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Boundedness and applications of singular integrals and square functions: a survey

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Abstract We present a survey of certain aspects of the theory of singular integrals and square functions, with emphasis on L^2 boundedness criteria and recent applications in partial differential equations.

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Contents

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1 Introduction

We survey those aspects of the theory of singular integral operators which have been obtained since the pioneering work of Zygmund, Calderón and Mikhlin, concerning the Calderón program as developed by Coifman and Meyer. Key results in this development include the Calderón commutator theorem, L^2 bounds on the higher commutators and on the Cauchy integral on Lipchitz curves, the solutions of the Painlevé problem on analytic capacity and the Kato square root problem for elliptic operators, along with further applications to analytic capacity and partial differential equations.

Our emphasis is on L^2 boundedness criteria for singular integrals, commonly known as $T1$, Tb and *local Tb theorems*, which have arisen from and contributed to the above-mentioned program. We conclude the survey with a discussion of some recent progress and applications.

For the classical theory of singular integrals and square functions, we refer the reader to the excellent monographs of Stein [86,87], and of Christ [32].

1.1 Singular integrals

A *singular integral operator* (SIO) in $\mathbb{R}^n$ (in the generalized sense of Coifman and Meyer) [37], is a linear mapping T from test functions $\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$ into distributions $\mathcal{D}'(\mathbb{R}^n)$, which is associated to a Calderón-Zygmund kernel $K(x, y)$, in the sense that

$$\langle T\varphi, \psi\rangle = \iint \psi(x)\, K(x, y)\, \varphi(y)\, dydx \tag{1.1}$$

whenever $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ with disjoint supports. A *Calderón-Zygmund kernel* is one which satisfies the standard size and Hölder bounds

$$|K(x, y)| \le C\,|x-y|^{-n} \qquad \text{and} \tag{1.2}$$

$$|K(x, y+h) - K(x, y)| + |K(x+h, y) - K(x, y)| \le C\,\frac{|h|^\alpha}{|x-y|^{n+\alpha}} \tag{1.3}$$

for some $\alpha \in (0, 1]$, whenever $2|h| \le |x-y|$. For now, let us take the point of view that $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\} \to \mathbb{C}$, although in the sequel we shall also mention the case that the range of K is, more generally, a Hilbert space.

We remark that, given a closed cube $Q \subset \mathbb{R}^n$, T extends to a bounded linear mapping from $L^2(Q)$ into $L^2(\mathbb{R}^n \backslash Q)$, with the representation

$$Tf(x) = \int_Q K(x, y) f(y)\, dy \tag{1.4}$$

for all $f \in L^2(Q)$ and all $x \in \mathbb{R}^n \backslash Q$. Indeed, this follows readily from the kernel estimate (1.2), and the Hardy inequality

$$\int_{\mathbb{R}^n \backslash Q} \left| \int_Q \frac{1}{|x-y|^n} |f(y)|\, dy \right|^2 dx \leq C_n \int_{\mathbb{R}^n} |f(x)|^2 dx,$$

along with (1.1) and a density argument.

The theory can be extended to settings other than Euclidean space, and there are worthwhile reasons for doing so, but for most of this survey we shall just consider functions defined on $\mathbb{R}^n$, for the sake of simplicity of exposition.

Let us now mention several examples. The *Hilbert transform*

$$Hf(x) := p.v. \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{x-y} f(y)\, dy := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\, dy, \tag{1.5}$$

relates the real and imaginary parts of a holomorphic function F in the half-space $\mathbb{C}_+ := \mathbb{R}^2_+ := \{(x, t) \in \mathbb{R} \times (0, \infty)\}$, by the formula $H(\Re e\, F(\cdot, t)) = -\Im m\, F(\cdot, t)$, assuming adequate integrability of F on horizontal slices (say $F(\cdot, t)$ is uniformly in $L^p(\mathbb{R})$ for some $p \in (1, \infty)$). Here, the convergence of the *principal value* limit holds pointwise a.e. and in L^p, for $f \in L^p$, $1 < p < \infty$. We shall not explore pointwise convergence further in the present survey, but see, e.g., [86], Chapters II–III, and [87], Chapter I, Section 7.

In higher dimensions, the operators analogous to H are the *Riesz transforms*

$$R_j f(x) := p.v. \frac{2}{\sigma_n} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \qquad j = 1, 2, \dots, n, \tag{1.6}$$

where σ_n is the volume of the unit n-sphere in $\mathbb{R}^{n+1}$. The Riesz transforms relate the tangential and normal derivatives of a harmonic function u in the half-space $\mathbb{R}^{n+1}_+ := \{(x, t) \in \mathbb{R}^n \times (0, \infty)\}$, via the formula $R_j(\partial_t u(\cdot, t)) = \partial_{x_j} u(\cdot, t)$, assuming, say, $u(\cdot, t) \in L^p(\mathbb{R}^n)$. They also arise naturally in the study of $W^{2,p}$ regularity of solutions of Poisson's equation $\Delta u = f$ in $\mathbb{R}^n$ (see [86], Chapter III).

We observe that the two examples (1.5) and (1.6) are both of convolution type, i.e., $K(x, y) = K(x - y)$. We shall discuss convolution operators further in Section 3.1.

We now mention some examples that are not of convolution type. The *Calderón Commutators* are the operators

$$C^k_A f(x) := p.v. \frac{i}{2\pi} \int_{\mathbb{R}} \left(\frac{A(x) - A(y)}{x-y} \right)^k \frac{1}{x-y} f(y)\, dy \tag{1.7}$$

where A is a Lipschitz function. Observe that, up to normalization, the case $k = 0$ is the Hilbert transform, and that at least formally, C_A^1 is a commutator:

$$C_A^1 f = \frac{i}{2}\left[\tfrac{d}{dx}H, A\right] f := \frac{i}{2}\left(\tfrac{d}{dx}(HAf) - A\,\tfrac{d}{dx}(Hf)\right) \tag{1.8}$$

while C_A^k is a higher commutator ($k = 2, 3, \ldots$):

$$C_A^k f = \frac{i}{k!2}\left[\ldots\left[\left[\frac{d^k}{dx^k}H, A\right]A\right]\ldots A\right] f.$$

The operator C_A^1 (and its higher dimensional analogues) arose in Calderón's construction of an algebra of SIOs suitable for the treatment of partial differential operators with merely Lipschitz coefficients, thus, a sort of pseudo-differential calculus which, in contrast to the classical pseudo-differential calculus, was applicable to operators with rather minimally smooth coefficients [26].

Moreover, the family of operators C_A^k arise in the power series expansion of the operator

$$\mathfrak{C}_A f(x) := p.v.\ \frac{i}{2\pi}\int_{\mathbb{R}} \frac{1}{x - y + i(A(x) - A(y))} f(y)\,dy, \tag{1.9}$$

namely

$$\mathfrak{C}_A = \sum_{k=0}^{\infty} (-i)^k C_A^k,$$

at least when $\|A'\|_\infty < 1$. In turn, the operator $\mathfrak{C}_A$ arises when writing the parametric representation of the *Cauchy singular integral operator* on a Lipschitz graph. More precisely, set

$$\mathcal{C}_\gamma g(z) := p.v.\ \frac{i}{2\pi}\int_\gamma \frac{1}{z - v}\, g(v)\,dv.$$

If γ is a Lipschitz curve in the complex plane $\mathbb{C}$ parametrized by $z = x + iA(x)$, then

$$\mathfrak{C}_A f(x) = \mathcal{C}_\gamma g(x + iA(x))\,, \qquad \text{where} \qquad f(y) := (1 + iA'(y))g(y + iA(y)). \tag{1.10}$$

Of course, the role of the Cauchy integral in complex function theory is well known. We observe that, for A Lipschitz, the kernels $K(x, y) = (A(x) - A(y))^k/(x - y)^{k+1}$ and $K(x, y) = (x - y + i(A(x) - A(y)))^{-1}$, corresponding to the operators C_A^k and $\mathfrak{C}_A$ respectively, satisfy the Calderón-Zygmund kernel conditions (1.2) and (1.3), as the reader may readily verify.

Calderón's lecture at the International Congress of Mathematicians in Helsinki in 1978 contains a clear account of the state of the art at that time concerning commutators, Cauchy integrals on Lipschitz curves and applications [28].

Let us note that for all of the operators (1.5), (1.6), (1.7) and (1.9), the kernel $K(x, y)$ is *anti-symmetric*, i.e.,

$$K(x, y) = -K(y, x). \tag{1.11}$$

For all anti-symmetric kernels which satisfy the pointwise kernel bound (1.2) $|K(x, y)| \leq C|x - y|^{-n}$, the associated principal value operator is always well defined, at least in the sense of distributions. Indeed, in that case, one may extend the representation (1.1) as follows. For all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ (with supports that are not necessarily disjoint), the principal value

$$\langle T\varphi, \psi\rangle := \lim_{\varepsilon\to 0} \iint_{|x-y|>\varepsilon} \psi(x)\, K(x, y)\, \varphi(y)\, dydx \tag{1.12}$$

exists. Moreover it satisfies the *Weak Boundedness Property* (WBP), i.e. there exists $C = C(K, n)$ such that

$$|\langle T\varphi, \psi\rangle| \leq CR^n\{\|\varphi\|_\infty + R\|\nabla\varphi\|_\infty\}\{\|\psi\|_\infty + R\|\nabla\psi\|_\infty\}. \tag{1.13}$$

for all $R > 0$ and $x \in \mathbb{R}^n$, and all test functions φ, ψ supported in the ball

$$B(x, R) := \{y \in \mathbb{R}^n : |x - y| < R\}.$$

Indeed, to verify (1.12) and (1.13), we use (1.11) and then a re-labelling of the variables to write

$$\begin{aligned}\langle T_\varepsilon \varphi, \psi\rangle &:= \iint_{|x-y|>\varepsilon} \psi(x)\, K(x, y)\, \varphi(y)\, dydx \\ &= -\iint_{|x-y|>\varepsilon} \psi(x)\, K(y, x)\, \varphi(y) dydx = -\iint_{|x-y|>\varepsilon} \psi(y)\, K(x, y)\, \varphi(x)\, dxdy,\end{aligned}$$

and thus

$$\langle T_\varepsilon \varphi, \psi\rangle = \frac{1}{2}\iint_{|x-y|>\varepsilon} K(x, y)\ (\psi(x)\varphi(y) - \psi(y)\varphi(x))\ dydx.$$

Written this way, the integrand is only weakly singular, in the sense that the kernel bound (1.2) $|K(x, y)| \leq C|x - y|^{-n}$ has been improved to

$$\begin{aligned}|K(x, y)(\psi(x)\varphi(y) - \psi(y)\varphi(x))| &\leq |K(x, y)\psi(x)(\varphi(y) - \varphi(x)) + \varphi(x)(\psi(x) - \psi(y))| \\ &\leq C'|x - y|^{-n+1}\{\|\psi\|_\infty\|\nabla\varphi\|_\infty + \|\varphi\|_\infty\|\nabla\psi\|_\infty\}\end{aligned}$$

from which it is easy to deduce convergence of the limit in (1.12), along with the bound (1.13).

We remark that the weak boundedness property (1.13) holds for any L^2 bounded operator T, with $C \approx \|T\|_{op}$ (by Cauchy-Schwarz), and for an SIO should be viewed as expressing some cancellation in the operator T.

We remark also that there are L^2 bounded singular integral operators T which do not satisfy (1.12). Indeed some care is needed, for in our definition of an SIO, nothing is said about the behaviour of $K(x, y)$ when $x = y$. So the L^2 bounded operator $T = I$ is associated to the kernel $K(x, y) \equiv 0$, as also is the unbounded operator $T = \frac{d}{dx}$, though the latter does not satisfy WBP.

1.2 Square functions

Closely related to SIOs are the *square functions*. These can arise in the analysis of SIOs, and in turn, may be viewed as singular integrals with kernels taking their values in a Hilbert space. They are also of interest in their own right, especially in applications to partial differential equations. We describe one particular set-up, but note that there are several others of interest, as e.g. in [85].

Following Christ and Journé [33] and Christ [32], we say that a family of kernels $\{\psi_t(x, y)\}_{t\in(0,\infty)}$, is a *standard Littlewood-Paley family* if, for some exponent $\alpha > 0$ and $C < \infty$, we have

$$|\psi_t(x, y)| \leq C \frac{t^\alpha}{(t + |x - y|)^{n+\alpha}} \qquad \text{and} \tag{1.14}$$

$$|\psi_t(x, y + h) - \psi_t(x, y)| \leq C \frac{|h|^\alpha}{(t + |x - y|)^{n+\alpha}}, \qquad |h| \leq t, \tag{1.15}$$

for all $x, y \in \mathbb{R}^n$ and $t > 0$. Often, one may also have local Hölder continuity in the x variable:

$$|\psi_t(x + h, y) - \psi_t(x, y)| \leq C \frac{|h|^\alpha}{(t + |x - y|)^{n+\alpha}}, \qquad |h| \leq t. \tag{1.16}$$

For any family $\{\psi_t\}$ which satisfies (1.14), one can show that

$$\left|\int_{\mathbb{R}^n} \psi_t(x, y) f(y)\, dy\right| \lesssim Mf(x),$$

where M denotes the Hardy-Littlewood maximal operator which is well known to be bounded on every L^p, $1 < p \leq \infty$. See (1.30). Thus, in particular, the linear operators Θ_t defined by

$$\Theta_t f(x) := \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy$$

are uniformly bounded in $L^\infty(\mathbb{R}^n)$ with

$$\|\Theta_t b\|_\infty \le C_{n,\alpha}\,\|b\|_\infty \tag{1.17}$$

for all $b \in L^\infty(\mathbb{R}^n)$ and all $t > 0$, and are uniformly bounded in $L^2(\mathbb{R}^n)$ with

$$\|\Theta_t f\|_2 \le C_{n,\alpha}\,\|f\|_2 \tag{1.18}$$

for all $f \in L^2(\mathbb{R}^n)$ and all $t > 0$.

Alternatively, one may observe that (1.14) implies uniform L^1 bounds

$$\max\left\{\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\psi_t(x,y)|\,dx,\ \sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|\psi_t(x,y)|\,dy\right\}$$
$$\le C\int_{\mathbb{R}^n}\frac{t^\alpha}{(t+|z|)^{n+\alpha}}\,dz = C\int_{\mathbb{R}^n}\frac{1}{(1+|z|)^{n+\alpha}}\,dz =: C_{n,\alpha}, \tag{1.19}$$

whence (1.17) follows directly, and then (1.18) follows either by duality and interpolation, or else as a consequence of the Schur estimate (1.32) below.

