt \\ &\geq \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2} dt \right)^2 \frac{1}{\int_{\mathbb{R}} \left(\sqrt{p_E^2 + v_E'^2} + \sqrt{p_{E^*}^2 + v_E'^2} \right) dt} \\ &\geq \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2} dt \right)^2 \frac{1}{P(E) + P(E^*)}, \end{aligned}$$

where the inequality before the last one follows from Hölder’s inequality. Since $D(E) \leq \delta_0 < 1$, we have $P(E^*) \leq P(E) \leq 2P(B)$. Therefore we get, observing that $p_E \geq p_{E^*}$,

$$\begin{aligned} \sqrt{D(E)} &\geq c \int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2} dt = c \int_{\mathbb{R}} \sqrt{p_E + p_{E^*}} \sqrt{p_E} \sqrt{(p_E - p_{E^*})/p_{E^*}} dt \\ &\geq \sqrt{2}c \int_{\mathbb{R}} p_{E^*} \sqrt{(p_E - p_{E^*})/p_{E^*}} dt, \end{aligned} \tag{4.29}$$

for some constant c depending only on n . Now assume that $n \geq 3$ and observe that since $(E^*)_t$ is a $(n - 1)$ -dimensional ball of radius $r_E(t)$ with \mathcal{H}^{n-1} measure equal to $\mathcal{H}^{n-1}(E_t)$, the ratio

$$\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}$$

is precisely the isoperimetric deficit in \mathbb{R}^{n-1} of E_t . Since by assumption, the quantitative isoperimetric inequality (4.3) holds true in \mathbb{R}^{n-1} , we have

$$\sqrt{\gamma(n-1)} \sqrt{\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}} \geq \alpha_{n-1}(E_t),$$

where $\alpha_{n-1}(E_t)$ is the $(n - 1)$ -dimensional Fraenkel asymmetry of E_t . But E_t is an $(n - 1)$ -symmetric set in \mathbb{R}^{n-1} and $(E^*)_t$ is the ball centered at the center of symmetry of E_t . Therefore from Lemma 4.4 we get

$$\sqrt{\gamma(n-1)} \sqrt{\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}} \geq \alpha_{n-1}(E_t) \geq \frac{1}{3} \frac{\mathcal{H}^{n-1}(E_t \Delta (E^*)_t)}{\mathcal{H}^{n-1}((E^*)_t)}.$$

Inserting this inequality in (4.29) we then get

$$\sqrt{D(E)} \geq c' \int_{\mathbb{R}} \frac{\mathcal{H}^{n-1}(E_t \Delta E_t^*)}{r_E(t)} dt \geq \frac{c'}{l} \int_{-l}^l \mathcal{H}^{n-1}(E_t \Delta E_t^*) dt = \frac{c'}{l} |E_t \Delta E_t^*|,$$

where the inequality before the last one follows from the inclusion $E \subset Q_l$ and the last equality is just Fubini's theorem. This proves (4.27). Hence the assertion follows when $n \geq 3$.

If $n = 2$, since E is 2-symmetric, either E_t is a symmetric interval (hence $E_t = E_t^*$) or E_t is the union of at least two intervals with no endpoints in common, hence $p_E(t) \geq 4$, while $p_{E^*}(t) = 2$. Moreover, since $E \subset Q_l$, then $\mathcal{H}^1(E_t \Delta E_t^*) \leq 2l$ for all $t \in \mathbb{R}$. Therefore, from (4.29) we get

$$\begin{aligned} \sqrt{D(E)} &\geq 2c \int_{\{t: E_t \neq E_t^*\}} \sqrt{p_E - p_{E^*}} dt \geq 2c \int_{\{t: E_t \neq E_t^*\}} \sqrt{2} dt \\ &\geq \frac{\sqrt{2}c}{l} \int_{\{t: E_t \neq E_t^*\}} \mathcal{H}^1(E_t \Delta E_t^*) dt = \frac{\sqrt{2}c}{l} |E \Delta E^*|, \end{aligned}$$

thus concluding the proof also in this case.

Finally, if (4.28) does not hold, we approximate E in measure by a sequence of sets E_h obtained by rotating a little E so that (4.28) holds true for each E_h . Then, the conclusion follows by observing that also E_h^* converge to E^* in measure and that (4.27) holds for all the sets E_h . □

For an axially symmetric set the proof of the quantitative isoperimetric inequality was given by a direct argument in [82, Th. 2]. A different proof can be also found in [76, Th. 4.2]. However both proofs are technically a bit complicate. Here we shall give a third proof due to Maggi [94, Sec. 7.2]. This last proof has two interesting features. First, it is based on a further and very natural reduction argument, which was not exploited in the original proof of (4.3) given in [76], the reduction to a connected set. This further reduction, is again a consequence of the strict concavity of the isoperimetric function. Another interesting feature is the use of some ideas from the mass transportation argument of Gromov's proof of the isoperimetric inequality, to which we will come back in the next section. So, let us start with the reduction lemma proved in [94, Th. 4.4].

Lemma 4.10 *There exists two positive constants, δ and C , depending only on the dimension n , such that if E is a bounded open set of class C^∞ with $D(E) \leq \delta$ there exists a connected component F of E such that $|F| > |E|/2$,*

$$\alpha(E) \leq \alpha(F) + CD(E), \quad D(F) \leq CD(E). \tag{4.30}$$

Proof Without loss of generality we may assume $|E| = |B|$. Since E is a smooth bounded open set, if E is not connected we may decompose it as the union of finitely many connected components $E_h, h \in I \subset \mathbb{N}$. Moreover, $P(E) = \sum_{h \in I} P(E_h)$. From the isoperimetric inequality (1.4) we have

$$\begin{aligned}
 D(E) &= \frac{1}{n|B|} \left(\sum_{h \in I} P(E_h) - P(B) \right) \geq \frac{1}{n|B|} \left(n|B|^{1/n} \sum_{h \in I} |E_h|^{\frac{n-1}{n}} - n|B| \right) \\
 &= \sum_{h \in I} a_h^{\frac{n-1}{n}} - 1,
 \end{aligned}$$

where $a_h := |E_h|/|B|$. Observe that $\sum_{h \in I} a_h = 1$ and assume that the a_h are arranged in decreasing order. If $a_1 > 1/2$, from inequality (4.9) we have

$$D(E) \geq a_1^{\frac{n-1}{n}} + (1 - a_1)^{\frac{n-1}{n}} - 1 \geq 2(2^{1/n} - 1)(1 - a_1), \tag{4.31}$$

hence $|E \setminus E_1| \leq C_0 D(E)$, with $C_0 = \omega_n / 2(2^{1/n} - 1)$. Thus, we set $F = E_1$ and we have, denoting by r the radius of the ball with the same measure of F ,

$$P(F) - P(B_r) \leq P(E) - P(B) + P(B) - P(B_r) \leq P(E) - P(B) + n\omega_n(1 - r^{n-1}).$$

From this inequality, since $\omega_n(1 - r^n) = |E \setminus E_1| \leq C_0 D(E)$ and $\omega_n r^n = |F| > |B|/2$, we easily get

$$D(F) \leq C D(E)$$

for some possibly bigger constant, still depending only on n . Similarly, denoting by $B_r(x_0)$ an optimal ball for $\alpha(F)$ we get

$$\begin{aligned}
 \omega_n \alpha(E) &\leq |E \Delta B(x_0)| \leq |E \Delta F| + |F \Delta B_r(x_0)| + |B_r(x_0) \Delta B(x_0)| \\
 &= |F \Delta B_r(x_0)| + 2|E \setminus F| \leq |F \Delta B_r(x_0)| + 2C_0 D(E).
 \end{aligned}$$

thus completing the proof of (4.30).

To conclude the proof of the lemma it is enough to show that if δ is sufficiently small, in dependence on n , then $a_1 > 1/2$. In fact, suppose that $a_1 \leq 1/2$ and denote by $N \geq 2$ the smallest integer such that $\sum_{h < N} a_h \leq 1/2$. Then, using again (4.9) and arguing as in the proof of (4.31), we have

$$\begin{aligned}
 D(E) &\geq \left(\sum_{h < N} a_h \right)^{\frac{n-1}{n}} + \left(\sum_{h \geq N} a_h \right)^{\frac{n-1}{n}} - 1 \geq 2(2^{1/n} - 1) \sum_{h < N} a_h \\
 D(E) &\geq \left(\sum_{h \leq N} a_h \right)^{\frac{n-1}{n}} + \left(\sum_{h > N} a_h \right)^{\frac{n-1}{n}} - 1 \geq 2(2^{1/n} - 1) \sum_{h > N} a_h
 \end{aligned}$$

Adding up these two inequalities we then conclude that

$$D(E) \geq (2^{1/n} - 1) \sum_{h \neq N} a_h = (2^{1/n} - 1)(1 - a_N) \geq (2^{1/n} - 1)(1 - a_1) > \frac{2^{1/n} - 1}{2},$$

which is impossible if we choose $\delta < 2/(2^{1/n} - 1)$. □

We may now proceed to the proof of the quantitative isoperimetric inequality. As we already said before, for the final argument, i.e. the proof of (4.3) for an axially and n -symmetric set, we follow the argument of Maggi [94].

Proof of Theorem 4.1 Step 1 As we observed at the beginning of this section, it is enough to prove the quantitative inequality (4.3) only when E satisfies the assumptions (4.26) for a sufficiently small δ_0 , depending only on the dimension, that will be specified during the proof. Observe that we may also assume without loss of generality that E is a bounded open set of class C^∞ . Otherwise, by a standard approximation procedure, see for instance the proof of [8, Th. 3.42], we may find a sequence of n -symmetric smooth open sets E_h converging in measure to E , $|E_h| = |E|$ for all h , $P(E_h) \rightarrow P(E)$, satisfying the assumptions (4.26) with l possibly replaced by $2l$. Then the quantitative isoperimetric inequality for E will follow from the same inequality for E_h .

Finally, observe that we may also assume that E is connected. In fact if E were not connected, taking δ_0 sufficiently small we could apply Lemma 4.10 and consider the open connected component F of E such that $|F| > |E|/2$, satisfying (4.30). Observe that also this connected component is n -symmetric. Indeed, consider for instance the hyperplane $H = \{x_1 = 0\}$ and assume by contradiction that there exists $x \in F$ such that its reflection $r_H(x) \notin F$. Then $F \cap H = \emptyset$, since otherwise x and $r_H(x)$ would belong to the same connected component of E , hence both would belong to F . Since E is n -symmetric, also $r_H(F)$ is contained in E . However, this is not possible since F is contained either in H^+ or in H^- and thus $|F \cup r_H(F)| = 2|F| > |E|$. This proves that $r_H(F) = F$. By repeating this argument for all the coordinate hyperplanes, we conclude that F is n -symmetric. Thus, we may replace E by λF , where $\lambda > 0$ is such that $\lambda^n |F| = |B|$. The set λF is now a bounded, smooth, open and connected set satisfying all the assumptions in (4.26), with l possibly replaced by $2l$.

Consider the strips $S = \{x : |x_n| < \sqrt{2}/2\}$ and $S' = \{x : |x_1| < \sqrt{2}/2\}$. Since $B \subset S \cup S'$ one of the two strips must contain at least one half of the measure of $B \setminus E$. To fix the ideas, let us assume that S does it, that is $|(B \setminus E) \cap S| \geq |B \setminus E|/2$. Therefore,

$$|E \Delta B| \leq 4|(B \setminus E) \cap S| \tag{4.32}$$

Denote by E^* is the Schwartz symmetral of E with respect to the x_n axis. We claim that if $D(E^*)$ is sufficiently small, then

$$|(B \Delta E^*) \cap S| \leq C(n)\sqrt{D(E^*)}. \tag{4.33}$$

Note that the conclusion of the proof will follow immediately from (4.33). Indeed, assume $n = 2$ or $n \geq 3$ and that (4.3) holds true in \mathbb{R}^{n-1} . Then, if $D(E)$ is sufficiently small, from Lemma 4.4, and (4.27), (4.32) and (4.33) we get

$$\begin{aligned} |B|\alpha(E) &\leq 3|E \Delta B| \leq 12|(B \setminus E) \cap S| \\ &\leq 12(|(B \Delta E^*) \cap S| + |E^* \Delta E|) \leq C(n)\sqrt{D(E^*)} \leq C(n)\sqrt{D(E)}. \end{aligned}$$

The rest of the proof will be devoted to prove (4.33).

Step 2 Set $v(t) := \mathcal{H}^{n-1}(E^* \cap \{x_n = t\})$. From our assumptions and Theorem 2.6 we have that $v \in BV(\mathbb{R})$, v is symmetric with respect to 0 and $\{v > 0\} = (-a, a)$ for some $a > 0$. Observe that we may also assume without loss of generality that $v \in W^{1,1}(\mathbb{R})$. Otherwise, we split v as $v = v^a + v^s$ where v^a and v^s are nonnegative, symmetric with respect to 0, $v^a \in W^{1,1}(\mathbb{R})$ and v^s is purely singular, see [8, Sect. 3.2], and we approximate v^s in $L^1(\mathbb{R})$ with a sequence $w_h \in C_c^1(\mathbb{R})$, with w_h nonnegative and symmetric with respect to 0, $w_h > 0$ in an interval $(-a', a')$ and such that $\int_{-a}^a |w'_h| \rightarrow |Dv^s|(\mathbb{R})$ as $h \rightarrow \infty$. Then denoting by E_h^* the axially symmetric sets such that $v_{E_h^*} = v^a + w_h$, from (2.7) we get immediately that the E_h^* converge to E^* in measure and that $P(E_h^*) \rightarrow P(E^*)$. Thus we prove (4.33) for E_h^* and then pass to the limit as $h \rightarrow \infty$.

So, let us assume that $v \in W^{1,1}(\mathbb{R})$ and in particular that it is continuous. We first show that if $D(E^*)$ is sufficiently small, then the strip S contains a fairly large amount of the mass of E^* . Precisely, we are going to show that there exist c_0 and δ_0 , depending only on the dimension n such that if $D(E^*) \leq \delta_0$, then

$$v(t) \geq c_0 \quad \text{for all } t \in \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right). \tag{4.34}$$

In order to prove this inequality we compare the distribution function v of E^* with the distribution function w of B , defined as $w(t) := \mathcal{H}^{n-1}(B \cap \{x_n = t\})$.

We set $c_0(n) = w(4/5)/3$ and then argue by contradiction. If (4.34) is not true, there exists $t_0 \in (-\sqrt{2}/2, \sqrt{2}/2) \subset (-4/5, 4/5)$ such that $v(t_0) < c_0$. Then, there exist $t_1 < t_0 < t_2$ such that (t_1, t_2) is the largest interval containing t_0 , contained in $(-4/5, 4/5)$ and such that

$$v(t) \leq 2c_0 \quad \text{for all } t \in (t_1, t_2).$$

By the definition of c_0 , recalling Lemma 4.4, we immediately have

$$c_0(t_2 - t_1) \leq \int_{\mathbb{R}} |w - v| dx = |E^* \Delta B| \leq 3|B|\alpha(E^*). \tag{4.35}$$

Then Lemma 4.8 implies that $t_2 - t_1$ is small if $D(E^*)$ is small. Therefore the closed interval $[t_1, t_2]$ is contained in $(-4/5, 4/5)$ and thus in particular $v(t_1) = v(t_2) = 2c_0$. Now we compare E^* with the axially symmetric set whose distribution function v_F coincides with v outside the interval (t_1, t_2) and such that $v_F \equiv 2c_0$ in (t_1, t_2) . Then, setting $\Sigma := \mathbb{R}^{n-1} \times (t_1, t_2)$,

$$P(F) = P(E^*) + P(F; \Sigma) - P(E^*; \Sigma).$$

Clearly $P(F; \Sigma) = c_1(n)(t_2 - t_1)$ with $c_1(n)$ depending only on $c_0(n)$ and from (2.7) we have

$$P(E^*; \Sigma) \geq \int_{t_1}^{t_2} |v'(t)| dt \geq |v(t_2) - v(t_0)| + |v(t_0) - v(t_1)| = 2c_0.$$

Thus, from the above inequalities, observing that $|F| \geq |E^*| = |B|$ and using the isoperimetric inequality, we get

$$P(E^*) \geq P(F) + 2c_0 - c_1(n)(t_2 - t_1) \geq P(B) + 2c_0 - c_1(n)(t_2 - t_1).$$

Since by (4.35) $t_2 - t_1 \rightarrow 0$ as $D(E^*) \rightarrow 0^+$, from this inequality we get a contradiction if $D(E^*)$ is sufficiently small.

Step 3 Recall that $\{v > 0\} = (-a, a)$ for some $a > 0$. Observe that (4.34) implies that $a > \sqrt{2}/2$. Then, we define a function $\tau : (-a, a) \rightarrow (-1, 1)$ which associates to every sublevel $E^* \cap \{x_n < t\}$ the sublevel $B \cap \{x_n < \tau(t)\}$ with the same measure. Precisely, for all $t \in (-a, a)$ we define $\tau(t)$ as the unique number in $(-1, 1)$ such that

$$\int_{-\infty}^t v(s) ds = \int_{-\infty}^{\tau(t)} w(s) ds.$$

Note that τ is a strictly increasing $C^1(-a, a)$ function and for all $t \in (-a, a)$

$$\tau'(t) = \frac{v(t)}{w(\tau(t))}. \tag{4.36}$$

Since $v \in W^{1,1}(\mathbb{R})$ the above equality implies that $\tau \in W_{loc}^{2,1}(-a, a)$. Moreover, since w is locally Lipschitz in $(-1, 1)$, setting $I := (-\sqrt{2}/2, \sqrt{2}/2)$, we get from (4.36)

$$\begin{aligned} |(E^* \Delta B) \cap S| &= \int_I |w(t) - v(t)| dt = \int_I |w(t) - w(\tau(t))\tau'(t)| dt \\ &\leq \int_I [|w(t) - w(\tau(t))| + |w(\tau(t)) - w(\tau(t))\tau'(t)|] dt \\ &\leq C(n) \int_I [|t - \tau(t)| + |1 - \tau'(t)|] dt \leq C(n) \int_I |1 - \tau'(t)| dt, \end{aligned}$$

for some constant $C(n)$ depending only on the Lipschitz constant of w in I . Note that in the last inequality we have used the fact that since $\tau(0) = 0$ by the n -symmetry of E , then $|t - \tau(t)| \leq \int_I |1 - \tau'|$. Thus, to achieve the proof of (4.33) we only need to show that

$$\int_I |\tau'(t) - 1| dt \leq C(n)\sqrt{D(E^*)}. \tag{4.37}$$

To this aim we introduce the map $T : (-a, a) \times \mathbb{R}^{n-1} \mapsto \mathbb{R}^n$ defined as

$$T(x) = \sum_{i=1}^{n-1} \left(\frac{w(\tau(x_n))}{v(x_n)} \right)^{\frac{1}{n-1}} x_i e_i + \tau(x_n) e_n.$$

Note that T maps every level set $E^* \cap \{x_n = t\}$ into the level set $B \cap \{x_n = \tau(t)\}$, and it is a one-to-one continuous map between the open sets E^* and B . Moreover for all $0 < b < a$, $T \in W^{1,1}(E^* \cap (\mathbb{R}^{n-1} \times (-b, b)); B)$. It is also easily checked that for a.e. $x \in E^* \cap (\mathbb{R}^{n-1} \times (-a, a))$

$$\operatorname{div} T(x) = \frac{n-1}{\tau'(x_n)^{1/(n-1)}} + \tau'(x_n) \geq n,$$

where we used that the function $s \rightarrow s + (n-1)s^{-1/(n-1)}$ has a strict minimum at $s = 1$ for $s > 0$. Therefore, using the divergence theorem in the sets $E^* \cap (\mathbb{R}^{n-1} \times (-b, b))$ and observing that $\lim_{b \rightarrow a^-} v(\pm b) = 0$, we have

$$\begin{aligned} P(E^*) &= \mathcal{H}^{n-1}(\partial^* E^*) \geq \int_{\partial E^*} T \cdot \nu^{E^*} d\mathcal{H}^{n-1} \\ &= \lim_{b \rightarrow a^-} \int_{\partial E^* \cap (\mathbb{R}^{n-1} \times (-b, b))} T \cdot \nu^{E^*} d\mathcal{H}^{n-1} \\ &= \lim_{b \rightarrow a^-} \left[\int_{\partial(E^* \cap (\mathbb{R}^{n-1} \times (-b, b)))} T \cdot \nu^{E^*} d\mathcal{H}^{n-1} - \tau(b)v(b) + \tau(-b)v(-b) \right] \\ &= \lim_{b \rightarrow a^-} \int_{E^* \cap (\mathbb{R}^{n-1} \times (-b, b))} \operatorname{div} T \, dx = \int_{E^*} \operatorname{div} T \, dx. \end{aligned}$$

