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Multilevel wavelet packets in sobolev space over local fields of positive characteristic

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Abstract
The concepts of multiresolution analysis (MRA) and wavelets in Sobolev space over local fields of positive characteristic ($H^s(\mathbb{K})$) are developed by Pathak and Singh [9]. In this paper, we constructed wavelet packets in Sobolev space $H^s(\mathbb{K})$ and derived their orthogonality at each level. By using convolution theory, an example of wavelet packets in $H^s(\mathbb{K})$ is presented

Keywords Wavelets · Multiresolution analysis · Local fields · Sobolev space · Fourier transforms · Wavelet packet

Mathematics Subject Classification 42C40 · 46E35 · 11F85 · 42A38

1 Introduction

The theory of wavelets on local fields and related groups is developed by Benedetto et al. in [2, 3]). Albeverio, Skopina, et al. (see [1, 6, 7]) constructed MRA-related wavelets on the p-adic field. Jiang et al. [5] discusses wavelets on local fields. Recently, Pathak and Singh modified the classical concept of MRA and constructed orthonormal wavelets in Sobolev space; their H^s-norm was translation invariant but not dilation invariant. Hence, they used different scaling functions at each level of dilation (see [8–16]). In this paper, we construct wavelet packets corresponding to such an MRA.

This article is divided into the following sections. In Sect. 1, we discuss some properties of local fields and Sobolev space over $\mathbb{K}$. In Sect. 2, we recall the MRA on $H^s(\mathbb{K})$, and one essential lemma, the splitting lemma, is proved. In Sect. 3, we construct wavelet packets and prove their orthogonality at each level. We also show that they form an orthonormal basis for $H^s(\mathbb{K})$. Finally, we construct wavelet packets in $H^s(\mathbb{K})$ at the jth level.

Throughout the paper, $\mathbb{K}$ denotes the local field of positive characteristic, χ is a fixed character on $\mathbb{K}^+$, $\mathfrak{p}$ is a fixed prime element in $\mathbb{K}$ used for dilation, and $u(k) \in \mathbb{K}$, , $k \in \mathbb{N}_0 = 0, 1, 2, 3, \ldots$ is used for translation. For more detail, we refer to [9].

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The Sobolev space $H^s(\mathbb{K}), s \in \mathbb{R}$, consists of all those $f \in \mathscr{S}'(\mathbb{K})$ (the space of continuous linear functionals on $\mathscr{S}(\mathbb{K})$, where $\mathscr{S}(\mathbb{K})$ is the space of all finite linear combinations of characteristic functions of balls in $\mathbb{K}$) which satisfy:

$$\|f\|^2_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\xi)|\hat{f}(\xi)|^2 d\xi, \quad \text{where } \hat{\gamma}^s(\xi) = (max(1, |\xi|))^s,$$

the corresponding inner product is defined by

$$\langle f, g\rangle_{H^s(\mathbb{K})} = \int_{\mathbb{K}} \hat{\gamma}^s(\xi)\hat{f}(\xi)\overline{\hat{g}(\xi)}d\xi,$$

where

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\overline{\chi_\xi(x)}dx, \ \xi \in \mathbb{K}.$$

For more detail, refer to [5, 9, 18]).

2 Multiresolution analysis on $H^s(\mathbb{K})$ and the splitting lemma

Pathak and Singh [9] modified the classical multiresolution analysis on $L^2(\mathbb{K})$. Now, we recall the theory of wavelet in Sobolev space over $\mathbb{K}$.

Definition 2.1 A multiresolution analysis of $H^s(\mathbb{K})$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of the closed linear subspaces of $H^s(\mathbb{K})$ such that

(a) $V_j \subset V_{j+1}$;

(b) $\overline{\cup_{j\in\mathbb{Z}} V_j} = H^s(\mathbb{K})$;

(c) $\cap_{j\in\mathbb{Z}} V_j = 0$;

(d) For each $j \in \mathbb{Z}$, there exists a function $\phi^{(j)} \in H^s(\mathbb{K})$ such that $\{\phi^{(j)}_{j,k}\}_{k\in\mathbb{N}_0}$, forms an orthonormal basis of V_j,

where

$$\phi^{(j)}_{j,k}(.) = q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}. - u(k)), \ k \in \mathbb{N}_0, \quad j \in \mathbb{Z}.$$

