A numerical block hybrid algorithm for solving systems of first-order initial value problems

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Abstract

Attracted by the importance of ordinary differential equations in many physical situations like, engineering, business and health care in particular, an effective and successful numerical algorithm is needed in order to explain many of the ambiguities about the phenomena in many fields of human endeavor. In this study, an interpolation and collocation technique are adopted in deriving a Block Hybrid Algorithm (BHA) for the numerical solution of systems of first-order Initial Value Problems (IVPs). To derive the BHA, the shifted Legendre polynomials was interpolated at two selected points and its derivative was collocated at seven selected points. This led to a continuous scheme which was eventually evaluated at some points to obtain the discrete schemes used in the numerical computation. Furthermore, some illustrative examples are introduced to show the applicability and validity of the proposed algorithm. It was observed that the proposed algorithm has the desired rate of convergence to the exact solution. The suggested method utilizes data at points other than the step numbers which is viewed as an important landmark; another major advantage of this algorithm is that it possesses remarkably small error constants (Table 2). Some graphical representations of the exact and numerical results are presented to show how accurate the numerical results agree with the exact solutions.

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1 Introduction

Differential equations are the best language for expressing many of the general laws of nature in quantum physics, electronics, computational chemistry and astronomy [1]. They are important because for many physical systems, one can subject it to suitable idealizations. They are important in mathematics and the sciences because they can be used to model a wide variety of real-world situations, [2, 3]. In physics, for example, differential equations can be used to model the motion of particles in a fluid, the trajectory of a projectile, calculate the movement or flow of electricity, motion of an object to and fro like a pendulum, to explain thermodynamics concepts. Also, in medical terms, they are used to check the growth of diseases in graphical representation [4, 5].

The primary purpose of the study of differential equation is the study of solutions that satisfy the equations and the properties of the solutions. Therefore, understanding the solutions of these equations is of paramount interest [6–8]. The exact solution is obtained analytically while the numerical solutions are demonstrated using some techniques, namely, the adaptive moving mesh and uniform mesh methods with the exact solution presented in a form of convergent power series [9, 10].

In this work, the block method approach (numerical solutions) will be considered because it provides a faster and easy way to solve problems among other advantages as compared to the analytic method of solution [11, 12]. These solutions are available only at selected (discrete) solution points, but not at all points covered by the functions as in the case with analytical solution methods [2, 13].

One of the major advantages of using block methods is that the methods do not require starting values [14–16]. Hence, they are cheaper to implement and it also has the capability of generating simultaneous approximate solutions at grid points within the interval of integration [17–19]. For more on block methods, see the works of [12, 19–21].

A number of methods have been proposed in literature for the solution of systems of first order IVPs. The authors in [9] proposed the adoption of hybrid block methods for the solution of first order IVPs of the form (1). They went further to implement the proposed method on some sets of first-order problems. The author in [22] formulated block hybrid methods with intra-step points for the solution of first order linear and nonlinear differential equations. They further used the absolute and residual error analysis to carry out the analysis of some basic properties of the methods. The authors in [23] derived a pair of three-step hybrid block methods of orders five and six for the solution of linear and nonlinear first-order systems using the power series polynomial as the basis function to derive the methods, the convergence analysis of the methods was also carried out. The authors in [24] formulated a sixth-order hybrid block method for the solution of first order IVPs. The Lagrange polynomial was adopted as the basis function in constructing the method and the method was found to be A-stable. The authors in [25] derived a class of hybrid block methods for the solution of first-order IVPs. They analysed the convergence properties of the methods. The authors in [26–29] also gave deep insight as to how to analyze algorithms for solving differential systems. Other authors that used hybrid block methods to solve first order IVPs are [8, 30–32].

Most of the conventional methods for solving systems of first-order IVPs of the form (1) have been reported to have some setbacks like low order of accuracy, large function evaluation and large error constants. The desire to address some of these challenges necessitated this research. The proposed BHA has the advantages of possessing fewer number of function evaluation per step, high order of accuracy and very small error constants.

The paper is structured into four sections as follows. Section 1 introduces the topic and the main aim of the article. In Sect. 2, the BHA is derived and analyzes for consistency, zero-stability and convergence. The applicability and validity of the algorithm is tested on some few systems of first order IVPs of ordinary differential equations in Sect. 3. Finally, some concluding remarks and suggestions are contained in Sect. 4.