Therefore we have by Fubini's theorem

$$\begin{aligned} P(E^*) - P(B) &\geq \int_{E^*} (\operatorname{div} T - n) \, dx = \int_{E^*} \left(\frac{n-1}{\tau'(x_n)^{1/(n-1)}} + \tau'(x_n) - n \right) dx \\ &= \int_{-a}^a v(t) \left(\frac{n-1}{\tau'(t)^{1/(n-1)}} + \tau'(t) - n \right) dx \\ &= \int_{-a}^a \frac{v(t)}{\tau'(t)^{1/(n-1)}} \left(n-1 + \tau'(t)^{n/(n-1)} - n\tau'(t)^{1/(n-1)} \right) dx \\ &\geq \frac{n}{2} \int_{-a}^a \frac{v(t)}{\tau'(t)^{1/(n-1)}} \left(\tau'(t)^{1/(n-1)} - 1 \right)^2 dt, \end{aligned}$$

where we used the elementary inequality $n - 1 + t^n - nt \geq n(t - 1)^2/2$ for all $t > 0$ and $n \geq 2$. Then from the previous chain of inequalities we get, setting $\sigma(t) := \tau'(t)^{1/(n-1)}$,

$$\int_{-a}^a v|\sigma - 1| dt \leq \sqrt{\int_{-a}^a \frac{v}{\sigma} (\sigma - 1)^2 dt} \sqrt{\int_{-a}^a v\sigma dt} \leq C(n)\sqrt{D(E^*)}, \tag{4.38}$$

where the last inequality follows by observing that, since $\int_{-a}^a \tau' dt = 2$, $\int_{-a}^a v dt = |E^*| = |B|$ and $v \leq (2l)^{n-1}$ by assumption (4.26), if $n \geq 3$ we have

$$\int_{-a}^a v\sigma \, dt \leq \left(\int_{-a}^a \tau' \, dt \right)^{\frac{1}{n-1}} \left(\int_{-a}^a v^{(n-1)/(n-2)} \, dt \right)^{\frac{n-2}{n-1}} \leq C(n).$$

A similar estimate holds also when $n = 2$. To conclude the proof we observe that there exists a constant $\lambda(n) \in (0, 1)$ such that

$$\sup_{t \in I} |\tau(t)| \leq \lambda(n). \tag{4.39}$$

To this aim, note that

$$|E^* \cap \{x_n > \sqrt{2}/2\}| \geq |B \cap \{x_n > \sqrt{2}/2\}| - 3|B|\alpha(E^*) \geq c(n) - 3|B|\alpha(E^*) \geq c_2(n),$$

for some positive constant $c_2(n)$, provided that $D(E^*)$, hence $\alpha(E^*)$ is sufficiently small. On the other hand, by the very definition of τ , we have

$$|E^* \cap \{x_n > \sqrt{2}/2\}| = |B \cap \{x_n > \tau(\sqrt{2}/2)\}| \leq c(n)(1 - \tau(\sqrt{2}/2)).$$

Combining the two previous inequalities we get that $1 - \tau(\sqrt{2}/2) \geq c_3(n)$ for a suitable positive constant depending only on n , hence $\tau(\sqrt{2}/2) \leq 1 - c_3(n)$. Since a similar inequality can be proved for $\tau(-\sqrt{2}/2)$, (4.39) follows. Observe that (4.39), together with (4.36) implies that $\sup_{t \in I} |\tau'(t)| \leq C(n)$ for a suitable constant depending only on n . Then, recalling (4.34) and (4.38) we may conclude that

$$\begin{aligned} \int_I |\tau' - 1| \, dt &\leq C \int_I |(\tau')^{1/(n-1)} - 1| \, dt = C \int_I |\sigma - 1| \, dt \\ &\leq C \int_I v|\sigma - 1| \, dt \leq C\sqrt{D(E^*)}, \end{aligned}$$

thus proving (4.37). □

5 Other proofs and extensions

5.1 Isoperimetric inequality via mass transport

In the previous section we have presented the proof of the the quantitative isoperimetric inequality given in [76]. As we have seen, that proof was based on symmetrization arguments. The same approach has been used in several other papers to obtain quantitative versions of the Sobolev inequality, of the isoperimetric inequality in Gauss space and of other relevant geometric and functional inequalities, see for instance [43, 44, 75, 78]. On the other hand there are situations where one considers inequalities which are realized by non symmetric sets or functions.

This is the case of the *anisotropic isoperimetric inequality*. In order to state it let us fix some notation. Let $\varphi : \mathbb{R}^n \rightarrow [0, \infty)$ be a positively 1-homogeneous, continuous

function such that $\varphi(x) > 0$ for all $x \neq 0$. To the function φ we may associate the *anisotropic perimeter*, defined for any set E of finite perimeter as

$$P_\varphi(E) := \int_{\partial^* E} \varphi(\nu^E(x)) d\mathcal{H}^{n-1}.$$

It is well known that the isoperimetric sets for P_φ are the homothetic and translated of the so called *Wulff shape set* associated to φ which is given by

$$W_\varphi := \{x \in \mathbb{R}^n : \langle x, \nu \rangle - \varphi(\nu) < 0 \text{ for all } \nu \in \mathbb{S}^{n-1}\}.$$

Then, the anisotropic isoperimetric inequality states that

$$P_\varphi(E) \geq P_\varphi(W_\varphi)$$

for all sets of finite perimeter such that $|E| = |W_\varphi|$, with equality holding if and only if E is a translated of the Wulff shape set W_φ (see [69, 70] and also [50] for a proof in two dimensions). Note that as in the case of the standard euclidean perimeter we have $P_\varphi(W_\varphi) = n|W_\varphi|$. However, differently from the usual perimeter, P_φ is not invariant under the action of either $O(n)$ or $SO(n)$. Moreover, in general $P_\varphi(E) \neq P_\varphi(\mathbb{R}^n \setminus E)$ unless φ , hence W_φ , is symmetric with respect to the origin.

The quantitative version of the anisotropic isoperimetric inequality is a remarkable result proved by Figalli et al. in [67]. It states that there exists a constant C , depending only on n , such that for any set of finite perimeter E such that $|E| = r^n |W_\varphi|$

$$\alpha_\varphi(E)^2 \leq C D_\varphi(E), \tag{5.1}$$

where

$$\alpha_\varphi(E) := \min_{x \in \mathbb{R}^n} \left\{ \frac{|E \Delta (x + rW_\varphi)|}{|E|} \right\}, \quad D_\varphi(E) := \frac{P_\varphi(E) - P_\varphi(rW_\varphi)}{P_\varphi(rW_\varphi)}$$

denote the *anisotropic asymmetry index* and the *anisotropic isoperimetric deficit*, respectively.

Since the Wulff shape W_φ can be any bounded open convex set, it is clear that no symmetrization argument can be used to prove the anisotropic isoperimetric inequality or its quantitative counterpart (5.1). And in fact the strategy used in [67] is completely different from the one we have seen in the last section, since it relies on an optimal mass transportation argument and on the proof of a very general trace inequality.

To simplify even further the presentation of the main ideas used in the proof of Figalli et al. we shall only consider the case of the standard perimeter where inequality (5.1) reduces to the more familiar quantitative isoperimetric inequality (4.3). The starting point is a variant of Gromov’s proof of the classical isoperimetric inequality where the *Knothe map* originally used in [100, App. 1] is replaced by the *Brenier map*. Its main properties are stated in the next theorem, which follows from the results in [26, 93].

Theorem 5.1 *Let μ and ν be two probability measures on \mathbb{R}^n with μ absolutely continuous with respect to Lebesgue measure. Then there exists a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, such that the map $T := \nabla\varphi$ transports μ onto ν , that is for every Borel function $h : \mathbb{R}^n \rightarrow [0, \infty]$*

$$\int_{\mathbb{R}^n} h(y) d\nu(y) = \int_{\mathbb{R}^n} h(T(x)) d\mu(x). \tag{5.2}$$

Observe that (5.2) can be restated by saying that the map T pushes forward the measure μ onto ν (for the definition of the push forward of a measure see for instance [8, Sect. 1.5]). In the particular case that E is a set of finite perimeter with $|E| = |B|$, setting $\mu = \frac{1}{\omega_n} \chi_E dx$ and $\nu = \frac{1}{\omega_n} \chi_B dx$, from the above theorem we may conclude (see also [97, Sect. 2.1]), that there exists a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that, setting $T := \nabla\varphi$, then $T(x) \in B$ for a.e. $x \in \mathbb{R}^n$ and for every Borel function $h : B \rightarrow [0, \infty]$

$$\int_B h(y) dy = \int_E h(T(x)) dx. \tag{5.3}$$

Note that there is a regularity issue here, since φ is convex, hence T is defined only a.e. and it is just a BV map. However, in order to avoid technical complications, see the discussion in [67, Sect. 2.2], let us assume that T is a Lipschitz map. Under this extra assumption, by a change of variable, one easily gets from (5.3) that $\det \nabla T(x) = 1$ for a.e. $x \in E$. Then we can give the

Gromov’s proof of the isoperimetric inequality For every $x \in E$ denote by $\lambda_i(x)$, $i = 1, \dots, n$, the eingenvales of the symmetric matrix $\nabla T(x)$. Using the geometric–arithmetic mean inequality and the divergence theorem, we have

$$\begin{aligned} P(B) &= n\omega_n = n \int_B dy = n \int_E (\det \nabla T)^{1/n} dx = n \int_E (\lambda_1 \cdots \lambda_n)^{1/n} dx \\ &\leq \int_E (\lambda_1 + \cdots + \lambda_n) dx = \int_E \operatorname{div} T dx = \int_{\partial E} T \cdot \nu^E d\mathcal{H}^{n-1} \leq P(E). \end{aligned}$$

Observe that if $P(E) = P(B)$ then $\lambda_1(x) = \lambda_2(x) = \cdots = \lambda_n(x) = 1$ for a.e. $x \in E$, since $\det \nabla T(x) = 1$. Hence, up to a translation, T is the identity map and E is a ball. □

Let us try to exploit this argument to prove the quantitative isoperimetric inequality. Since, by definition,

$$P(E) = P(B) + |B|D(E),$$

from the inequalities above we immediately get that

$$\int_E \left[(\lambda_1 + \cdots + \lambda_n)/n - (\lambda_1 \cdots \lambda_n)^{1/n} \right] dx \leq \omega_n D(E), \tag{5.4}$$

$$\int_{\partial^* E} (1 - T \cdot \nu^E) d\mathcal{H}^{n-1} \leq n\omega_n D(E). \tag{5.5}$$

It is not too difficult to show (see [67, Corollary 2.4]) that (5.4) implies that there exists a constant C depending only on n such that, if $D(E) \leq 1$, then

$$\int_E |\nabla T - I| dx \leq C(n)\sqrt{D(E)}, \tag{5.6}$$

where I stands for the identity matrix. Let us assume for a moment that the set E is so good to satisfy a Poincaré inequality for some universal constant depending only on n and let us see what information we may deduce from (5.6). Indeed, it can be proved that if $D(E)$ is smaller than some positive δ depending only on n one may remove from E a small critical set, whose measure is controlled by $D(E)$, such that outside this critical set a Poincaré inequality holds true with a constant depending only on n . Therefore, by a reduction argument, we may assume that

$$\inf_{c \in \mathbb{R}^N} \int_E |f - c| dx \leq C(n) \int_E |\nabla f| dx \quad \text{for all } f \in C_c^1(\mathbb{R}^n).$$

Therefore, by translating E we have from (5.6) that

$$\int_E |T(x) - x| dx \leq C(n)\sqrt{D(E)}.$$

Therefore, given $\varepsilon > 0$, from the previous inequality we get

$$\begin{aligned} \omega_n \alpha(E) &\leq |E \Delta B| = 2|E \setminus B| \leq 2(|E \setminus B_{1+\varepsilon}| + |B_{1+\varepsilon} \setminus B|) \\ &\leq C \left(\varepsilon^{-1} \int_E |Tx - x| + \varepsilon \right) \leq C(n) \left(\varepsilon + \varepsilon^{-1} \sqrt{D(E)} \right). \end{aligned}$$

Minimizing the right hand side of this inequality with respect to ε we then get

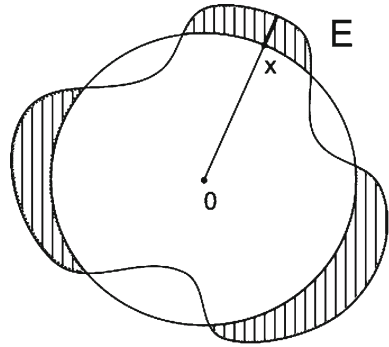
$$\alpha(E)^4 \leq C(n)D(E),$$

that is the quantitative isoperimetric inequality with the not optimal exponent 4. Note however that this argument can never lead to a proof of the quantitative isoperimetric inequality: firstly, because even if E is a connected open set the constant of the Poincaré inequality may blow up in presence of small cusps; secondly because in the above argument we are not taking into account the information contained in the inequality (5.5) derived from Gromov’s proof.

Indeed the strategy followed in [67] is more subtle. Namely, one can show that if E has small deficit, then (see [67, Th. 3.4 and Lemma 3.1]), up to removing a small critical set from E whose measure is controlled by $D(E)$, one may assume that E satisfies the following trace inequality

$$\inf_{c \in \mathbb{R}^n} \int_{\partial^* E} |f - c| d\mathcal{H}^{n-1} \leq \tau(n) \int_E |\nabla f| dx \quad \text{for all } f \in C_c^1(\mathbb{R}^n),$$

Fig. 6 $|E \Delta B_1| \leq C \int_{\partial^* E} |1 - |x||$



for some constant τ depending only on n . Then, by applying the previous inequality to the map $T - Id$, up to translating E conveniently, we have, recalling (5.6),

$$\int_{\partial^* E} |T(x) - x| d\mathcal{H}^{n-1} \leq \tau(n) \int_E |\nabla T - I| dx \leq C(n) \sqrt{D(E)}.$$

Combining this inequality with (5.5), we obtain, assuming $D(E) \leq 1$,

$$\begin{aligned} \int_{\partial^* E} |1 - |x|| d\mathcal{H}^{n-1} &\leq \int_{\partial^* E} [|1 - |T(x)|| + ||T(x)| - |x||] d\mathcal{H}^{n-1} \\ &\leq \int_{\partial^* E} \left[(1 - T(x) \cdot \nu^E(x)) + |T(x) - x| \right] d\mathcal{H}^{n-1} \\ &\leq C [D(E) + \sqrt{D(E)}] \leq C(n) \sqrt{D(E)}. \end{aligned}$$

The proof of the quantitative isoperimetric inequality (4.3) then immediately follows from this estimate since, see [67, Lemma 3.5], there exists a constant $C(n)$ such that, see Fig. 6,

$$|E \Delta B| \leq C(n) \int_{\partial E} |1 - |x|| d\mathcal{H}^{n-1}.$$

Beside providing an alternative proof of the quantitative isoperimetric inequality in the wider framework of anisotropic perimeter, the paper by Figalli et al. contains several interesting results. In particular, Theorem 3.4 which states that given any set of finite perimeter E with small deficit one may always extract from E a maximal set for which a trace inequality holds with a universal constant. This is a new and deep result that may have several applications. Moreover, the mass transportation approach used in [67] has been also successfully used to obtain the quantitative versions of other important inequalities, see [43,66,68].

5.2 Quantitative isoperimetric inequality via regularity

Another very interesting proof of the quantitative isoperimetric inequality has been recently given by Cicalese and Leonardi in [45]. The starting point is the quantitative inequality (3.2) proved by Fuglede for nearly spherical sets and the observation that all known examples suggest that the quantitative inequality becomes really critical only when the set E is a small perturbation of a ball. Therefore their idea is to reduce the general case to the case of nearly spherical sets via a contradiction argument. Precisely, they start by assuming that there exists a sequence of sets E_j , converging in measure to the unit ball, for which the quantitative inequality does not hold. Then they replace it with a different sequence of sets F_j , still not satisfying the quantitative inequality, but converging to B in C^1 , thus contradicting Fuglede’s Theorem 3.1 for nearly spherical sets. The sets F_j are constructed as the solutions of certain minimum problems and their convergence in C^1 to the unit ball is a consequence of the a priori estimates for *perimeter almost minimizers* established in the theory of minimal surfaces.

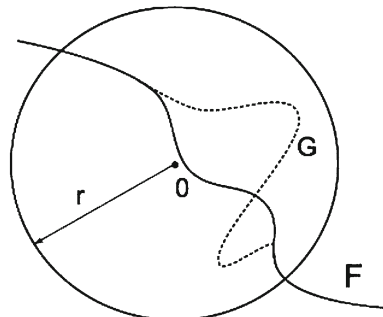
Though the approach of Cicalese and Leonardi to the quantitative isoperimetric inequality is based on the results of a difficult and deep theory, it has the advantage of providing a short proof that has been successfully applied to several other inequalities, see [1, 18, 19, 24, 47, 48, 74]. The proof we are going to present here is a further simplification of the original proof by Cicalese and Leonardi which has been developed in a more general context by Acerbi et al. in [1]. To this aim, let us quickly recall the definition and the regularity properties of the perimeter almost minimizers.

Let ω, r_0 be positive numbers. A set F of finite perimeter is an (ω, r_0) -almost minimizer if, for all balls $B_r(x)$ with $r < r_0$ and all measurable sets G such that $F \Delta G \subset\subset B_r(x)$, we have

$$P(F) \leq P(G) + \omega r^n, \tag{5.7}$$

see (Fig. 7). Thus, an almost minimizer locally minimizes the perimeter up to a higher order error term. The main properties of almost minimizers are contained in the following statement which is essentially due to Tamanini, see [114, Sect. 1.9 and 1.10] and also [95, Th. 26.5 and 26.6].

Fig. 7 A perimeter almost minimizer F



Theorem 5.2 *If E is an (ω, r_0) -almost minimizer, then ∂^*E is a $C^{1,1/2}$ manifold, $\partial E \setminus \partial^*E$ is relatively closed in ∂E and $\mathcal{H}^s(\partial E \setminus \partial^*E) = 0$ for all $s > n - 8$. Moreover, if E_j is a sequence of equibounded (ω, r_0) -almost minimizers converging in measure to a C^2 open set E , then for j large each E_j is of class $C^{1,1/2}$ and the sequence E_j converges to E in $C^{1,\alpha}$ for all $0 < \alpha < 1/2$.*

As we said above the starting point of the proof is the Fuglede estimate for nearly spherical sets. Recall that Theorem 3.1 states that there exist two positive constants $\varepsilon(n), c_0(n)$ such that if E is an open set with $|E| = |B|$, the barycenter of E is at the origin and $\partial E = \{x + u(x) : x \in \partial B\}$ for a Lipschitz function u such that $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, then the following estimate holds

$$P(E) - P(B) \geq c_0(n)|E \Delta B|^2. \tag{5.8}$$

We also need the following simple lemma.

Lemma 5.3 *If $\Lambda > n$, the unique solution up to translations of the problem*

$$\min\{P(F) + \Lambda||F| - |B|| : F \subset \mathbb{R}^n\} \tag{5.9}$$

is the unit ball.

Proof By the isoperimetric inequality it follows that in order to minimize the functional in (5.9), we may restrict to the balls B_r . Thus the above problem is equivalent to minimizing in $[0, \infty)$ the function $r \rightarrow nr^{n-1} + \Lambda|r^n - 1|$, which has a unique minimum for $r = 1$, if $\Lambda > n$. □

We are now ready to give the proof of the quantitative isoperimetric inequality (4.3) via regularity. Before that we need also to introduce the *non-rescaled asymmetry index* by setting for any measurable set E of finite measure

$$A(E) := \min_{x \in \mathbb{R}^n} \{|E \Delta B(x)|\}.$$

Proof of the quantitative isoperimetric inequality via regularity Step 1 Thanks to Lemma 4.2 we may assume, without loss of generality, that $|E| = |B|$ and that $E \subset B_{R_0}$, for some R_0 depending only on n . As we have already observed in the previous section, in order to prove (4.3) it is enough to show that

Claim *There exists $\delta_0 > 0$ such that if $E \subset B_{R_0}, |E| = |B|$ and $P(E) - P(B) \leq \delta_0$, then*

$$\frac{c_0}{2}A(E)^2 \leq P(E) - P(B), \tag{5.10}$$

where c_0 is the constant in (5.8).

To this aim we argue by contradiction assuming that there exist a sequence $E_j \subset B_{R_0}, |E_j| = |B|$, with $P(E_j) \rightarrow P(B)$ and

$$P(E_j) < P(B) + \frac{c_0}{2}A(E_j)^2. \tag{5.11}$$

Since $P(E_j) \rightarrow P(B)$, by the compactness Theorem 2.3 we may assume that up to a subsequence the sets E_j converge in measure to some set E . Then, by the lower semicontinuity of the perimeter we get that $P(E) = P(B)$. Thus E is a ball of radius 1 and we may conclude that $A(E_j) \rightarrow 0$ as $j \rightarrow \infty$. Now, to achieve the proof of (5.10), we would like to replace the E_j with a sequence of sets converging to B in C^1 and contradicting inequality (5.8).