Such function $\phi^{(j)}$ are called scaling function. The condition $V_j \subset V_{j+1}$; for $j \in \mathbb{Z}$ is equivalent to the existence of integral-periodic function $m^{(j)}_0 \in L^2(\mathfrak{D})$ such that the following scale relation holds.

$$\hat{\phi}^{(j)}(\xi) = m^{(j+1)}_0(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi), \tag{2.1}$$

these functions $m^{(j+1)}_0$ are called low pass filter. Define $\psi^{(j)}_r$, $j \in \mathbb{Z}$ and $r \in D_1 = \{0, 1, 2, 3, 4...q-1\}$, by the formula

$$\hat{\psi}^{(j)}_r(\xi) = m^{(j+1)}_r(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi), \quad j \in \mathbb{Z}, \quad r \in D_1, \tag{2.2}$$

where $m_t^{(j+1)}(t = 1, 2, 3, ..., q-1)$ are called high pass filters such that the matrix $M^{(j)}(\xi) = [m_{r_1}^{(j)}(\mathfrak{p}\xi + \mathfrak{p}u(r_2))]_{r_1,r_2=0}^{q-1}$ is unitary.

We get $\{\psi_{r,j,k}^{(j)}\}_{j\in\mathbb{Z},k\in\mathbb{N}_0,r\in D_1}$ form an orthonormal basis for $H^s(\mathbb{K})$, where

$$\psi_{r,j,k}^{(j)}(.) = q^{\frac{j}{2}}\psi_r^{(j)}(\mathfrak{p}^{-j}. - u(k)),\ j, k \in \mathbb{Z}, r \in D_1. \tag{2.3}$$

Theorem 2.2 *If $s \in \mathbb{R}$, $\phi^{(j)} \in H^s(\mathbb{K})$ then the distribution $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)), k \in \mathbb{Z}\}$ are orthonormal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k=0}^{\infty}\hat{\gamma}^s(\mathfrak{p}^{-j}(\xi + u(k)))|\hat{\phi}^{(j)}(\xi + u(k))|^2 = 1 \quad a.e. \tag{2.4}$$

Moreover, we also have

$$|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)| \le \hat{\gamma}^{-\frac{s}{2}}(\xi). \tag{2.5}$$

Theorem 2.3 *Let $\{\phi^{(j)}\}_{j\in\mathbb{Z}}$ be a sequence of functions of $H^s(\mathbb{K})$ such that, for every j, the distributions*

$$\phi_{j,k}(x) = q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}x - u(k)), k \in \mathbb{N}_0, \tag{2.6}$$

are orthonormal in $H^s(\mathbb{K})$ and $V_j = \overline{\{\phi_{j,k}(\xi) : k \in \mathbb{N}_0\}}$.

If,

$$\lim_{j\to\infty}|\hat{\phi}^{(j)}(\mathfrak{p}^j\xi)| = \hat{\gamma}^{-\frac{s}{2}}(\xi), \tag{2.7}$$

holds then, $\overline{\cup_{j\in\mathbb{Z}}V_j} = H^s(\mathbb{K})$. Moreover, for every $j \le 0$, then $\cap_{j\in\mathbb{Z}}V_j = \{0\}$.

For construction of wavelet packets the following splitting lemma is required.

Lemma 2.4 *Let $\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}. - u(m)) : m \in \mathbb{N}_0\}$ be an orthonormal system in $H^s(\mathbb{K})$ and $V_j = span\{q^{\frac{j}{2}}\phi^{(j)}(\mathfrak{p}^{-j}. - u(m)) : m \in \mathbb{N}_0\}$. Let $\hat{\psi}_r^{(j)}(\xi) = m_r^{(j+1)}(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi)$, $0 \le r \le q-1$. Then $\{\psi_{r,j,m}^{(j)}(.) : 0 \le r \le q-1, m \in \mathbb{N}_0\}$ is an orthonormal basis in V_j if and only if the matrix*

$$M^{(j)}(\xi) = [m_{r_1}^{(j)}(\mathfrak{p}\xi + \mathfrak{p}u(r_2))]_{r_1,r_2=0}^{q-1}$$

is unitary for a.e. $\xi \in \mathfrak{D}$.