2 Derivation and convergence analysis of the BHA

Consider the IVP of first order ordinary differential equation of the form

$$y'(x) = f(x, y(x)), \; y(x_0) = y_0 \tag{1}$$

where $y(x)$ is the unknown function to be determined. The idea here is to approximate the exact solution $y(x)$ of (1) in the partition $I_n = [a = x_0 < x_1 < x_2 < \ldots < x_n = b]$ of the integration interval $[a, b]$ with a constant step size $h = x_i - x_{i-1}, \; i = 1, \ldots, n$ by a shifted Legendre polynomial basis function of degree $s + r - 1$ of the form;

$$y(x) = \sum_{i=0}^{s+r-1} c_i P_i(t) \tag{2}$$

where $P_i(t) = \sum_{k=0}^{i}(-1)^{(i+k)}\frac{(i+k)!t^k}{(i-k)!(k!)^2 1^k}$, $c_i \in \mathbb{R}$, $y \in C^1(a, b)$ and $t = (x - x_n)$. The shifted Legendre polynomial was used as basis function in deriving the new block hybrid algorithm. This is in contrast to the power series polynomial basis function that is conventionally used in most literatures.

The first derivative of (2) is then substituted into (1) to obtain a differential system of the form

$$y'(x) = \sum_{i=0}^{s+r-1} c_i P_i'(t) = f(x, y(x)) \tag{3}$$

Now interpolating (2) at x_{n+s}, $s = 0$, $\frac{4}{5}$ and collocating (3) at x_{n+r}, $r = 0(\frac{1}{5})1$ where s and r represents the interpolation and collocation points respectively, the continuous scheme of the form below is obtained;

$$y(x) = \alpha_0(x)y_n + \alpha_{\frac{4}{5}}(x)y_{n+\frac{4}{5}} + h\left(\sum_{\tau=0}^{1}\beta_\tau(x)f(x_{n+\tau}, y_{n+\tau}) + \sum_{v=\frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}}^{1}\beta_v(x)f(x_{n+v}, y_{n+v})\right) \tag{4}$$

where

$$\alpha_0(x) = 1 - \frac{7875}{32h^2}x^2 + \frac{119875}{64h^3}x^3 - \frac{2953125}{512h^4}x^4 + \frac{1115625}{128h^5}x^5 - \frac{1640625}{256h^6}x^6 + \frac{234375}{128h^7}x^7$$

$$\alpha_{\frac{4}{5}}(x) = \frac{7875}{32h^2}x^2 - \frac{119875}{64h^3}x^3 + \frac{2953125}{512h^4}x^4 - \frac{1115625}{128h^5}x^5 + \frac{1640625}{256h^6}x^6 - \frac{234375}{128h^7}x^7$$

$$\beta_0(x) = x - \frac{1009}{48h}x^2 + \frac{38065}{288h^2}x^3 - \frac{292625}{768h^3}x^4 + \frac{107125}{192h^4}x^5 - \frac{464375}{1152h^5}x^6 + \frac{21875}{192h^6}x^7$$

$$\beta_{\frac{1}{5}}(x) = \frac{17255}{36h^2}x^3 - \frac{148625}{96h^3}x^4 + \frac{9625}{4h^4}x^5 - \frac{259375}{144h^5}x^6 + \frac{3125}{6h^6}x^7 - \frac{115}{2h}x^2$$

$$\beta_{\frac{2}{5}}(x) = \frac{19735}{72h^2}x^3 - \frac{147625}{192h^3}x^4 + \frac{51125}{48h^4}x^5 - \frac{209375}{288h^5}x^6 + \frac{3125}{16h^6}x^7 - \frac{155}{4h}x^2$$

$$\beta_{\frac{3}{5}}(x) = \frac{8615}{18h^2}x^3 - \frac{72625}{48h^3}x^4 + \frac{14125}{6h^4}x^5 - \frac{128125}{72h^5}x^6 + \frac{3125}{6h^6}x^7 - \frac{185}{3h}x^2$$

$$\beta_{\frac{4}{5}}(x) = \frac{39665}{288h^2}x^3 - \frac{316625}{768h^3}x^4 + \frac{38375}{64h^4}x^5 - \frac{484375}{1152h^5}x^6 + \frac{21875}{192h^6}x^7 - \frac{295}{16h}x^2$$

$$\beta_1(x) = \frac{1}{2h}x^2 - \frac{125}{36h^2}x^3 + \frac{875}{96h^3}x^4 - \frac{125}{12h^4}x^5 + \frac{625}{144h^5}x^6$$

In order to obtain the discrete BHA, Eq. (4) is evaluated at $x = x_n, x_{n+\frac{1}{5}}, \ldots, x_{n+1}$ and its first derivative evaluated at $x_{n+\frac{3}{10}}$, to give the following discrete BHA in Table 1 below.