To build this new sequence, for every j we consider a minimizer F_j of the problem:

$$\min \{P(F) + |A(F) - A(E_j)| + \Lambda ||F| - |B|| : F \subset B_{R_0}\}, \tag{5.12}$$

where $\Lambda > n$ is a fixed constant. Using again the compactness Theorem 2.3 we may assume that the sets F_j converge in measure to a set F . Moreover, the lower semicontinuity of the perimeter immediately yields that F is a minimizer of the problem:

$$\min \{P(E) + A(E) + \Lambda ||E| - |B|| : E \subset B_{R_0}\}.$$

Therefore by Lemma 5.3 we may conclude that the sequence F_j converge in measure to a ball $B(x_0)$. Let us now show that this convergence holds indeed in C^1 . To this aim, by Theorem 5.2 it is enough to prove that each F_j is an (ω, R_0) -almost minimizer for some $\omega > 0$. To prove this take a ball $B_r(x)$ with $r < R_0$ and a set G such that $F_j \Delta G \subset\subset B_r(x)$. Two cases may occur.

Case 1 $G \subset \bar{B}_{R_0}$. Then, by the minimality of F_j we get

$$\begin{aligned} P(F_j) &\leq P(G) + |A(G) - A(E_j)| - |A(F_j) - A(E_j)| \\ &\quad + \Lambda [||G| - |B|| - ||F_j| - |B||] \\ &\leq P(G) + |A(G) - A(F_j)| + \Lambda ||G| - |F_j|| \\ &\leq P(G) + (\Lambda + 1)|G \Delta F_j| \leq P(G) + (\Lambda + 1)\omega_n r^n. \end{aligned}$$

Case 2 $|G \setminus \bar{B}_{R_0}| > 0$. In this case we split G as follows

$$P(F_j) - P(G) = [P(F_j) - P(G \cap B_{R_0})] + [P(G \cap B_{R_0}) - P(G)].$$

Since $G \cap B_{R_0} \subset B_{R_0}$, as before we have

$$P(F_j) - P(G \cap B_{R_0}) \leq (\Lambda + 1)\omega_n r^n,$$

while

$$P(G \cap B_{R_0}) - P(G) = P(B_{R_0}) - P(G \cup B_{R_0}) \leq 0$$

by the isoperimetric inequality. Therefore we may conclude that the sets F_j are all $((\Lambda + 1)\omega_n, R_0)$ -almost minimizers and that they converge to a ball $B(x_0)$ in $C^{1,\alpha}$ for all $\alpha < 1/2$.

Step 2 By the minimality of the F_j , recalling (5.11) and using Lemma 5.3, we get

$$\begin{aligned}
 P(F_j) + \Lambda \left| |F_j| - |B| \right| + |A(F_j) - A(E_j)| &\leq P(E_j) \\
 &\leq P(B) + \frac{c_0}{2} A(E_j)^2 \leq P(F_j) + \Lambda \left| |F_j| - |B| \right| + \frac{c_0}{2} A(E_j)^2.
 \end{aligned}
 \tag{5.13}$$

Therefore, we have that $|A(F_j) - A(E_j)| \leq \frac{c_0}{2} A(E_j)^2$ and since $A(E_j) \rightarrow 0$ we get that $A(F_j)/A(E_j) \rightarrow 1$ as $j \rightarrow \infty$.

To conclude the proof we need only to adjust the volumes of the sets F_j . For this reason we set $\tilde{F}_j = \lambda_j F_j$, where λ_j is chosen so that $|\tilde{F}_j| = |B|$. Note that $\lambda_j \rightarrow 1$ since the F_j are converging in C^1 to $B(x_0)$. Observe also that, since $P(F_j) \rightarrow P(B)$ and $\Lambda > n$, for j large we have $P(F_j) < \Lambda |F_j|$. Therefore for j large we have

$$\begin{aligned}
 |P(\tilde{F}_j) - P(F_j)| &= P(F_j) |\lambda_j^n - 1| \leq P(F_j) |\lambda_j^n - 1| \\
 &\leq \Lambda |\lambda_j^n - 1| |F_j| = \Lambda \left| |\tilde{F}_j| - |F_j| \right|.
 \end{aligned}$$

From this estimate, recalling (5.13) we get that

$$P(\tilde{F}_j) \leq P(F_j) + \Lambda \left| |\tilde{F}_j| - |F_j| \right| = P(F_j) + \Lambda \left| |F_j| - |B| \right| \leq P(B) + \frac{c_0}{2} A(E_j)^2.
 \tag{5.14}$$

However, since $A(F_j)/A(E_j) \rightarrow 1$ as $j \rightarrow \infty$ we have $A(E_j)^2 < 2A(\tilde{F}_j)^2$ for j large. Therefore, from (5.14) we obtain

$$P(\tilde{F}_j) - P(B) < c_0 A(\tilde{F}_j)^2,$$

which is a contradiction to (5.8) since, up to translations, the sets \tilde{F}_j have all barycenter at the origin and are converging in C^1 to the unit ball. This contradiction proves the Claim, thus concluding the proof of the quantitative inequality. \square

5.3 An improved version of the quantitative isoperimetric inequality

Let E be a nearly spherical set and let us look back at the estimate (3.2) stated in Fuglede’s Theorem 3.1. Observe that in the previous argument we have only used part of the information provided by (3.2), since we have not exploited the presence of the full norm of u in $H^1(\mathbb{S}^{n-1})$.

The fact that in the quantitative isoperimetric inequality (4.3) we are throwing away some valuable information encoded in the isoperimetric deficit $D(E)$ can be understood by looking at the two sets E and F in Fig. 8. Indeed, E and F have the same measure, the same Fraenkel asymmetry, but $D(E) \ll 1$, while $D(F) \gg 1$. Therefore the quantitative inequality (4.3) gives a sharp information on E while it is useless when applied to F . The reason is that the isoperimetric deficit depends strongly on the oscillation of the boundary of the set, while the Fraenkel asymmetry only looks at the distance in measure of a set from a ball.

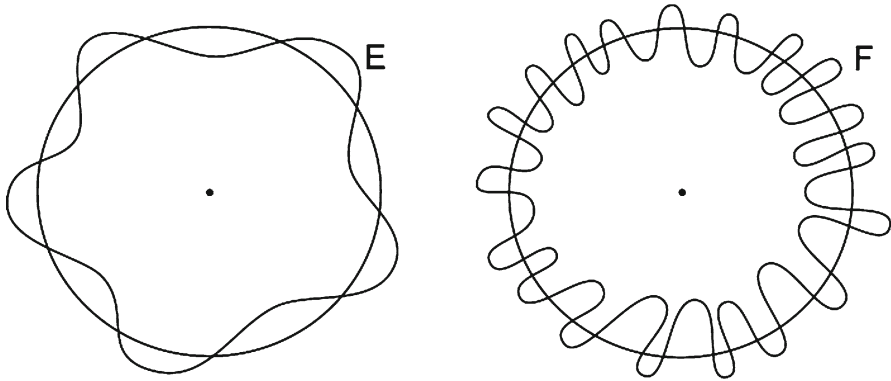


Fig. 8 E and F have the same measure and the same asymmetry

This suggests that we should introduce a more precise index which takes into account also the oscillation of the normals. To this aim, given a set of finite perimeter E and a ball $B_r(y)$ with the same volume as E , we are going to measure the distance from E to the ball in the following way, see Fig. 9. For every point $x \in \partial^*E$ we take the projection $\pi_{y,r}(x)$ of x on the boundary of $B_r(y)$ and consider the distance $|v^E(x) - v^{r,y}(\pi_{y,r}(x))|$ from the exterior normal to E at the point x to the exterior normal to $B_r(y)$ at the projection point $\pi_{y,r}(x)$. Then, we take the L^2 norm of this distance and minimize the resulting norm among all possible balls, thus getting

$$\beta(E) := \min_{y \in \mathbb{R}^n} \left\{ \left(\frac{1}{2} \int_{\partial^*E} |v^E(x) - v^{r,y}(\pi_{y,r}(x))|^2 d\mathcal{H}^{n-1}(x) \right)^{1/2} \right\}. \tag{5.15}$$

We shall refer to $\beta(E)$ as to the *oscillation index* (or *excess*) of the set E .

Observe that Fuglede’s Theorem 3.1 provides indeed an estimate for the oscillation index. In fact, if E is a nearly spherical set satisfying (3.1), $|E| = |B|$ and the $W^{1,\infty}$ norm of u is sufficiently small, by (3.14) at every point $x \in \partial^*E$ the exterior normal to E is given by

$$v^E(x) = \frac{z(1 + u(z)) - \nabla_\tau u(z)}{\sqrt{(1 + u(z))^2 + |\nabla_\tau u(z)|^2}},$$

where $z = x/|x|$ and thus $x = z(1 + u(z))$. Hence, from (3.2) we have

$$\begin{aligned} \beta(E)^2 &\leq \frac{1}{2} \int_{\partial^*E} \left| v^E(x) - \frac{x}{|x|} \right|^2 d\mathcal{H}^{n-1} = \int_{\partial^*E} \left(1 - v^E(x) \cdot \frac{x}{|x|} \right) d\mathcal{H}^{n-1} \\ &\leq C \int_{\mathbb{S}^{n-1}} \left(1 - \frac{1 + u(z)}{\sqrt{(1 + u)^2 + |\nabla_\tau u|^2}} \right) d\mathcal{H}^{n-1} \\ &= C \int_{\mathbb{S}^{n-1}} \frac{\sqrt{(1 + u)^2 + |\nabla_\tau u|^2} - (1 + u)}{\sqrt{(1 + u)^2 + |\nabla_\tau u|^2}} d\mathcal{H}^{n-1} \end{aligned} \tag{5.16}$$

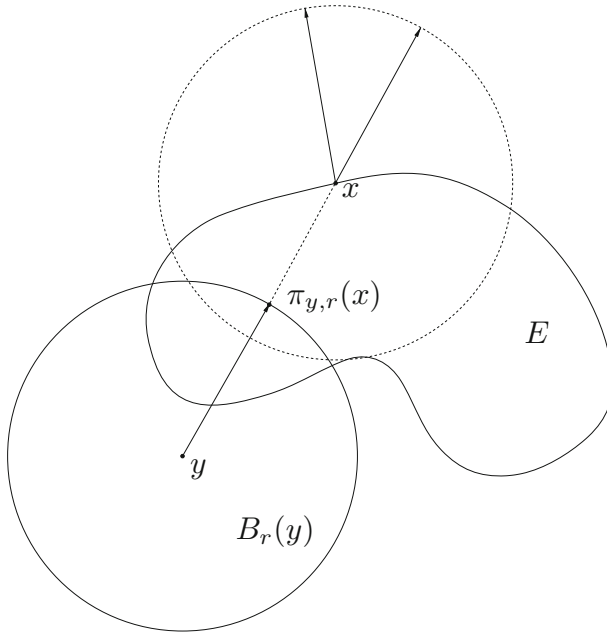


Fig. 9 The construction of the oscillation index

$$\leq C \int_{\mathbb{S}^{n-1}} |\nabla_{\tau} u|^2 d\mathcal{H}^{n-1} \leq \tilde{C}(P(E) - P(B)),$$

for some constant \tilde{C} depending only on n . The next result, proved by Julin and the author in [74], is an improved version of the quantitative isoperimetric inequality.

Theorem 5.4 *There exists a constant $\kappa(n)$, such that for any set of finite perimeter E , with $|E| = |B_r|$ for some $r > 0$,*

$$\beta(E)^2 \leq \kappa(n)(P(E) - P(B_r)). \tag{5.17}$$

Inequality (5.17) is stronger than (4.3), since a Poincaré type inequality shows that the asymmetry index α is always controlled by β , see Proposition 5.5. Before proving this, let us give a closer look to the new index. Observe that by the divergence theorem we immediately have

$$\begin{aligned} \frac{1}{2} \int_{\partial^* E} |v^E(x) - v^{r,y}(\pi_{y,r}(x))|^2 d\mathcal{H}^{n-1} &= \int_{\partial^* E} \left(1 - v^E(x) \cdot \frac{x-y}{|x-y|} \right) d\mathcal{H}^{n-1} \\ &= P(E) - \int_E \frac{n-1}{|x-y|} dx. \end{aligned}$$

Therefore, we may write

$$\beta(E)^2 = P(E) - (n-1)\gamma(E), \tag{5.18}$$

where we have set

$$\gamma(E) := \max_{y \in \mathbb{R}^n} \int_E \frac{1}{|x - y|} dx. \tag{5.19}$$

If $y \in \mathbb{R}^n$ is the center of an optimal ball for $\beta(E)$ then y maximizes the integral on the right hand side of (5.19). Such a point will be called a *center* for the set E . Note that in general such a center is not unique. The next result is contained in [74, Prop. 1.2].

Proposition 5.5 *There exists a constant $C(n)$ such that if E is a set of finite perimeter, with $|E| = |B_r|$ for some $r > 0$, then*

$$\alpha(E)^2 + D(E) \leq Cr^{1-n} \beta(E)^2.$$

Proof Let E be a set of finite perimeter. Without loss of generality we may assume that $|E| = |B|$ and that E is centered at the origin, i.e.,

$$\beta(E)^2 = \int_{\partial^* E} \left(1 - \nu^E \cdot \frac{x}{|x|} \right) d\mathcal{H}^{n-1}.$$

Since $\beta(B) = 0$, hence $P(B) = (n - 1) \int_E \frac{1}{|x|} dx$, from the equality above and (5.18) we have

$$\beta(E)^2 = P(B)D(E) - \int_{E \setminus B} \frac{n - 1}{|x|} dx + \int_{B \setminus E} \frac{n - 1}{|x|} dx. \tag{5.20}$$

Let us estimate the last two terms in (5.20). Since $|E| = |B|$ we have

$$|E \setminus B| = |B \setminus E| =: a. \tag{5.21}$$

Denote by $A(R, 1) = B_R \setminus B_1$ and $A(1, r) = B_1 \setminus B_r$ two annuli such that $|A(R, 1)| = |A(1, r)| = a$. Hence,

$$R = \left(1 + \frac{a}{\omega_n} \right)^{1/n} \quad \text{and} \quad r = \left(1 - \frac{a}{\omega_n} \right)^{1/n}.$$

By construction $|A(R, 1)| = |E \setminus B|$. Therefore, since $\frac{1}{|x|}$ is radially decreasing,

$$\int_{E \setminus B} \frac{n - 1}{|x|} dx \leq \int_{A(R, 1)} \frac{n - 1}{|x|} dx.$$

Similarly, we have

$$\int_{B \setminus E} \frac{n - 1}{|x|} dx \geq \int_{A(1, r)} \frac{n - 1}{|x|} dx.$$

Therefore we may estimate (5.20) by

$$\begin{aligned} \beta(E)^2 &\geq P(B)D(E) - \int_{A(R,1)} \frac{n-1}{|x|} dx + \int_{A(1,r)} \frac{n-1}{|x|} dx \\ &= P(B)D(E) - n \left[\omega_n(R^{n-1} - 1) - \omega_n(1 - r^{n-1}) \right] \\ &= P(B)D(E) + n\omega_n \left(2 - \left(1 + \frac{a}{\omega_n} \right)^{\frac{n-1}{n}} - \left(1 - \frac{a}{\omega_n} \right)^{\frac{n-1}{n}} \right). \end{aligned} \tag{5.22}$$

The function $f(t) = (1 + t)^{\frac{n-1}{n}}$ is uniformly concave in $[-1, 1]$ and there exists a positive constant $c(n)$ such that for all $s, t \in [-1, 1]$

$$\frac{1}{2} (f(t) + f(s)) \leq f\left(\frac{t}{2} + \frac{s}{2}\right) - c(n)|t - s|^2.$$

Therefore, recalling (5.21), we may estimate (5.22) by

$$\beta(E)^2 \geq P(B)D(E) + \frac{8nc(n)}{\omega_n} a^2 = P(B)D(E) + \frac{2nc(n)}{\omega_n} |E \Delta B|^2.$$

Hence, the assertion follows. □

The strategy for proving inequality (5.17) is the same that we have seen in the previous section and the starting point is again Fuglede’s inequality (3.2) which implies in particular (5.17) if E is a nearly spherical set sufficiently close in $W^{1,\infty}$ to the unit ball. However, beside using the regularity properties of perimeter almost minimizers, we need to introduce another notion of minimality. We say that a set of locally finite perimeter E is an area (K, r_0) -quasi-minimizer if for every F , such that $F \Delta E \subset\subset B_r(x)$, with $r \leq r_0$, the following inequality holds

$$P(E; B_r(x)) \leq K P(F; B_r(x)).$$

We observe that this definition is the counterpart in the framework of sets of finite perimeter of the notion of quasi-minimum introduced by Giaquinta and Giusti in [79] in the context of variational integrals. Therefore it is not surprising that although quasi-minimality is a very weak property, yet area quasi-minimizers have some kind of mild regularity. This is indeed the content of the next result which was first proved by David and Semmes in [51] and then extended by Kinnunen et al. [88] to the metric spaces setting.

Theorem 5.6 *Suppose that E is an area (K, r_0) -quasi-minimizer. Then there exists $C > 1$ such that for any $0 < r < r_0$ and every $x \in \partial E$ there are points $y, z \in B_r(x)$ for which*

$$B_{r/C}(y) \subset E \text{ and } B_{r/C}(z) \subset \mathbb{R}^n \setminus E.$$

We are now ready to prove the main result of this section.

Proof of Theorem 5.4 Step 1 As in the previous section, we start by observing that in order to prove (5.17) it is enough to assume $|E| = |B|$ and that $D(E)$ is smaller than a given constant δ_0 to be chosen later. Moreover, the same reduction argument used to prove Lemma 4.2 shows that also in this case we may assume without loss of generality that $E \subset B_{R_0}$, where the radius R_0 depends only on the dimension n , see [74, Lemma 3.2].

Let $c_0 > 0$ be a constant which will be chosen at the end of the proof. From what we have just observed it is enough to show that there exists $\delta_0 > 0$ such that, if $D(E) \leq \delta_0$, $E \subset B_{R_0}$ and $|E| = |B|$, then

$$c_0\beta(E)^2 \leq P(E) - P(B).$$

We argue by contradiction assuming that there exists a sequence of sets $E_j \subset B_{R_0}$ such that $|E_j| = |B|$, $D(E_j) \rightarrow 0$ and

$$P(E_j) < P(B) + c_0\beta(E_j)^2. \tag{5.23}$$

By the compactness Theorem 2.3 it follows that, up to a subsequence, $E_j \rightarrow E_\infty$ in measure and by the lower semicontinuity of the perimeter we immediately conclude that E_∞ is a ball of radius one. It is easily checked that the functional γ defined in (5.19) is continuous with respect to the convergence in measure. Therefore, since the E_j are converging to a ball of radius one and $P(E_j) \rightarrow P(B)$, we have that

$$\beta(E_j)^2 = P(E_j) - (n - 1)\gamma(E_j) \rightarrow 0.$$

As in the previous section we replace each set E_j by a minimizer F_j of the following problem

$$\min \left\{ P(F) + \Lambda||F| - |B|| + \frac{1}{4}|\beta(F)^2 - \beta(E_j)^2|, \quad F \subset B_{R_0} \right\}$$

for some fixed $\Lambda > n$. It is not too hard to show that the above functional is lower semicontinuous with respect to the convergence in measure, see [74, Lemma 3.4], and therefore a minimizer exists. Moreover, up to a subsequence, we may assume that $F_j \rightarrow F_\infty$ in measure. From the minimality of F_j , (5.23) and Lemma 5.3 we have that

$$\begin{aligned} P(F_j) + \Lambda||F_j| - |B|| + \frac{1}{4}|\beta(F_j)^2 - \beta(E_j)^2| &\leq P(E_j) < P(B) + c_0\beta(E_j)^2 \\ &\leq P(F_j) + \Lambda||F_j| - |B|| + c_0\beta(E_j)^2. \end{aligned}$$

Hence $|\beta(F_j)^2 - \beta(E_j)^2| \leq 4c_0\beta(E_j)^2$, which implies $\beta(F_j) \rightarrow 0$ and

$$\beta(E_j)^2 \leq \frac{1}{1 - 4c_0}\beta(F_j)^2. \tag{5.24}$$

Therefore F_∞ is a minimizer of the problem

$$\min \left\{ P(F) + \Lambda ||F| - |B|| + \frac{1}{4}\beta(F)^2 : F \subset B_{R_0} \right\}.$$

Thus from Lemma 5.3 we conclude that F_∞ is a ball $B(x_0)$, for some $x_0 \in \mathbb{R}^n$.

Step 2 We claim that for any $\varepsilon > 0$, $B_{1-\varepsilon}(x_0) \subset F_j \subset B_{1+\varepsilon}(x_0)$ for j large enough. To this aim we show that the sets F_j are area (K, r_0) -quasi-minimizers and use Theorem 5.6. Let $G \subset \mathbb{R}^n$ be such that $G \Delta F_j \subset \subset B_r(x)$, $r \leq r_0$.