Proof Let $M^{(j)}(\xi)$ is unitary. Then, we have

$$\begin{aligned}
&\left\langle\psi_{r_1,j,m}^{(j)}(.),\ \psi_{r_2,j,n}^{(j)}(.)\right\rangle \\
&= \int_{\mathbb{K}}\hat{\gamma}^s(\xi)q^{-\frac{j}{2}}\hat{\psi}_{r_1}^{(j)}(\mathfrak{p}^j\xi)\bar{\chi}_m(\mathfrak{p}^j\xi)q^{-\frac{j}{2}}\overline{\hat{\psi}_{r_2}^{(j)}}(\mathfrak{p}^j\xi)\chi_n(\mathfrak{p}^j\xi)d\xi \\
&= \int_{\mathfrak{D}}\sum_{l\in\mathbb{N}_0}\hat{\gamma}^s(\mathfrak{p}^{-j}(\xi + u(l)))\hat{\psi}_{r_1}^{(j)}(\xi + u(l))\overline{\hat{\psi}_{r_2}^{(j)}}(\xi + u(l))\bar{\chi}_m(\xi)\chi_n(\xi)d\xi \\
&= \int_{\mathfrak{D}}\sum_{l\in\mathbb{N}_0}\gamma^s(\mathfrak{p}^{-j}(\xi + u(l)))m_{r_1}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(l))
\end{aligned}$$

$$
\begin{aligned}
&\times \hat{\phi}^{j+1}(\mathfrak{p}\xi + \mathfrak{p}u(l))\overline{m_{r_2}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(l))}\,\overline{\hat{\phi}^{j+1}(\mathfrak{p}\xi + \mathfrak{p}u(l))}\bar{\chi}_m(\xi)\chi_n(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1} \\
&(\mathfrak{p}\xi + \mathfrak{p}u(ql+i)))m_{r_1}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(ql+i)))\overline{m_{r_2}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(ql+i)))} \\
&\times |\hat{\phi}^{j+1}(\mathfrak{p}\xi + \mathfrak{p}u(ql+i)))|^2\bar{\chi}_m(\xi)\chi_n(\xi)d\xi \\
&= \int_{\mathfrak{D}} \{\sum_{i=0}^{q-1} m_{r_1}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i)))\overline{m_{r_2}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i)))}\}\bar{\chi}_m(\xi)\chi_n(\xi)d\xi \\
&= \int_{\mathfrak{D}} \delta_{r_1,r_2}\bar{\chi}_m(\xi)\chi_n(\xi)d\xi \\
&= \delta_{r_1,r_2}\delta_{m,n}.
\end{aligned}
$$

Therefore, $\{q^{\frac{j}{2}}\psi_r^{(j)}(\mathfrak{p}^{-j}.-u(m)) : 0 \le r \le q-1, m \in \mathbb{N}_0\}$ is an orthonormal system in V_j. For proving it basis, suppose $h \in V_j$ be such that it is orthonormal to $\psi_r^{(j)}(\mathfrak{p}^{-j}.-u(m))\ \forall r = 0, 1, ..., q-1; m \in \mathbb{N}_0$. We claim that $h = 0$ a.e.
Since $h \in V_j$, therefore $h \in V_{j+1}$. Hence $h(x)$ can be written as

$$
h(x) = \sum_{k\in\mathbb{N}_0} q^{\frac{j+1}{2}} c_k^{(j+1)}\phi^{(j+1)}(\mathfrak{p}^{-j-1}x - u(k)),\ \text{ for some } \{c_k^{(j+1)} : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0).
$$

Therefore,

$$
\begin{aligned}
\hat{h}(\xi) &= q^{-\frac{j+1}{2}} \sum_{k\in\mathbb{N}_0} c_k^{(j+1)}\hat{\phi}^{(j+1)}(\mathfrak{p}^{j+1}\xi)\bar{\chi}_k(\mathfrak{p}^{j+1}\xi) \\
&= m_h^{(j+1)}(\mathfrak{p}^{j+1}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}^{j+1}\xi),
\end{aligned} \tag{2.8}
$$

where, $m_h^{(j+1)}(\xi) = q^{-\frac{j+1}{2}}\sum_{k\in\mathbb{N}_0} c_k^{(j+1)}\bar{\chi}_k(\xi)$, i.e., $m_h^{(j+1)}$ is integral periodic and is in $L^2(\mathfrak{D})$.
For $r = 0, 1, 2, ..., q-1$ and $m \in \mathbb{N}_0$, we have