Equation (4) is the continuous scheme while Table 1 gives the corresponding block discrete schemes for the one-step hybrid block method.

2.1 Order and error constant

Expanding each term of the block discrete schemes in Table 1 in Taylor's series using for instance scheme 1 given by;

$$y_{n+1} = \frac{1375}{512}y_{n+\frac{4}{5}} - \frac{233}{2304}hf_n - \frac{863}{512}y_n + \frac{19}{288}hf_{n+1}$$
$$- \frac{145}{288}hf_{n+\frac{1}{5}} - \frac{65}{576}hf_{n+\frac{2}{5}} - \frac{85}{144}hf_{n+\frac{3}{5}} + \frac{215}{2304}hf_{n+\frac{4}{5}}$$

Now, expanding each term of the above discrete scheme in Taylor's series and adding together the coefficients of hy_n in all the terms of the scheme, we have

$$\frac{863}{512}y_n = \frac{863}{512}\sum_{j=0}^{10}\frac{(0)^j h^j}{j!}y_n^{(j)} = \frac{863}{512}$$

$$y_{n+1} = \sum_{j=0}^{8}\frac{(1)^j h^j}{j!}y_n^{(j)}$$
$$= \left(\frac{1}{362880}\right)h^9 y_{\{n\}}^9 + \left(\frac{1}{40320}\right)h^8 y_{\{n\}}^8 + \left(\frac{1}{5040}\right)h^7 y_{\{n\}}^7 + \left(\frac{1}{720}\right)h^6 y_{\{n\}}^6$$
$$+ \left(\frac{1}{120}\right)h^5 y_{\{n\}}^5 + \left(\frac{1}{24}\right)h^4 y_{\{n\}}^4 + \left(\frac{1}{6}\right)h^3 y_{\{n\}}^3 + \left(\frac{1}{2}\right)h^2 y_{\{n\}}^2 + hy_{\{n\}} + 1$$

$$-\frac{1375}{512}y_{n+\frac{4}{5}} = -\frac{1375}{512}\sum_{j=0}^{8}\frac{\left(\frac{4}{5}\right)^j h^j}{j!}y_n^{(j)} = -\left(\frac{11}{984375}\right)h^8 y_{\{n\}}^8 - \left(\frac{22}{196875}\right)h^7 y_{\{n\}}^7$$
$$- \left(\frac{11}{11250}\right)h^6 y_{\{n\}}^6 - \left(\frac{11}{1500}\right)h^5 y_{\{n\}}^5 - \left(\frac{11}{240}\right)h^4 y_{\{n\}}^4$$
$$- \left(\frac{11}{48}\right)h^3 y_{\{n\}}^3 - \left(\frac{55}{64}\right)h^2 y_{\{n\}}^2 - \left(\frac{275}{128}\right)hy_{\{n\}} - \left(\frac{1375}{512}\right)$$

$$\frac{233}{2304}hf_n = \frac{233}{2304}h\sum_{j=0}^{8}\frac{(0)^j h^j}{j!}y_n^{(j+1)} = \left(\frac{233}{2304}\right)hy_{\{n\}}$$

Table 1 The BHA

		y_n	$y_{n+\frac{4}{5}}$	hf_n	$hf_{n+\frac{1}{5}}$	$hf_{n+\frac{3}{10}}$	$hf_{n+\frac{2}{5}}$	$hf_{n+\frac{3}{5}}$	$hf_{n+\frac{4}{5}}$	hf_{n+1}
$1.\, y_{n+1}$	$=$	$-\frac{863}{512}$	$\frac{1375}{512}$	$-\frac{233}{2304}$	$-\frac{145}{288}$	0	$-\frac{65}{576}$	$-\frac{85}{144}$	$\frac{215}{2304}$	$\frac{19}{288}$
$2.\, y_{n+\frac{1}{5}}$	$=$	$-\frac{351}{512}$	$\frac{863}{512}$	$-\frac{249}{6400}$	$-\frac{9}{32}$	0	$-\frac{93}{320}$	$-\frac{33}{80}$	$-\frac{33}{256}$	$\frac{3}{800}$
$3.\, y_{n+\frac{3}{10}}$	$=$	$-\frac{3175}{8192}$	$\frac{11367}{8192}$	$-\frac{93}{4096}$	$-\frac{129}{1024}$	0	$-\frac{219}{1024}$	$-\frac{177}{512}$	$-\frac{429}{4096}$	$\frac{3}{1024}$
$4.\, y_{n+\frac{2}{5}}$	$=$	$-\frac{5}{32}$	$\frac{37}{32}$	$-\frac{7}{720}$	$-\frac{19}{450}$	0	$-\frac{83}{900}$	$-\frac{67}{225}$	$-\frac{307}{3600}$	$\frac{1}{450}$
$5.\, y_{n+\frac{3}{5}}$	$=$	$-\frac{271}{512}$	$\frac{783}{512}$	$-\frac{201}{6400}$	$-\frac{129}{800}$	0	$-\frac{33}{1600}$	$-\frac{117}{400}$	$-\frac{777}{6400}$	$\frac{3}{800}$
$6.\, y_{n+\frac{4}{5}}$	$=$	1	0	$\frac{262}{4725}$	$\frac{608}{1575}$	$-\frac{8192}{33075}$	$\frac{488}{1575}$	$-\frac{1024}{4725}$	$\frac{26}{315}$	$-\frac{32}{11025}$