Case 1 Suppose that $B_r(x) \subset B_{R_0}$. By the minimality of F_j we obtain

$$P(F_j) \leq P(G) + \frac{1}{4}|\beta(F_j)^2 - \beta(G)^2| + \Lambda ||F_j| - |G|| \tag{5.25}$$

Assume that $\beta(F_j) \geq \beta(G)$ (otherwise the argument is similar) and denote by y_G a center of G . Then we get

$$\begin{aligned} |\beta(F_j)^2 - \beta(G)^2| &\leq \int_{\partial^* F_j} \left(1 - \nu^{F_j} \cdot \frac{z - y_G}{|z - y_G|} \right) d\mathcal{H}^{n-1}(z) \\ &\quad - \int_{\partial^* G} \left(1 - \nu^G \cdot \frac{z - y_G}{|z - y_G|} \right) d\mathcal{H}^{n-1}(z) \\ &= \int_{\partial^* F_j \cap B_r(x)} \left(1 - \nu^{F_j} \cdot \frac{z - y_G}{|z - y_G|} \right) d\mathcal{H}^{n-1}(z) \\ &\quad - \int_{\partial^* G \cap B_r(x)} \left(1 - \nu^G \cdot \frac{z - y_G}{|z - y_G|} \right) d\mathcal{H}^{n-1}(z) \\ &\leq 2[P(F_j; B_r(x)) + P(G; B_r(x))]. \end{aligned}$$

Therefore, from (5.25) we get

$$P(F_j; B_r(x)) \leq 3P(G; B_r(x)) + 2\Lambda ||F_j| - |G||.$$

From the above inequality the (K, r_0) -quasi-minimality immediately follows observing that

$$|F_j \Delta G| \leq \omega_n^{1/n} r^{1/n} |F_j \Delta G|^{\frac{n-1}{n}} \leq C(n)r^{1/n} [P(F_j; B_r(x)) + P(G; B_r(x))]$$

and choosing r_0 sufficiently small.

Case 2 If $|B_r(x) \setminus B_{R_0}| > 0$, we may write

$$\begin{aligned} P(F_j; B_r(x)) - P(G; B_r(x)) &= P(F_j; B_r(x)) - P(G \cap B_{R_0}; B_r(x)) \\ &\quad + P(G \cap B_{R_0}; B_r(x)) - P(G; B_r(x)) \\ &= P(F_j; B_r(x)) - P(G \cap B_{R_0}; B_r(x)) \\ &\quad + P(B_{R_0}) - P(G \cup B_{R_0}) \\ &\leq P(F_j; B_r(x)) - P(G \cap B_{R_0}; B_r(x)). \end{aligned}$$

From Case 1 we have that this term is less than $(K - 1)P(G \cap B_{R_0}; B_r(x))$ which in turn is smaller than $(K - 1)P(G; B_r(x))$. Hence, all F_j are (K, r_0) -quasi-minimizers with uniform constants K and r_0 . The claim then follows from the theory of (K, r_0) -quasi-minimizers and the fact that $F_j \rightarrow B(x_0)$ in L^1 . Indeed, arguing by contradiction, assume that there exists $0 < \varepsilon_0 < 2r_0$ such that for infinitely many j one can find $x_j \in \partial F_j$ for which

$$x_j \notin B_{1+\varepsilon_0}(x_0) \setminus B_{1-\varepsilon_0}(x_0).$$

Let us assume that $x_j \in B_{1-\varepsilon_0}(x_0)$ for infinitely many j (otherwise, the argument is similar). From Theorem 5.6 it follows that there exist $y_j \in B_{\frac{\varepsilon_0}{2}}(x_j)$ such that $B_{\frac{\varepsilon_0}{2C}}(y_j) \subset B(x_0) \setminus F_j$. This implies

$$|B(x_0) \setminus F_j| \geq |B_{\frac{\varepsilon_0}{2C}}| > 0,$$

which contradicts the fact that $F_j \rightarrow B(x_0)$ in L^1 , thus proving the claim.

Step 3 Let us now translate F_j , for j large, so that the resulting sets, still denoted by F_j , are contained in B_{R_0} , have their barycenters at the origin and converge to B . We are going to use Theorem 5.2 to show that the F_j are $C^{1,1/2}$ and converge to B in $C^{1,\alpha}$ for all $\alpha < 1/2$. To this aim, fix a small $\varepsilon > 0$. From Step 2 we have that for j large

$$B_{1-\varepsilon} \subset F_j \subset B_{1+\varepsilon}. \tag{5.26}$$

We want to show that when j is large F_j is a (Λ', r_0) -almost minimizer for some constants Λ', r_0 independent of j . To this aim, fix a set $G \subset \mathbb{R}^n$ such that $G \Delta F_j \subset\subset B_r(y)$, with $r < r_0$.

If $B_r(y) \subset B_{1-\varepsilon}$, from (5.26) it follows that $G \Delta F_j \subset\subset F_j$ for j large enough. This immediately yields $P(F_j) \leq P(G)$.

If $B_r(y) \not\subset B_{1-\varepsilon}$, choosing r_0 and ε sufficiently small we have that

$$B_r(y) \cap B_{1/2} = \emptyset. \tag{5.27}$$

Denote by y_{F_j} and y_G the centers of F_j and G , respectively. If ε is sufficiently small, from (5.26) and Lemma 5.7 below we have that for j large

$$|y_{F_j}| \leq \frac{1}{4} \quad \text{and} \quad |y_G| \leq \frac{1}{4}. \tag{5.28}$$

By the minimality of F_j we have

$$P(F_j) \leq P(G) + \frac{1}{4}|P(F_j) - P(G)| + \Lambda||F_j| - |G|| + \frac{n-1}{4}|\gamma(F_j) - \gamma(G)|,$$

which immediately implies

$$P(F_j) \leq P(G) + 2\Lambda|F_j \Delta G| + (n - 1)|\gamma(F_j) - \gamma(G)|. \tag{5.29}$$

We may estimate the last term simply by

$$\gamma(F_j) - \gamma(G) \leq \int_{F_j} \frac{1}{|x - y_{F_j}|} dx - \int_G \frac{1}{|x - y_{F_j}|} dx \leq \int_{F_j \Delta G} \frac{1}{|x - y_{F_j}|} dx$$

and

$$\gamma(G) - \gamma(F_j) \leq \int_G \frac{1}{|x - y_G|} dx - \int_F \frac{1}{|x - y_G|} dx \leq \int_{F_j \Delta G} \frac{1}{|x - y_G|} dx.$$

Therefore, recalling (5.27) and (5.28), we have

$$|\gamma(F_j) - \gamma(G)| \leq 4|F_j \Delta G|.$$

From this estimate and inequality (5.29) we may then conclude that

$$P(F_j) \leq P(G) + (2\Lambda + 4(n - 1)) |F_j \Delta G| \leq P(G) + \Lambda' r^n.$$

Hence, the sets F_j are (Λ', r_0) -almost minimizers with uniform constants Λ' and r_0 . Thus, Theorem 5.2 yields that the F_j are $C^{1,1/2}$ and that, for j large,

$$\partial F_j = \{z(1 + u_j(z)) : z \in \partial B\}$$

for some $u_j \in C^{1,1/2}(\partial B)$ such that $u_j \rightarrow 0$ in $C^1(\partial B_1)$.

Step 4 By the minimality of F_k , (5.23) and (5.24) we have

$$P(F_j) + \Lambda ||F_j| - |B|| \leq P(E_j) < P(B) + c_0 \beta(E_j)^2 \leq P(B) + \frac{c_0}{1 - 4c_0} \beta(F_j)^2. \tag{5.30}$$

We are almost in a position to use (5.16) to obtain a contradiction. We only need to rescale the F_j so that the volume constraint is satisfied. Thus, set $F'_j := \lambda_j F_j$, where λ_j is such that $\lambda_j^n |F_j| = |B|$. Then $\lambda_j \rightarrow 1$ and also the sets F'_j converge to B in C^1 and have their barycenters at the origin. Therefore, since $\Lambda > n$, $P(F_j) \rightarrow n|B|$ and $|F_j| \rightarrow |B|$, we have that for j sufficiently large

$$|P(F'_j) - P(F_j)| = |\lambda_j^{n-1} - 1| P(F_j) \leq \Lambda |\lambda_j^n - 1| |F_j| = \Lambda ||F'_j| - |F_j||. \tag{5.31}$$

Then (5.30) and (5.31) yield

$$\begin{aligned} P(F'_j) &\leq P(F_j) + \Lambda ||F_j| - |B|| < P(B) + \frac{c_0}{1 - 4c_0} \beta(F_j)^2 \\ &= P(B) + \frac{c_0 \lambda_j^{1-n}}{1 - 4c_0} \beta(F'_j)^2. \end{aligned}$$

which contradicts (5.16) if $2c_0/(1 - 4c_0) < 1/\tilde{C}$ and j is large. □

Lemma 5.7 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $F \subset B_{R_0}$ and $|F \Delta B| < \delta$, then $|y_F| < \varepsilon$ for every center y_F of F .*

Proof We argue by contradiction and assume that there exist $F_j \subset B_{R_0}$ such that $F_j \rightarrow B$ in measure and $y_{F_j} \rightarrow y_0$ with $|y_0| \geq \varepsilon$, for some $\varepsilon > 0$. Then we would have

$$\int_{F_j} \frac{1}{|x|} dx \leq \int_{F_j} \frac{1}{|x - y_{F_j}|} dx.$$

Letting $j \rightarrow \infty$, by the dominated convergence theorem the left hand side converges to $\int_B \frac{1}{|x|} dx$, while the right hand side converges to $\int_B \frac{1}{|x - y_0|} dx$. Thus we have

$$\int_B \frac{1}{|x|} dx \leq \int_B \frac{1}{|x - y_0|} dx.$$

By the divergence theorem we conclude that

$$\int_{\partial B} 1 dx \leq \int_{\partial B} x \cdot \frac{x - y_0}{|x - y_0|} dx$$

and this inequality may only hold if $y_0 = 0$, thus leading to a contradiction. □

We conclude this section quoting a recent paper by Neumayer [103] dealing with the same kind of improved quantitative estimate that we have discussed here, but in the case of the anisotropic perimeter. It turns out that if the function φ which defines the anisotropic perimeter is of class C^2 and satisfies a suitable form of ellipticity, then one can prove an inequality of the type (5.17) with β replaced by a suitable *anisotropic oscillation index*. In this case, see [103, Th. 1.5] one may get the stability estimate with the exponent 2 and a constant depending on the function φ . In the general case, i.e., when no assumptions on φ are made, a stability estimate is still obtained with a constant depending only on the dimension, but with a not optimal power [103, Th. 1.1].

5.4 The isoperimetric inequality in higher codimension

The isoperimetric inequality in higher codimension goes back to a celebrated paper by Almgren [3]. In that paper he proved the manifold counterpart of the classical isoperimetric inequality established by De Giorgi in [53]. Though stated in the framework of currents, in the particular case of a smooth $(n - 1)$ -dimensional manifolds $\Gamma \subset \mathbb{R}^{n+k}$ without boundary, spanning an area minimizing smooth surface M , his inequality states that

$$\mathcal{H}^{n-1}(\Gamma) \geq \mathcal{H}^{n-1}(\partial D),$$

where D is an n -dimensional flat disk with the same area as M . Moreover equality occurs if and only if Γ is the boundary of a flat disk. The stability of this inequality has

been addressed, in the context of currents, in a recent paper by Bögelein et al. [18]. In order to describe the result we restrict to the case of smooth $(n - 1)$ -dimensional closed surfaces Γ in \mathbb{R}^{n+k} and give the definitions of the quantities that come into play.

Denoting by $Q(\Gamma)$ an area minimizing n -dimensional surface with boundary Γ the *isoperimetric deficit* is defined as

$$\mathbf{D}(\Gamma) := \frac{\mathcal{H}^{n-1}(\Gamma) - \mathcal{H}^{n-1}(\partial D_r)}{\mathcal{H}^{n-1}(\partial D_r)},$$

where D_r is an n -dimensional flat disk in \mathbb{R}^{n+k} with the same area as $Q(\Gamma)$, i.e., $\mathcal{H}^n(D_r) = \mathcal{H}^n(Q(\Gamma))$. Note that the area minimizing surface $Q(\Gamma)$ may have singularities even if Γ is smooth. It is then clear why the use of currents with finite mass becomes unavoidable even to give the basic definitions. This is even more true in the case of the *asymmetry index* $\alpha(\Gamma)$ whose precise definition is more technical and requires the use of a certain seminorm defined for integer multiplicity currents [18, Sect. 2]. However, the underlying geometric idea is easy to describe. Given any flat disk D_r with the same area as $Q(\Gamma)$, first one considers an area minimizing cylindrical type surface $\Sigma(D_r)$ spanned by the boundary components Γ and ∂D_r , and afterwards one takes the infimum of the surface area $\mathcal{H}^n(\Sigma(D_r))$ among all possible disks D_r :

$$\alpha(\Gamma) := r^{-n} \inf \{ \mathcal{H}^n(\Sigma(D_r)) : \mathcal{H}^n(D_r) = \mathcal{H}^n(Q(\Gamma)) \}.$$

With these definitions in hand we may state the following quantitative isoperimetric inequality proved in [18, Th. 2.1].

Theorem 5.8 *Let $n \geq 2$ and $k \geq 0$. There exists a constant $C = C(n, k) > 0$ such that for any $(n - 1)$ -dimensional closed surface $\Gamma \subset \mathbb{R}^{n+k}$ the following inequality holds:*

$$\alpha(\Gamma)^2 \leq C\mathbf{D}(\Gamma). \tag{5.32}$$

Note that if Γ is the boundary of a smooth open set E contained in an n -dimensional hyperplane, then the asymmetry index $\alpha(\Gamma)$ coincides with the Fraenkel asymmetry index $\alpha(E)$ defined in (4.1). Hence, (5.32) reduces to (4.3). In particular this shows that the exponent 2 on the left hand side of the inequality cannot be improved, since we know that it is already optimal for (4.3).

As in the two previous sections the strategy to prove Theorem 5.8 is to show first a Fuglede type inequality and then to reduce the general case to it via a regularity argument. However, here the situation is more delicate and involved due to the higher codimension. First of all, the analogue of Fuglede’s result deals with a *spherical graph over \mathbb{S}^{n-1} in \mathbb{R}^{n+k}* , i.e., a manifold Γ which can be parametrized by a map $X: \mathbb{S}^{n-1} \rightarrow \mathbb{R}^{n+k}$ of the form

$$X(x) := (1 + u(x))(x, 0) + (0, v(x)) \quad x \in \mathbb{S}^{n-1},$$

where $u \in C^1(\mathbb{S}^{n-1})$ and $v \in C^1(\mathbb{S}^{n-1}, \mathbb{R}^k)$ have both small C^1 -norms. In the present case a substantial difficulty arises from the fact that, beside imposing the volume

constraint $\mathcal{H}^n(Q(\Gamma)) = \omega_n$ and that the barycenter of Γ is at the origin, we have also to fix some second order quantities. This can be done for instance by assuming that they are all equal to zero, i.e.,

$$\int_{\Gamma} z_i z_j d\mathcal{H}^{n-1} = 0 \tag{5.33}$$

for any choice of $i = 1, \dots, n$ and $j = n + 1, \dots, n + k$. Differently from the case $k = 0$ considered in Sect. 3, in which v does not appear, the conditions (5.33) play a crucial role in the estimation of the $n \cdot k$ first order Fourier coefficients of v . Instead, the bounds on the first order Fourier coefficients of u and the zero order Fourier coefficient of v follow from the barycenter condition, while the zero order Fourier coefficient of u is controlled by using the constraint $\mathcal{H}^n(Q(\Gamma)) = \omega_n$. Then, under the above assumptions on u and v one has the following inequality, see [18, Th. 4.1],

$$\mathcal{H}^{n-1}(\Gamma) - \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \geq c(n) [\|u\|_{H^1(\mathbb{S}^{n-1})}^2 + \|v\|_{H^1(\mathbb{S}^{n-1}, \mathbb{R}^k)}^2], \tag{5.34}$$

where $c(n)$ is a constant depending only on n , provided the C^1 norms of u and v are sufficiently small. Note also that

$$\alpha(\Gamma) \leq c(n) [\|u\|_{L^2(\mathbb{S}^{n-1})}^2 + \|v\|_{L^2(\mathbb{S}^{n-1}, \mathbb{R}^k)}^2].$$

Combining this inequality with (5.34) one then gets the quantitative isoperimetric inequality (5.32) in this case.

The next step is to reduce the general case to the previous one by the same contradiction argument that we have seen in Sect. 5.2. Beside the technical complications due to the fact that one is now dealing with higher codimension, the main ingredients come from the theory of currents that ‘almost’ minimize certain elliptic variational integrals. The regularity results needed in the proof were obtained in [20, 54].

5.5 The isoperimetric inequality on the sphere

The isoperimetric property of geodesic balls on the sphere and in hyperbolic spaces goes back to Schmidt [108]. Before stating it, let us fix some basic notation. Recall that if x, y are two points in the n -dimensional sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, $n \geq 1$, the *geodesic distance* between them is given by

$$\text{dist}_{\mathbb{S}^n}(x, y) := \arccos(x \cdot y).$$

Thus, the open *geodesic ball* with center at a point $p \in \mathbb{S}^n$ and radius $\vartheta \in (0, \pi)$ is defined by setting

$$B_{\vartheta}(p) := \{x \in \mathbb{S}^n : \text{dist}_{\mathbb{S}^n}(x, p) < \vartheta\}.$$

We just write B_ϑ if the center of the geodesic ball is at the north pole e_{n+1} . Then, the isoperimetric inequality on the sphere states that if $E \subset \mathbb{S}^n$ is a Borel set such that $\mathcal{H}^n(E) = \mathcal{H}^n(B_\vartheta)$ then

$$\mathbf{P}(B_\vartheta) \leq \mathbf{P}(E), \tag{5.35}$$

with the equality occurring if and only if E is a geodesic ball. Here, $\mathbf{P}(E)$ denotes the *perimeter* of E on the sphere, see the definition below, which coincides with the \mathcal{H}^{n-1} -measure of the (relative) boundary of E on the sphere if E is a C^1 manifold. Note that from (5.35) we immediately have that if $E \subset \mathbb{S}^n$ is a set of finite perimeter such that $\mathbf{P}(E) = \mathbf{P}(B_\vartheta)$ for some $0 < \vartheta < \pi$, then

$$|E| \leq |B_\vartheta| \quad \text{or} \quad |\mathbb{S}^n \setminus E| \leq |B_\vartheta|.$$

Also in this case equality occurs if and only if E is a geodesic ball.

We recall that if $E \subset \mathbb{S}^n$ is a Borel set, the perimeter of E on the sphere can be defined similarly to the euclidean case by setting

$$\mathbf{P}(E) = \sup \left\{ \int_E \operatorname{div}_\tau \varphi \, dx : \varphi \in C^\infty(\mathbb{S}^n; \mathbb{R}^{n+1}), \varphi(x) \cdot x = 0 \quad \text{for all } x \in \mathbb{S}^n, \|\varphi\|_\infty \leq 1 \right\}.$$

If $\mathbf{P}(E) < \infty$ by Riesz’s representation theorem it follows that the tangential gradient of χ_E can be represented as a vector valued Radon measure $D_\tau \chi_E$. Thus, as in the euclidean case one may consider the set $\partial^* E$ of all points in \mathbb{S}^n such that the following limit exists

$$v_E(x) := - \lim_{\vartheta \downarrow 0} \frac{D_\tau \chi_E(B_\vartheta(x))}{|D_\tau \chi_E|(B_\vartheta(x))}$$

and satisfies $|v_E(x)| = 1$ and $v_E(x) \in T_x \mathbb{S}^n$, where $T_x \mathbb{S}^n$ stands for the tangential space to \mathbb{S}^n at the point x . Then the De Giorgi structure theorem on the sphere states that $\partial^* E$ is countably $(n - 1)$ -rectifiable and that the total variation measure $|D_\tau \chi_E|$ coincides with $\mathcal{H}^{n-1} \llcorner \partial^* E$. Moreover, if g is a C^1 tangential vector-field on \mathbb{S}^n we have the following Gauss–Green formula, see [110, (7.6)]

$$\int_E \operatorname{div}_{\mathbb{S}^n} g \, d\mathcal{H}^n = \int_{\partial^* E} g \cdot v_E \, d\mathcal{H}^{n-1}. \tag{5.36}$$

Note that when E is a smooth open subset of \mathbb{S}^n , the unit vector $v_E(x) \in T_x \mathbb{S}^n$ coincides with the *outer unit normal vector* to ∂E at x . In the case of a geodesic ball $B_\vartheta(p)$ it can be easily checked that the outer unit normal vector field along its boundary is given by

$$v_{B_\vartheta(p)}(x) = \frac{(x \cdot p)x - p}{\sqrt{1 - (x \cdot p)^2}}. \tag{5.37}$$

In view of the results that we have seen in Sect. 4, the natural counterpart of (4.3) would be the inequality

$$\frac{\mathbf{P}(E) - \mathbf{P}(B_\vartheta)}{\mathbf{P}(B_\vartheta)} \geq c(n)\alpha(E)^2, \tag{5.38}$$

where now the Fraenkel asymmetry index is defined by

$$\alpha(E) := \min_{p \in \mathbb{S}^n} \left\{ \frac{\mathcal{H}^n(E \Delta B_\vartheta(p))}{\mathcal{H}^n(B_\vartheta)} : \mathcal{H}^n(E) = \mathcal{H}^n(B_\vartheta) \right\}.$$

When compared with (4.3) inequality (5.38), even if it looks similar, has a completely different nature. In fact (4.3) is scaling invariant while there is no scaling at all on \mathbb{S}^n . It would be quite easy to adapt one of the different arguments that we have seen in Sect. 4 and in Sect. 5.2 in order to prove (5.38) with a constant depending additionally on the volume of the set E , but possibly blowing up as $\vartheta \downarrow 0$. Indeed, the difficult case is when the set E has a small volume sparsely distributed over the sphere. In this situation a localization argument aimed to reduce the problem to the flat Euclidean estimate (4.3) does not work. However, as in Sect. 5.3 one can prove a stronger inequality by replacing $\alpha(E)$ with an oscillation index which is the counterpart in our new setting of the one defined in (5.15).