We now define two *square function operators*:

$$G_\psi f(x) := \left(\int_0^\infty |\Theta_t f(x)|^2\,\frac{dt}{t}\right)^{1/2} \qquad \text{and} \tag{1.20}$$

$$S_\psi f(x) := \left(\iint_{\Gamma(x)} |\Theta_t f(y)|^2\,\frac{dydt}{t^{n+1}}\right)^{1/2}, \tag{1.21}$$

where $\Gamma(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ is the standard cone with vertex at x. For obvious reasons, G_ψ and S_ψ are typically referred to as *vertical* and *conical* square functions, respectively. We remark that they have equivalent L^2 norms:

$$\|S_\psi f\|_2^2 = \int_{\mathbb{R}^n}\int_0^\infty\int_{|x-y|<t}|\Theta_t f(y)|^2\,dy\,\frac{dt}{t^{n+1}}\,dx$$
$$= \int_0^\infty\int_{\mathbb{R}^n}\left(\frac{1}{t^n}\int_{|x-y|<t}dx\right)|\Theta_t f(y)|^2\,dy\,\frac{dt}{t} = \omega_n\|G_\psi f\|_2^2 \tag{1.22}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. The choice of aperture 1 in the cone defining S_ψ is merely a normalization, and one could just as well integrate over the cone with any other fixed aperture $\beta > 0$, i.e., $\Gamma_\beta(x) := \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < \beta t\}$.

The prototypical example is

$$\psi_t(x,y) = \psi_t(x-y) = t\,\frac{\partial}{\partial t}\,p_t(x-y), \tag{1.23}$$

where

$$p_t(x) = \frac{2\,t}{\sigma_n \left(t^2 + |x|^2\right)^{(n+1)/2}}$$

is the classical Poisson kernel for the Laplacian in the half-space $\mathbb{R}^{n+1}_+$, with σ_n denoting the volume of the unit n-sphere in $\mathbb{R}^{n+1}$. Thus,

$$\Theta_t f = \psi_t * f = t\,\frac{\partial}{\partial t}\,\mathcal{P}_t f,$$

where $\mathcal{P}_t := e^{-t\sqrt{-\Delta}}$ is the Poisson semigroup, so that the Fourier transform is

$$\widehat{\Theta_t f}(\xi) = \widehat{\psi}(t\xi) f(\xi) = -2\pi t|\xi| e^{-2\pi t|\xi|} f(\xi)$$

(See Section 3.1.) Note that in this case the operator Θ_t is of convolution type. Not surprisingly, the corresponding square function operators G_ψ and S_ψ play a fundamental role in the theory of harmonic functions in $\mathbb{R}^{n+1}_+$.

A general class of convolution examples is provided by choosing $\psi \in C_0^\infty(\mathbb{R}^n)$ (or more generally, in the Schwarz class $\mathcal{S}$), with $\int_{\mathbb{R}^n} \psi(x)\,dx = 0$, and setting

$$\psi_t(x) := \frac{1}{t^n}\,\psi\left(\frac{x}{t}\right), \qquad Q_t f := \psi_t * f. \tag{1.24}$$

It is an easy exercise to verify that the functions $\psi_t(x, y) := \psi_t(x - y)$ form a standard Littlewood-Paley family with $\alpha = 1$. In this situation, it is the usual notational convention to use Q_t in place of Θ_t, thus $Q_t f := \psi_t * f$, $G_\psi f = \left(\int_0^\infty |Q_t f|^2 dt/t\right)^{1/2}$, and so on.

An important class of non-convolution examples arises in the theory of divergence form elliptic operators. Let A be an $n \times n$ matrix with bounded measurable entries

$$A_{jk} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad j = 1, \ldots, n, \quad k = 1, \ldots, n, \tag{1.25}$$

satisfying the *ellipticity* condition

$$\lambda|\xi|^2 \le \Re e A(x)\xi \cdot \bar{\xi} \quad \text{and} \quad |A(x)\xi \cdot \bar{\zeta}| \le \Lambda|\xi||\zeta|, \quad \forall \xi, \zeta \in \mathbb{C}^n,\ x \in \mathbb{R}^n \text{ (a.e.)} \tag{1.26}$$

for some constants $0 < \lambda \le \Lambda < \infty$. Let L denote the second order divergence form operator defined by

$$Lf := -\mathrm{div}(A\nabla f) = -\sum_{1 \le j,k \le n} \frac{\partial}{\partial x_j}\left(A_{jk}\frac{\partial f}{\partial x_k}\right), \tag{1.27}$$

which we interpret in the usual weak sense via a sesquilinear form, and let $p_t^L(x, y)$ and $\mathcal{P}_t^L := e^{-t\sqrt{L}}$ be the associated Poisson kernel and Poisson semigroup, respectively. Generalizing (1.23), we then set

$$\psi_t(x, y) := t \frac{\partial}{\partial t} p_t^L(x - y), \qquad \Theta_t := t \frac{\partial}{\partial t} \mathcal{P}_t^L. \tag{1.28}$$

If the coefficient matrix A has real entries, then this kernel ψ_t satisfies the bounds (1.14), (1.15) and (1.16) by the classical De Giorgi/Nash regularity theory [43,78]. The De Giorgi/Nash estimates remain true for complex coefficients when $n = 2$ [16], or when A is sufficiently close (in L^∞) to a real matrix [1,18]. On the other hand, in general, $\psi_t(x, y)$ need not satisfy the pointwise bounds (1.14), (1.15) nor (1.16), yet even then it is still possible to develop some aspects of the theory. We refer the reader to [3] and to [11] for a discussion of the latter situation.

1.3 Notation

The *Lebesgue measure* of a measurable subset $S \subset \mathbb{R}^n$ is denoted by $|S|$, while its *indicator function* 1_S is defined by $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ if $x \in \mathbb{R}^n \setminus S$.

For each cube $Q \subset \mathbb{R}^n$, the *mean value* of a function f over Q is defined to be

$$[f]_Q := |Q|^{-1} \int_Q f(x)\, dx, \tag{1.29}$$

while $\ell(Q)$ denotes its *side length*, and κQ denotes the concentric dilate of Q by a factor of $\kappa > 0$. Our cubes always have sides parallel to the coordinate axes.

For each integer N, the symbol $\mathbb{D}(N)$ denotes the collection of all *dyadic cubes* in $\mathbb{R}^n$ with $\ell(Q) = 2^N$, that is, the collection of cubes $2^N\tilde{k} + (0, 2^N]^n$ with $\tilde{k} \in \mathbb{Z}^n$, and $\mathbb{D} := \bigcup_{N\in\mathbb{Z}} \mathbb{D}(N)$.

For each $x \in \mathbb{R}^n$ and $r > 0$, $B(x, r)$ denotes the open ball with centre x and radius r.

The *Hardy-Littlewood maximal function* of a measurable function f defined on $\mathbb{R}^n$ is $Mf(x) := \sup_{r>0} |B(x, r)|^{-1} \int_{B(x,r)} f(y)\, dy$. We use the well-known result that, when $1 < p \leq \infty$, there exists $C_{n,p} < \infty$ such that

$$\|Mf\|_p \leq C_{n,p}\|f\|_p \tag{1.30}$$

for all $f \in L^p(\mathbb{R}^n)$. (This is the first theorem in [86].)

We use the notation $X \lesssim Y$ to mean that there exists a constant $C > 0$ such that $X \leq CY$. The notation $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$. The value of C varies from one usage to the next.

For later use, we note *Chebychev's inequality* with exponent $p \in [1, \infty)$: For each $g \in L^p(\mathbb{R}^n)$ and $\lambda > 0$,

$$\lambda^p|\{x \in \mathbb{R}^n\, ;\ |g(x)| > \lambda\}| \leq \|g\|_p^p, \tag{1.31}$$

which is easily verified, as is the *Schur estimate* for integral operators $Sf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$ with weakly singular kernels:

$$\|Sf\|_2 \leq \left\{\sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dx\right\}^{1/2}\left\{\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|\,dy\right\}^{1/2}\|f\|_2. \quad (1.32)$$

The theory of SIOs deals with integral operators with kernels which are not absolutely integrable in this sense, but whose boundedness depends on some cancelation properties of the kernel.

2 L^p and endpoint theory of SIOs and square functions

In this section, we discuss L^p and endpoint bounds that are satisfied by SIOs and square functions, *assuming* boundedness on L^2. The fundamental result is that of Calderón and Zygmund [40].

Theorem 2.1 (Calderón-Zygmund Theorem) [40]. *Let T be an SIO associated to a Calderón-Zygmund kernel $K(x,y)$ satisfying the standard bounds* (1.2) *and* (1.3). *Suppose also that T extends to a bounded linear operator on $L^2(\mathbb{R}^n)$, i.e., that for all $f,g\in C_0^\infty(\mathbb{R}^n)$, we have*

$$|\langle Tf,g\rangle| \leq C\,\|f\|_2\|g\|_2.$$

Then T extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1<p<\infty$.

To be historically accurate, we should observe that the original paper [40] treated only the convolution case, but the argument there extends essentially verbatim to the general setting described above. Let us now give the proof, following [40]. We begin with a fundamental stopping time lemma, in which an L^1 function f is decomposed into a *good part* g and a *bad part* b.

Lemma 2.2 (Calderón-Zygmund decomposition). *Suppose that $f\in L^1(\mathbb{R}^n)$, and let $\lambda>0$. Then there is a family $\mathcal{F} := \{Q_j\}_{j=1}^\infty$ of non-overlapping dyadic cubes, and a decomposition $f=b+g$ such that*

(1) $\lambda \leq |Q_j|^{-1}\int_{Q_j}|f(x)|\,dx < 2^n\lambda$, *for every $Q_j\in\mathcal{F}$;*
(2) $\sum_j|Q_j| \leq \lambda^{-1}\|f\|_1$;
(3) $b=\sum_j b_{Q_j}$, *with* $\operatorname{supp}(b_{Q_j})\subset Q_j$, *and* $\int b_{Q_j}=0$, $\forall Q_j\in\mathcal{F}$;
(4) $g\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, *with* $\|g\|_1\leq\|f\|_1$ *and* $\|g\|_2^2\leq C_n\lambda\|f\|_1$.

Proof of Lemma 2.2 We start with an initial grid $\mathbb{D}(N)$ of dyadic cubes of side length 2^N, chosen so large that for $Q\in\mathbb{D}(N)$, we have

$$\frac{1}{|Q|}\int_Q|f(x)|\,dx \leq \frac{1}{|Q|}\|f\|_1 < \lambda.$$

We then subdivide each Q in $\mathbb{D}(N)$ dyadically, stopping the first time we reach a sub-cube $Q' \subset Q$ for which

$$\frac{1}{|Q'|}\int_{Q'} |f(x)|\, dx \geq \lambda. \tag{2.3}$$

The family $\mathcal{F}$ is simply the collection of cubes that are maximal with respect to the property (2.3). The left hand inequality in (1) then holds by definition, while the right hand bound follows from the maximality of $Q_j \in \mathcal{F}$. In turn, (2) follows directly from the left hand estimate in (1), since the cubes Q_j are non-overlapping (by maximality).

To establish (3) and (4), we set $E := \cup_j Q_j$, and define

$$g := f 1_{\mathbb{R}^n \setminus E} + \sum_j 1_{Q_j} \frac{1}{|Q_j|}\int_{Q_j} f(x)\, dx, \qquad b := f - g.$$

Then (3) holds by definition, with

$$b_{Q_j} = (f - [f]_{Q_j}) 1_{Q_j},$$

recalling that $[f]_{Q_j}$ denotes the mean value of f over Q_j (1.29).

To prove (4), we observe first that the claimed L^1 bound for g is trivial, while the L^2 bound may be obtained from the L^1 bound and the fact that $\|g\|_\infty \leq 2^n\lambda$. In turn, the latter estimate follows directly from Lebesgue's differentiation theorem (to control g on $\mathbb{R}^n \setminus E$), and the right hand inequality in (1). □

Proof of Theorem 2.1 We shall obtain this result by interpolation and duality, after first establishing the weak-type (1,1) bound:

$$\sup_{\lambda>0} \left(\lambda \left|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right|\right) \leq C\, \|f\|_1, \tag{2.4}$$

where C depends only on dimension, the kernel bounds (1.2) and (1.3), and the $L^2 \to L^2$ operator norm of T. It is enough to prove (2.4) for $f \in L^2 \cap L^1$, as one may then extend by continuity to all of L^1. To this end, we fix $\lambda > 0$, and write $f = b + g$ as in Lemma 2.2. Then

$$\left|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}\right| \leq \left|\{x \in \mathbb{R}^n : |Tb(x)| > \lambda/2\}\right| + \left|\{x \in \mathbb{R}^n : |Tg(x)| > \lambda/2\}\right|.$$

The desired bound for Tg follows directly from Chebychev's inequality with exponent 2 and the L^2 bound for g in Lemma 2.2:

$$\left|\left\{x\, ;\, |Tg(x)| > \frac{\lambda}{2}\right\}\right| \leq \frac{4}{\lambda^2}\|Tg\|_2^2 \leq \frac{4C_n}{\lambda}\|T\|_{op}^2 \|f\|_1.$$

To handle Tb, we set $E_* := \cup_j(5Q_j)$. Then

$$\left|\{x \in \mathbb{R}^n : |Tb(x)| > \lambda/2\}\right| \leq \sum_j |5Q_j| + \left|\{x \in \mathbb{R}^n \setminus E_* : |Tb(x)| > \lambda/2\}\right| =: I + II.$$

By Lemma 2.2 (2), we have that $I \leq 5^n \lambda^{-1} \|f\|_1$, as desired.

It now remains to treat term II. We proceed as follows. First, by our qualitative assumption that $f \in L^2 \cap L^1$, we have in particular that the sum in Lemma 2.2 converges in L^2. Since $T : L^2 \to L^2$, it therefore follows that

$$|Tb(x)| \leq \sum_j |Tb_{Q_j}(x)| \quad \text{a.e.},$$

so that by Chebychev's inequality we have

$$II \leq \frac{2}{\lambda} \sum_j \int_{\mathbb{R}^n \setminus E_*} |Tb_{Q_j}(x)|\, dx. \tag{2.5}$$

Next, we let y_j denote the center of Q_j, and note that by (1.4), the fact that b_{Q_j} has mean value 0, and (1.3), we have

$$\begin{aligned} |Tb_{Q_j}(x)| &= \left| \int K(x, y) b_{Q_j}(y)\, dy \right| = \left| \int \big(K(x, y) - K(x, y_j)\big)\, b_{Q_j}(y)\, dy \right| \\ &\leq C \frac{\ell(Q_j)^\alpha}{\big(\ell(Q_j) + |x - y_j|\big)^{n+\alpha}} \int |b_{Q_j}(y)|\, dy, \quad \forall x \in \mathbb{R}^n \setminus 5Q_j, \end{aligned} \tag{2.6}$$

where $\ell(Q)$ denotes the side length of Q. Combining (2.5) and (2.6), we obtain

$$II \leq \frac{C_{n,\alpha}}{\lambda} \sum_j \int |b_{Q_j}(y)|\, dy = \frac{C_{n,\alpha}}{\lambda} \int |b(y)|\, dy \leq 2 \frac{C_{n,\alpha}}{\lambda} \|f\|_1,$$

thus completing the proof of (2.4).