$$\frac{145}{288}hf_{n+\frac{1}{5}} = \frac{145}{288}h\sum_{j=0}^{8}\frac{\left(\frac{1}{5}\right)^j h^j}{j!}y_n^{(j+1)}$$

$$= \left(\frac{29}{907200000000}\right)h^9 y_{\{n\}}^9 + \left(\frac{29}{22680000000}\right)h^8 y_{\{n\}}^8$$

$$+ \left(\frac{29}{648000000}\right)h^7 y_{\{n\}}^7 + \left(\frac{29}{21600000}\right)h^6 y_{\{n\}}^6 + \left(\frac{29}{864000}\right)h^5 y_{\{n\}}^5$$

$$+ \left(\frac{29}{43200}\right)h^4 y_{\{n\}}^4 + \left(\frac{29}{2880}\right)h^3 y_{\{n\}}^3 + \left(\frac{29}{288}\right)h^2 y_{\{n\}}^2 + \left(\frac{145}{288}\right)h y_{\{n\}}$$

$$\frac{65}{576}hf_{n+\frac{2}{5}} = \frac{65}{576}h\sum_{j=0}^{8}\frac{\left(\frac{2}{5}\right)^j h^j}{j!}y_n^{(j+1)} = \left(\frac{13}{7087500000}\right)h^9 y_{\{n\}}^9 + \left(\frac{13}{354375000}\right)h^8 y_{\{n\}}^8$$

$$+ \left(\frac{13}{20250000}\right)h^7 y_{\{n\}}^7 + \left(\frac{13}{1350000}\right)h^6 y_{\{n\}}^6 + \left(\frac{13}{108000}\right)h^5 y_{\{n\}}^5$$

$$+ \left(\frac{13}{10800}\right)h^4 y_{\{n\}}^4 + \left(\frac{13}{1440}\right)h^3 y_{\{n\}}^3 + \left(\frac{13}{288}\right)h^2 y_{\{n\}}^2 + \left(\frac{65}{576}\right)h y_{\{n\}}$$

$$\frac{85}{144}hf_{n+\frac{3}{5}} = \frac{85}{144}h\sum_{j=0}^{8}\frac{\left(\frac{3}{5}\right)^j h^j}{j!}y_n^{(j+1)} = \left(\frac{1377}{5600000000}\right)h^9 y_{\{n\}}^9 + \left(\frac{459}{140000000}\right)h^8 y_{\{n\}}^8$$

$$+ \left(\frac{153}{4000000}\right)h^7 y_{\{n\}}^7 + \left(\frac{153}{400000}\right)h^6 y_{\{n\}}^6 + \left(\frac{51}{16000}\right)h^5 y_{\{n\}}^5$$

$$+ \left(\frac{17}{800}\right)h^4 y_{\{n\}}^4 + \left(\frac{17}{160}\right)h^3 y_{\{n\}}^3 + \left(\frac{17}{48}\right)h^2 y_{\{n\}}^2 + \left(\frac{85}{144}\right)h y_{\{n\}}$$

$$-\frac{215}{2304}hf_{n+\frac{4}{5}} = -\frac{215}{2304}h\sum_{j=0}^{8}\frac{\left(\frac{4}{5}\right)^j h^j}{j!}y_n^{(j+1)}$$

$$= -\left(\frac{86}{221484375}\right)h^9 y_{\{n\}}^9 - \left(\frac{172}{44296875}\right)h^8 y_{\{n\}}^8$$

$$- \left(\frac{43}{1265625}\right)h^7 y_{\{n\}}^7 - \left(\frac{43}{168750}\right)h^6 y_{\{n\}}^6 - \left(\frac{43}{27000}\right)h^5 y_{\{n\}}^5$$