The idea is the same as in the euclidean case. Given a geodesic ball $B_\vartheta(p)$ with the same \mathcal{H}^n -measure of E , for every point $x \in \partial^*E$ we want to measure the distance between $\nu_E(x)$ and the unit normal vector $\nu_{B_\vartheta(p)}(\pi_{\vartheta,p}(x))$, where $\pi_{\vartheta,p}(x)$ is the closest point in $\partial B_\vartheta(p)$ in the sense of geodesic distance. However, since the two normals $\nu_E(x)$ and $\nu_{B_\vartheta(p)}(\pi_{\vartheta,p}(x))$ lie in two different tangent spaces in order to measure their distance correctly we have to parallel transport the normal $\nu_{B_\vartheta(p)}(\pi_{\vartheta,p}(x))$ to the tangent space $T_x\mathbb{S}^n$. Since the parallel transport on the sphere is just translation along great circles, the normal $\nu_{B_\vartheta(p)}(\pi_{\vartheta,p}(x))$ is thus carried into the normal $\nu_{B_{\vartheta(x)}(p)}(x)$ at the point x to the geodesic ball with center p and radius $\vartheta(x) := \text{dist}_{\mathbb{S}^n}(x, p)$. So, as in Sec. 5.3, we measure the L^2 distance between the normals $\nu_E(x)$ and $\nu_{B_{\vartheta(x)}(p)}(x)$ and then minimize the resulting norm among all possible geodesic balls having the same measure of E . This leads to the following definition of oscillation index for the set E

$$\beta(E) := \min_{p \in \mathbb{S}^n} \left[\frac{1}{2} \int_{\partial^*E} |\nu_E(x) - \nu_{B_{\vartheta(x)}(p)}(x)|^2 d\mathcal{H}^{n-1} \right]^{\frac{1}{2}}.$$

The quantitative isoperimetric inequality proved in [19, Th. 1.1] reads as follows.

Theorem 5.9 *There exists a constant $c(n)$ such that for any Borel set $E \subset \mathbb{S}^n$ with $\mathcal{H}^n(E) = \mathcal{H}^n(B_\vartheta)$ for some $\vartheta \in (0, \pi)$, the following inequality holds*

$$\mathbf{P}(E) - \mathbf{P}(B_\vartheta) \geq c(n)\beta^2(E). \tag{5.39}$$

As in the euclidean case, see the discussion made in Sect. 5.3, inequality above implies (5.38), since a Poincaré type inequality still holds in the case, allowing to control $\alpha(E)$ by the oscillation index $\beta(E)$, see [19, Lemma 2.7].

Lemma 5.10 *There exists a constant $c = c(n) > 0$ such that for any Borel set $E \subset \mathbb{S}^n$ with $\mathcal{H}^n(E) = \mathcal{H}^n(B_\vartheta)$ for some $\vartheta \in (0, \pi)$ there holds*

$$\beta(E)^2 \geq c(n)\mathbf{P}(B_\vartheta)\alpha(E)^2.$$

As in Sect. 5.3 the starting point for the proof of Theorem 5.9 is a Fuglede-type stability result aimed to establish (5.39) in the special case of sets $E \subset \mathbb{S}^n$ whose boundary can be written as a radial graph over the boundary of a geodesic ball $B_\vartheta(p)$ with the same volume. To establish such a result one could follow in principle the strategy used in the euclidean case in Theorem 3.1. However, to deduce (5.39) for radial graphs with a constant not depending on the volume needs much more care in the estimations, see the proof of Theorem 3.1 in [19]. The main difficulty arises when passing from the special situation of radial graphs to arbitrary sets. To deal with this issue one needs to change significantly the strategies that we have seen in the previous sections.

To explain where the major difficulties come from, we observe that as in the euclidean case the oscillation index can be re-written as the difference of $\mathbf{P}(E)$ and a suitable potential on E . To this aim, note that

$$\begin{aligned} \frac{1}{2} \int_{\partial^* E} |v_E(x) - v_{B_\vartheta(x)(p)}(x)|^2 d\mathcal{H}^{n-1} &= \int_{\partial^* E} 1 - v_E(x) \cdot v_{B_\vartheta(x)(p)}(x) d\mathcal{H}^{n-1} \\ &= \mathbf{P}(E) - \int_{\partial^* E} v_E(x) \cdot v_{B_\vartheta(x)(p)}(x) d\mathcal{H}^{n-1}. \end{aligned}$$

To proceed further, we recall (5.37) and set

$$X(x) := v_{B_\vartheta(x)(p)}(x) = \frac{(x \cdot p)x - p}{\sqrt{1 - (x \cdot p)^2}}.$$

Note that X is a tangential vector field, that is $X(x) \cdot x = 0$ for all $x \in \partial^* E$. Hence, by the Gauss–Green formula (5.36) we have

$$\frac{1}{2} \int_{\partial^* E} |v_E(x) - v_{B_\vartheta(x)(p)}(x)|^2 d\mathcal{H}^{n-1} = \mathbf{P}(E) - \int_E \operatorname{div}_{\mathbb{S}^n} X d\mathcal{H}^n.$$

Computing the tangential divergence $\operatorname{div}_{\mathbb{S}^n} X$ we obtain, denoting by $\{\tau_1, \dots, \tau_n\}$ an orthonormal base for $T_x \mathbb{S}^n$,

$$\begin{aligned} \operatorname{div}_{\mathbb{S}^n} X &= \sum_{i=1}^n \tau_i \cdot \left[\frac{(\tau_i \cdot p)x + (x \cdot p)\tau_i}{\sqrt{1 - (x \cdot p)^2}} + ((x \cdot p)x - p) \frac{(x \cdot p)(\tau_i \cdot p)}{\left(\sqrt{1 - (x \cdot p)^2}\right)^3} \right] \\ &= \sum_{i=1}^n \left[\frac{x \cdot p}{\sqrt{1 - (x \cdot p)^2}} - \frac{(x \cdot p)(\tau_i \cdot p)^2}{\left(\sqrt{1 - (x \cdot p)^2}\right)^3} \right] = \frac{(n - 1)(x \cdot p)}{\sqrt{1 - (x \cdot p)^2}}. \end{aligned}$$

Inserting this equality in the formula above, we conclude that

$$\beta^2(E) = \mathbf{P}(E) - \gamma(E),$$

where

$$\gamma(E) := (n - 1) \max_{p \in \mathbb{S}^n} \int_E \frac{x \cdot p}{\sqrt{1 - (x \cdot p)^2}}.$$

From this formula it is clear that the core of the proof is to provide estimates independent of the volume of E for the potential

$$\int_E \frac{x \cdot p}{\sqrt{1 - (x \cdot p)^2}} d\mathcal{H}^{n-1} \tag{5.40}$$

and for its maximum with respect to p . This requires some technically involved ideas and strategies. In particular, in the contradiction argument used to deduce (5.39) for general sets from the case of a radial graph we need to show that all the constants are independent of the volume of E . The arguments become particularly delicate when the volume of E is small. In this case inequality (5.39) shows a completely different nature depending on the size of the ratio $\beta^2(E)/\mathbf{P}(B_\vartheta)$. In fact, if $\mathcal{H}^n(E) \rightarrow 0$ and also $\beta^2(E)/\mathbf{P}(B_\vartheta) \rightarrow 0$, then E behaves asymptotically like a flat set, i.e., a set in \mathbb{R}^n and inequality (5.39) can be proven by reducing to the euclidean case, rescaling and then arguing as when E has large volume. However, the most difficult situation to deal with is when $\mathcal{H}^n(E) \rightarrow 0$ and $\beta^2(E)/\mathbf{P}(B_\vartheta)$ converge to a strictly positive number. This case has to be treated with ad hoc estimates for the potential (5.40).

5.6 The fractional perimeter

The fractional s -perimeter has recently attracted the interest of several authors. In particular, Caffarelli et al. have initiated in [31] the study of Plateau-type problems in the fractional setting, see also [32]. Recall that for $s \in (0, 1)$ and a Borel set $E \subset \mathbb{R}^n$, $n \geq 2$, the *fractional s -perimeter* is defined by setting

$$P_s(E) := \int_E \int_{E^c} \frac{1}{|x - y|^{n+s}} dx dy.$$

Note that if $P_s(E) < \infty$, then

$$P_s(E) = \frac{1}{2} [\chi_E]_{W^{\sigma,p}(\mathbb{R}^n)}^p,$$

for $p \geq 1$ and $\sigma p = s$, where $[\chi_E]_{W^{\sigma,p}(\mathbb{R}^n)}^p$ denotes the Gagliardo $W^{\sigma,p}$ -seminorm of the characteristic function of E . The functional $P_s(E)$ can be thought as a $(n - s)$ -dimensional perimeter in the sense that $P_s(\lambda E) = \lambda^{n-s} P_s(E)$ for any $\lambda > 0$. Note that since for $s \in (0, 1)$ $BV(\mathbb{R}^n)$ is imbedded in $W^{\frac{s(n-1)}{n-s}, \frac{n-s}{n-1}}(\mathbb{R}^n)$, see [81, Theorem 1.4.4.1], the s -perimeter of E is finite if E has finite (standard) perimeter and finite measure. On the other hand, $P_s(E)$ can be finite even if the Hausdorff dimension of ∂E is strictly greater than $n - 1$, see for instance [107]. Observe also that if E, F are two bounded sets of non trivial and finite s -perimeter in general

$$P_s(E \cup F) < P_s(E) + P_s(F).$$

However, the s -perimeter provides an approximation of the standard euclidean perimeter since it can be shown that for any bounded set of finite perimeter E

$$\lim_{s \uparrow 1} (1 - s) P_s(E) = \omega_{n-1} P(E). \tag{5.41}$$

Formula (5.41) originates from a paper by Bourgain, Brezis and Mironescu [23, Th. 3' and Rem 4]. It was completed by Dávila [52], see also [115] for a simpler proof. Subsequently, the same formula was proved in [32] and [7] under additional smoothness assumptions on E . On the other hand, as a consequence of [98, Th. 3], we have that, for any set E of finite measure and finite s -perimeter for all $s \in (0, 1)$,

$$\lim_{s \downarrow 0} s P_s(E) = n \omega_n |E|.$$

Also for the s -perimeter balls are the isoperimetric sets, since it can be proved that if $E \subset \mathbb{R}^n$ is a measurable set with $|E| = |B_r|$ for some $r > 0$, then

$$P_s(B_r) \leq P_s(E) \tag{5.42}$$

with equality holding if and only if E is a ball. Inequality (5.42) can be deduced from a symmetrization result due to Almgren and Lieb [4], while the cases of equality have been determined in [71]. Note also, see for instance [62, (2.11)], that the s -perimeter of the unit ball is given by

$$P_s(B) = \frac{2^{1-s} \pi^{\frac{n-1}{2}} P(B)}{s(n-s)} \cdot \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{n-s}{2})},$$

where Γ is the so-called *gamma function*. In order to state a quantitative version of the isoperimetric inequality (5.42), if E is a measurable set such that $|E| = |B_r|$ and $s \in (0, 1)$, we define the *s-isoperimetric deficit* as

$$D_s(E) := \frac{P_s(E) - P_s(B_r)}{P_s(B_r)}.$$

The next result has been proven in [62, Th. 1.1].

Theorem 5.11 *For every $n \geq 2$ and $s_0 \in (0, 1)$ there exists a positive constant $C(n, s_0)$ such that*

$$\alpha(E)^2 \leq C(n, s_0) D_s(E), \tag{5.43}$$

whenever $s \in [s_0, 1]$ and E has positive, finite measure.

The interesting feature of inequality (5.43) is that it holds uniformly with respect to s , at least if s is bounded away from zero. Therefore, thanks to (5.41), it implies in particular the quantitative inequality (4.3) for the standard perimeter. As we have seen in the previous sections, the starting point is a Fuglede type result, see [62, Th. 2.1], which now states that there exist two positive constants ε, c_0 , depending only on n , such that if E is a nearly spherical set as in Theorem 3.1, with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, then for all $s \in (0, 1)$

$$P_s(E) - P_s(B) \geq c_0 \left([u]_{H^{\frac{1+s}{2}}(\mathbb{S}^{n-1})}^2 + s P_s(B) \|u\|_{L^2(\mathbb{S}^{n-1})}^2 \right), \tag{5.44}$$

where the Gagliardo seminorm $[u]_{H^{\frac{1+s}{2}}(\mathbb{S}^{n-1})}$ is given by

$$[u]_{H^{\frac{1+s}{2}}(\mathbb{S}^{n-1})} := \left(\iint_{\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+s}} d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1} \right)^{\frac{1}{2}}.$$

Since it can be proved, see [23, Cor. 2] and also [62, (8.4)] for a different proof, that

$$\lim_{s \uparrow 1} (1 - s) [u]_{H^{\frac{1+s}{2}}(\mathbb{S}^{n-1})}^2 = \omega_{n-1} \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2,$$

one immediately recovers the Fuglede’s estimate (3.2) from (5.44). As we have seen already in different situations, in order to pass from a nearly spherical set to a general set of finite perimeter one has to use the right notion of perimeter almost minimizer and the right regularity results. The definition of perimeter almost minimizer given in (5.7) can be immediately extended to the s -perimeter in the obvious way and the counterpart in this new setting of Theorem 5.2 has been proved by Caputo and Guillen in [34]. However, in order to prove Theorem 5.11 one needs to show that all the regularity estimates needed in the proof are in fact independent of s , at least if s is bounded away from zero. This requires the use of more technically involved arguments and of more delicate estimates, see Sect. 4 in [62].

5.7 The isoperimetric inequality in Gauss space

The Gauss space is the space \mathbb{R}^n , $n \geq 1$, endowed with the measure γ given by

$$\gamma(E) = \frac{1}{(2\pi)^{n/2}} \int_E e^{-\frac{|x|^2}{2}} dx,$$

for any Lebesgue measurable set $E \subset \mathbb{R}^n$. Note that (\mathbb{R}^n, γ) is a probability space, since $\gamma(\mathbb{R}^n) = 1$. Clearly, γ is invariant under a rotation around the origin, while is not invariant under translation. Similarly to what we do in the euclidean space, if E is measurable and $\Omega \subset \mathbb{R}^n$ is an open set we define the *Gaussian perimeter* of E in Ω as

$$P_\gamma(E; \Omega) = \sqrt{2\pi} \sup \left\{ \int_E (\operatorname{div}\varphi - x \cdot \varphi) d\gamma : \varphi \in C_c^\infty(\Omega; \mathbb{R}^n), \|\varphi\|_\infty \leq 1 \right\}.$$

It turns out that if E has finite Gaussian perimeter in Ω then it has also locally finite perimeter in Ω in the Euclidean sense and

$$P_\gamma(E; \Omega) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial^* E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}.$$

The perimeter of E in \mathbb{R}^n will be simply denoted by $P_\gamma(E)$.

The isoperimetric inequality in the Gauss space asserts that among all subsets of \mathbb{R}^n with prescribed Gaussian measure, half-spaces have the least Gaussian perimeter. Precisely, for any $\nu \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$ denote by $H_{\nu,s}$ the half space

$$H_{\nu,s} := \{x \in \mathbb{R}^n : x \cdot \nu > s\}.$$

Then we have $\gamma(H_{\nu,s}) = \Phi(s)$ and $P_\gamma(H_{\nu,s}) = e^{-\frac{s^2}{2}}$, where Φ is the strictly decreasing function defined for all $s \in \mathbb{R}$ by setting

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_s^\infty e^{-\frac{t^2}{2}} dt.$$

Then the analytic form of the Gaussian isoperimetric inequality states that if $\gamma(E) = \gamma(H_{\nu,s}) = \Phi(s)$ for some $s \in \mathbb{R}$, then

$$P_\gamma(E) \geq P_\gamma(H_{\nu,s}) \quad \text{or equivalently} \quad P_\gamma(E) \geq e^{-\frac{s^2}{2}} = e^{-\frac{(\Phi^{-1}(\gamma(E)))^2}{2}}, \quad (5.45)$$

with the equality holding if and only if $E = H_{\nu,s}$ for some $\nu \in \mathbb{S}^{n-1}$. This inequality was independently established by Borell in [22] and by Sudakov and Cirel'son in [112]. However, the characterization of half-spaces as the unique minimizers in the isoperimetric Gaussian problem is a relatively recent result of Carlen and Kerce [36], relying on the theory of rearrangements as well as on probabilistic techniques involving the Mehler semigroup of contractions. A different proof has been recently given in [44],

where the Ehrhard symmetrization [55] plays the same role of Steiner symmetrization in De Giorgi’s proof [53] of the isoperimetric inequality.

The stability of the isoperimetric property of half spaces in (5.45) has been addressed for the first time in [44]. In order to state the result proved therein we introduce the *Gaussian isoperimetric deficit* $D_\gamma(E)$ and the *Gaussian asymmetry* $\alpha_\gamma(E)$ by setting, respectively,

$$D_\gamma(E) := P_\gamma(E) - e^{-\frac{s^2}{2}}, \quad \alpha_\gamma(E) := \min_{\nu \in \mathbb{S}^{n-1}} \gamma(E \Delta H_{\nu,s}),$$

whenever E is a measurable set with $\gamma(E) = \Phi(s)$, $n \geq 1$. With this definitions in hand, the stability result proved in [44, Th. 1.1] reads as follows.

Theorem 5.12 *Let $n \geq 2$. For any $0 < \lambda < 1$ there exists a positive constant $C(n, \lambda)$ depending only on n and λ such that*

$$\alpha_\gamma(E)^2 \leq C(n, \lambda) D_\gamma(E). \tag{5.46}$$

for every measurable set $E \subset \mathbb{R}^n$ such that $\gamma(E) = \lambda$.

Note that also in (5.46) the power 2 is sharp. To see this one can consider in dimension 2 the family of sets constructed in [44, Lemma 6.1]. When $n = 1$, the quantitative version of the isoperimetric inequality (5.46) takes a somewhat different form. In fact one can prove that if $n = 1$, for any $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$ such that

$$\alpha_\gamma(E) \sqrt{\log \left(\frac{1}{\alpha_\gamma(E)} \right)} \leq C(\lambda) D_\gamma(E) \tag{5.47}$$

for every measurable set $E \subset \mathbb{R}$ such that $\gamma_1(E) = \lambda$. Also in this case the result is sharp, in the sense that the left hand side of (5.47) cannot be replaced by any function of $\alpha_\gamma(E)$ decaying slower as $\alpha_\gamma(E) \rightarrow 0$, see the sets defined in [44, Lemma 3.5].

As in the proof of (4.3) that we discussed in Sect. 4, the strategy to prove (5.46) is to reduce, via subsequent simplifications, to classes of sets E enjoying special additional geometric properties. A first reduction consists in showing that it suffices to prove (5.46) for $(n - 1)$ -symmetric sets E , namely sets which are symmetric about $(n - 1)$ mutually orthogonal hyperplanes containing the origin. This is obtained by proving a counterpart of Theorem 4.6 in the Gaussian setting. Once this reduction is performed, in order to prove (5.46) in dimension $n = 2$ it is enough to deal only with sets that are symmetric with respect to one of the two coordinate axes. In this case the proof follows by using ad hoc geometric constructions, together with a careful study of a particular class of almost optimal sets. Finally, by combining Ehrhard symmetrization with the reduction result to $(n - 1)$ -symmetric sets, one may pass from a generic set E to a Cartesian product of \mathbb{R}^{n-2} times an epigraph in \mathbb{R}^2 . Thanks to the tensorial properties of the Gaussian density, this class of sets can be handled by the two-dimensional result.