By the Marcinkiewicz interpolation theorem, as presented e.g. in Theorem 5, Ch. I of [86], it follows from the L^2 bound for T and the weak-type (1,1) bound (2.4), that T is bounded in L^p for all $p \in (1, 2]$. By the symmetry of the kernel conditions (1.2) and (1.3), the same is true for its transpose ${}^tT$. So, by duality, T is bounded in L^p for all $p \in [2, \infty)$, and the proof is complete.

We shall not discuss L^p theory for square functions explicitly, but let us simply note that, assuming (1.14)–(1.16), the square function operators G_ψ and S_ψ may be viewed as SIOs with standard kernels taking values in an appropriate Hilbert space, and thus bounded on L^p, given L^2 boundedness. For example, in the case of G_ψ, we set

$$K(x, y) = \{K(x, y)(t)\}_t := \{\psi_t(x, y)\}_t,$$

and observe that $K(x, y)$ satisfies the kernel bounds (1.2), (1.3), if the modulus $|\cdot|$ is replaced by the Hilbert space norm

$$\|h\|_{\mathcal{H}} := \left(\int_0^\infty |h(t)|^2 \frac{dt}{t}\right)^{1/2}.$$

We leave this observation as an exercise for the reader. In this context, the proof of the Calderón-Zygmund Theorem 2.1 carries over *mutatis mutandi*.

It is worth emphasizing that the L^p bounds in Theorem 2.1 were obtained by interpolating between the assumed L^2 estimate and an endpoint estimate, in this case a weak-type (1,1) bound. However, there is another type of endpoint estimate which may also serve via interpolation to obtain L^p bounds, namely, the fact that an L^2 bounded SIO maps $L^\infty(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, the space of functions of *bounded mean oscillation*.

We recall that $BMO(\mathbb{R}^n)$ is the Banach space of locally integrable functions modulo constants for which the norm

$$\|b\|_* = \sup |Q|^{-1} \int_Q |b(x) - [b]_Q|\, dx$$

is finite, where the supremum runs over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the co-ordinate axes (though balls would work just as well). The fundamental result about *BMO* is the John-Nirenberg Theorem [67], which implies in particular that, when $1 \le p < \infty$,

$$\|b\|_* \approx \sup \left(|Q|^{-1} \int_Q |b(x) - [b]_Q|^p dx\right)^{1/p}, \tag{2.7}$$

where the implicit constants depend only on p and dimension. We also note the elementary fact that the mean value $[b]_Q$ is essentially the optimal constant. More precisely,

$$\frac{1}{2}\|b\|_* \le \sup \inf |Q|^{-1} \int_Q |b(x) - c_Q| dx \le \|b\|_*, \tag{2.8}$$

where the infimum runs over all constants c_Q, and the supremum again runs over all cubes $Q \subset \mathbb{R}^n$.

The following result was obtained independently by Peetre [81], Spanne [83] and Stein [84]:

Theorem 2.9 (Peetre-Spanne-Stein Theorem). *Let T be an L^2 bounded SIO, associated to a standard Calderón-Zygmund kernel $K(x, y)$ (cf. (1.2), (1.3).) Then the mapping $T : L^\infty(\mathbb{R}^n) \to BMO(\mathbb{R}^n)$ is bounded.*

Proof Fix Q, and let $f \in L^\infty$. We write $f = f_1 + f_2$, where $f_1 := f 1_{5Q}$. Let x_Q denote the center of Q, and set $c_Q := T f_2(x_Q)$. We then have

$$\int_Q \left|Tf(x) - c_Q\right| dx \le \int_Q |Tf_1(x)|\, dx + \int_Q \left|Tf_2(x) - c_Q\right| dx =: I + II.$$

By Cauchy-Schwarz and the L^2 boundedness of T, we have that

$$I \leq \|T\|_{op} |Q|^{1/2} \|f_1\|_2 \leq 5^{n/2} \|T\|_{op} |Q| \, \|f\|_\infty.$$

Also, by (1.3) we have, for $x \in Q$:

$$\begin{aligned} \left|Tf_2(x) - c_Q\right| &= \left| \int_{\mathbb{R}^n \setminus 5Q} \big(K(x,y) - K(x_Q,y)\big) f(y)\, dy \right| \\ &\leq C \, \|f\|_\infty \int_{\mathbb{R}^n} \frac{\ell(Q)^\alpha}{\big(\ell(Q) + |x_Q - y|\big)^{n+\alpha}} \, dy = C_{n,\alpha} \, \|f\|_\infty, \end{aligned}$$

whence $II \leq C\|f\|_\infty |Q|$. The conclusion of the theorem then follows by (2.8). □

Note that, implicitly, Tf is only defined modulo constants for general $f \in L^\infty$.

A few remarks are in order. Suppose that T is an L^2 bounded SIO. By the symmetry of the kernel conditions (1.2) and (1.3), the same is true for its transpose ${}^tT$. Then by Theorem 2.9, both T and ${}^tT$ map L^∞ into *BMO*, whence by Fefferman's duality theorem [51], T also maps the Hardy space $H^1(\mathbb{R}^n)$ into L^1. By the Fefferman-Stein interpolation theorem [51], we then have that $T : L^p \to L^p$, $1 < p < \infty$, thus providing an alternative proof of Theorem 2.1.

Moreover, the same line of reasoning leads to a more subtle observation, which in turn serves to clarify the nature of the results in the next section. Notice that if we *start* with the hypothesis that T and ${}^tT$ both map L^∞ into *BMO*, then the same duality and interpolation arguments yield that T is bounded on L^2. The L^2 boundedness criteria that we shall discuss in Section 3 state that, rather than testing T and ${}^tT$ on all of L^∞, it is enough to verify that each of them maps *one particular function* in L^∞ into *BMO*.

Finally, Theorem 2.9, and also the Calderón Zygmund Theorem 2.1, say that given L^2 boundedness of an SIO associated to a standard kernel, one then automatically knows L^p and endpoint estimates for the operator. Thus, the question of L^2 boundedness is paramount. We remark that one can also obtain weighted L^p estimates. See, e.g. [53]. The weighted theory is also of considerable interest, and has, in fact, enjoyed a renaissance of late. However, we shall not touch on this subject in the present survey.

3 L^2 boundedness criteria

3.1 The convolution case

The theory of convolution integral operators is closely related to Fourier theory. Suppose $\psi \in L^1(\mathbb{R}^n)$. Its *Fourier transform* $\hat{\psi}$, defined by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} \, \psi(x)\, dx$$

is a continuous function which satisfies $\|\widehat{\psi}\|_\infty \leq \|\psi\|_1$ and the *Plancherel identity* $\|\widehat{\psi}\|_2 = \|\psi\|_2$ provided also that $\psi \in L^2(\mathbb{R}^n)$. The transform of convolution is multiplication, in the sense that $\widehat{\psi * f}(\xi) = \widehat{\psi}(\xi)\widehat{f}(\xi)$ for all $\xi \in \mathbb{R}^n$ when $f \in L^1(\mathbb{R}^n)$. Hence

$$\|\psi * f\|_2 = \|\widehat{\psi}\widehat{f}\|_2 \leq \|\widehat{\psi}\|_\infty \|\widehat{f}\|_2 \leq \|\psi\|_1 \|f\|_2$$

for all $\psi \in L^1(\mathbb{R}^n)$ and $f \in L^1 \cap L^2(\mathbb{R}^n)$.

The question of L^2 boundedness of a *convolution* SIO with kernel $K(x, y) = K(x - y)$ is straightforward: by Plancherel's Theorem, it is sufficient to verify that the associated *Fourier multiplier*

$$m(\xi) := \lim_{\varepsilon \to 0} \int_{\varepsilon < |x| < \varepsilon^{-1}} e^{-2\pi i \xi \cdot x} K(x)\, dx$$

belongs to L^∞. In turn, it is not hard to establish the boundedness of m, given the standard kernel estimates (1.2), (1.3) and sufficient cancellation, say that K has mean value 0 on every annulus $0 < a < |x| < b < \infty$. Clearly, the latter cancellation condition holds for any odd kernel K, for example as one encounters in the Hilbert and Riesz transforms. In this case, we have formally that $Tf = K * f$, or $\widehat{Tf}(\xi) = m(\xi)\widehat{f}(\xi)$, so T can be written in terms of the bounded functional calculus of commuting self-adjoint operators:

$$(Tf)(x) = m\left(\frac{1}{2\pi i}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\right) f(x)$$

with $\|T\|_{op} = \|m\|_\infty$ as $\|Tf\|_2 = \|K * f\|_2 = \|m\widehat{f}\|_2$. We must be careful though. When, for example $m(\xi) = |\xi|^i$, then the corresponding SIO is not given by a principal value integral, though it can be obtained as a limit of a particular sequence of $\epsilon_n \to 0$. (See p. 51 of [86]).

There is an excellent account of convolution SIOs in the classic [86].

For square functions, Plancherel's theorem is also applicable in the convolution case when $Q_t f = \psi_t * f$ with $\psi_t(x) = t^{-n}\psi(x)/t$ where $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$ as discussed in (1.24). The key point is that the Fourier transform $\widehat{\psi}$ belongs to the test space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing C^∞ functions and $\widehat{\psi}(0) = \int_0^\infty \psi(x)\, dx = 0$, so that

$$|\widehat{\psi}(\xi)| \lesssim \min\left\{|\xi|, |\xi|^{-1}\right\}$$

for all $\xi \in \mathbb{R}^n$. Moreover $\widehat{\psi_t}(\xi) = \widehat{\psi}(t\xi)$. Then, using (1.22),

$$\begin{aligned}
\|S_\psi f\|_2 \approx \|G_\psi f\|_2 &= \iint_{\mathbb{R}^{n+1}_+} |Q_t f(x)|^2 \frac{dxdt}{t} \\
&= \int_0^\infty \int_{\mathbb{R}^n} |\widehat{\psi}(t\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \frac{dt}{t} = \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 \left(\int_0^\infty |\widehat{\psi}(t\xi/|\xi|)|^2 \frac{dt}{t}\right) d\xi \\
&\lesssim \|f\|^2_{L^2(\mathbb{R}^n)} \left(\int_0^1 t^2 \frac{dt}{t} + \int_1^\infty t^{-2} \frac{dt}{t}\right) \lesssim \|f\|_2. \qquad (3.1)
\end{aligned}$$

We remark for later use that if ψ is a non-trivial real-valued radial function, then $\widehat{\psi}$ has the same properties, so with a slight abuse of notation we have

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = \int_0^\infty |\widehat{\psi}(t|\xi|)|^2 \frac{dt}{t} = \int_0^\infty |\widehat{\psi}(t)|^2 \frac{dt}{t} = c < \infty,$$

and by renormalizing, we may suppose that $c = 1$. In this case $\|G_\psi f\|_2 = \|f\|_2$.

On noting that Q_t is self-adjoint in this case, we obtain that

$$\int_0^\infty Q_s^2 \frac{ds}{s} = \mathfrak{I}, \tag{3.2}$$

in the strong operator topology on L^2, where $\mathfrak{I}$ is the identity operator on $L^2(\mathbb{R}^n)$. The identity (3.2) is referred to as the *Calderón reproducing formula*.

3.2 The non-convolution case

The non-convolution case is much more subtle. The original proofs of the L^2 boundedness of the prototypical non-convolution examples (1.7) and (1.9), had each been a real tour de force, obtained by a variety of methods. See [25] for the first Calderón commutator, [35] for the second commutator, [36] for the higher commutators, [27] for the Cauchy integral on a Lipschitz graph with small Lipschitz constant, and [39] for the Cauchy integral on all Lipschitz graphs. Thus, the search for general L^2 boundedness criteria was driven in large part by the desire to better understand these fundamentally important examples and to treat them in a more systematic way.

The first such criterion was the $T1$ *Theorem* of David and Journé [44]:

Theorem 3.3 (T1 Theorem) [44]. *Suppose that T is a singular integral operator associated to a standard kernel $K(x, y)$ satisfying* (1.2), (1.3). *Then T extends to a bounded operator on L^2 if and only if T satisfies WBP* (1.13), *and $T1 \in BMO$ and ${}^tT1 \in BMO$.*

Remark If $K(x, y)$ is antisymmetric, and T is the associated principal value operator given by (1.12), then ${}^tT = -T$, and WBP is satisfied automatically, as we have observed above (cf. (1.12), (1.13)). Thus, in that case, matters reduce to the simple statement that $T : L^2 \to L^2 \iff T1 \in BMO$.

Remark Note that the "only if" direction of the $T1$ Theorem was already known: given that T is L^2 bounded, one obtains that $T1$, ${}^tT1 \in BMO$ by the Peetre-Spanne-Stein Theorem, and as we have noted earlier, WBP follows by Cauchy-Schwarz.

Remark The $T1$ Theorem yields a simple proof of the L^2 boundedness of the first Calderón commutator C_A^1, for A a Lipschitz function. Here is a formal proof, which can easily be made rigorous. By antisymmetry, it is enough to verify that $C_A^1 1 \in BMO$. By the representation (1.8), and the fact that $(d/dx \circ H)1 = 0$, we have that $C_A^1 1 = \frac{1}{i} HA'$. By the Peetre-Spanne-Stein Theorem, the latter belongs to BMO, since $A' \in L^\infty$ and $H : L^2 \to L^2$. Similarly, one may handle the higher order commutators by induction, reducing $C_A^k 1$ to $C_A^{k-1} A'$.

We now present the proof of the $T1$ Theorem, following an approach made explicit in [33], although many of the essential ideas were already implicit in [38]. We begin with an analogue of Theorem 3.3 for square functions, which appears in [33].

Theorem 3.4 (T1 Theorem for square functions) [33]. *Let*

$$\Theta_t f(x) := \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy,$$

where $\psi_t(x, y)$ *satisfies* (1.14), (1.15). *Suppose that* $d\mu(x, t) := |\Theta_t 1(x)|^2 dx dt/t$ *is a Carleson measure, i.e., that*

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Theta_t 1(x)|^2 \frac{dx dt}{t} =: \|\mu\|_{\mathcal{C}} < \infty. \tag{3.5}$$

Then the following square function estimate holds:

$$\|G_\psi f\|^2_{L^2(\mathbb{R}^n)} = \iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx dt}{t} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.6}$$

Remark The converse direction (i.e. that (3.6) implies (3.5)) is essentially due to Fefferman and Stein [51].