$$- \left(\frac{43}{5400}\right)h^4 y_{\{n\}}^4 - \left(\frac{43}{1440}\right)h^3 y_{\{n\}}^3 - \left(\frac{43}{576}\right)h^2 y_{\{n\}}^2 - \left(\frac{215}{2304}\right)h y_{\{n\}}$$

$$-\frac{19}{288}hf_{n+1} = -\frac{19}{288}h\sum_{j=0}^{8}\frac{(1)^j h^j}{j!}y_n^{(j+1)} = -\left(\frac{19}{11612160}\right)h^9 y_{\{n\}}^9 - \left(\frac{19}{1451520}\right)h^8 y_{\{n\}}^8$$

$$- \left(\frac{19}{207360}\right)h^7 y_{\{n\}}^7 - \left(\frac{19}{34560}\right)h^6 y_{\{n\}}^6 - \left(\frac{19}{6912}\right)h^5 y_{\{n\}}^5$$

$$- \left(\frac{19}{1728}\right)h^4 y_{\{n\}}^4 - \left(\frac{19}{576}\right)h^3 y_{\{n\}}^3 - \left(\frac{19}{288}\right)h^2 y_{\{n\}}^2 - \left(\frac{19}{288}\right)h y_{\{n\}}$$

Let C_0 be the coefficient of $(hy_n)^0$, C_1 be the coefficient of $(hy_n)^1$ and C_2 be the coefficient of $(hy_n)^2$ respectively and so on. Summing up these coefficients we, obtain the following;

$$C_0 = \left(\frac{863}{512}\right) + 1 - \left(\frac{1375}{512}\right) = 0$$

$$C_1 = 1 - \left(\frac{275}{128}\right) + \left(\frac{233}{2304}\right) + \left(\frac{145}{288}\right) + \left(\frac{65}{576}\right) + \left(\frac{85}{144}\right) - \left(\frac{215}{2304}\right) - \left(\frac{19}{288}\right) = 0$$

$$C_2 = \left(\frac{1}{2}\right) - \left(\frac{55}{64}\right) + \left(\frac{29}{288}\right) + \left(\frac{13}{288}\right) + \left(\frac{17}{48}\right) - \left(\frac{43}{576}\right) - \left(\frac{19}{288}\right) = 0$$

$$C_3 = -\left(\frac{19}{576}\right) - \left(\frac{43}{1440}\right) + \left(\frac{17}{160}\right) + \left(\frac{13}{1440}\right) + \left(\frac{29}{2880}\right) - \left(\frac{11}{48}\right) + \left(\frac{1}{6}\right) = 0$$

$$C_4 = -\left(\frac{19}{1728}\right) - \left(\frac{43}{5400}\right) + \left(\frac{17}{800}\right) + \left(\frac{13}{10800}\right) + \left(\frac{29}{43200}\right) - \left(\frac{11}{240}\right) + \left(\frac{1}{24}\right) = 0$$

$$C_5 = -\left(\frac{19}{6912}\right) - \left(\frac{43}{27000}\right) + \left(\frac{51}{16000}\right) + \left(\frac{13}{108000}\right) + \left(\frac{29}{864000}\right) - \left(\frac{11}{1500}\right) + \left(\frac{1}{120}\right)$$
$$= 0$$

$$C_6 = -\left(\frac{19}{34560}\right) - \left(\frac{43}{168750}\right) + \left(\frac{153}{400000}\right) + \left(\frac{13}{1350000}\right)$$
$$+ \left(\frac{29}{21600000}\right) - \left(\frac{11}{11250}\right) + \left(\frac{1}{720}\right) = 0$$

$$C_7 = -\left(\frac{19}{207360}\right) - \left(\frac{43}{1265625}\right) + \left(\frac{153}{4000000}\right) + \left(\frac{13}{20250000}\right)$$
$$+ \left(\frac{29}{648000000}\right) - \left(\frac{22}{196875}\right) + \left(\frac{1}{5040}\right) = 0$$

$$C_8 = -\left(\frac{19}{1451520}\right) - \left(\frac{172}{44296875}\right) + \left(\frac{459}{140000000}\right) + \left(\frac{13}{354375000}\right) + \left(\frac{29}{22680000000}\right)$$
$$- \left(\frac{11}{984375}\right) + \left(\frac{1}{40320}\right) = -\left(\frac{11}{378000000}\right) = -2.91005291005 \times 10^{-8}$$

Hence, the order of the discrete scheme 1 is $C_p = C_7 \rightarrow p = 7$ and error constant $C_{p+1} = C_8 \neq 0 = -2.9101 \times 10^{-8}$ as shown in Table 2 below and the order and error constants of the remaining schemes are computed using the same approach.