The fact that the Gaussian isoperimetric inequality (5.45) is independent of the dimension suggests that also the constant in the quantitative inequality (5.46) should only depend on λ . Beside its intrinsic mathematical interest, having a constant independent of n in (5.45) would have some interesting applications in probability, as pointed out in [101, Sect. 1.4]. Indeed in [101] a stability estimate with a non sharp logarithmic dependence on the isoperimetric deficit $D_\gamma(E)$ was proven, but with a dimension free constant. This last result was later on greatly improved by Eldan [56] who proved inequality (5.46), actually an even stronger one, with a dimensionless constant and an almost optimal dependence on $D_\gamma(E)$, actually just a logarithm below the optimal growth. However, it was only a very recent paper by Barchiesi et al. [10] that settled the issue in a complete and satisfactory way.

To state properly their result we have to introduce a sharper index, which in the gaussian context plays the role of the one that we have introduced in Sect. 5.3. To this aim, for any measurable set $E \subset \mathbb{R}^n$ we define the non-renormalized barycenter of E by setting $b(E) := \int_E x \, d\gamma$. Then, following [10] we define a *strong asymmetry* by setting

$$\beta_\gamma(E) := \min_{v \in \mathbb{S}^{n-1}} |b(E) - b(H_{v,s})|.$$

In [10] the following result is proved.

Theorem 5.13 *There exists an absolute constant C such that for every $s \in \mathbb{R}$ and for every set $E \subset \mathbb{R}^n$ with $\gamma(E) = \Phi(s)$ the following estimate holds*

$$\beta_\gamma(E) \leq C(1 + s^2)D_\gamma(E). \tag{5.48}$$

Observe that this inequality is extremely interesting for several reasons. First, the constant C is dimension free, second, the dependence on the mass of E through s is also optimal, see [10, Rem. 1]. Moreover, a Poincaré type inequality shows that for any measurable set E with $\gamma(E) = \Phi(s)$ then

$$\beta_\gamma(E) \geq \frac{e^{\frac{s^2}{2}}}{4} \alpha_\gamma(E)^2.$$

Finally, the new asymmetry index is clearly related to the Gaussian counterpart of the oscillation index defined in (5.15) since one can prove that

$$2D_\gamma(E) + 2\sqrt{2\pi} \beta_\gamma(E) = \min_{v \in \mathbb{S}^{n-1}} \frac{1}{(2\pi)^{(n-1)/2}} \int_{\partial^* E} |v^E(x) - v|^2 \, d\mathcal{H}^{n-1}.$$

The key point in the proof of Theorem 5.13 is to study the minimization problem

$$\min \left\{ P_\gamma(E) + \frac{\varepsilon}{2} |b(E)|^2 + \Lambda |\gamma(E) - \Phi(s)| : E \subset \mathbb{R}^n \right\}$$

and to show that for ε sufficiently small, and Λ large, the only minima are the half spaces $H_{v,s}$. A part from the penalization term that eventually forces the minimizers to

have mass equal to $\Phi(s)$, the other two terms in the energy functional are somewhat of opposite nature. Indeed, for a given mass, half spaces minimize the Gaussian perimeter while maximize the absolute value of the barycenter. The reason why proving that the half spaces $H_{v,s}$ minimize the above functional leads to the quantitative inequality (5.48) can be understood on observing that $b(H_{v,s}) = e^{-\frac{s^2}{2}} v/\sqrt{2\pi}$. Therefore, from the minimality of $H_{v,s}$ one has that if $\gamma(E) = \Phi(s)$, then

$$\begin{aligned} P_\gamma(E) - P_\gamma(H_{v,s}) &\geq \frac{\varepsilon}{2} (|b(H_{v,s})|^2 - |b(E)|^2) = \frac{\varepsilon}{2} (|b(H_{v,s})| + |b(E)|)\beta_\gamma(E) \\ &\geq \frac{\varepsilon}{2\sqrt{2\pi}} e^{-\frac{s^2}{2}} \beta_\gamma(E), \end{aligned}$$

from which is not difficult to deduce (5.48), see the argument at the beginning of Section 4 of [10].

The proof that half spaces are minimizers of the above problem is based on a PDE rigidity argument that is certainly new in this context. The idea in [10] is to derive the second variation for the above functional and then to deduce the minimality of a half space by a powerful and elegant choice of the test function, see Step 3 of the proof of the main result contained in Sect. 4 of [10].

We conclude this section by observing that while the result by Barchiesi et al. proves the quantitative Gaussian isoperimetric inequality with an optimal dependence on the dimension and on the mass, the dependence on the dimension of the constant $\gamma(n)$ which appears in (4.3) is not known. However, inequality (3.2) implies that for a nearly spherical set E sufficiently close to a ball one has

$$\alpha(E)^2 \leq 8nD(E).$$

This suggests that $\gamma(n)$ should grow like n as $n \rightarrow \infty$. Unfortunately, looking more carefully at the proof of the quantitative isoperimetric inequality (4.3) given in Sect. 4 one gets a constant growing exponentially fast with n , while the contradiction argument used in the Sect. 5.2 gives no clue about $\gamma(n)$. However, in [67, (1.12)] it is observed that $\gamma(n)$ has at most a polynomial growth, though the power that one obtains with their methods, even if one may be possibly improve it a bit, seems still far away from being optimal. The value of the constant in dimension $n = 2$ for small values of the asymmetry was first given by Hall et al. in [83,84] in the convex case, and then extended by Cicalese and Leonardi in [45] for general sets of finite perimeter. Their result states that

$$D(E) \geq \frac{\pi}{8(4 - \pi)} \alpha(E)^2 + o(\alpha(E)^2).$$

Other, interesting estimates of the quantitative isoperimetric constant in dimension 2 for special classes of convex sets, not necessarily with small asymmetry, are studied in [6,46].

6 Other functional inequalities

6.1 Stability of Sobolev inequality

In this section we shall discuss the stability of Sobolev inequality, which states that if $n \geq 2$, $1 \leq p < n$ and $p^* = np/(n - p)$, then

$$S(p, n) \|f\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} \tag{6.1}$$

for every function f in the homogeneous Sobolev space $W^{1,p}(\mathbb{R}^n)$. Throughout this section by the *homogeneous Sobolev space* we mean the completion of $C_c^1(\mathbb{R}^n)$ with respect to the $L^p(\mathbb{R}^n)$ norm of the gradient.

If $p = 1$ it is well known that (6.1) is equivalent to the isoperimetric inequality, see [59, p. 192], and thus $S(1, n) = n\omega_n^{1/n}$. Indeed, a simple approximation argument shows that if f is a function in $BV(\mathbb{R}^n)$, the homogeneous space of all functions in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ whose distributional gradient is a measure with finite total variation, then

$$n\omega_n^{1/n} \|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} \leq |Df|(\mathbb{R}^n),$$

where $|Df|(\mathbb{R}^n)$ denotes the total variation, with equality holding if and only if f is the characteristic function of a ball. If $p > 1$ the best constant in (6.1) was found independently by Aubin [9] and Talenti [113] and it turns out to be equal to

$$S(p, n) = \sqrt{\pi} n^{1/p} \left(\frac{n-p}{p-1}\right)^{(p-1)/p} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(1+n/2)\Gamma(n)}\right)^{1/n}.$$

In this case the family of extremals in (6.1) is given by the functions g_{a,b,x_0} defined as

$$g_{a,b,x_0}(x) = \frac{a}{(1+b|x-x_0|^{p'})^{(n-p)/p}} \text{ for } x \in \mathbb{R}^n \tag{6.2}$$

for some $a \neq 0$, $b > 0$, $x_0 \in \mathbb{R}^n$, where $p' = p/(p - 1)$. When $p = 2$, as a consequence of a celebrated result by Gidas et al. [80] applied to the Euler equation of the functional $\|\nabla f\|_{L^2(\mathbb{R}^n)} / \|f\|_{L^{2^*}(\mathbb{R}^n)}$, one can show that the above functions are the only ones for which equality holds in (6.1). However if $p > 1$, $p \neq 2$ the characterization of the functions of the form (6.2) as the only ones attaining equality in (6.1) was shown much later by Cordero-Erausquin et al. [49] with a proof of the Sobolev inequality (6.1) via mass transportation in the same spirit of Gromov’s proof of the isoperimetric inequality that we have seen in Sect. 5.1. We reproduce here their proof.

Proof of the sharp Sobolev inequality via mass transport Observe that in order to prove (6.1) it is enough to assume that f is nonnegative since $|\nabla|f|| = |\nabla f|$. Therefore, if f and g are two nonnegative functions from $W^{1,p}(\mathbb{R}^n)$ such that $\|f\|_{p^*} = \|g\|_{p^*} = 1$, by applying Theorem 5.1 with $\mu = f^{p^*} dx$ and $\nu = g^{p^*} dy$ we

find that there exists a convex function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that, setting $T := \nabla\varphi$ one has

$$\int_{\mathbb{R}^n} h(y)g^{p^*}(y) dy = \int_{\mathbb{R}^n} h(T(x))f^{p^*}(x) dx, \tag{6.3}$$

for all nonnegative Borel functions h . To avoid some technical details that could make the idea of the proof less clear, let us assume that f and g have compact support and that T is Lipschitz. Then from the above formula we deduce that

$$f^{p^*}(x) = g^{p^*}(T(x)) \det \nabla T(x) \text{ for a.e. } x \in \mathbb{R}^n.$$

From this inequality, using the geometric-arithmetic mean inequality, we have that for a.e. x in the support of f

$$\begin{aligned} g^{-\frac{p^*}{n}}(T(x)) &= f^{-\frac{p^*}{n}}(x)(\det \nabla T(x))^{1/n} \leq \frac{1}{n} f^{-\frac{p^*}{n}}(x) \text{Tr}(\nabla T(x)) \\ &= \frac{1}{n} f^{-\frac{p^*}{n}}(x) \text{div} T(x), \end{aligned}$$

where $\text{Tr}(T)$ denotes the trace of T . Multiplying both sides of this inequality by $f^{p^*}(x)$ and integrating by parts we get

$$\begin{aligned} \int_{\mathbb{R}^n} g^{-\frac{p^*}{n}}(T(x))f^{p^*}(x) dx &\leq \frac{1}{n} \int_{\mathbb{R}^n} f^{p^*(1-1/n)}(x) \text{div} T(x) dx \\ &= -\frac{p^*(n-1)}{n^2} \int_{\mathbb{R}^n} f^{\frac{p^*}{p'}}(x) \nabla f(x) \cdot T(x) dx. \end{aligned} \tag{6.4}$$

Using Hölder inequality and recalling (6.3) again we have

$$\begin{aligned} \int_{\mathbb{R}^n} f^{\frac{p^*}{p'}}(x) \nabla f(x) \cdot T(x) dx &\leq \|\nabla f\|_p \left(\int_{\mathbb{R}^n} f^{p^*}(x) |T(x)|^{p'} dx \right)^{1/p'} \\ &= \|\nabla f\|_p \left(\int_{\mathbb{R}^n} g^{p^*}(y) |y|^{p'} dy \right)^{1/p'}. \end{aligned}$$

Combining this inequality with (6.4) and recalling (6.3) we then conclude that

$$\int_{\mathbb{R}^n} g^{p^*(1-1/n)}(y) dy \leq \frac{p(n-1)}{n(n-p)} \|\nabla f\|_p \left(\int_{\mathbb{R}^n} g^{p^*}(y) |y|^{p'} dy \right)^{1/p'}. \tag{6.5}$$

Now we take $b = 1$ and $a > 0$ such that the function $h_p := g_{a,1,0}$ has L^{p^*} norm in \mathbb{R}^n equal to 1 and we argue as before, choosing $f = g = h_p$. Since in this case the map T is just the identity map, the above argument simplifies a lot and one can easily check that all previous inequalities are indeed equalities. Therefore we conclude by getting that

$$\int_{\mathbb{R}^n} h_p^{p^*(1-1/n)}(y) dy = \frac{p(n-1)}{n(n-p)} \|\nabla h_p\|_p \left(\int_{\mathbb{R}^n} h_p^{p^*}(y) |y|^{p'} dy \right)^{1/p'}$$

From this equality, recalling (6.5) one gets that if $\|f\|_{p^*} = \|h_p\|_{p^*} = 1$, then

$$\|\nabla f\|_p \geq \|\nabla h_p\|_p.$$

This inequality, as one can check with a few computations is precisely the Sobolev inequality (6.1) with the sharp constant $S(n, p)$. □

In the argument above we have chosen to put ourself in the simple situation when the functions f and g have compact support and T is Lipschitz. With a bit of technical adjustments, see [49, Sect. 4] the above argument can be justified also in the general case. Thus, if f and g are two functions for which inequality (6.5) holds as an equality one can prove [49, Prop. 6], that there exist two positive constants α and λ such that $f(x) = \alpha g(\lambda(x - x_0))$ for some $x_0 \in \mathbb{R}^n$. From this the characterization of the functions in (6.2) as the only extremals in the Sobolev inequality immediately follows.

In [27, (c) p. 75] Brezis and Lieb raised the following stability problem: ‘is there a way to bound $\|\nabla f\|_2 - S(2, n)\|f\|_{2^*}$ from below in terms of a ‘distance’ of f from the set of optimal functions (6.2)?’ The answer to this question was provided a few years later in a nice paper by Bianchi and Egnell [16]. Given a function f in the homogeneous Sobolev space $W^{1,2}(\mathbb{R}^n)$ they considered the distance μ from f to the manifold \mathcal{M} made up by all the functions g_{a,b,x_0} given by (6.2) with $p = 2$:

$$\mu(f) = \inf_{a,b,x_0} \|\nabla f - \nabla g_{a,b,x_0}\|_{L^2(\mathbb{R}^n)}. \tag{6.6}$$

They proved the following result.

Theorem 6.1 *There exist a positive constant $c(n)$ depending only on n such that for all $f \in W^{1,2}(\mathbb{R}^n)$*

$$\|\nabla f\|_2^2 - S(2, n)^2 \|f\|_{2^*}^2 \geq c(n)\mu(f)^2. \tag{6.7}$$

Note that in inequality (6.7) the power 2 is optimal, since it cannot be replaced, up to a rescaling, by any smaller power. The key point in the proof is a lemma, see [16, Lemma 1], where they show the inequality when $\mu(f)$ is sufficiently small. The proof of this lemma is elegant and simple. Given a function f , they consider the function g_{a,b,x_0} minimizing $\mu(f)$. Due to the Hilbert structure of $W^{1,2}(\mathbb{R}^n)$, one immediately has that $f - g_{a,b,x_0}$ is orthogonal to the tangent space $T\mathcal{M}_{g_{a,b,x_0}}$. Then the proof consists in relating $T\mathcal{M}_{g_{a,b,x_0}}$ to the first and second eigenspace of the operator $g_{a,b,x_0}^{2-2^*} \Delta$ on the weighted L^2 space in \mathbb{R}^n with weight $g_{a,b,x_0}^{2-2^*}$.

The situation is completely different when dealing with $p \neq 2$, where one cannot take advantage of the Hilbert structure of the space. Indeed, the results proved in this case in [43, 75] are weaker and not completely satisfactory. Let us first describe the content of the last paper, where the case $1 < p < n$ is consider. The first big difference between the result proved by Bianchi and Egnell and the one proved in [43] is that one has to replace the distance considered in (6.6) with a weaker kind of distance or

asymmetry. Indeed, given a function f in the homogeneous Sobolev space $W^{1,p}(\mathbb{R}^n)$ one sets

$$\lambda(f) = \inf_{a,b,x_0} \left\{ \frac{\|f - g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)}^{p^*}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}^{p^*}} : \|g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)} = \|f\|_{L^{p^*}(\mathbb{R}^n)} \right\}. \tag{6.8}$$

A part from the rescaling factor at the denominator and the constraint $\|g_{a,b,x_0}\|_{p^*} = \|f\|_{p^*}$, the new distance $\lambda(f)$ is clearly controlled from above by $\mu(f)$ due to the Sobolev inequality itself. Indeed, if one goes back to the asymmetry indices defined in Sects. 4 and 5, it is clear that $\lambda(f)$ and $\mu(f)$ play the role of the indices $\alpha(E)$, see (4.1), and $\beta(E)$, see (5.15), respectively. Moreover in the quantitative estimate proved in [43, Th. 1] the power is far from being optimal.

Theorem 6.2 *Let $n \geq 2$ and let $1 < p < n$. There exist two positive constants α and κ , depending only on p and n , such that for every $f \in W^{1,p}(\mathbb{R}^n)$*

$$S(p, n)\|f\|_{L^{p^*}(\mathbb{R}^n)} (1 + \kappa\lambda(f)^\alpha) \leq \|\nabla f\|_{L^p(\mathbb{R}^n)}. \tag{6.9}$$

The power α in (6.9) can be retrieved from [43, (2.54), Cor. 4, (4.1) and the proof of Th. 1], but it is clearly non optimal. A comparison with the statement of Theorem 6.1 above and the statement of Theorem 3.3 in [43] suggests that the optimal power could be $\alpha = \max\{2, p\}$, but we have no examples in this direction. Observe also that (6.9) can be rewritten in equivalent way as

$$\kappa(n)\lambda(f)^\alpha \leq \frac{\|\nabla f\|_{L^p(\mathbb{R}^n)}}{S(p, n)\|f\|_{L^{p^*}(\mathbb{R}^n)}} - 1 := \delta(f). \tag{6.10}$$

We shall refer to the term $\delta(f)$ at the right hand side of the previous equation as to the *Sobolev deficit* of the function f . Note also that in order to have an inequality of the type (6.10) both the asymmetry $\lambda(f)$ and the deficit $\delta(f)$ must be rescaling invariant. However we observe that the constraint in definition of $\lambda(f)$ is not really needed. Indeed one could define a smaller asymmetry by setting

$$\tilde{\lambda}(f) := \inf_{a,b,x_0} \frac{\|f - g_{a,b,x_0}\|_{L^{p^*}(\mathbb{R}^n)}^{p^*}}{\|f\|_{L^{p^*}(\mathbb{R}^n)}^{p^*}}.$$

Observe that $\tilde{\lambda}(f) \leq \lambda(f) \leq 2^{p^*}\tilde{\lambda}(f)$, hence we could indifferently use one of the two asymmetries.

The proof of the inequality follows a strategy very similar to the one used in Sect. 4 aimed to reduce the general case in (6.9) to the case where f is a more and more symmetric function. But before discussing that proof in more detail, let us recall the notion of *spherically symmetric decreasing rearrangement* of a function $f : \mathbb{R}^n \rightarrow [0, \infty)$ such that $|\{x : f(x) > t\}| < \infty$ for all $t > 0$. If f has this property, for every $x \in \mathbb{R}^n$ we set

$$f^*(x) =: \sup\{t \geq 0 : |\{y \in \mathbb{R}^n : f(y) > t\}| > \omega_n |x|^n\}.$$

Note that f^* is a decreasing function depending on the norm of x and that from the definition it follows that $|\{f^* > t\}| = |\{f > t\}|$. Thus, in particular, one has that

$$\|f^*\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all } p \geq 1. \tag{6.11}$$

Moreover, the *Pólya–Szegő principle* states that if f is a nonnegative function from $W^{1,p}(\mathbb{R}^n)$, with $p \geq 1$, then also f^* belongs to $W^{1,p}(\mathbb{R}^n)$ and moreover, see for instance [28,87],

$$\int_{\mathbb{R}^n} |\nabla f^*|^p dx \leq \int_{\mathbb{R}^n} |\nabla f|^p dx. \tag{6.12}$$

The proof of this inequality is a simple application of the coarea formula for functions (2.8) and of the isoperimetric inequality.