Proof of Theorem 3.4 We start with the special case when $\Theta_t 1 \equiv 0$ for all $t > 0$. We shall show that, in this case, Θ_t satisfies the *orthogonality* condition

$$\|\Theta_t Q_s\|_{L^2 \to L^2} \lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^\eta \qquad \text{for some } \eta > 0, \tag{3.7}$$

with respect to convolution operators Q_s defined by $Q_s f := \zeta_s * f$ with $\zeta_s(x) := s^{-n}\zeta(x/s)$ with $\zeta \in C_0^\infty(B(0, 1))$ and $\int \zeta = 0$. To do this, we shall apply the Schur estimate (1.32) to the kernel $K_{ts}(x, y) = \int \psi_t(x, z)\zeta_s(z - y)\, dz$ of $\Theta_t Q_s$.

When $s \leq t$, we use the smoothness (1.15) of $\psi_t(x, y)$ and cancellation of ζ_s to estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} |K_{ts}(x, y)|\, dy &= \int_{\mathbb{R}^n} \left| \int_{|z-y| \leq s} (\psi_t(x, z) - \psi_t(x, y))\zeta_s(z - y)\, dz \right| dy \\
&\lesssim \int_{\mathbb{R}^n} \sup_{\{z; |z-y| \leq s\}} |\psi_t(x, z) - \psi_t(x, y)|\, dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{s^\alpha}{(t + |x - y|)^{n+\alpha}}\, dy \lesssim \frac{s^\alpha}{t^\alpha}
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Also, using (1.19),

$$\int_{\mathbb{R}^n} |K_{ts}(x, y)|\, dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\psi_t(x, z)|\, |\zeta_s(z - y)|\, dx\, dz \lesssim 1$$

for all $y \in \mathbb{R}^n$, so the Schur estimate (1.32) gives $\|\Theta_t Q_s\|_{op} \lesssim (s/t)^{\alpha/2}$.

When $t \le s$, we reverse the roles of smoothness and cancellation, so in fact here is where we use $\Theta_t 1 \equiv 0$. For all $y \in \mathbb{R}^n$:

$$\begin{aligned}\int_{\mathbb{R}^n} |K_{ts}(x,y)|\,dx &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \psi_t(x,z)(\zeta_s(z-y) - \zeta_s(x-y))\,dz \right| dx \\ &\le \int_{\mathbb{R}^n} \int_{|x-z|>s} |\psi_t(x,z)\zeta_s(z-y)|\,dz\,dx \\ &\quad + \int_{\mathbb{R}^n} \int_{|x-z|>s} |\psi_t(x,z)\zeta_s(x-y)|\,dz\,dx \\ &\quad + \int_{\mathbb{R}^n} \int_{|x-z|\le s} |\psi_t(x,z)|\,|\zeta_s(z-y) - \zeta_s(x-y)|\,dz\,dx,\end{aligned}$$

so that in turn,

$$\begin{aligned}\int_{\mathbb{R}^n} |K_{ts}(x,y)|\,dx \lesssim{}& \int_{\mathbb{R}^n} \frac{s}{(s+|z-y|)^{n+1}} \left(\int_{|x-z|>s} \frac{t^\alpha}{|x-z|^{n+\alpha}}\,dx \right) dz \\ &+ \int_{\mathbb{R}^n} \frac{s}{(s+|x-y|)^{n+1}} \left(\int_{|x-z|>s} \frac{t^\alpha}{|x-z|^{n+\alpha}}\,dz \right) dx \\ &+ \frac{t^{\alpha/2}}{s^{\alpha/2}} \int_{\mathbb{R}^n} \frac{s^{\alpha/2}}{(s+|x-y|)^{n+\alpha/2}} \left(\int_{\mathbb{R}^n} \frac{t^{\alpha/2}|x-z|^{\alpha/2}}{(t+|x-z|)^{n+\alpha}}\,dz \right) dx \lesssim \frac{t^{\alpha/2}}{s^{\alpha/2}}.\end{aligned}$$

(We have used the fact, noted after (1.24), that the family $\{\zeta_s(x-y)\}$ satisfies the Littlewood Paley bounds (1.14) and (1.15) for all $\alpha \le 1$.)

Continuing as above we obtain $\|\Theta_t Q_s\|_{op} \lesssim (t/s)^{\alpha/4}$, thus completing the proof of the orthogonality condition (3.7).

We further recall that $\|G_\zeta f\|_2 = \int_0^\infty \|Q_s f\|_2^2 \lesssim \|f\|_2$, as we have shown in (3.1). On choosing ζ to be a non-trivial real-valued radial function, appropriately normalised, we have the Calderón reproducing formula

$$\int_0^\infty Q_s^2 \frac{ds}{s} = \mathfrak{I},$$

as shown in (3.2).

We are now ready to dispose of the special case when $\Theta_t 1 = 0$. Indeed

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dx\,dt}{t} &= \int_0^\infty \|\Theta_t f\|_2^2 \frac{dt}{t} \\ &= \int_0^\infty \left\| \int_0^\infty (\Theta_t Q_s) Q_s f \frac{ds}{s} \right\|_2^2 \frac{dt}{t} \\ &\le \sup_{t>0} \int_0^\infty \|\Theta_t Q_s\|_{op} \tfrac{ds}{s} \sup_{s>0} \int_0^\infty \|\Theta_t Q_s\|_{op} \tfrac{dt}{t} \\ &\quad \times \int_0^\infty \|Q_s f\|_2^2 \frac{ds}{s} \\ &\lesssim \|f\|_2^2\end{aligned}$$

where we have used (3.2) in the second line, a variant of the Schur inequality (1.32) in the third line, and (3.7) and (3.1) in the final one.

We now return to the proof of Theorem 3.4 in the general case. To this end, we use a technique from [38], which has become standard in this subject. Let P_t be a *nice approximate identity*, i.e., $P_t f := \varphi_t * f$, where $\varphi_t(x) := t^{-n}\varphi(x/t)$, and $0 \le \varphi \in C_0^\infty(B(0,1))$, with $\int \varphi = 1$, so that $P_t 1 = 1$. Write

$$\Theta_t = (\Theta_t - (\Theta_t 1)P_t) + (\Theta_t 1)P_t =: R_t + (\Theta_t 1)P_t. \tag{3.8}$$

Then $R_t 1 \equiv 0$ for all $t > 0$, and, since $\Theta_t 1 \in L^\infty(\mathbb{R}^n)$ uniformly in $t > 0$ by (1.17), the kernel of R_t continues to satisfy (1.14), (1.15). The contribution of R_t may therefore be handled by the special case that we have just proved.

The term $(\Theta_t 1)P_t$ may be treated by Carleson's embedding lemma [29]:

$$\begin{aligned}\iint_{\mathbb{R}^{n+1}_+} |\Theta_t 1(x)|^2\, |P_t f(x)|^2 \frac{dxdt}{t} &=: \iint_{\mathbb{R}^{n+1}_+} |P_t f(x)|^2\, d\mu(x,t)\\ &\le C\, \|\mu\|_{\mathcal{C}}\, \|N_*(P_t f)\|^2_{L^2(\mathbb{R}^n)},\end{aligned} \tag{3.9}$$

where μ and $\|\mu\|_{\mathcal{C}}$ are defined in the statement of the theorem and

$$N_* F(x) := \sup_{\{(y,t)\colon |x-y|<t\}} |F(y,t)|$$

is the *non-tangential maximal function*. In turn, it is a routine matter to verify that $N_*(P_t f)(x) \lesssim Mf(x)$ whence $\|N_*(P_t f)\|_{L^2(\mathbb{R}^n)} \lesssim \|Mf\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}$ by (1.30). The proof of Theorem 3.4 is now complete. □

Remark 3.10 The lemma of Carleson which we have just applied, is of fundamental importance in analysis. There is a proof in Chapter 2, Section 2, of [87].

Remark 3.11 For later use, we note that our proof of the $T1$ theorem for square functions, Theorem 3.4, remains valid when the Hölder condition (1.15) is replaced by the following weaker integral condition: There exists $\alpha > 0$, $C < \infty$, such that for all $x \in \mathbb{R}^n$ and all $0 < s \le t < \infty$,

$$\int_{\mathbb{R}^n} \sup_{\{z;|z-y|\le s\}} |\psi_t(x,z) - \psi_t(x,y)|\, dy \le C\, \frac{s^\alpha}{t^\alpha}. \tag{3.12}$$

Proof of Theorem 3.3 With Theorem 3.4 in hand, we now turn to the proof of the $T1$ theorem. As in the proof of Theorem 3.4, let Q_s denote convolution operators of the form $Q_s f = \zeta_s * f$ with $\zeta_s(x) := s^{-n}\zeta(x/s)$ for $s > 0$, where ζ is a real-valued radial function in $C_0^\infty(B(0,1))$ with $\int \zeta = 0$, normalised so that (3.2) holds. The interested reader may readily verify that the operators

$$P_t := \int_t^\infty Q_s^2 \frac{ds}{s} \tag{3.13}$$

form a nice approximate identity in the sense that $P_t f = \varphi_t * f$ with $\varphi_t(x) = t^{-n}\varphi(x/t)$, where $\varphi \in C_0^\infty(B(0,2))$ with $\int \varphi = 1$.

Now, for T as in Theorem 3.3, $f, g \in C_0^\infty$, and P_t as in (3.13), we may write

$$\langle Tf, g\rangle = \lim_{\varepsilon\to 0}\langle TP_\varepsilon f, P_\varepsilon g\rangle = -\lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} \frac{d}{dt}\langle TP_t f, P_t g\rangle\, dt$$
$$= \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} \langle TP_t f, Q_t^2 g\rangle \frac{dt}{t} + \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} \langle TQ_t^2 f, P_t g\rangle\, \frac{dt}{t} =: I + II,$$

where we have used the WBP (1.13) to obtain the the fact that, as $R \to \infty$,

$$|\langle TP_R f, P_R g\rangle| \lesssim R^n(\|P_R f\|_\infty + R\|\nabla P_R f\|_\infty)(\|P_R g\|_\infty + R\|\nabla P_R g\|_\infty)$$
$$\lesssim R^{-n}\|f\|_1\|g\|_1 \to 0.$$

(For the L^∞ bounds, use $\|P_R f\|_\infty = \|\phi_R * f\|_\infty \le \|\phi_R\|_\infty\|f\|_1 \lesssim R^{-N}\|f\|_1$, etc.)

The dual of term II is the same as term I, except with ${}^tT$ in place of T and the roles of f and g reversed, so it is enough to treat term I. Since Q_t, having a radial kernel, is its own transpose, we therefore have (applying (3.1)) that

$$|I| \le \left(\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2\, \frac{dxdt}{t}\right)^{1/2}\left(\iint_{\mathbb{R}^{n+1}_+} |Q_t g(x)|^2\, \frac{dxdt}{t}\right)^{1/2}$$
$$\lesssim \left(\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2\, \frac{dxdt}{t}\right)^{1/2} \|g\|_2$$

where $\Theta_t = Q_t T P_t$, i.e., $\Theta_t f(x) := \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy$, and

$$\psi_t(x, y) := \langle \zeta_t(x - \cdot), T\varphi_t(\cdot - y)\rangle.$$

Here, as above, ζ_t and φ_t are the kernels of Q_t and P_t, respectively.

What remains is to bound the first factor by $\|f\|_2$, for we can then conclude that T is a bounded operator in $L^2(\mathbb{R}^n)$. We do this by applying the $T1$ Theorem for square functions, Theorem 3.4.

For this we must first show that $\psi_t(x, y)$ satisfies the standard kernel conditions (1.14) and (1.15). When $|x - y| \le 8t$ then (1.14) is a consequence of (1.13) as we now show:

$$|\psi_t(x, y)| = |\langle \zeta_t(x - \cdot), T\varphi_t(\cdot - y)\rangle| \lesssim t^n(\|\zeta_t\|_\infty + t\|\nabla\zeta_t\|_\infty)(\|\varphi_t\|_\infty + t\|\nabla\varphi_t\|_\infty)$$
$$\le ct^{-n} \approx \frac{t^\alpha}{(t + |x - y|)^{n+\alpha}};$$

while (1.15) can be verified in a similar way. When $|x - y| > 8t$, we may use the representation (1.1) along with (1.3) and the fact that $\int \zeta_s = 0$ to argue as in (2.6) to prove the claim:

$$|\psi_t(x, y)| = \left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \zeta_t(x - w)(K(w, z) - K(x, z))\varphi_t(z - y)\, dw\, dz\right|$$
$$\le C \sup\left\{\frac{|w - x|^\alpha}{|w - z|^{n+\alpha}} : |x - w| \le t, |z - y| \le 2t\right\}$$

$$\times \int_{\mathbb{R}^n} |\zeta_t(x-w)|dw \int_{\mathbb{R}^n} |\varphi_t(z-y)|dz$$
$$\lesssim \frac{t^\alpha}{|x-y|^{n+\alpha}} \approx \frac{t^\alpha}{(t+|x-y|)^{n+\alpha}} \qquad \text{(using } |w-z| > \tfrac{1}{2}|x-y|\text{);}$$

while (1.15) can be verified in a similar way.

Finally, we need to establish the Carleson measure bound (3.5):

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Theta_t 1(x)|^2 \frac{dxdt}{t} =: \|\mu\|_{\mathcal{C}} < \infty$$

In fact, since $P_t 1 = 1$, we have

$$\Theta_t 1 = Q_t(T1) = Q_t b$$

where $b := T1 \in BMO$ by hypothesis. It is enough to invoke a simple version of the Fefferman-Stein inequality [51]:

$$\frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |Q_t b(x)|^2 \frac{dxdt}{t}$$
$$= \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q \left| Q_t \left(\left(b - [b]_{3Q} \right) 1_{3Q} \right)(x) \right|^2 \frac{dxdt}{t}$$
$$\leq C \frac{1}{|Q|} \int_{3Q} \left| b - [b]_{3Q} \right|^2 \leq C \|b\|_*^2,$$

where in the middle line we have used that $Q_t 1 = 0$ and that $\zeta_t(x - \cdot)$ is supported in the ball $B(x, t)$ with $t \leq \ell(Q)$, and in the last line we have used the L^2 bound (3.1) and the consequence (2.7) of the John-Nirenberg inequality.