Hence, the BHA has uniform order $\check{p} = 7$ as can be seen in Table 2 below.

2.2 Consistency

The block discrete schemes in Table 1 is said to be consistent if the following conditions hold:

(i)   It has order $\check{p} \geq 1$,
(ii) $\sum_{j=0}^{k} \check{\alpha}_j = 0$,
(iii) $\sum_{j=0}^{k} j\hat{\alpha}_j = \sum_{j=0}^{k} \check{\beta}_j$,
(iv) $\rho(1) = 0$ and $\rho'(1) = \sigma(1)$,

Table 2 Order and error constants of individual members of the BHA

Algorithm no	Order	Error constants
1	$P = 7$	-2.9101×10^{-8}
2	$P = 7$	-7.4286×10^{-9}
3	$P = 7$	-5.3013×10^{-9}
4	$P = 7$	-3.3862×10^{-9}
5	$P = 7$	-7.4286×10^{-9}
6	$P = 7$	7.7400×10^{-9}

where $\rho(r)$ and $\sigma(r)$ are the first and the second characteristic polynomials of the block discrete schemes in Table 1. Following [2, 3, 9, 10], condition (i) is a sufficient condition for the block discrete schemes in Table 1 to be consistent. Since $\hat{p} = 7 > 1$, hence the BHA is consistent.

2.3 Zero-stability

The block discrete schemes in Table 1 is said to be zero stable if the roots $z_r; r = 1, \ldots, n$ of the first characteristic polynomial $p(z)$, defined by

$$p(z) = det|zQ - T|$$

satisfies $|z_r| \leq 1$ and every root with $|z_r| = 1$ has multiplicity not exceeding the order of the differential equation in the limit as $h \to 0$, [13].

Calculations from all available information revealed that the BHA has the following roots

$$z^5(z - 1) = 0 \Rightarrow z = (0, 0, 0, 0, 0, 1)$$

Hence, the BHA is zero-stable, since all roots with modulus one does not have multiplicity exceeding the order of the differential equation in the limit as $h \to 0$.

2.4 Convergence

According to [10, 33], we can safely assert the convergence of the block discrete schemes in Table 1 since the BHA is consistent and zero-stable.

3 Numerical experiments

In this section, some systems of first-order IVPs shall be solved using the BHA. The absolute errors of the BHA shall be compared with those of some existing methods. The following notations shall be used in Tables 3, 4 and 5 and Figs. 1, 2 and 3.

x: point of evaluation.

y_i: solution component.

$N_e = |y_E - y_N|$: absolute error.

BHA: the proposed block hybrid algorithm.

Problem 1 *Consider a mildly stiff system problem which was solved by [30]*

Table 3 Comparison of the performance of the BHA on Problem 1

x	y_i	Absolute error in BHA $N_e = \|y_E - y_N\|$	Absolute error in [30] $N_e = \|y_E - y_N\|$
5	y_1	4.9294×10^{-14}	1.3920×10^{-11}
	y_2	2.4647×10^{-14}	6.9700×10^{-12}
40	y_1	3.6082×10^{-15}	3.3628×10^{-12}
	y_2	1.7972×10^{-15}	1.6818×10^{-12}
70	y_1	1.9732×10^{-16}	2.9325×10^{-13}
	y_2	9.8662×10^{-17}	1.4664×10^{-13}

Table 4 Comparison of the performance of the BHA on Problem 2

x	y_i	Absolute error in BHA $N_e = \|y_E - y_N\|$	Absolute error in [8] $N_e = \|y_E - y_N\|$
0.2	y_1	7.050000×10^{-12}	7.740000×10^{-5}
	y_2	5.040000×10^{-14}	7.660000×10^{-6}
0.4	y_1	6.340000×10^{-12}	1.722546×10^{-3}
	y_2	1.520000×10^{-13}	1.47976×10^{-4}
0.6	y_1	4.270000×10^{-12}	7.992946×10^{-3}
	y_2	3.420000×10^{-13}	1.695327×10^{-3}
0.8	y_1	2.560000×10^{-12}	1.9194892×10^{-2}
	y_2	6.810000×10^{-13}	1.2429885×10^{-2}
1.0	y_1	1.440000×10^{-12}	2.2159840×10^{-3}
	y_2	1.270000×10^{-12}	5.3353238×10^{-2}