$$
\begin{aligned}
0 &= \left\langle h, \psi_r^{(j)}(\mathfrak{p}^{-j}. - u(m))\right\rangle \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\xi)\hat{h}(\xi)q^{-j}\overline{\hat{\psi}_r^{(j)}}(\mathfrak{p}^j\xi)\chi_m(\mathfrak{p}^j\xi)d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi)m_h^{(j+1)}(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi)\overline{m_r^{(j+1)}}(\mathfrak{p}\xi)\overline{\hat{\phi}^{(j+1)}}(\mathfrak{p}\xi)\chi_m(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\xi + u(l)))m_h^{(j+1)}(\mathfrak{p}(\xi + u(l)))\overline{m_r^{(j+1)}}(\mathfrak{p}(\xi + u(l))) \\
&\times|\hat{\phi}^{(j+1)}(\mathfrak{p}(\xi + u(l)))|^2\chi_m(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{l\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\xi + \mathfrak{p}u(i) + u(l)))|\hat{\phi}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i) + u(l))|^2 \\
&\times m_h^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i))\overline{m_r^{(j+1)}}(\mathfrak{p}\xi + \mathfrak{p}u(i))\chi_m(\xi)d\xi
\end{aligned}
$$

$$= \int_{\mathfrak{D}} \{\sum_{i=0}^{q-1} m_h^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i))\overline{m_r^{(j+1)}}(\mathfrak{p}\xi + \mathfrak{p}u(i))\}\chi_m(\xi)d\xi.$$

Hence,

$$\sum_{i=0}^{q-1} m_h^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i))\overline{m_r^{(j+1)}}(\mathfrak{p}\xi + \mathfrak{p}u(i)) = 0, \text{ for all } r = 0, 1, ..., q-1.$$

That is the vector $(m_h^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i)))_{i=0}^{q-1} \in \mathbb{C}^q$ is orthogonal to each row vector of the unitary matrix $M^{(j)}(\xi)$. Therefore it is zero for a.e. ξ. In particular, $m_h^{(j+1)}(\mathfrak{p}^{j+1}(\mathfrak{p}^{-j}\xi)) = 0$ a.e. This implies that $\hat{h} = 0$ a.e. Therefore $h = 0$ a.e. By reversing the above steps, the converse part can also be proved. □

3 wavelet packets in Sobolev space

If we apply splitting lemma to V_j, then we see that $\{q^{\frac{j}{2}}\psi_r^{(j)}(\mathfrak{p}^{-j}. - u(m)) : 0 \leq r \leq q-1, m \in \mathbb{N}_0\}$ is an orthonormal basis for V_j. Define a sequence of functions $\{w_n^{(j)} : n \geq 0\}$ such that

$$w_0^{(j)} = \phi^{(j)} \text{ and } w_n^{(j)} = \psi_n^{(j)} \ (1 \leq n \leq q-1),$$

where

$$\psi_t^{(j)}(\xi) = m_t^{(j+1)}(\mathfrak{p}\xi)\hat{\phi}^{(j+1)}(\mathfrak{p}\xi) \ (1 \leq t \leq q-1).$$

In general, let $w_n^{(j)}$ be defined for every integer $n \geq 0$ by

$$w_{r+qn}^{(j)}(\mathfrak{p}^{-j}x) = q^{\frac{j+1}{2}} \sum_{m\in\mathbb{N}_0} h_{m,r}^{(j+1)} w_n^{(j+1)}(\mathfrak{p}^{-j-1}x - u(m)), \text{ for } 0 \leq r \leq q-1.$$

Taking the Fourier transform, we get

$$\hat{w}_{r+qn}^{(j)}(\xi) = m_r^{(j+1)}(\mathfrak{p}\xi)\hat{w}_n^{(j+1)}(\mathfrak{p}\xi).$$

We can also define $w_n^{(j)}$ for every integer $n \geq 0$ by its Fourier transform as (here $[x]$ denotes greatest integer less than or equal to x)

$$\hat{w}_n^{(j)}(\xi) = m_r^{(j+1)}(\mathfrak{p}\xi)\hat{w}_{\left[\frac{n}{q}\right]}^{(j+1)}(\mathfrak{p}\xi),$$

where r is given by

$$r = n - q[\frac{n}{q}]. \tag{3.1}$$

Definition 3.1 The set of functions $\{w_n^{(j)} : n \geq 0\}$ defined as above are said to be wavelet packets associated with the MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $H^s(\mathbb{K})$.