This concludes the proof of the $T1$ Theorem. □

The $T1$ theorem shed considerable light on the behavior of the Calderón commutators C_A^k (1.7). Indeed, as we have remarked above, the $T1$ theorem provided a general framework which incorporated the fundamental result of Calderón [25] concerning the L^2 boundedness of the first commutator C_A^1. Moreover, an induction scheme based on the $T1$ theorem produces an operator norm of the order of c^k (for some constant c) for the k$^{\underline{\text{th}}}$ commutator C_A^k. Thus, expanding the operator $\mathfrak{C}_A$ in (1.9) as a series of commutators:

$$\mathfrak{C}_A = \sum_{k=0}^{\infty} (-i)^k C_A^k,$$

one obtains a proof of the boundedness of the Cauchy integral on a Lipschitz curve with small Lipschitz constant, first proved by Calderón in [27]. However, the $T1$ theorem does not yield a direct proof of the L^2 boundedness of the Cauchy integral operator on an arbitrary Lipschitz curve, which was first established by Coifman, McIntosh and Meyer, by showing that the norms of the commutators C_A^k actually depend polynomially on k. [39]. The desire to rectify this shortcoming of the $T1$ theorem, and to develop a general theory that would encompass the Cauchy integral operator, led to the so-called *Tb Theorem*.

The *Tb* theorem is an extension of the $T1$ theorem, in which the conditions $T1$, ${}^tT1 \in BMO$ are replaced by the conditions Tb_1, ${}^tTb_2 \in BMO$ for suitable functions $b_1, b_2 \in L^\infty(\mathbb{R}^n)$. It was first proved in a special case by McIntosh and Meyer [75], and in general by David, Journé and Semmes [45].

Theorem 3.14 (Tb Theorem) [45]. *Suppose that* $b_1, b_2 \in L^\infty$ *are accretive, i.e., there is a constant* $\delta > 0$ *such that*

$$\Re e(b_i) > \delta, \quad i = 1, 2. \tag{3.15}$$

Let T *be a mapping from* $b_1 C_0^\infty$ *into* $(b_2 C_0^\infty)'$, *associated to a standard kernel* $K(x, y)$ *(equivalently,* $b_2 T b_1$ *is a mapping from test functions to distributions associated to the kernel* $b_2(x)K(x, y)b_1(y)$.*) Suppose also that* $b_2 T b_1$ *satisfies WBP* (1.13) *and that* Tb_1 *and* ${}^tTb_2$ *are in BMO. Then* T *extends to a bounded operator on* $L^2(\mathbb{R}^n)$.

Remark For the principal value operator associated to an antisymmetric kernel, it is enough to verify that there is a single accretive b such that $Tb \in BMO$. In this case WBP holds automatically for bTb.

Remark The accretivity condition (3.15) may be relaxed to *pseudo-accretivity*:

$$\inf_Q \left|[b]_Q\right| \geq \delta,$$

or even *para-accretivity*, a relaxed version of pseudo-accretivity in which nondegeneracy of the average over each given cube is replaced by nondegeneracy of the average over some sub-cube of comparable size.

Remark Somewhat earlier, it was proved in [75] that when Tb_1 and ${}^tTb_2$ both satisfy WBP, and $Tb_1 = 0 = {}^tTb_2$, then T extends to a bounded operator on $L^2(\mathbb{R}^n)$. (In fact, this paper only mentions the case $b_1 = b_2$, though the same proof holds when b_1 and b_2 are different functions [73].)

Both papers [75] and [45] provide alternative proofs of the Cauchy integral theorem of [39]. Indeed, let γ be the graph of a Lipschitz function A, and observe that $b := 1 + iA'$ is accretive. By definition, (cf. (1.9), (1.10)),

$$\mathfrak{C}_A b(x) = \left(\mathcal{C}_\gamma 1\right)(x + iA(x)),$$

and at least formally, by the formula of Plemelj, $\mathfrak{C}_A b = 0$ in the sense of *BMO*. Moreover, the requisite operators satisfy WBP. (Some care must be taken in interpreting the Plemelj formula on an infinite graph, but this can be managed.)

The *Tb* theorem also provided a more direct proof of *n* dimensional analogues of the Cauchy integral theorem, which had initially been proved by using the Calderón rotation method to extend the one dimensional theory to higher dimensions. In particular, such results include L^2 bounds for double layer potential operators, and derivatives of single layer potential operators, on strongly Lipschitz surfaces, thus allowing the methods of potential theory to be used to solve boundary value problems for harmonic functions on domains with such boundaries. There is a vast literature on singular integrals and potential theory, as developed for example by the school of Mikhlin [76] and many others. Calderón's paper [27] provided L^2 bounds for potential operators on surfaces with small Lipschitz constants, and was used by Fabes, Jodeit and Riviére to [50] to solve Dirichlet and Neumann problems on C^1 domains. After the publication of [39], Verchota [89] showed how to solve these boundary value problems for harmonic functions on all regions in $\mathbb{R}^n$ with strongly Lipschitz boundaries, using appropriate Rellich identities to invert the boundary potential operators, thus complementing the approach, via harmonic measures, of Dahlberg, Jerison, Kenig and others.

We shall not present the proof of the *Tb* theorem in the present survey, but here is the basic idea: one constructs a discrete variant of the Calderón reproducing formula (3.2) that is adapted to the accretive functions b_1 and b_2, and for which discrete square function estimates still hold for the adapted Q_t operators. In the discrete version, they are now Q_k's. The main outline of the proof then follows that of the $T1$ Theorem. We refer the reader to [34] or to [32], pp 64–67, for the details of an argument that is somewhat simpler than the original one in [45].

We shall however, present the proof of a square function version of the *Tb* theorem, due to Semmes [82], as it contains the germ of an idea that has turned out to be quite useful and to which we will return in the next section. The theorem generalizes Theorem 3.4, and the latter will be used in the course of the proof.

Theorem 3.16 (*Tb* Theorem for square functions) [82] *Let*

$$\Theta_t f(x) := \int \psi_t(x,y) f(y)\, dy\ ,$$

and assume that ψ_t satisfies the standard kernel conditions (1.14)*,* (1.15)*. Suppose there exists an accretive function b such that $d\mu(x,t) := |\Theta_t b(x)|^2 dxdt/t$ is a Carleson measure, i.e., that*

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t b(x)|^2 \frac{dxdt}{t} =: \|\mu\|_{\mathcal{C}} < \infty. \tag{3.17}$$

Then the square function estimate (3.6) *holds:*

$$\|G_\psi f\|^2_{L^2(\mathbb{R}^n)} = \iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2\, \frac{dxdt}{t} \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.$$

Proof By Theorem 3.4, it is enough to show that $|\Theta_t 1(x)|^2 \frac{dxdt}{t}$ is a Carleson measure. By accretivity, we have that

$$|\Theta_t 1| \le C|\,(\Theta_t 1)\; P_t b|,$$

where P_t is a nice approximate identity (e.g., as in (3.8)), with kernel supported in a ball of radius t. For each cube Q, let $b_Q := b1_{2Q}$, so that when $x \in Q$ and $t \le \ell(Q)$, then $P_t b(x) = P_t b_Q(x)$. Therefore

$$\frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t 1|^2 \frac{dxdt}{t} \le \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |(\Theta_t 1) P_t b_Q|^2 \frac{dxdt}{t}$$
$$\lesssim \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t b_Q|^2 \frac{dxdt}{t} + \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |R_t b_Q|^2 \frac{dxdt}{t} = I + II$$

where, again following [38] as in (3.8), we let $R_t = \Theta_t - (\Theta_t 1) P_t$. Now the kernels of R_t satisfy (1.14), (1.15), and $R_t(1) = 0$, so by the $T1$ theorem for square functions,

$$II \le \frac{1}{|Q|}\iint_{\mathbb{R}^{n+1}_+} |R_t b_Q|^2 \frac{dxdt}{t} \lesssim \frac{1}{|Q|}\|b_Q\|_2^2 \lesssim 1.$$

To bound the first term I, apply the assumption (3.17) together with the estimate

$$\int_0^{\ell(Q)}\int_Q |\Theta_t(b - b_Q)|^2 \frac{dxdt}{t} = \int_0^{\ell(Q)}\int_Q \left|\int_{\mathbb{R}^n\setminus 2Q} \psi_t(x-y)b(y)\,dy\right|^2 \frac{dxdt}{t}$$
$$\lesssim \int_0^{\ell(Q)}\int_Q \left|\int_{\mathbb{R}^n\setminus 2Q} \frac{t^\alpha}{|x-y|^{n+\alpha}}\,dy\right|^2 \frac{dxdt}{t}\,\|b\|_\infty^2 \lesssim |Q|.$$

Thus $|\Theta_t 1(x)|^2 \,\frac{dxdt}{t}$ is a Carleson measure, and the proof is complete. □

Remark 3.18 Observe that this argument carries over if b is allowed to vary with Q, i.e. if we have a system $\{b_Q\}$, indexed on the dyadic cubes, satisfying

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t 1|^2 \,\frac{dxdt}{t} \le C \sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\,(\Theta_t 1)\; P_t b_Q|^2 \,\frac{dxdt}{t} \tag{3.19}$$

and

$$\sup_Q \frac{1}{|Q|}\int_0^{\ell(Q)}\int_Q |\Theta_t b_Q|^2 \,\frac{dxdt}{t} \le C\,. \tag{3.20}$$

This observation is essentially due to Auscher and Tchamitchian [18], and is the starting point for the solution of the Kato problem, which we shall discuss in the next section.

Thus, it is natural to pose the question: when does (3.19) hold? In fact, the solution to the Kato problem provided a sufficient condition which answers this question (see in particular Theorem 4.7 (i), (iii), and Theorem 4.15 (i), (iii) below). Moreover, the question is related to some previous work of Christ [31], who gave the first example of a class of results which have come to be referred to as local *Tb* theorems. Here is a brief overview of the latter notion.

In some applications, it may not be at all evident that there is a single accretive (or pseudo-accretive) b for which *Tb* is well behaved (where we should think of T as either an SIO or a square function operator.) On the other hand, in such cases it is sometimes possible to find a family $\{b_Q\}$, indexed by dyadic cubes Q, such that Tb_Q behaves well locally on Q as e.g. in (3.20). This motivates the introduction of the notion of a *Local Tb Theorem*, in which good *local* control of T, on each member of a family of suitably non-degenerate functions b_Q, one for each dyadic cube Q, still suffices to deduce *global* L^2 boundedness of T. Such results are the topic of the next section.

4 Local *Tb* theorems and applications

The first local *Tb* theorem was proved by Christ [31], in connection with the theory of analytic capacity. The appropriate version of non-degeneracy in this context is as follows: a *pseudo-accretive system* is a collection of functions $\{b_Q\}$, indexed by the dyadic cubes, with b_Q supported in Q and integrable, such that for some $\delta > 0$, we have that

$$\left|\int_Q b_Q\right| \geq \delta|Q|.$$

Theorem 4.1 [31] *Suppose that T is a singular integral operator associated to a standard kernel $K(x, y)$, which in addition we assume to be in L^∞. Suppose also that there are constants $\delta > 0$ and $C_0 < \infty$, and pseudo-accretive systems $\{b^1_Q\}, \{b^2_Q\}$, with supp $b^i_Q \subseteq Q$, $i = 1, 2$, such that for each dyadic cube Q,*

(i) $\|b^1_Q\|_{L^\infty(Q)} + \|b^2_Q\|_{L^\infty(Q)} \leq C_0$
(ii) $\|Tb^1_Q\|_{L^\infty(Q)} + \|{}^tTb^2_Q\|_{L^\infty(Q)} \leq C_0$
(iii) $\min\left\{\left|\int_Q b^1_Q\right|, \left|\int_Q b^2_Q\right|\right\} \geq \delta|Q|.$

Then T extends to a bounded operator on L^2, with bound depending on n, δ, C_0 and the kernel constants in (1.2), (1.3), but not on the L^∞ norm of $K(x, y)$.

A few remarks are in order. The assumption that $K \in L^\infty$ is merely qualitative, and is satisfied, e.g., by smooth truncations of a standard kernel. This assumption allows one to make certain formal manipulations with impunity, during the course of the proof. Christ actually proved this theorem in the setting of a space $\mathcal{X}$ endowed with a pseudo-metric ρ and a *doubling measure* μ (meaning that $\mu(B(x, 2r)) \leq C\mu(x, r)$ for all $x \in \mathcal{X}$ and $r > 0$ and some constant C), which, as he demonstrated, necessarily possesses a suitable version of a *dyadic cube* structure. Christ's theorem and the technique of its proof are related to the solution of Painlevé's problem concerning the characterization of those compact sets $K \subset \mathbb{C}$ for which there exist non-constant bounded analytic functions on $\mathbb{C}\setminus K$. We will not discuss the latter subject in detail, nor the deep related work on extending *Tb* theory to the non-doubling setting. See Section 5 for some further results in this vein.

Instead, we shall concentrate on extensions of Christ's result in another direction, in which L^∞ control of b_Q and Tb_Q is replaced by local, scale invariant L^2 control. Moreover, our emphasis will be on local *Tb* theory for square functions, as opposed to singular integrals, although in Section 5 we shall discuss briefly some recent progress in the latter case. It turns out that the square function setting is somewhat technically simpler, yet, to date, it is that setting which has been more fruitful in terms of applications.