Table 5 Comparison of the performance of the BHA on Problem 3

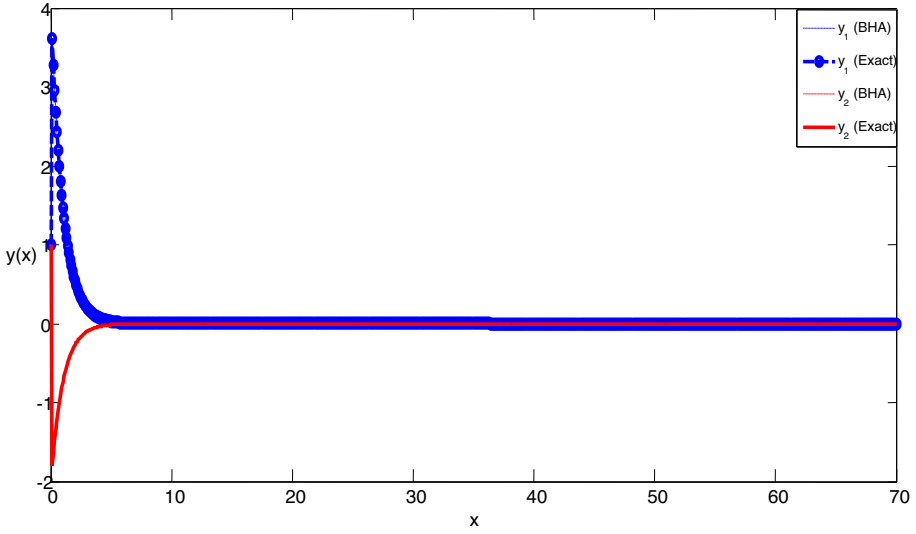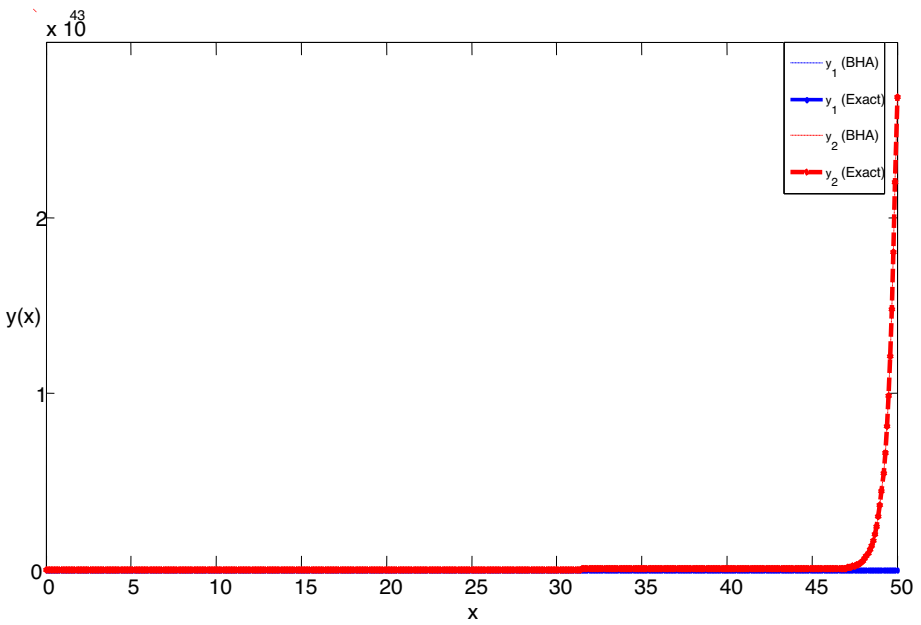| x	y_i	Absolute error in BHA $N_e = \|y_E - y_N\|$	Absolute error in [34] $N_e = \|y_E - y_N\|$
1.0	y_1	2.4516×10^{-17}	0.0821×10^{-16}
	y_2	1.6721×10^{-17}	0.0975×10^{-16}
1.5	y_1	1.3509×10^{-18}	0.2136×10^{-16}
	y_2	1.8218×10^{-17}	0.0845×10^{-16}
2.0	y_1	3.1722×10^{-18}	0.3981×10^{-16}
	y_2	2.1892×10^{-18}	0.0565×10^{-16}
2.5	y_1	4.8192×10^{-18}	0.4564×10^{-16}
	y_2	6.2886×10^{-18}	0.0389×10^{-16}
3.0	y_1	3.1782×10^{-19}	0.8126×10^{-16}
	y_2	3.1902×10^{-18}	0.0156×10^{-16}