Definition 3.2 For every $n \in \mathbb{N}_0$ and $0 \leq r \leq q-1$, the wavelet packet spaces at jth level are given by

$$W_j^{[\frac{n}{q}],r} = span\{q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\} \cap H^s(\mathbb{K}),$$

where r is given by (3.1).

Definition 3.3 Suppose $w_n^{(j)}(x)$ be a wavelet packet corresponding to the scaling function $\phi^{(j)}(x)$. Then the translates and dilates form of wavelet packet functions for integer j and $k \in \mathbb{N}_0$ are defined as

$$w_{j,k,n}^{(j)}(x) = q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}x - u(k)). \tag{3.2}$$

Proposition 3.4 *Let the unique expansion for an integer $m \geq 1$ in the base q is*

$$m = \lambda_1 + \lambda_2 q + \lambda_3 q^2 + \cdots + \lambda_k q^{k-1} = \sum_{i=1}^{k} \lambda_i q^{i-1}, \tag{3.3}$$

where $\lambda_k \neq 0$ and $0 \leq \lambda_i < q$ for all $i = 1, 2, ..., k$. Then

$$\hat{w}_m^{(j)}(\xi) = \prod_{i=1}^{\infty} m_{\lambda_i}^{(j+i)}(\mathfrak{p}^i \xi)\hat{\gamma}^{-\frac{s}{2}}(\xi)$$

Proof By using the induction hypothesis and Eq. (2.7), it can be easily proved. □

We can view the decomposition process in Fig. 1.

Lemma 3.5 *For $j \in \mathbb{Z}$, let $w_n^{(j+1)} \in H^s(\mathbb{K})$, then the distribution $\left\{q^{\frac{j+1}{2}} w_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - u(k)) : k \in \mathbb{N}_0\right\}$ are orthonormal in $H^s(\mathbb{K})$ if and only if*

$$\sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\xi + u(k)))|\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\xi + u(k))|^2 = 1.$$

Proof Let

$$S^{(j+1)}(\xi) = \sum_{k\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\xi + u(k)))|\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\xi + u(k))|^2.$$

Since $w_n^{(j+1)} \in H^s(\mathbb{K})$, then the above series converges almost everywhere and belongs to $L^1_{Loc}(\mathfrak{D})$.
Moreover, for every $l \in \mathbb{N}_0$, we have

$$\begin{aligned}
\int_{\mathfrak{D}} S^{(j+1)}\bar{\chi}_k(\xi)\chi_l(\xi)d\xi &= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\xi))|\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\xi)|^2\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\xi)q^{-\frac{j+1}{2}}\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}\xi)\bar{\chi}_k(\mathfrak{p}^{j+1}\xi)q^{-\frac{j+1}{2}}\overline{\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}\xi)}\chi_l(\mathfrak{p}^{j+1}\xi)d\xi \\
&= \left\langle q^{\frac{j+1}{2}} w_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - u(k)), q^{\frac{j+1}{2}} w_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}^{j+1}x - u(l))\right\rangle \\
&= 1 \text{ if } l = k \text{ and } S^{(j+1)}(\xi) = 1.
\end{aligned}$$