Before presenting the local *Tb* theorem for square functions, we introduce the *dyadic averaging operator* A_t, defined by

$$A_t f(x) := \frac{1}{|Q(x,t)|}\int_{Q(x,t)} f(y)dy, \tag{4.2}$$

where $Q(x,t) \in \mathbb{D}$ denotes the minimal dyadic cube containing x, with side length at least t. Just as the nice approximate identity P_t defined before (3.8) filters out frequencies higher than $1/t$, so does A_t, the difference being that $A_t f$ approximates f by a piecewise constant function, whereas $P_t f$ approximates f by a smooth function. These approximations are close in the following sense:

Lemma 4.3

$$\iint_{\mathbb{R}^{n+1}_+} |(A_t - P_t)f(x)|^2 \frac{dxdt}{t} \lesssim \|f\|_2^2 \tag{4.4}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof Write the operator A_t as an integral operator $A_t f(x) = \int_{\mathbb{R}^n} \chi_t(x,y) f(y)\,dy$ with

$$\chi_t(x,y) := \frac{1}{|Q(x,t)|} 1_{Q(x,t)}(y),$$

where $Q(x,t)$ is the unique dyadic cube containing x with $\ell(Q) \geq t > \frac{1}{2}\ell(Q)$. It is easy to check that $\chi_t(x,y)$ satisfies the first Littlewood-Paley estimate (1.14), and although it does not satisfy (1.15), it does satisfy the integral version (3.12) presented in Remark 3.11 with $\alpha = 1$. This is because, when $|z-y| \leq s \leq t$, then $\chi_t(x,z) - \chi_t(x,y)$ can only be non-zero when $\operatorname{dist}(y, \partial Q(x,t)) \leq s$. Here $\partial Q(x,t)$ denotes the boundary of $Q(x,t)$. Therefore

$$\begin{aligned}\int_{\mathbb{R}^n} \sup_{\{z;|z-y|\leq s\}} |\chi_t(x,z) - \chi_t(x,y)|\,dy &\leq \int_{\operatorname{dist}(y,\partial Q(x,t))\leq s} \frac{1}{|Q(x,t)|}\,dy \lesssim \frac{st^{n-1}}{|Q(x,t)|} \\ &\approx \frac{s}{t}\end{aligned}$$

as required. The kernel $\varphi_t(x,y) = \varphi_t(x-y)$ of P_t satisfies the standard Littlewood-Paley estimates (1.14) and (1.15), so the kernel $(\chi_t - \varphi_t)(x,y)$ of $\Theta_t := A_t - P_t$ satisfies (1.14) and (3.12). Moreover $\Theta_t 1(x) = \int \chi_t(x,y)\,dy - 1 = 0$ for all $x \in \mathbb{R}^n$, so by the special case when $\Theta_t 1 \equiv 0$ of the $T1$ Theorem for square functions as generalised in Remark 3.11, the square function estimate (4.4) holds. □

Before coming to the local *Tb* theorem for square functions, we make two preliminary observations about Carleson measures. A *Carleson measure* is a Borel measure μ on $\mathbb{R}^{n+1}_+$ such that, for some constant $C_\mu < \infty$, $\mu(R_Q) \le C_\mu |Q|$ for all cubes $Q \subset \mathbb{R}^n$, where R_Q denotes the *Carleson box* $R_Q := Q \times (0, \ell(Q)) \subset \mathbb{R}^{n+1}_+$.

In the first observation, we note that it is sufficient to check the Carleson measure condition on dyadic cubes:

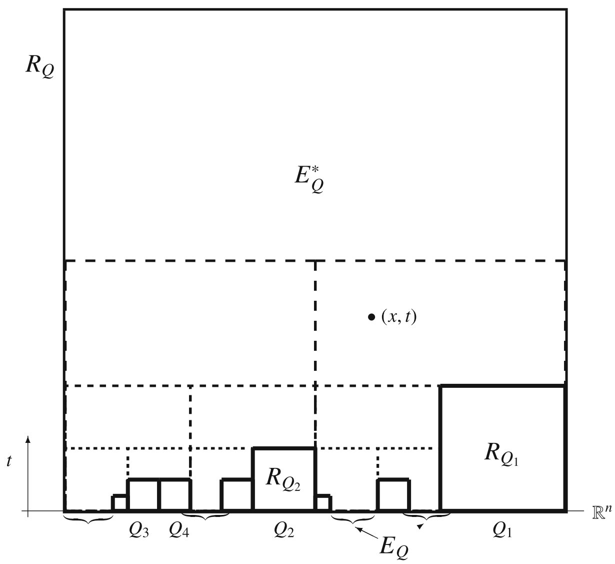
Lemma 4.5 *Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$ such that $\mu(R_Q) \le C|Q|$ for all $Q \in \mathbb{D}$, where $R_Q := Q \times (0, \ell(Q))$. Then $\mu(R_Q) \le 2^{2n} C|Q|$ for all cubes $Q \subset \mathbb{R}^n$.*

Proof For a cube $Q \subset \mathbb{R}^n$, cover Q with $Q_j \in \mathbb{D}$, $j = 1, 2, \cdots, N$, with $N \le 2^n$ and $\ell(Q) \le \ell(Q_j) < 2\ell(Q)$. Then $R_Q \subset \bigcup_j R_{Q_j}$, so that

$$\mu(R_Q) \le \sum \mu(R_{Q_j}) \le C \sum |Q_j| \le C 2^n 2^n |Q|.$$

□

In the second, we show that when proving that a measure μ on $\mathbb{R}^{n+1}_+$ is a Carleson measure, it suffices to prove a bound for μ on an η*-ample sawtooth* region of each Carleson box R_Q for some $\eta > 0$. This is a region of the form $E^*_Q := R_Q \backslash \left(\bigcup_j R_{Q_j}\right)$, where $\{Q_j\}$ is a collection of non-overlapping dyadic sub-cubes of Q such that $E_Q := Q \backslash \left(\bigcup_j Q_j\right)$ has Lebesgue measure $|E_Q| \ge \eta |Q|$. This result is known as a *John-Nirenberg type lemma for Carleson measures*.



Lemma 4.6 ("John-Nirenberg" lemma for Carleson measures). *Let μ be a Borel measure on $\mathbb{R}^{n+1}_+$. Suppose that there exist $\eta > 0$, $C_1 < \infty$ such that for every dyadic cube Q, there is a collection $\{Q_j\}$ of non-overlapping dyadic sub-cubes of Q satisfying*

$$|E_Q| := |Q \setminus \left(\cup_j Q_j\right)| \geq \eta |Q|,$$

for which the η-ample sawtooth $E^*_Q := R_Q \setminus \left(\bigcup_j R_{Q_j}\right)$ satisfies

$$\mu\left(E^*_Q\right) \leq C_1 |Q| \, .$$

Then μ is a Carleson measure, with

$$\sup_{Q \in \mathbb{D}} \frac{\mu\left(R_Q\right)}{|Q|} \leq \frac{C_1}{\eta} \, .$$

Sketch of proof Start with a dyadic cube Q. Then $R_Q = E^*_Q \cup \bigcup_j R_{Q_j}$ (disjoint union), and $\sum_j |Q_j| = |\bigcup_j Q_j| \leq (1-\eta)|Q|$. Iterating, we decompose $R_{Q_j} = E^*_{Q_j} \cup \bigcup_k R_{Q_{j,k}}$ (disjoint union) and

$$\sum_{j,k} |Q_{j,k}| \leq \sum_j (1-\eta)|Q_j| \leq (1-\eta)^2 |Q|$$

and so on. Therefore

$$\begin{aligned}
\mu(R_Q) &= \mu(E^*_Q) + \sum_j \mu(R_{Q_j}) \\
&\leq C_1|Q| + \sum_j \mu(E^*_{Q_j}) + \sum_{j,k} \mu(R_{Q_{j,k}}) \\
&\leq C_1|Q| + C_1 \sum_j |Q_j| + \sum_{j,k} \mu(E^*_{Q_{j,k}}) + \cdots \\
&\leq C_1(|Q| + (1-\eta)|Q| + (1-\eta)^2|Q| + \cdots) \;=\; \frac{C_1|Q|}{\eta}.
\end{aligned}$$

□

4.1 Local *Tb* theorems for square functions

We begin with a local *Tb* theorem for square functions, which extends the global version, Theorem 3.16. This result is essentially contained in the solution of the Kato problem: see [60,56,9]. The stated version is formulated explicitly in [2] and [54].

Theorem 4.7 *Let $\Theta_t f(x) := \int \psi_t(x,y) f(y) dy$, where $\psi_t(x,y)$ satisfies* (1.14), (1.15). *Suppose that there exist constants $\delta > 0$, $C_0 < \infty$, and a system $\{b_Q\}$ of functions indexed by dyadic cubes Q in $\mathbb{R}^n$ such that for each dyadic cube Q:*

(i) $\int_{\mathbb{R}^n} |b_Q(x)|^2\, dx \le C_0|Q|$;
(ii) $\int_0^{\ell(Q)}\int_Q |\Theta_t b_Q(x)|^2 \frac{dxdt}{t} \le C_0|Q|$;
(iii) $\left|\int_Q b_Q(x)\, dx\right| \ge \delta|Q|$.

Then the following square function estimate holds:

$$\iint_{\mathbb{R}^{n+1}_+} |\Theta_t f(x)|^2 \frac{dxdt}{t} \lesssim \|f\|_2^2. \tag{4.8}$$

Remark 4.9 On dividing each function b_Q by an appropriate complex constant, we may replace (iii) by

(iii′) $\int_Q b_Q(x)\, dx = |Q|$

noting that the constant C_0 in (i), (ii) needs to be modified. We shall assume (i), (ii) and (iii′) in the proof.

Proof We follow the outline of the proof of Semmes' result Theorem 3.16 (cf. [18] and Remark 3.18), but with an additional stopping time argument, in the spirit of that used in Christ's proof [31] of Theorem 4.1, to exploit the *pseudo-accretive system* condition, which is weaker than global pseudo-accretivity of a *single* function b.

As in the proof of Theorem 3.16, again by Theorem 3.4, it suffices to verify that $|\Theta_t 1|^2 dxdt/t$ is a Carleson measure, now given the existence of a family $\{b_Q\}$ satisfying hypotheses (i), (ii) and (iii′). To this end, we first observe that, as in [82] and [18], it is enough to verify the bound

$$\sup_{Q\in\mathbb{D}} \frac{1}{|Q|}\iint_{R_Q} |\Theta_t 1|^2 \frac{dxdt}{t} \le C_2 \sup_{Q\in\mathbb{D}} \frac{1}{|Q|}\iint_{R_Q} |(\Theta_t 1)A_t b_Q|^2 \frac{dxdt}{t}, \tag{4.10}$$

where $R_Q := Q \times (0, \ell(Q))$ is the Carleson box above Q, and A_t is the dyadic averaging operator defined in (4.2). Indeed, suppose momentarily that (4.10) holds. Then to obtain (3.5), and thus also the conclusion of the theorem, it suffices to show that the right hand side of (4.10) is bounded. Once again let P_t define a nice approximate identity as defined before (3.8), and following [38], write

$$(\Theta_t 1)P_t b_Q = \big((\Theta_t 1)P_t b_Q - \Theta_t b_Q\big) + \Theta_t b_Q =: R_t b_Q + \Theta_t b_Q$$

where $R_t := (\Theta_t 1)P_t - \Theta_t$ satisfies $R_t 1 \equiv 0$. Therefore

$$\begin{aligned}\iint_{R_Q} |(\Theta_t 1)A_t b_Q(x)|^2 \frac{dxdt}{t} &\le \iint_{\mathbb{R}^{n+1}_+} |(\Theta_t 1)(A_t - P_t)b_Q(x)|^2 \frac{dxdt}{t}\\ &\quad + \iint_{\mathbb{R}^{n+1}_+} |R_t b_Q(x)|^2 \frac{dxdt}{t} + \iint_{R_Q} |\Theta_t b_Q(x)|^2 \frac{dxdt}{t}\\ &\lesssim \int_{\mathbb{R}^n} |b_Q(x)|^2\, dx + \int_{\mathbb{R}^n} |b_Q(x)|^2\, dx + |Q| \lesssim |Q|\end{aligned}$$

where the bound on the first of the three integrals follows from Lemma 4.3 and (1.17), the bound on the second follows from the special case of Theorem 3.4 since $R_t 1 \equiv 0$, and the bound on the third is hypothesis (ii). The final estimate needs (i).

We turn now to the proof of (4.10). In order to apply Lemma 4.6, it suffices to show that there are constants $\eta > 0$, $C < \infty$, such that for each $Q \in \mathbb{D}$, there is a dyadic sawtooth region

$$E_Q^* := R_Q \setminus (\cup_j R_{Q_j}), \tag{4.11}$$

where $\{Q_j\}$ are non-overlapping dyadic sub-cubes of Q, with

$$|Q \setminus (\cup_j Q_j)| \geq \eta |Q|$$

and

$$\iint_{E_Q^*} |\Theta_t 1(x)|^2 \frac{dxdt}{t} \leq 4 \iint_{E_Q^*} |(\Theta_t 1(x))(A_t b_Q(x))|^2 \frac{dxdt}{t}. \tag{4.12}$$

We prove (4.12) via the same stopping time argument as in [60,56,9]. See also [31], where a similar idea had previously appeared. Our starting point is (iii′). We sub-divide Q dyadically, to select a family of non-overlapping cubes $\{Q_j\}$ which are maximal with respect to the property that

$$\Re e \frac{1}{|Q_j|} \int_{Q_j} b_Q \leq 1/2. \tag{4.13}$$

If E_Q^* is defined as in (4.11) with respect to this family $\{Q_j\}$, then by construction, if $(x, t) \in E_Q^*$, it follows that

$$\Re e \, A_t b_Q(x) \geq 1/2$$

so that (4.12) holds. It remains only to verify that there exists $\eta > 0$ such that

$$|E_Q| \geq \eta |Q|, \tag{4.14}$$

where $E_Q := Q \setminus (\bigcup_j Q_j)$. By (iii′) we have that

$$\begin{aligned} |Q| = \int_Q b_Q = \Re e \int_Q b_Q = \Re e \int_{E_Q} b_Q + \Re e \sum_j \int_{Q_j} b_Q \\ \leq |E_Q|^{\frac{1}{2}} \left(\int_Q |b_Q|^2 \right)^{\frac{1}{2}} + \tfrac{1}{2} \sum |Q_j|, \end{aligned}$$

when in the last step we have used (4.13). From hypothesis (i), we then obtain

$$|Q| \leq \sqrt{C_0} |E_Q|^{\frac{1}{2}} |Q|^{\frac{1}{2}} + \tfrac{1}{2} |Q|,$$

and (4.14) now follows readily with $\eta = \frac{1}{4C_0}$.

To conclude the proof, we apply Lemma 4.6 with

$$d\mu = |\Theta_t 1(x)|^2 \frac{dxdt}{t} \qquad \text{and}$$
$$C_1 = 4C_2 \sup_{Q\in\mathbb{D}} \frac{1}{|Q|} \iint_{R_Q} |(\Theta_t 1) A_t b_Q|^2 \frac{dxdt}{t}.$$

□

The previous Theorem has an extension to the matrix valued setting. We explain in the next subsection why this is interesting. Let $\mathbb{M}^N$ denote the space of $N \times N$ matrices with complex entries.