Proof of the Pólya–Szegő inequality (6.12) Let f be a nonnegative function from the homogeneous Sobolev space $W^{1,p}(\mathbb{R}^n)$ and assume without proving it that also f^* is in $W^{1,p}(\mathbb{R}^n)$ (see [92, Th. 73]). We set for all $t > 0$

$$\mu(t) := |\{x \in \mathbb{R}^n : |f(x)| > t\}|. \tag{6.13}$$

The function μ is called the *distribution function* of f . Since by construction the level sets of f and f^* have the same measure, the two functions share the same distribution function μ . Note that μ is a decreasing function. Moreover, an easy application of the coarea formula (2.8) shows that for a.e. $t > 0$

$$-\mu'(t) = \frac{\mathcal{H}^{n-1}(\{f^* = t\})}{|\nabla f^*|_{\{f^*=t\}}}, \tag{6.14}$$

and that

$$-\mu'(t) \geq \int_{\{f=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla f|}, \tag{6.15}$$

see for instance [41, Lemmas 2.4 and 2.6]. Note that in (6.14) we have implicitly used the fact that $|\nabla f^*|$ is constant on $\{f^* = t\}$. Then from coarea formula (2.8), (6.14), the fact that $|\nabla f^*|$ is constant on $\{f^* = t\}$ and the isoperimetric inequality applied to the sublevel sets of f , we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f^*|^p dx &= \int_0^\infty dt \int_{\{f^*=t\}} |\nabla f^*|^{p-1} d\mathcal{H}^{n-1} = \int_0^\infty \frac{\mathcal{H}^{n-1}(\{f^* = t\})^p}{\left(\int_{\{f^*=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla f^*|}\right)^{p-1}} dt \\ &= \int_0^\infty \frac{\mathcal{H}^{n-1}(\{f^* = t\})^p}{(-\mu'(t))^{p-1}} dt \leq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{f = t\})^p}{(-\mu'(t))^{p-1}} dt. \end{aligned}$$

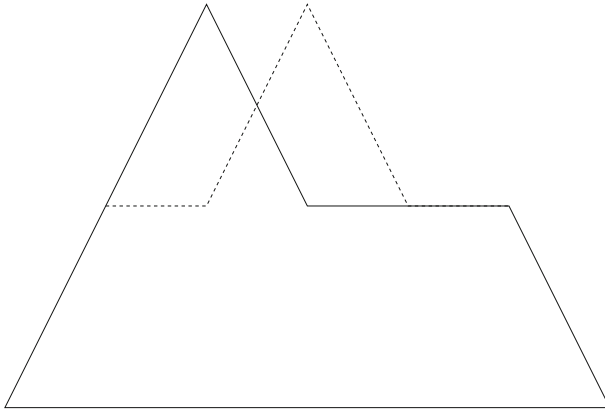


Fig. 10 An example showing the necessity of condition (6.16) to deduce that f is a translated of f^*

From this inequality, using (6.15), Hölder inequality and coarea formula again we get

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla f^*|^p dx &\leq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{f = t\})^p}{(-\mu'(t))^{p-1}} dt \leq \int_0^\infty \frac{\mathcal{H}^{n-1}(\{f = t\})^p}{\left(\int_{\{f=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla f|}\right)^{p-1}} dt \\ &\leq \int_0^\infty dt \int_{\{f=t\}} |\nabla f|^{p-1} d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} |\nabla f|^p dx, \end{aligned}$$

thus concluding the proof. □

Note that from the above proof it is clear that when equality holds in (6.12) then for a.e. $t > 0$

$$\{f > t\} \text{ is equivalent to a ball and } |\nabla f| = |\nabla f^*|_{|\{f^*=t\}} \quad \mathcal{H}^{n-1}\text{-a.e. on } \{f = t\}.$$

However this information is not enough to conclude that f coincides with f^* up to a translation, see Fig. 10. The equality cases in the Pólya–Szegő inequality have been fully characterized first by Brothers and Ziemer in [28], see also [41, 61]. Their result reads as follows.

Theorem 6.3 *Let $p > 1$ and $f \in W^{1,p}(\mathbb{R}^n)$ a nonnegative function such that*

$$|\{\nabla f^* = 0\} \cap \{0 < f^* < \text{ess sup } f\}| = 0. \tag{6.16}$$

If equality holds in (6.12), then there exists $x_0 \in \mathbb{R}^n$ such that $f(x) = f^(x - x_0)$ for all $x \in \mathbb{R}$.*

We observe that if $|\{\nabla f = 0\} \cap \{0 < f < \text{ess sup } f\}| = 0$, then (6.16) holds. The reverse implication is not in general true.

As we already observed, it may happen that $\|\nabla f\|_p$ almost agrees with $\|\nabla f^*\|_p$ without f being close to any translated of f^* . The presence of large sets where $|\nabla f|$

is zero or very small is responsible of this phenomenon. Therefore the quantitative versions of (6.12) available require either a control on the size of the set where $|\nabla f|$ is small, see [40, 42], or further assumptions on the class of functions under consideration, see [12, Th. 1.3].

Going back to the Sobolev inequality, it is clear the role played by the Pólya–Szegő inequality (6.12). Indeed observe that we may always assume, without loss of generality, that $f \geq 0$. In fact, replacing f by $|f|$ changes neither the L^p norm of the function nor the L^p norm of the gradient, since $|\nabla|f|(x)| = |\nabla f(x)|$ for a.e. $x \in \mathbb{R}^n$. Therefore, if one wants to find the equality cases in the Sobolev inequality it is clear that these will be realized by spherically symmetric and decreasing functions. Thus Pólya–Szegő inequality plays in the proof of the quantitative Sobolev inequality given in [43] the same role played by the perimeter inequality for Schwartz symmetrization in the proof of (4.3) that we have seen in Sect. 4.

Let us describe briefly how the proof of inequality (6.9) goes. The idea is to reduce to spherically symmetric decreasing functions exactly as in Sect. 4 we reduced the proof of the quantitative isoperimetric inequality to axially symmetric sets. Observe that if $f \in W^{1,p}(\mathbb{R}^n)$ is nonnegative, spherically symmetric and decreasing, then there exists a decreasing function $u \in W_{loc}^{1,p}(\mathbb{R})$ such that $f(x) = u(|x|)$. In this case the Sobolev inequality (6.1) is equivalent to the one-dimensional Bliss inequality [17]

$$S(p, n) \left(n\omega_n \int_0^\infty u(r)^{p^*} r^{n-1} dr \right)^{1/p^*} \leq \left(n\omega_n \int_0^\infty (-u'(r))^p r^{n-1} dr \right)^{1/p} \tag{6.17}$$

for every decreasing, locally absolutely continuous function $u : [0, \infty) \rightarrow [0, \infty)$. The extremals in (6.17) are the one dimensional profiles of the functions defined in (6.2), see for instance [17, 49, 113], that is

$$v_{a,b}(r) = \frac{a}{(1 + br^{p'})^{(n-p)/p}} \quad \text{for } r \geq 0,$$

for some $a > 0, b > 0$. Moreover, if we set for a measurable function $w : (0, \infty) \rightarrow \mathbb{R}$,

$$\|w\|_{p^*, r^{n-1}} := \left(\int_0^\infty |w|^{p^*}(r) r^{n-1} dr \right)^{\frac{1}{p^*}},$$

and for every nonnegative function u in $(0, \infty)$ we define

$$\lambda(u) = \inf_{a,b>0} \left\{ \frac{\|u - v_{a,b}\|_{p^*, r^{n-1}}^{p^*}}{\|u\|_{p^*, r^{n-1}}^{p^*}} : \|u\|_{p^*, r^{n-1}} = \|v_{a,b}\|_{p^*, r^{n-1}} \right\},$$

Theorem 6.2 for spherically symmetric functions is equivalent to the following quantitative Bliss inequality.

Theorem 6.4 *Let $n \geq 2$ and let $1 < p < n$. Then there exist constants β and κ , depending only on p, n , such that*

$$\begin{aligned}
 S(p, n) & \left(n\omega_n \int_0^\infty u(r)^{p^*} r^{n-1} dr \right)^{1/p^*} (1 + \kappa \lambda(u)^\beta) \\
 & \leq \left(n\omega_n \int_0^\infty (-u'(r))^p r^{n-1} dr \right)^{1/p}
 \end{aligned}
 \tag{6.18}$$

for every decreasing, locally absolutely continuous function $u : [0, \infty) \rightarrow [0, \infty)$.

The proof of this theorem is a bit long and technically involved, but the underlying idea is quite simple. First of all, by an approximation, rescaling and normalization argument one may always assume that u is a nonnegative decreasing function with support in $[0, 1]$ such that

$$n\omega_n \int_0^\infty u^{p^*}(r)r^{n-1} dr = 1.$$

Then, one tries to compare this function with the function $v := v_{a,1}$, where a is chosen in such a way that also

$$n\omega_n \int_0^\infty v^{p^*}(r)r^{n-1} dr = 1.$$

In order to compare the two functions, following the argument that we have seen in the proof of the Sobolev inequality given above, one defines a transport map $T : [0, 1) \rightarrow [0, \infty)$ by setting

$$\int_0^r u(s)^{p^*} s^{n-1} ds = \int_0^{T(r)} v(s)^{p^*} s^{n-1} ds.$$

From this definition one gets immediately that T is a $C^1(0, 1)$ strictly increasing function such that $T(0) = 0$, $\lim_{r \rightarrow 1^-} T(r) = \infty$, and

$$u(r)^{p^*} = v(T(r))^{p^*} T(r)^{n-1} T'(r) r^{1-n} \quad \text{for all } r \in (0, 1).
 \tag{6.19}$$

In particular, Eq. (6.19) implies that

$$\int_0^1 h(T(r))u(r)^{p^*} r^{n-1} dr = \int_0^\infty h(r)v(r)^{p^*} r^{n-1} dr,$$

for every Borel function $h : [0, \infty) \rightarrow [0, \infty]$. Therefore the function T can be regarded as a transport map carrying the measure $u(r)^{p^*} r^{n-1} dr$ into $v(r)^{p^*} r^{n-1} dr$.

Note that when $T(r) = kr$ for some $k > 0$, one has $u(r) = k^{(n-p)/p} v(kr)$, thus proving that u is an extremal function in the Bliss inequality (6.17). Thus, the idea is

to show that when the difference between the two sides of the Bliss inequality is small, then an interval $[r_1, r_2] \subset [0, 1]$ can be chosen in such a way that $T(r)$ is close to some linear function kr for $r \in [r_1, r_2]$, and simultaneously the integral of $u(r)^p r^{n-1}$ outside $[r_1, r_2]$ is small. These facts lead to conclude that u is close to $k^{(n-p)/p} v(kr)$ in the quantitative way stated in (6.18).

As observed in [43, (2.54)] one can in fact choose β in (6.18) to be the (non optimal) exponent

$$\beta = 3 + 4p - \frac{3p + 1}{n}. \tag{6.20}$$

The next step in the proof of Theorem 6.2 is to take advantage of the quantitative Bliss inequality (6.18) to reduce the general case to the case of a spherically symmetric and decreasing function, via the Pólya–Szegő inequality (6.12). In fact, from that inequality, recalling also (6.11), one has

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} - \|\nabla f^*\|_{L^p(\mathbb{R}^n)} \leq \|\nabla f\|_{L^p(\mathbb{R}^n)} - S(p, n)\|f\|_{L^{p^*}(\mathbb{R}^n)}. \tag{6.21}$$

for every $f \in W^{1,p}(\mathbb{R}^n)$. The idea at this point is to estimate from below the difference between the L^p norm of ∇f and ∇f^* in terms of the L^{p^*} distance of f from a suitable translated of f^* . Unfortunately, as we already observed, it is not possible in general to control the distance between f and f^* in terms of the left hand side of (6.21), since if the measure of the set $\{\nabla f = 0\}$ is large the L^p norms of ∇f and ∇f^* may be equal, without f or f^* being close. However, if f belongs to a sufficiently rigid class of functions, this estimate is still possible. Indeed, this is the case of n -symmetric functions, for which it is not too hard to show, see [43, Th. 3], the following stability result for the Pólya–Szegő inequality.

Theorem 6.5 *Let $n \geq 2$ and let $1 < p < n$. Set $q = \max\{p, 2\}$. Then a positive constant C exists such that*

$$\int_{\mathbb{R}^n} |f - f^*|^{p^*} \leq C \left(\int_{\mathbb{R}^n} |f|^{p^*} \right)^{p/n} \left(\int_{\mathbb{R}^n} |\nabla f^*|^p \right)^{1/q'} \left(\int_{\mathbb{R}^n} |\nabla f|^p - \int_{\mathbb{R}^n} |\nabla f^*|^p \right)^{1/q}$$

for every nonnegative $f \in W^{1,p}(\mathbb{R}^n)$ which is symmetric with respect to the coordinate hyperplanes.

The proof of this theorem consists in a careful revisit of the proof of the Pólya–Szegő inequality that we have given above, using the quantitative isoperimetric inequality (4.3) in place of the standard isoperimetric inequality and taking advantage of the fact that since f is n -symmetric then also the level sets $\{f > t\}$ are n -symmetric and thus their asymmetry $\alpha(\{f > t\})$ is equivalent to the symmetric difference with the ball centered at the origin with the same measure, see Lemma 4.4. As a consequence of this theorem and of the quantitative Bliss inequality (6.18) one can easily deduce (6.9).

Proposition 6.6 *Let $n \geq 2$ and let $1 < p < n$. Then there exists a constant $\kappa > 0$ such that (6.9) holds for every nonnegative n -symmetric function $f \in W^{1,p}(\mathbb{R}^n)$, with $\alpha = \beta$, β as in (6.20).*

Proof As we already observed (6.9) is equivalent to (6.10). To prove this last one, since both $\delta(f)$ and $\lambda(f)$ are invariant by rescaling and multiplication by a constant, we may assume, without loss of generality, that $\|f\|_{L^{p^*}(\mathbb{R}^n)} = 1$. Observe also that $\lambda(f) \leq 2^{p^*}$, hence we may also assume that $\delta(f) \leq 1/S(p, n)$, since otherwise (6.10) follows with $\kappa(n) = 1/(2^{\alpha p^*} S(n, p))$. Then, from (6.11) and (6.12) we have

$$S(p, n) \leq \|\nabla f^*\|_p \leq \|\nabla f\|_p \leq 1 + S(p, n). \tag{6.22}$$

We have

$$\begin{aligned} \lambda(f) &\leq 2^{p^*-1} \left(\lambda(f^*) + \|f - f^*\|_{p^*}^{p^*} \right) \\ &\leq C \left[(\|\nabla f^*\|_p - S(p, n))^{1/\beta} + \|\nabla f^*\|_p^{p/q'} (\|\nabla f\|_p^p - \|\nabla f^*\|_p^p)^{1/q} \right], \end{aligned} \tag{6.23}$$

for some constant C , where the first inequality is just a consequence of the triangle inequality, and the second one follows from Theorems 6.4 and 6.5. Inequalities (6.22) ensure that

$$\|\nabla f\|_p^p - \|\nabla f^*\|_p^p \leq C(\|\nabla f\|_p - \|\nabla f^*\|_p), \tag{6.24}$$

for some constant C . Combining (6.23), (6.24), and (6.21) yields

$$\lambda(f) \leq C(\delta(f))^{1/\beta} + \delta(f)^{1/q}$$

for some constant C . Hence, inequality (6.9) follows with $\alpha = \beta$, since by (6.20) one has $\beta > q$. □

At this point the full proof of Theorem 6.2 follows from a reduction argument of the same kind of the one we stated in Theorem 4.6. However, the reduction to n -symmetric functions although related to a similar construction employed in Sect. 4, entails the overcoming of new serious obstacles in the present setting, mainly due to the nonlinear growth of the functional $\|\nabla f\|_{L^p(\mathbb{R}^n)}^p$. The precise statement of the reduction theorem reads as follows, see [43, Th. 6].

Theorem 6.7 *Let $n \geq 2$ and let $1 < p < n$. There exists a positive constant C , depending only on n and p such that for every $f \in W^{1,p}(\mathbb{R}^n)$ there exists a nonnegative n -symmetric function \tilde{f} with the property that*

$$\lambda(f) \leq C\lambda(\tilde{f}), \quad \delta(\hat{f}) \leq C\delta(f)^{1/\beta},$$

where β is given by (6.20).

We conclude this discussion on the quantitative Sobolev inequality for $1 < p < n$ by noting that, in view of the results of [16,75], the question arises of the optimal exponent α in equality (6.9). Furthermore, the result of [16] also leaves open the problem of whether the distance of f from the family of extremals in $L^{p^*}(\mathbb{R}^n)$ can be replaced by the distance in $W^{1,p}(\mathbb{R}^n)$ in Theorem 6.2.

In the special case $p = 1$ the Sobolev inequality (6.1) takes the form

$$n\omega_n^{1/n} \|f\|_{L^{n'}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\nabla f| dx$$

for all f in the homogeneous Sobolev space $W^{1,1}(\mathbb{R}^n)$. Here $n' := n/(n - 1)$ stands for the Sobolev exponent 1^* . By approximation it can be readily extended to the homogenous space $BV(\mathbb{R}^n)$. For these functions it takes the form

$$n\omega_n^{1/n} \|f\|_{L^{n'}(\mathbb{R}^n)} \leq |Df|(\mathbb{R}^n). \tag{6.25}$$

As we already observed, the above inequality is equivalent to the isoperimetric inequality, Indeed if E is a set of finite measure, taking $f = \chi_E$ in the inequality above one gets exactly the isoperimetric inequality. It is well-known that equality holds in (6.25) if and only if $f = a\chi_{B_r(x_0)}$ for some $a \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $r > 0$. Therefore, in analogy with (6.8) it is natural to introduce the asymmetry of a function $f \in BV(\mathbb{R}^n)$ by setting

$$\lambda(f) := \inf \left\{ \frac{\|f - a\chi_{B_r(x)}\|_{L^{n'}}^{n'}}{\|f\|_{L^{n'}}^{n'}} : |a|^{n'} r^n \omega_n = \|f\|_{L^{n'}}^{n'}, a \in \mathbb{R}, x \in \mathbb{R}^n \right\}.$$

Indeed, it can be proved that the above infimum is attained, see [75, Lemma B.1]. In the case $p = 1$ the quantitative version of the Sobolev inequality (6.25) has been obtained with the sharp exponent 2 [75, Th. 1.1].

Theorem 6.8 *There exists a dimensional constant $C = C(n)$ such that*

$$n\omega_n^{1/n} \|f\|_{L^{n'}(\mathbb{R}^n)} \left(1 + \frac{\lambda(f)^2}{C(n)} \right) \leq |Df|(\mathbb{R}^n) \tag{6.26}$$

for every $f \in BV(\mathbb{R}^n)$.

The proof of this result follows the same pattern of the proof of the quantitative isoperimetric inequality that we have seen in Sect. 4. A part from a series of technical difficulties due to the fact that one deals with functions instead of sets, the only point where the proof really differs from the one we have already seen is when one has to show inequality (6.26) for a spherically symmetric and decreasing function, see [75, Sect. 3].

I would like also to point out that very recently Figalli et al. in [68] proved a similar stability estimate for the anisotropic Sobolev (and log-Sobolev) inequality for functions of bounded variation. Also in this case the stability estimate is obtained with the sharp exponent 2, see [68, Th. 1.1]. The proof combines symmetrization arguments with the Gromov’s idea for proving the isoperimetric inequality, and thus also the Sobolev inequality for BV functions, with a mass transport argument.

The Morrey–Sobolev imbedding theorem states that if $p > n$ a function from the homogeneous Sobolev space $W^{1,p}(\mathbb{R}^n)$ is essentially bounded and actually Hölder continuous. In particular, if the support of f is bounded the following inequality holds

$$S(p, n)\|f\|_{L^\infty(\mathbb{R}^n)} \leq |\text{supp } f|^{\frac{1}{n}-\frac{1}{p}}\|\nabla f\|_{L^p(\mathbb{R}^n)}, \tag{6.27}$$

where $\text{supp } f$ denotes the support of f and the sharp constant $S(n, p)$ is given by

$$S(p, n) = n^{1/p}\omega_n^{1/n}\left(\frac{p-n}{p-1}\right)^{1/p'}.$$

In this case the extremals are given by the family of functions

$$g_{a,b,x_0} := \begin{cases} a\left(b^{\frac{p-n}{p-1}} - |x-x_0|^{\frac{p-n}{p-1}}\right) & \text{if } |x-x_0| \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

for some $a \in \mathbb{R}$, $b \geq 0$ and $x_0 \in \mathbb{R}^n$. The stability for (6.27) has been proved by Cianchi and stated in terms of the L^∞ distance from the extremals. Precisely, see [39, Th. 1.1], we have the following quantitative estimate, where $\delta(f)$ is defined as in (6.10) for $p = \infty$.