□

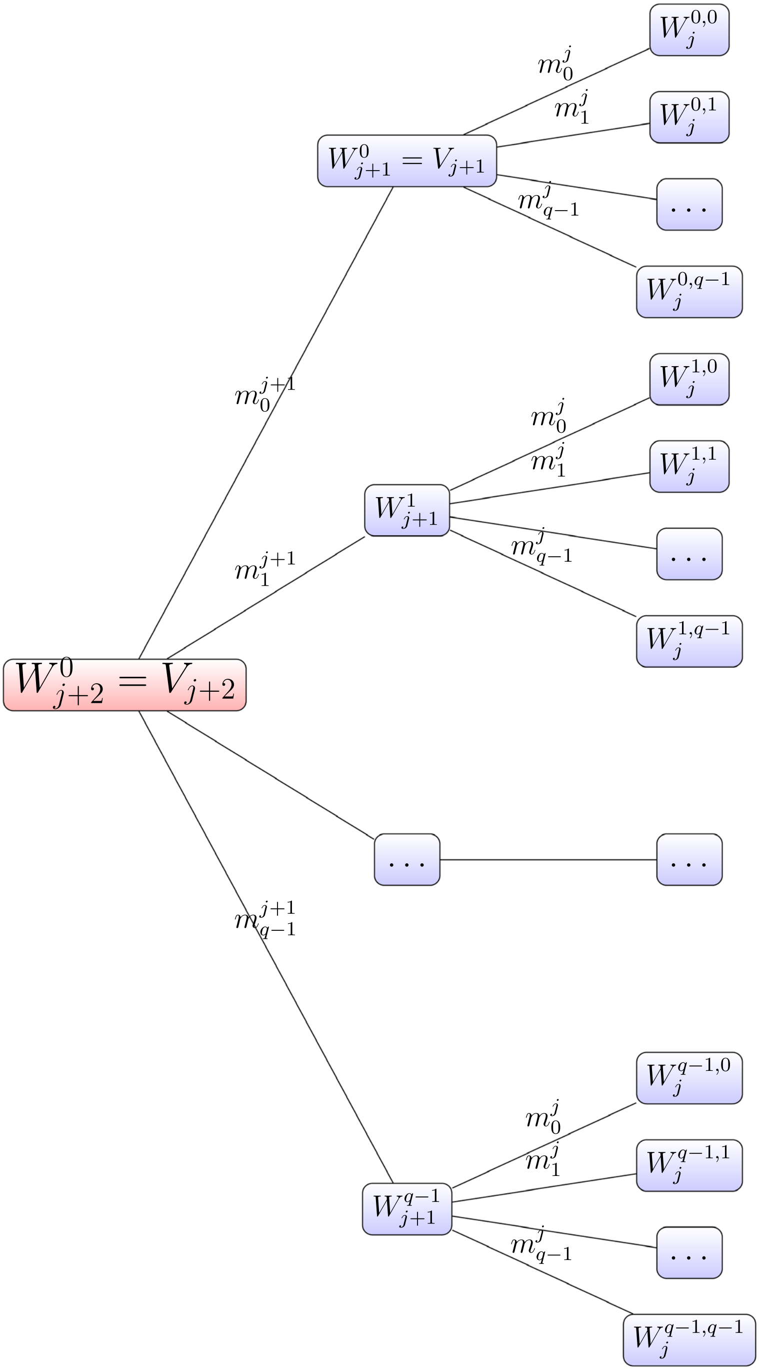

Fig. 1 Two iterations of the decomposition in $H^s(\mathbb{K})$ at $(j+2)^{th}$ level

3.1 Orthogonality of wavelet packets at *j*th level

In the following theorems, we obtain the orthogonality at jth level.

Theorem 3.6 *Let $j \in \mathbb{Z}$ and $k, l, n \in \mathbb{N}_0$. Then*

$$\langle w_{j,k,n}^{(j)}, w_{j,l,n}^{(j)} \rangle = \delta_{k,l}.$$

Proof

$$\begin{aligned}
\left\langle w_{j,k,n}^{(j)}, w_{j,l,n}^{(j)} \right\rangle &= \left\langle q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}. - u(k)), q^{\frac{j}{2}} w_n^{(j)}(\mathfrak{p}^{-j}. - u(l)) \right\rangle \\
&= q^{-j} \int_{\mathbb{K}} \hat{\gamma}^s(\xi) \hat{w}_n^{(j)}(\mathfrak{p}^j \xi) \bar{\chi}_k(\mathfrak{p}^j \xi) \overline{\hat{w}_n^{(j)}(\mathfrak{p}^j \xi)} \chi_l(\mathfrak{p}^j \xi) d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi) \hat{w}_n^{(j)}(\xi) \overline{\hat{w}_n^{(j)}(\xi)} \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi) |\hat{w}_n^{(j)}(\xi)|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi) |m_r^{(j+1)}(\mathfrak{p}\xi) \hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\xi)|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathfrak{D}} \sum_{n \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\xi + u(n)) |m_r^{(j+1)}(\mathfrak{p}(\xi + u(n))|^2 \\
&\quad \times |\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\xi + u(n))|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} \sum_{n \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\xi + u(qn + i)) |m_r^{(j+1)}(\mathfrak{p}(\xi + u(qn + i))|^2 \\
&\quad \times |\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}(\xi + u(qn + i))|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} \sum_{n \in \mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\xi + \mathfrak{p}u(i) + u(n))) |\hat{w}_{[\frac{n}{q}]}^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i) + u(n))|^2 \\
&\quad \times |m_r^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i))|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} |m_r^{(j+1)}(\mathfrak{p}\xi + \mathfrak{p}u(i))|^2 \bar{\chi}_k(\xi) \chi_l(\xi) d\xi \\
&= \int_{\mathfrak{D}} \bar{\chi}_k(\xi) \chi_l(\xi) d\xi = \delta_{k,l}.
\end{aligned}$$