Theorem 4.15 *Suppose that* $\mathbf{\Psi}_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^N$ *satisfies the standard kernel conditions* (1.14), (1.15). *Define, for* $\mathbf{f} : \mathbb{R}^n \to \mathbb{C}^N$, *the operator*

$$\mathbf{\Theta_t} \cdot \mathbf{f}(x) := \int \mathbf{\Psi}_t(x, y) \cdot \mathbf{f}(y)dy. \tag{4.16}$$

Suppose also that there are constants $\delta > 0,\ C_0 < \infty$ *and a system of matrix valued functions* $\mathbf{b}_Q : \mathbb{R}^n \to \mathbb{M}^N$, *indexed by the dyadic cubes, such that*

(i) $\int_{\mathbb{R}^n} |\mathbf{b}_Q(x)|^2\, dx \le C_0|Q|$;
(ii) $\int_0^{\ell(Q)}\int_Q |\mathbf{\Theta}_t \mathbf{b}_Q(x)|^2 \frac{dxdt}{t} \le C_0|Q|$;
(iii) $\Re e\ \xi \cdot \left(|Q|^{-1} \int_Q \mathbf{b}_Q(x)\, dx\right) \overline{\xi} \ge \delta|\xi|^2$.

where the ellipticity condition (iii) holds for all $\xi \in \mathbb{C}^N$, *and where the action of* $\mathbf{\Theta}_t$ *on the matrix valued function* $\mathbf{b}_Q$ *is defined in the obvious way as in* (4.16) *by viewing the kernel* $\mathbf{\Psi}_t(x, y)$ *as a* $1 \times N$ *matrix which multiplies the* $N \times N$ *matrix* $\mathbf{b}_Q$. *By* $|\mathbf{b}_Q(x)|$ *is meant the operator norm of the matrix* $\mathbf{b}_Q(x)$. *Then the following square function estimate holds:*

$$\iint_{\mathbb{R}^{n+1}_+} |\mathbf{\Theta}_t \cdot \mathbf{f}|^2 \frac{dxdt}{t} \lesssim \|\mathbf{f}\|_2^2. \tag{4.17}$$

Remark 4.18 It turns out that a variant of this theorem lies at the heart of the solution of the Kato problem [9,56,60]. See also [18], where a similar result is given but with (3.19) in lieu of (iii). Thus, the present result, along with the scalar version Theorem 4.7, addresses the question posed immediately following Remark 3.18.

We now sketch the proof, which is essentially the same as the argument used to establish the Kato conjecture. Let $\mathbf{1}$ denote the $N \times N$ identity matrix. Since

$$\mathbf{\Theta}_t \mathbf{1} = (\Theta_t^1 1, \Theta_t^2 1, \ldots, \Theta_t^N 1),$$

Proposition 3.4 therefore implies that it is enough to show that $|\mathbf{\Theta}_t \mathbf{1}|^2 \, t^{-1} dxdt$ is a Carleson measure. For ϵ small, but fixed, cover $\mathbb{C}^N$ by cones

$$\Gamma_k^\epsilon = \left\{ z \in \mathbb{C}^N : \left| \frac{z}{|z|} - \nu_k \right| < \epsilon \right\}$$

where $\nu_k \in \mathbb{M}^N$ with $|\nu_k| = 1$ for $k = 1, 2, 3, \ldots, K(\epsilon, N)$. We see that

$$\int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \frac{dxdt}{t} \le \sum_{k=1}^{K} \int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma_k^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t}.$$

Thus, it suffices to show that there is a constant $C_1 = C_1(\epsilon, \delta, C_0, n, N)$ such that

$$\sup_{Q \in \mathbb{D}} |Q|^{-1} \int_0^{\ell(Q)} \int_Q |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t} \le C_1,$$

for each fixed cone Γ^ϵ, provided that ϵ is small enough, but fixed.

To this end, normalizing so that $\delta = 1$, and fixing Q, we follow the stopping time argument of the previous theorem, in the present case extracting dyadic subcubes $Q_j \subset Q$ which are maximal with respect to the property that at least one of the following holds:

$$\int_{Q_j} |\mathbf{b}_Q| \ge \frac{1}{4\epsilon} \tag{4.19}$$

or

$$\Re e \;\; \nu \cdot \left(|Q_j|^{-1} \int_{Q_j} \mathbf{b}_Q \right) \overline{\nu} \le \frac{3}{4}, \tag{4.20}$$

where $\nu \in \mathbb{C}^N$ is the unit vector in the direction of the central axis of Γ^ϵ, i.e.,

$$\Gamma^\epsilon = \left\{ z \in \mathbb{C}^N : \left| \frac{z}{|z|} - \nu \right| < \epsilon \right\}.$$

As in the proof of the previous theorem, one may check that

$$|E_Q| := |Q \backslash (\cup_j Q_j)| \ge \eta |Q|,$$

for some fixed $\eta > 0$. Moreover, for $(x, t) \in E_Q^* := R_Q \backslash (\bigcup_j R_{Q_j})$, and for $z \in \Gamma^\epsilon$, we claim that

$$|z \cdot A_t \mathbf{b}_Q(x) \overline{\nu}| \ge \frac{1}{2} |z|, \tag{4.21}$$

where again A_t denotes the dyadic averaging operator defined in (4.2). Indeed, since the opposite inequalities to (4.19) and (4.20) hold in E_Q^*, we have that

$$|\omega \cdot A_t \mathbf{b}_Q(x)\overline{\nu}| \geq |\nu \cdot A_t \mathbf{b}_Q(x)\overline{\nu}| - |(\omega - \nu) \cdot A_t \mathbf{b}_Q(x)\overline{\nu}| \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

when $|\omega - \nu| < \epsilon$ and $(x, t) \in E_Q^*$. Taking $\omega = z/|z|$ with $z \in \Gamma^\epsilon$, we obtain (4.21).

Consequently, we have that

$$\iint_{E_Q^*} |\mathbf{\Theta}_t \mathbf{1}|^2 \, 1_{\Gamma^\epsilon}(\mathbf{\Theta}_t \mathbf{1}) \frac{dxdt}{t} \leq 4 \iint_{E_Q^*} |\mathbf{\Theta}_t \mathbf{1} \cdot A_t \mathbf{b}_Q \overline{\nu}|^2 \, \frac{dxdt}{t},$$

and the rest of the proof follows as in the previous theorem.

4.2 Application to the Kato square root problem

As mentioned above, a variant of the preceding theorem leads to the solution of the Kato problem. We recall the statement of the problem. Let A be an $n \times n$ matrix of complex-valued L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or accretivity) condition (1.26). Then the associated divergence form operator $L = -\operatorname{div} A\nabla$ defined as in (1.27), considered as an unbounded operator in the Hilbert space $L^2(\mathbb{R}^n)$ with *inner product* $(u, v) := \langle u, \overline{v} \rangle$, has both its *spectrum* $\sigma(L)$ and its *numerical range* $\{(Lu, u) \in \mathbb{C}\, ; u, \nabla u, Lu \in L^2\}$ contained in a sector $\{\zeta \in \mathbb{C}\, ; |\arg \zeta| \leq \omega\}$ for some $\omega \in [0, \pi/2)$. Such an operator, called ω-*accretive*, generates a contraction semigroup $\{e^{-tL}\}_{t>0}$ and has unique $\alpha\omega$-accretive fractional powers L^α when $0 < \alpha < 1$. In particular, L has a unique square root $\sqrt{L} := L^{1/2}$ satisfying $\sqrt{L}\sqrt{L} = L$. See [69,68].

We note for later use (see e.g. [8]) that such an ω-accretive operator L also satisfies the uniform bounds

$$\|\tau L e^{-\tau L} u\|_2 \lesssim \|u\| \qquad \text{for all } \tau > 0 \tag{4.22}$$

and (since L is one-one) quadratic estimates such as

$$\int_0^\infty \|\sqrt{\tau L} e^{-\tau L} u\|^2 \, \frac{d\tau}{\tau} \approx \|u\|^2. \tag{4.23}$$

When A, and hence L, is self-adjoint, then it is easy to see that the domain of the operator $\sqrt{L}$ is the Sobolev space $W^{1,2}(\mathbb{R}^n)$, because

$$\|\sqrt{L}u\|_2 = (Lu, u)^{1/2} = (A\nabla, \nabla u)^{1/2} \approx \|\nabla u\|_2.$$

The *Kato square root problem* is to establish the same equivalence of norms

$$\|\sqrt{L}u\|_2 \approx \|\nabla u\|_2 \tag{4.24}$$

for non self-adjoint operators, with C depending only on n, λ and Λ. When the dimension $n = 1$, the latter estimate is closely related to the L^2 boundedness of the Cauchy singular integral operator on a Lipschitz curve, and indeed it was solved affirmatively in the same paper [39].

The initial affirmative results for $n \geq 2$ concerned small perturbations of the Laplacian, i.e. $\|A - I\|_\infty < \epsilon$ for some small ϵ [30,49,66]. After that came various partial results (see [74] for a survey up to this point), but the main achievement for a long time after was the writing of the book [18] by Auscher and Tchamitchian, consolidating and extending prior results, and relating them to local *Tb* square function estimates (cf. Remark 4.18.) This led on to affirmative solutions, first in 2 dimensions [60], then in all dimensions for small perturbations of real symmetric operators [10], and for operators which satisfy Gaussian heat kernel bounds [56], and finally for all divergence form operators with bounded measurable coefficients [9].

In order to give some feel for the connection with local *Tb* theorems for square functions, let us make the additional assumption of heat kernel bounds (G):

$$e^{-\tau L}u(x) = \int_{\mathbb{R}^n} k_\tau(x, y)u(y)dy \qquad \text{for all } u \in L^2(\mathbb{R}^n)$$

where the heat kernel $k_\tau(x, y)$ satisfies the *Gaussian kernel bounds*:

$$|k_\tau(x, y)| \leq \frac{\beta}{t^{n/2}} e^{-\frac{|x-y|^2}{\alpha\tau}} \quad \text{and}$$
$$|k_\tau(x+h, y)-k_\tau(x, y)|+|k_\tau(x, y+h)-k_\tau(x, y)| \leq \beta \frac{|h|^\alpha}{t^{(\alpha+n)/2}} e^{-\frac{|x-y|^2}{\alpha\tau}} \quad \forall |h| \leq t,$$

for some $\alpha,\ \beta > 0$. We remark that the classes of operators which are stated to satisfy Poisson kernel bounds in the paragraph following equation (1.28), also satisfy heat kernel bounds.

What follows is an outline of the proof under this additional assumption, essentially following the relevant parts of the book [18] and the paper [56].

As L^* has the same form as L, only the direction

$$\|\sqrt{L}u\|_2 \lesssim \|\nabla u\|_2 \tag{4.25}$$

needs be shown, because $\|\sqrt{L^*}u\|_2 \leq C\|\nabla u\|_2$ implies that $\|\nabla u\|_2 \lesssim \|\sqrt{L}u\|_2$, for

$$\|\nabla u\|_2^2 \leq \tfrac{1}{\lambda}(A\nabla u, \nabla u) = \tfrac{1}{\lambda}(\sqrt{L}u, \sqrt{L^*}u) \leq \tfrac{C}{\lambda}\|\sqrt{L}u\|_2\|\nabla u\|_2.$$

(We are leaving out technical considerations concerning domains of operators, etc.)

Now (4.25) is equivalent to the square function estimate:

$$\iint_{\mathbb{R}^{n+1}_+} |tLe^{-t^2L}u(x)|^2 \, \frac{dx\, dt}{t} \lesssim \|\nabla u\|_2^2, \tag{4.26}$$

because, by (4.23),

$$\iint_{\mathbb{R}^{n+1}_+} \left|tLe^{-t^2L}u(x)\right|^2 \frac{dx\,dt}{t} = \frac{1}{2}\int_0^\infty \left\|\sqrt{\tau L}e^{-\tau L}(\sqrt{L}u)\right\|_2^2 \frac{d\tau}{\tau} \approx \|\sqrt{L}u\|_2^2.$$

The square function estimate (4.26) has the form of equation (4.17) in Theorem 4.15 with

$$\boldsymbol{\Theta}_t = te^{-t^2L}\operatorname{div} A$$

and $\mathbf{f} = \nabla u$. However we cannot apply this theorem as stated because $\boldsymbol{\Theta}_t$ does not satisfy the standard kernel estimates (1.14) and (1.15), and indeed the square function estimate (4.17) can, in this case, only be expected to hold for gradient vector fields $\mathbf{f} = \nabla u$. To proceed, the structure of the operator L is used to show that the operators $\{Z_t\}$, defined by

$$\begin{aligned} Z_t u(x) :&= \boldsymbol{\Theta}_t \nabla u(x) - \Theta_t \mathbf{1}(x)\cdot(P_t\nabla u)(x) \\ &= (tLe^{-t^2L}u)(x) - (tLe^{-t^2L}\boldsymbol{\phi}(x))\cdot(P_t\nabla u)(x), \end{aligned}$$

(where $\boldsymbol{\phi}(x) := x$) satisfy

$$\iint_{\mathbb{R}^{n+1}_+} |Z_t u(x)|^2 \frac{dx\,dt}{t} \lesssim \|\nabla u\|_2^2. \tag{4.27}$$

Once we have proved that $d\mu(x) := |\boldsymbol{\Theta}_t\mathbf{1}(x)|^2\,\frac{dxdt}{t}$ is a Carleson measure on $\mathbb{R}^{n+1}_+$, we can then verify (4.26) as follows:

$$\begin{aligned} \iint_{\mathbb{R}^{n+1}_+} \left|tLe^{-t^2L}u(x)\right|^2 \frac{dxdt}{t} &= \iint_{\mathbb{R}^{n+1}_+} |Z_t u(x) + \boldsymbol{\Theta}_t\mathbf{1}(x)\cdot(P_t\nabla u)(x)|^2 \frac{dxdt}{t} \\ &\lesssim \iint_{\mathbb{R}^{n+1}_+} |Z_t u(x)|^2 \frac{dxdt}{t} + \iint_{\mathbb{R}^{n+1}_+} |P_t\nabla u(x)|^2\,|\Theta_t\mathbf{1}(x)|^2 \frac{dxdt}{t} \lesssim \|\nabla u\|_2, \end{aligned}$$

by (4.27) and Carleson's Theorem as used in (3.9). (We shall not include a proof of (4.27), though remark that it is somewhat similar to the "$T1$"-type reductions of square function bounds to Carleson measure bounds which we have already considered.)

The next step is to proceed exactly as in Theorem 4.15, to show that the Carleson estimate is a consequence of an estimate of the form:

$$\iint_{R_Q} |\Theta_t\mathbf{1}(x)\cdot A_t\mathbf{b}_Q(x)|^2 \frac{dxdt}{t} \le C_2|Q| \quad \text{for all dyadic cubes } Q\subset\mathbb{R}^n, \tag{4.28}$$

provided we can find a system of matrix valued functions $\mathbf{b}_Q : \mathbb{R}^n \to \mathbb{M}^N$, $Q\in\mathbb{D}$, which satisfies hypotheses (i) and (iii) of that theorem. For this purpose, we define $\mathbf{b}_Q : \mathbb{R}^n \to \mathbb{M}^N$ for each dyadic cube $Q\subset\mathbb{R}^n$ by

$$\mathbf{b}_Q = \nabla e^{-\epsilon^2 \ell(Q)^2 L} \boldsymbol{\phi}_Q =: \nabla \mathbf{f}_Q, \tag{4.29}$$

with $\epsilon > 0$ to be chosen, small enough, and

$$\boldsymbol{\phi}_Q(x) = \eta_Q(x)(x - x_Q),$$

where $\eta_Q \in C_0^\infty(5Q)$ with $\eta \equiv 1$ on Q, and x_Q denotes the centre of Q.