Fig. 1 Solution curves for Problem 1
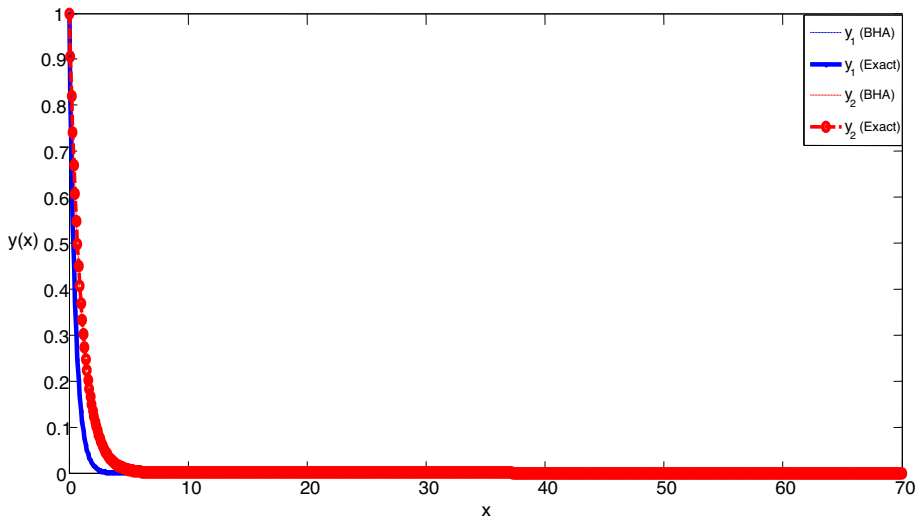

Fig. 2 Solution curves for Problem 2

Fig. 3 Solution curves for Problem 3

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} 998 & 1998 \\ -999 & -1999 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \textit{ with initial conditions } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \textit{ and the}$$

exact solution of the systems is given as $\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 4e^{-x} - 3e^{-1000x} \\ -2e^{-x} + 3e^{-1000x} \end{bmatrix}.$

The computed results are shown in Table 3, while the theoretical and numerical results are plotted graphically in Fig. 1.

Problem 2 *Consider the following linear differential equation solved by [8],*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -4y_1 \\ 2y_2 \end{bmatrix}, \textit{ with initial conditions } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \textit{ and the exact solution of this}$$

problem is $\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} e^{-4x} \\ e^{2x} \end{bmatrix}.$ *The numerical results are shown in Table 4 while Fig. 2*

shows the solution curves for the Problem.

Problem 3 *Consider the following nonlinear stiff problem*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1002y_1(x) + 1000y_2^2(x) \\ y_1(x) - y_2(x) - y_2^2(x) \end{bmatrix}, \textit{ with initial conditions } \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \textit{ The}$$

exact solution of this problem is $\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} e^{-2x} \\ e^{-x} \end{bmatrix}$ *which was solved by [34]. The*

computed results are shown in Table 5, while Fig. 3 shows the solution curves for the Problem.

From the numerical and graphical results presented in Tables 3, 4 and 5 and Figs. 1, 2 and 3, it is clear that the proposed BHA is computationally reliable. This explains why the absolute errors obtained (at selected values of x) on the applications of the BHA (on the first-order IVPs) are by far smaller than those of the existing methods we compared our results with. In essence, this implies that the proposed BHA is more accurate. Furthermore, the solution

curves obtained in Figs. 1, 2 and 3 show a measure of convergence of the approximate solution (using the BHA) to the exact solutions.

4 Conclusion

In this article, the BHA derived was tested and found to be accurate, consistent, zero-stable and convergent. The method was implemented on some systems of linear and nonlinear initial value problems of ordinary differential equations and the numerical results were found to be accurate when compared with the exact solutions of other numerical methods as contained in Tables 3, 4 and 5 and their respective solution curves. It was observed that the BHA is computationally reliable for solving ordinary differential equations of the first-order. The close relation between the exact and numerical results is confirmed by the plotted graphs. The new BHA is therefore a suitable candidate for all forms (linear and nonlinear) of first-order initial value problems of ordinary differential equations. The BHA proposed in this research is therefore recommended for the solution of systems of first-order IVPs. It is important to state that the BHA derived in this research is limited to the solution of first-order IVPs only. Further study could extend this work to the numerical solution of higher order IVPs. The possibility of exploiting other basis function than the shifted Legendre polynomial is also a viable option.

Author contributions This work was carried out in collaboration among the authors. Author Nathaniel Mahwash Kamoh proposed, derived and implemented the method. Author Bwebum Cleofas Dang analyzed the method while Author Joshua Sunday presented the numerical results graphically. All the authors managed the literature searches, read and approved the final manuscript.

Data availability The numerical data used to support the findings of this study are included within the article.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this article**.**

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