Theorem 6.9 *Let $n \geq 2$ and $p > n$. There exist two positive constants α and C , depending only on p and n , such that for every $f \in W^{1,p}(\mathbb{R}^n)$*

$$\left(\inf_{a,b,x_0} \frac{\|f - g_{a,b,x_0}\|_{L^\infty(\mathbb{R}^n)}}{\|f\|_{L^\infty(\mathbb{R}^n)}}\right)^\alpha \leq C\delta(f).$$

6.2 The Faber–Krahn inequality

The *Faber–Krahn* inequality goes back to Lord Rayleigh who in 1877 in his book ‘The theory of sound’ [111] conjectured that the gravest principal tone of a vibrating membrane is obtained by a circular one. In other words, if $\Omega \subset \mathbb{R}^n$ is an open set with $|\Omega| = |B_r|$ for some $r > 0$ then

$$\lambda(\Omega) \geq \lambda(B_r), \tag{6.28}$$

with the equality holding if and only if Ω is a ball. Here by $\lambda(\Omega)$ we denote the *first eigenvalue* of the Laplacian which is defined as

$$\lambda(\Omega) := \min \left\{ \int_\Omega |\nabla f|^2 dx : \|f\|_{L^2(\Omega)} = 1, f \in H_0^1(\Omega) \right\}. \tag{6.29}$$

Note that a function f realizing the minimum on the right hand side is called an *eigenfunction* for the Laplacian and satisfies the equation

$$-\Delta f = \lambda f \quad \text{in } \Omega, \quad f = 0 \quad \text{on } \partial\Omega. \tag{6.30}$$

Inequality (6.28) was proved in two dimensions by Faber [60] and extended for any $n \geq 2$ by Krahn [91]. Indeed, it can be proved as an immediate consequence of the Pólya–Szegő inequality (6.12) by observing that if f is a minimizer in $H_0^1(\Omega)$ of the above variational problem then

$$\lambda(\Omega) = \int_{\Omega} |\nabla f|^2 dx \geq \int_{B_r} |\nabla f^*|^2 dx \geq \lambda(B_r). \tag{6.31}$$

The characterization of the equality cases in (6.28) can be also easily recovered using Theorem 6.3. Observing that $\lambda(r\Omega) = r^{-2}\lambda(\Omega)$, inequality (6.28) may be rewritten in the following scaling invariant form

$$|\Omega|^{\frac{2}{n}}\lambda(\Omega) \geq |B|^{\frac{2}{n}}\lambda(B). \tag{6.32}$$

The stability of balls with respect to this inequality has been investigated by several authors. As for the quantitative isoperimetric inequality people tried first to understand the convex case. In particular, it was proved by Melas in [99] that if $\Omega \subset \mathbb{R}^n$ is a bounded, convex open set then one can find two balls $B' \subset \Omega \subset B''$ such that

$$\frac{\max\{|\Omega \setminus B'|, |B'' \setminus \Omega|\}}{|\Omega|}$$

is controlled by a suitable power, depending on n , of the *Faber–Krahn deficit*

$$\delta(\Omega) := |\Omega|^{\frac{2}{n}}\lambda(\Omega) - |B|^{\frac{2}{n}}\lambda(B). \tag{6.33}$$

A similar result was later on obtained by Hansen and Nadirashvili in [85] where the *inner* and the *outer radius* of a convex set are used to control the distance from a ball in terms of the Faber–Krahn deficit. For a general open set, it was conjectured by Bhattacharya and Weitsman in [15] and by Nadirashvili in [102] that the following *quantitative Faber–Krahn inequality* holds

$$\alpha(\Omega)^2 \leq C(n)\delta(\Omega), \tag{6.34}$$

for any open set $\Omega \subset \mathbb{R}^n$ of finite measure, where α is the Fraenkel asymmetry index defined in (4.1). An inequality of this kind, dealing with general open sets, was obtained first in dimension two by Bhattacharya in [14] with the exponent 3 in place of 2 and in any dimension n by Maggi et al. in [77] with the exponent 4. Indeed, in [77] a more general inequality was proved, related to the *first eigenvalue of the p -Laplacian*, $p > 1$ which is defined, similarly to (6.29), by setting for any open set Ω with finite measure

$$\lambda_p(\Omega) := \min \left\{ \int_{\Omega} |\nabla f|^p dx : \|f\|_{L^p(\Omega)} = 1, f \in W_0^{1,p}(\Omega) \right\}.$$

Note that the same argument used to show Faber–Krahn inequality (6.32) implies that for any open set of finite measure one has

$$|\Omega|^{\frac{p}{n}} \lambda(\Omega) \geq |B|^{\frac{p}{n}} \lambda(B).$$

Then, the following quantitative estimate was proved in [77, Th. 1].

Theorem 6.10 *Let $n \geq 2$ and $p > 1$. There exists a positive constant $c(n, p)$ such that for every open set $\Omega \subset \mathbb{R}^n$ of finite measure*

$$c(n, p)\alpha(\Omega)^{2+p} \leq |\Omega|^{\frac{p}{n}} \lambda_p(\Omega) - |B|^{\frac{p}{n}} \lambda_p(B).$$

The proof of the above inequality makes use of the following observation. Assuming for the sake of simplicity that $p = 2$ and that Ω has the same measure of the unit ball B , let $f \in H_0^1(\Omega)$ be the function minimizing the right hand side of (6.29). Then, recalling (6.31), one has that

$$\int_{\Omega} |\nabla f|^2 dx - \int_{B_r} |\nabla f^*|^2 dx \leq \lambda(\Omega) - \lambda(B).$$

In turn, denoting as in the previous section by μ the distribution function of f defined as in (6.13) one has, see [77, (32)],

$$c(n) \int_0^\infty \alpha(\{f > t\})^2 \frac{\mu(t)^{2(n-1)/n}}{|\mu'(t)|} dt \leq \int_{\Omega} |\nabla f|^2 dx - \int_{B_r} |\nabla f^*|^2 dx,$$

for some positive constant $c(n)$. Therefore, one may conclude that

$$c(n) \int_0^\infty \alpha(\{f > t\})^2 \frac{\mu(t)^{2(n-1)/n}}{|\mu'(t)|} dt \leq \lambda(\Omega) - \lambda(B).$$

This inequality shows indeed that the Faber–Krahn deficit controls in an integral form the asymmetry of the level sets of the eigenfunctions. However, to recover from this information the sharp quantitative estimate of the asymmetry for Ω does not seem easy.

The above conjecture on the optimal power in inequality (6.34) has been proved in a recent paper by Brasco et al. [24]. Here is their result.

Theorem 6.11 *There exists a positive constant $C(n)$ such that (6.34) holds true for every open set $\Omega \subset \mathbb{R}^n$ of finite measure.*

The proof is deep and technically involved. However, we shall try to explain the main ideas. The first key point of their strategy is to prove a sharp quantitative estimate for a weaker inequality related to the torsional rigidity. To explain this inequality we have to introduce, for an open set Ω of finite measure, the following quantity

$$E(\Omega) := \min \left\{ \frac{1}{2} \int_{\Omega} |\nabla f|^2 dx - \int_{\Omega} f dx : f \in H_0^1(\Omega) \right\}.$$

Note that the function f_Ω minimizing the integral functional on the right hand side coincides with the unique solution of the Dirichlet problem

$$-\Delta f_\Omega = 1 \quad \text{in } \Omega, \quad f_\Omega = 0 \quad \text{on } \partial\Omega.$$

Therefore, multiplying the above equation by f_Ω and integrating by parts one gets

$$E(\Omega) = -\frac{1}{2} \int_\Omega |\nabla f_\Omega|^2 dx = -\frac{1}{2} \int_\Omega f_\Omega dx.$$

From this equality one easily gets a reformulation of $E(\Omega)$ as

$$E(\Omega) := -\frac{1}{2} \max \left\{ \left(\int_\Omega f dx \right)^2 : \|\nabla f\|_{L^2(\Omega)} = 1 \right\}.$$

The maximum of the integral on the right hand side of the previous formula is the *torsional rigidity* of Ω . Observe that E satisfies the scaling law: $E(r\Omega) = r^{-n-2} E(\Omega)$ for all $r > 0$ and all open sets Ω with finite measure. Then, the same argument used to show inequality (6.28) immediately yields the following inequality

$$|\Omega|^{-\frac{n+2}{n}} E(\Omega) \geq |B|^{-\frac{n+2}{n}} E(B), \tag{6.35}$$

with the equality holding if and only if Ω is a ball. A deeper inequality, due to Kohler-Jobin, see [89,90], relates the first eigenvalue of the Laplacian to the torsional rigidity.

Theorem 6.12 *For every open set $\Omega \subset \mathbb{R}^n$ of finite measure we have*

$$\lambda(\Omega)(-E(\Omega))^{\frac{2}{n+2}} \geq \lambda(B)(-E(B))^{\frac{2}{n+2}}, \tag{6.36}$$

with the equality holding if and only if B is a ball.

As we already said, the first key point is to observe that in view of (6.36) one can deduce (6.34) from a quantitative version of (6.35), see [24, Prop. 2.1].

Proposition 6.13 *Assume that there exists a constant $c_0(n)$ such that for every open set $\Omega \subset \mathbb{R}^n$ of finite measure*

$$|\Omega|^{-\frac{n+2}{n}} E(\Omega) - |B|^{-\frac{n+2}{n}} E(B) \geq c_0(n)\alpha(\Omega)^2. \tag{6.37}$$

Then, there exists another constant $c_1(n)$ such that one has also

$$|\Omega|^{\frac{2}{n}} \lambda(\Omega) - |B|^{\frac{2}{n}} \lambda(B) \geq c_1(n)\alpha(\Omega)^2.$$

Proof Assume $|\Omega| = |B|$ and observe that from the Kohler-Jobin inequality (6.36) we have

$$\frac{\lambda(\Omega)}{\lambda(B)} - 1 \geq \left(\frac{E(B)}{E(\Omega)} \right)^{\frac{2}{n+2}} - 1. \tag{6.38}$$

Since $E(\Omega)$ is a negative quantity, from (6.35) we have that $E(B)/E(\Omega) \geq 1$. Therefore, if the ratio $E(B)/E(\Omega) \geq 2$, from (6.38) we get

$$\frac{\lambda(\Omega)}{\lambda(B)} - 1 \geq 2^{\frac{2}{n+2}} - 1 \geq \frac{2^{\frac{2}{n+2}} - 1}{4} \alpha(\Omega)^2.$$

On the other hand, if $1 \leq E(B)/E(\Omega) \leq 2$, from (6.38) and the assumption (6.37) we have

$$\frac{\lambda(\Omega)}{\lambda(B)} - 1 \geq c(n) \left(\frac{E(B)}{E(\Omega)} - 1 \right) \geq \frac{c(n)c_0}{|E(B)|} \alpha(\Omega)^2,$$

where $c(n)$ is such that $t^{\frac{2}{n+2}} - 1 \geq c(n)(t - 1)$ for all $t \in [1, 2]$. □

With this proposition in hands it is now clear that the strategy followed in [24] is to prove a stability inequality for $E(\Omega)$ of the type (6.37). Namely they follow the pattern that we have discussed in Sect. 5.2 of proving first the stability estimate for nearly spherical sets and then to extend it to general open sets by a contradiction argument via regularity. The proof for nearly spherical sets is essentially a second variation argument and it leads to the following Fuglede type result, see [24, Th. 3.3].

Theorem 6.14 *Let $\sigma \in (0, 1]$. There exists a positive constant δ , depending only on n and γ , such that if Ω is a nearly spherical set as in (3.1), $|\Omega| = |B|$, the barycenter of Ω is at the origin and $\|u\|_{C^{2,\sigma}(\mathbb{S}^{n-1})} \leq \delta$, then*

$$E(\Omega) - E(B) \geq \frac{1}{32n^2} \|u\|_{H^{1/2}(\mathbb{S}^{n-1})}. \tag{6.39}$$

As for Fuglede’s estimate (3.2) this inequality immediately implies (6.37) for nearly spherical sets. Note however that the perimeter deficit controls the H^1 norm of u , while the difference $E(\Omega) - E(B)$, hence the Faber–Krahn deficit (6.33), only controls the weaker $H^{1/2}$ norm of u . This is not surprising if one thinks that one may easily find open sets Ω_h for which the perimeter $P(\Omega_h)$ goes to ∞ while the first eigenvalue of the Laplacian $\lambda(\Omega_h)$ remains bounded. Indeed it is enough to take

$$\Omega_h := \{y \in \mathbb{R}^n : y = tx(1 + u_h(x)), \text{ with } x \in \mathbb{S}^{n-1}, 0 \leq t < 1\},$$

with $u_h : \mathbb{S}^{n-1} \rightarrow (-1, 1)$ chosen so that $\|u_h\|_{H^{1/2}(\mathbb{S}^{n-1})}$ is bounded and $\|u_h\|_{H^1(\mathbb{S}^{n-1})} \rightarrow \infty$.

Coming to the proof of the stability of the functional E , after proving Theorem 6.14 one would like to argue as we did in Sect. 5.2. So let us try to imitate the contradiction argument used in Sect. 5.2 by assuming that there exists a sequence Ω_j of open sets with $|\Omega_j| = |B|$, $E(\Omega_j) \rightarrow E(B)$ and

$$E(\Omega_j) - E(B) \leq C_0(n)\alpha(\Omega_j)^2, \tag{6.40}$$

for some big constant C_0 to be chosen later. Then, one would like to replace the sets Ω_j with a sequence of sets, say U_j , minimizing a functional of the type we introduced in (5.12)

$$E(\Omega) + |\alpha(\Omega) - \alpha(\Omega_j)| + \Lambda \left| |\Omega| - |B| \right|,$$

to show that they still satisfy a ‘wrong’ inequality as (6.40) and that they converge in $C^{2,\sigma}$ to the a ball, so to get a contradiction to (6.39). This is the point where the proof in [24] becomes really complicate and requires some new ideas and deep arguments. To understand why observe that minimizing the functional above is equivalent to finding a minimizer f_j of the functional

$$\frac{1}{2} \int_{\Omega} |\nabla f|^2 dx - \int_{\Omega} f dx + |\alpha(\{f > 0\}) - \alpha(\Omega_j)| + \Lambda \left| |\{f > 0\}| - |B| \right| \tag{6.41}$$

among all functions with a support of finite measure. It turns out that to get the desired regularity one has to modify in a non trivial way the theory developed by Alt and Caffarelli in [5]. But even this is not enough. In fact the functional in (6.41) is not sufficiently smooth to ensure that the support of the minimizers f_j are of class $C^{2,\sigma}$. Well, one could replace it with an essentially equivalent and smoother one, such as

$$\frac{1}{2} \int_{\Omega} |\nabla f|^2 dx - \int_{\Omega} f dx + \sqrt{\alpha(\Omega_j)^2 + (\alpha(\{f > 0\}) - \alpha(\Omega_j))^2} + \Lambda |\{f > 0\}|,$$

but even this functional would not work. And the reason is that the Fraenkel asymmetry is not smooth enough. So, another delicate point in [24] is the replacement of the asymmetry index with a new distance which on one side dominates the square of the Fraenkel asymmetry, on the other side is much smoother since it behaves like the square of an L^2 norm.

6.3 Further results

In the previous sections we have seen the natural extension of the ideas developed for the study of the stability of the quantitative isoperimetric inequality to two important inequalities. Indeed, several other inequalities have been investigated in the last years in this direction. For instance, in [77] the same argument used to prove Theorem 6.10 is also used to derive a quantitative inequality for the *isocapacitary inequality* stating that for any open set of finite measure Ω one has

$$|\Omega|^{\frac{p-n}{n}} \text{Cap}_p(\Omega) \geq |B|^{\frac{p-n}{n}} \text{Cap}_p(B),$$

where the p capacity of Ω , for $p \in (1, n)$, $n \geq 2$ is defined as

$$\text{Cap}_p(\Omega) := \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^p dx : f \geq \chi_{\Omega}, f \in L^{p^*}(\mathbb{R}^n), |\nabla f| \in L^p(\mathbb{R}^n) \right\}.$$

Quantitative inequalities for the so called *Cheeger constant* have been also obtained in [77] as a limit case of Theorem 6.10 and in [65] with the optimal exponent. To this aim we recall that if Ω is an open set with finite measure, $n \geq 2$, and

$$m > \frac{1}{n'}, \quad \text{where } n' = \frac{n}{n-1},$$

then the m -Cheeger constant of Ω is defined by setting

$$c_m(\Omega) := \inf \left\{ \frac{P(E)}{|E|^m} : E \subset \Omega, |E| > 0 \right\}.$$

When $m = 1$ a set E minimizing the right hand side is called a *Cheeger set* of Ω . The *Cheeger inequality* states that balls minimize the Cheeger constant among all open sets with the same measure. Thus, taking into account the right scaling factor, we have that

$$|\Omega|^{m-\frac{1}{n'}} c_m(\Omega) \geq |B|^{m-\frac{1}{n'}} c_m(B),$$

with the equality holding if and only if Ω is a ball. Using the quantitative isoperimetric inequality it has been shown by Figalli et al. in [65] that the following stability estimate holds. Given $m > 1/n'$, there exist a positive constant $\kappa(m, n)$ such that for every open set Ω of finite measure one has

$$\kappa \alpha(\Omega)^2 \leq |\Omega|^{m-\frac{1}{n'}} c_m(\Omega) - |B|^{m-\frac{1}{n'}} c_m(B).$$

The results discussed in the previous section have been extended by Brasco and Pratelli in [25] to the second eigenvalue of the Laplacian. To this aim, we recall that if Ω is an open set with finite measure the second eigenvalue $\lambda_2(\Omega)$ is defined by minimizing the Dirichlet integral among all functions that are orthogonal to the first eigenfunctions. Precisely, denoting by f_Ω a non trivial eigenfunction, that is a nontrivial solution to Eq. (6.30). We have

$$\lambda_2(\Omega) := \min \left\{ \int_\Omega |\nabla f|^2 dx : \|f\|_{L^2(\Omega)} = 1, f \in H_0^1(\Omega), \int_\Omega f f_\Omega dx = 0 \right\}.$$

Then *Krahn–Szegő inequality* states that among all open sets of given measure the unique minimizer of λ_2 is given by the union of two disjoint balls of equal measure. In other words, setting $\Theta := B' \cup B''$ where the balls B' and B'' are disjoint and $|B'| = |B''| = |B|/2$ we have, taking into account the rescaling law,

$$|\Omega|^{\frac{2}{n}} \lambda_2(\Omega) \geq |B|^{\frac{2}{n}} \lambda_2(\Theta),$$

with the equality holding if and only if Ω is the union of two disjoint balls of equal measure. As a consequence of Theorem 6.11 and of [25, Th. 3.5 and Rem. 3.6] one has

Theorem 6.15 *There exists a constant $\kappa(n)$ depending only on the dimension such that for any open set Ω of finite measure one has*

$$|\Omega|^{\frac{2}{n}} \lambda_2(\Omega) - |B|^{\frac{2}{n}} \lambda_2(\Theta) \geq \kappa(n) \tilde{\alpha}(\Omega)^{n+1},$$

where

$$\tilde{\alpha}(\Omega) := \inf \left\{ \frac{|\Omega \cap (B'_r(x') \cup B''_r(x'))|}{|\Omega|}, B'_r(x') \cap B''_r(x') = \emptyset, |B_r| = \frac{1}{2} |\Omega| \right\}.$$

It is not known whether the power $n + 1$ in the statement of Theorem 6.15 is optimal, though it is clear that in this case the optimal power must depend on the dimension.

Things are better understood with the second, or the first non trivial, *eigenvalue of the Laplacian under Neumann boundary conditions* $\mu_2(\Omega)$ which is defined by setting for every open set Ω of finite measure

$$\mu_2(\Omega) := \min \left\{ \int_{\Omega} |\nabla f|^2 dx : \|f\|_{L^2(\Omega)} = 1, f \in H_0^1(\Omega), \int_{\Omega} f dx = 0 \right\}.$$

In this case the *Szegő–Weinberger inequality* states that among all open sets of given measure the unique maximizer of λ_2 is given by a ball. This inequality can be written in a scaling invariant form as

$$|B|^{\frac{2}{n}} \mu_2(B) \geq |\Omega|^{\frac{2}{n}} \mu_2(\Omega).$$

It turns out that a quantitative version of this inequality can be proved as well, see [25, Th. 4.1].

Theorem 6.16 *There exists a constant $\kappa(n)$ depending only on the dimension such that for every open set with Lipschitz boundary one has*

$$\kappa(n) \alpha(\Omega)^2 \leq |B|^{\frac{2}{n}} \mu_2(B) - |\Omega|^{\frac{2}{n}} \mu_2(\Omega).$$

It is interesting to observe that in the inequality above the exponent 2 is sharp. The proof of this inequality uses a simple geometric argument. However proving the sharpness of the exponent 2 is harder, see [25, Sect. 6].

We conclude this section by quickly quoting a few other inequalities for which stability results have been obtained recently.

We recall that the *Brunn–Minkowski inequality* (1.2) states that if $H, K \subset \mathbb{R}^n$ are compact sets then

$$|H + K|^{\frac{1}{n}} \geq |H|^{\frac{1}{n}} + |K|^{\frac{1}{n}},$$

with the equality holding if and only if: $|H + K| = 0$, either H or K consists of a single point, H and K are two convex bodies homothetic to each other, i.e., there exist $x_0 \in \mathbb{R}^n$ and $\lambda > 0$ such that $H = x_0 + \lambda K$, see [29, Th. 8.1.1]. A quantitative

version of the inequality above, in the case when H and K are convex bodies, has been obtained in [66] and with an improved constant in [109]. See also a recent paper by Eldan and Klartag [57]. The general case of measurable sets is still open apart from a very recent result by Figalli and Jerison [64].

The *isodiametric inequality* states that among all sets of given diameter the ball is the one enclosing the biggest volume. Equivalently, if E is a measurable set of finite measure in \mathbb{R}^n then

$$|E| \leq \omega_n \left(\frac{\text{diam}(E)}{2} \right)^n,$$

with the equality holding if and only if E is a ball. The stability of this inequality has been studied by Maggi et al. in [96] where the following result is proved.

Theorem 6.17 *If $E \subset \mathbb{R}^n$ is a set with $\text{diam}(E) = 2$, there exists a ball $B(x)$ such that*

$$\kappa(n)|E \Delta B(x)|^2 \leq \frac{|B| - |E|}{|E|},$$

for some positive constant κ depending only on the dimension n .

Paper [63] is devoted to the stability of the *relative isoperimetric inequality in a cone* while [11] contains a characterization of the equality cases in the *perimeter inequality for Steiner symmetrization* in any codimension, as well as a stability result for this inequality for a class of convex sets. A similar stability result, concerning concave and log-concave functions has been proved in [12]. Finally, a quantitative version of the *Gagliardo–Sobolev–Nirenberg inequality*, has been recently proved by Carlen and Figalli in [35], starting from the Bianchi and Egnell stability result that we have discussed in Sect. 6.1.

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