Before checking the hypotheses (i)–(iii), we note some consequences of the heat kernel bounds (G), namely that $\boldsymbol{\Theta}_t \mathbf{1} \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ with a uniform bound

$$\|\boldsymbol{\Theta}_t \mathbf{1}\|_\infty = \|tLe^{-t^2 L} \boldsymbol{\phi}\|_\infty \le M_1 < \infty \qquad \text{for all } t > 0\text{ ; and} \tag{4.30}$$

$$\|(e^{-t^2 L} - I)\boldsymbol{\phi}_Q\|_\infty \le M_2\, t \qquad \text{for all } t > 0. \tag{4.31}$$

We now show that the hypotheses (i), (ii), (iii) of Theorem 4.15 are satisfied by the system of matrix valued functions $\mathbf{b}_Q$.

(i) The estimate $\|\mathbf{b}_Q\|_2^2 \le C_0 |Q|$ is a consequence of the more general fact that a homogeneous elliptic operator L in divergence form generates a bounded semigroup with respect to the homogeneous Sobolev norm $\|\nabla u\|_2$. Thus

$$\|\mathbf{b}_Q\|_2 = \|\nabla e^{-\epsilon^2 \ell(Q)^2 L} \boldsymbol{\phi}_Q\|_2 \lesssim \|\nabla \boldsymbol{\phi}_Q\|_2 \approx |Q|^{1/2}.$$

(ii) Noting that $\boldsymbol{\Theta}_t \mathbf{b}_Q(x) = te^{-t^2 L} L e^{-\epsilon^2 \ell(Q)^2 L} \boldsymbol{\phi}_Q$, we have

$$\begin{aligned}
\int_0^{\ell(Q)} \int_Q |\boldsymbol{\Theta}_t \mathbf{b}_Q(x)|^2 \frac{dx\,dt}{t} &\le \int_0^\infty \|Le^{-(t^2 + \epsilon^2 \ell(Q)^2)L} \boldsymbol{\phi}_Q\|_2^2\, t\, dt \\
&= \tfrac{1}{2} \int_{\epsilon^2 \ell(Q)^2}^\infty \|\tau L e^{-\tau L} \boldsymbol{\phi}_Q\|_2^2 \frac{d\tau}{\tau^2} \\
&\qquad \text{where } \tau = t^2 + \epsilon^2 \ell(Q)^2 \\
&\le c_0 \left(\frac{1}{\epsilon^2 \ell(Q)^2} \right) \|\boldsymbol{\phi}_Q\|_2^2 \le \frac{c_1}{\epsilon^2} |Q|,
\end{aligned}$$

using (4.22) to obtain the uniform operator bounds.

(iii)

$$\begin{aligned}
\Re e\ \xi \cdot \left(|Q|^{-1} \int_Q \mathbf{b}_Q(x)\, dx \right) \overline{\xi} &= |\xi|^2 + \Re e\ \xi \cdot \left(|Q|^{-1} \int_Q (\mathbf{b}_Q - \mathbf{1})(x)\, dx \right) \overline{\xi} \\
&\ge |\xi|^2 - |Q|^{-1} \left\| \int_Q \nabla (e^{-\epsilon^2 \ell(Q)^2 L} - I) \boldsymbol{\phi}_Q(x)\, dx \right\|_{op} |\xi|^2 \\
&\ge |\xi|^2 - \frac{c_1}{\ell(Q)} \left\| (e^{-\epsilon^2 \ell(Q)^2 L} - I) \boldsymbol{\phi}_Q \right\|_\infty |\xi|^2 \\
&\ge |\xi|^2 (1 - c_1 M_2 \epsilon) \qquad \text{(by (4.31))} \\
&\ge \tfrac{1}{2} |\xi|^2
\end{aligned}$$

provided ϵ is chosen sufficiently small. We fix such a value of ϵ, and in particular use it back in part (ii).

All that remains is to check (4.28). To this end, as in the proofs of Theorems 4.7 and 4.15 above, we follow [38] to write

$$\mathbf{\Theta}_t\mathbf{1}\cdot(A_t\mathbf{b}_Q)=\mathbf{\Theta}_t\mathbf{1}\cdot\big((A_t-P_t)\mathbf{b}_Q\big)+(\mathbf{\Theta}_t\mathbf{1}\cdot P_t-\mathbf{\Theta}_t)\,\mathbf{b}_Q+\mathbf{\Theta}_t\mathbf{b}_Q,$$

and we note that, in view of (4.29), $(\mathbf{\Theta}_t\mathbf{1}\cdot P_t-\mathbf{\Theta}_t)\mathbf{b}_Q=-Z_t\mathbf{f}_Q$. Thus,

$$\iint_{R_Q}|\mathbf{\Theta}_t\mathbf{1}(x)\cdot(A_t\mathbf{b}_Q)(x)|^2\,\frac{dxdt}{t}\lesssim\|\mathbf{b}_Q\|_2^2+\|\mathbf{b}_Q\|_2^2+C_0|Q|\;\le\;C_2|Q|$$

by (4.30), (4.4), (4.27) and (ii).

That completes our description of the proof of the Kato square root problem for elliptic operators which satisfy pointwise heat kernel bounds. As we have said, the result holds without this additional assumption. One may use, in lieu of the Gaussian heat kernel bounds (G), the “Davies-Gaffney” off-diagonal estimates, which hold for every divergence form elliptic operator L as in (1.25)–(1.27). We refer the reader to [9] for a complete proof of the Kato square root estimate in this general setting.

The question whether accretive operators satisfy the estimate (4.24) was originally asked by T. Kato, J.-L. Lions and others in an attempt to better understand the equivalence between the operators and their associated sesquilinear (or energy) forms. This question arose again during Kato’s study of hyperbolic wave equations with time-varying coefficients, as it is connected with the question whether the mapping $A\to L^{1/2}$ is analytic. See [72]. Another application, and an easier one to describe, was noted by Kenig [70, Remark 2.5.6]: On $\mathbb{R}^{n+1}_+:=\mathbb{R}^n\times(0,\infty)$ consider the Dirichlet problem:

$$\begin{cases}\frac{\partial^2}{\partial t^2}U(\cdot,t)-LU(\cdot,t)=0\\ \qquad\qquad\qquad U(\cdot,t)=u(\cdot)\in\mathcal{D}(\sqrt{L})\end{cases}$$

where still $L=-\sum_{j,k=1}^n\frac{\partial}{\partial x_j}(A_{jk}\frac{\partial}{\partial x_k})$. Then the solution $U(x,t):=e^{-t\sqrt{L}}u(x)$ also satisfies the Neumann boundary condition

$$\frac{\partial U}{\partial t}\Big|_{t=0}=-\sqrt{L}u$$

if $\sqrt{L}u\in L^2(\mathbb{R}^n)$. Hence the Kato estimate $\|\sqrt{L}u\|_2\approx\|\nabla_x u\|_2$ is equivalent to

$$\left\|\frac{\partial U}{\partial t}|_{t=0}\right\|_2\approx\|\nabla_x u\|_2.$$

This alternative form of the Kato estimate is also known as a *Rellich inequality* or a *Dirichlet-Neumann inequality*.

For an excellent survey of the Kato square root problem, see Kenig’s featured review [71].

Before completing this section, we comment briefly on L^p estimates. For the various *Tb* results proved up to Section 4.1, the estimates also hold with L^p norms in place of the L^2 norms, as follows from Theorem 2.1 (to treat the case $2 < p < \infty$, this requires either that we work with conical square functions, or that we impose some regularity in the x-variable, of the kernel $\psi_t(x, y)$; see, e.g., [11] for a more detailed discussion of this point.) On the other hand, the operators $\mathbf{\Theta}_t$ used in the current section, arising in the proof of Kato's square root estimate, do not typically satisfy standard kernel bounds, even if L does have heat kernel bounds, so we cannot expect the Kato estimates to remain true for all $p \neq 2$. However, a variety of means have been used to prove bounds in specific cases. One method, involving weak-type (1-1) bounds via an adaptation of the Calderón-Zygmund theorem [46], was used to obtain

$$\|\nabla u\|_p \lesssim \|\sqrt{L}u\|_p, \tag{4.32}$$

when $1 < p < 2$, provided L has pointwise heat kernel bounds (i.e. satisfies the first equation in (G)) [47], and, via a duality argument,

$$\|\sqrt{L}u\|_p \lesssim \|\nabla u\|_p, \tag{4.33}$$

when $2 < p < \infty$.Another, involving the use of Hardy spaces, was used in [9] to show that if L has pointwise heat kernel bounds, then (4.33) holds when $1 < p < 2$. In the absence of pointwise kernel bounds, an extension of the Calderón-Zygmund method of [46] was developed independently in [20,21], and in [57], to prove the "Riesz transform" estimate (4.32) for a (necessarily) restricted range of p; the Kato estimate (4.33) was proved for a sharp (and again restricted) range of p in [3]. We refer the reader to [3] for a rather complete discussion of the L^p theory in the absence of pointwise kernel bounds. Another approach to L^p estimates is mentioned in Section 5.1.3 below.

5 Further results and recent progress

In this section we briefly discuss some recent advances in this subject.

5.1 *Tb* theory for SIOs

5.1.1 Analytic capacity

As mentioned above, Christ's local *Tb* theorem (Theorem 4.1) was motivated in part by its connection with the theory of analytic capacity and the Painlevé problem, which was eventually solved in the remarkable work of Tolsa [88]. See also the earlier work of Mattila, Melnikov and Verdera [77], and David [41,42]. Analytic capacity is connected with *Tb* theory via the Cauchy integral: the existence of non-constant bounded analytic functions may be used to produce a testing function b which yields L^2 bounds for the Cauchy integral. These bounds, in turn, encode geometric information via the so-called *Menger curvature*. In practice, this program was quite difficult to carry out, especially in the general situation in which the underlying measure may be non-doubling. Some extensions of either local or global *Tb* theorems to the non-doubling setting have been obtained by David [41] and by Nazarov, Treil and Volberg [79,80]. The latter, especially, played a useful role in Tolsa's solution of the Painlevé problem and the related Vitushkin conjecture concerning the semi-additivity of analytic capacity. Finally, much of this theory has been extended to higher dimensions by Volberg [90].

5.1.2 Local Tb theory for SIOs

Theorem 4.1 has been extended in another direction, closer in spirit to the results that we described above in Section 4, in which one requires weaker quantitative control on b_Q and Tb_Q. In [13], Conditions (i) and (iii) of Theorem 4.1 are relaxed to

$$(i)\ \int_Q |b_Q^i|^p \le C_0|Q|, \qquad (ii)\ \int_Q |Tb_Q^1|^{p'} \le C_0|Q|,\ \int_Q |{}^tTb_Q^2|^{p'} \le C_0|Q|,$$

respectively, for $1 < p < \infty$, assuming that T is a perfect dyadic SIO. In the case of standard SIOs, the same result was obtained with $p = 2$ in [19], but it remains an open problem, in general, to treat the case $1 < p < 2$. The current state of the art appears in [17], where the case $1 < p < 2$ is handled in the presence of additional hypotheses in the spirit of WBP, and in [62], where the theory is extended to a class of non-doubling measures.

5.1.3 Tb theory for vector valued functions

Many applications require the study of SIOs acting on spaces of vector valued functions $f : \mathbb{R}^n \to X$, where X is a Banach space. As mentioned in the Introduction, many proofs in this survey carry over to this context provided X is a Hilbert space. However, the situation is much more complicated when X is not isomorphic to a Hilbert space, and even the Hilbert transform may then be unbounded. In the 1980's, Bourgain [22] and Burkholder [23] proved that the Hilbert transform is bounded on $L^2(\mathbb{R}^n; X)$ if and only if X has the UMD (Unconditional Martingale Difference) property. See [24] for a survey. The connection between SIOs and martingales turns out to be the key to the vector valued theory, but is also an important tool for the scalar valued theory, for example in obtaining best constants. The first UMD valued $T1$ theorem was obtained in the influential paper [48] by Figiel. Over the past ten years, applications to PDEs such as maximal regularity, have motivated the study of SIOs with operator valued kernels $K : \mathbb{R}^n \times \mathbb{R}^n \backslash \{x = y\} \to B(X)$, where $B(X)$ denotes the set of bounded linear operators from X to itself, and X is a UMD space. In this direction, a $T1$ theorem was obtained by Hytönen and Weis [64], and a *Tb* theorem followed in [65]. A type of local *Tb* theorem for square functions, used to solve Kato's problem in $L^p(\mathbb{R}^n)$ and more generally in a UMD valued context, was then proven in [63]. Currently the vector valued theory is also being developed in contexts where the space of variables $\mathbb{R}^n$ is replaced by a more general metric measure space; see [61].

5.2 Local *Tb* theory for square functions and applications

5.2.1 L^p control on b_Q

In contrast to the situation for SIOs, the local *Tb* theory for square functions has been generalized to permit $b_Q \in L^p$, with $1 < p < 2$, as in (5.1.2). See [55], where this is done in the Euclidean case (i.e., to obtain the square function estimate (4.8) in the upper half space $\mathbb{R}^{n+1}_+$). A further extension to the case that the half-space is replaced by $\mathbb{R}^{n+1} \setminus E$, where E is a closed Ahlfors-David regular set of Hausdorff dimension n, appears in [52]. The latter extension has been used to prove a result of *free boundary* type, in which higher integrability of the Poisson kernel, in the presence of certain natural background hypotheses, is shown to be equivalent to a quantitative rectifiability of the boundary [58,59]. In the spirit of the work mentioned above on analytic capacity, local *Tb* theory enters by allowing one to relate Poisson kernel estimates to square function bounds for harmonic layer potentials, which in turn are tied to quantitative rectifiability of the boundary. In this case, the "b_Q's" are normalized Poisson kernels.

5.2.2 Extensions of the Kato problem and elliptic PDEs

The circle of ideas involved in the solution of the Kato problem, including local *Tb* theory for square functions, has been used to establish certain generalizations of the Kato problem with applications to complex elliptic PDEs and systems. These include L^2 bounds for layer potentials associated to complex divergence form elliptic operators [4,55], and the development of an L^2 functional calculus of certain perturbed Dirac operators and other first order elliptic systems [5–7,14]. The layer potential bounds, and the existence of a bounded holomorphic functional calculus for first order elliptic systems, were each then applied to obtain L^2 solvability results for elliptic boundary value problems.

The local *Tb* theory for square functions has also been used to establish other generalizations of the Kato problem, such as to higher order elliptic operators and systems [12] and to elliptic operators on Lipschitz domains [15].

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