□

Theorem 3.7 *Let $n \in \mathbb{N}_0$ and $1 \leq t \leq q - 1$. Then, we have*

$$\langle w_{j,k,qn}^{(j)}, w_{j,l,t+qn}^{(j)} \rangle = 0.$$

Proof With the help of change of variable trick, we have

$$\begin{aligned}
&\left\langle w_{j,k,qn}^{(j)}, w_{j,l,t+qn}^{(j)} \right\rangle \\
&= \left\langle q^{\frac{j}{2}} w_{qn}^{(j)}(\mathfrak{p}^{-j}. - u(k)), q^{\frac{j}{2}} w_{t+qn}^{(j)}(\mathfrak{p}^{-j}. - u(l)) \right\rangle \\
&= q^{-j} \int_{\mathbb{K}} \hat{\gamma}^s(\xi) \hat{w}_{qn}^{(j)}(\mathfrak{p}^j \xi) \overline{\hat{w}_{t+qn}^{(j)}(\mathfrak{p}^j \xi)} \bar{\chi}_k(\mathfrak{p}^j \xi) \chi_l(\mathfrak{p}^j \xi) d\xi
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi)\hat{w}^{(j)}_{qn}(\xi)\overline{\hat{w}^{(j)}_{t+qn}(\xi)}\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi)m_0^{(j+1)}(\mathfrak{p}\xi)\hat{w}^{(j+1)}_{[\frac{n}{q}]}(\mathfrak{p}\xi)\overline{m_t^{(j+1)}}(\mathfrak{p}\xi)\overline{\hat{w}^{(j+1)}_{[\frac{n}{q}]}}(\mathfrak{p}\xi)\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathbb{K}} \hat{\gamma}^s(\mathfrak{p}^{-j}\xi)|\hat{w}^{(j+1)}_{[\frac{n}{q}]}(\mathfrak{p}\xi)|^2 m_0^{(j+1)}(\mathfrak{p}\xi)\overline{m_t^{(j+1)}}(\mathfrak{p}\xi)\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\xi+u(n)))|\hat{w}^{(j+1)}_{[\frac{n}{q}]}(\mathfrak{p}(\xi+u(n)))|^2 m_0^{(j+1)}(\mathfrak{p}(\xi+u(n))) \\
&\quad \times \overline{m_t^{(j+1)}}(\mathfrak{p}(\xi+u(n)))\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j}(\xi+u(qn+i)))|\hat{w}^{(j+1)}_{[\frac{n}{q}]}(\mathfrak{p}(\xi+u(qn+i)))|^2 m_0^{(j+1)}(\mathfrak{p}(\xi+u(qn+i))) \\
&\quad \times \overline{m_t^{(j+1)}}(\mathfrak{p}(\xi+u(qn+i)))\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1}\sum_{n\in\mathbb{N}_0} \hat{\gamma}^s(\mathfrak{p}^{-j-1}(\mathfrak{p}\xi+u(n)+\mathfrak{p}u(i))|\hat{w}^{(j+1)}_{[\frac{n}{q}]}(\mathfrak{p}\xi+u(n)+\mathfrak{p}u(i)))|^2 \\
&\quad \times m_0^{(j+1)}(\mathfrak{p}\xi+u(n)+\mathfrak{p}u(i))\overline{m_t^{(j+1)}}(\mathfrak{p}\xi+u(n)+\mathfrak{p}u(i))\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= \int_{\mathfrak{D}} \sum_{i=0}^{q-1} m_0^{(j+1)}(\mathfrak{p}\xi+\mathfrak{p}u(i))\overline{m_t^{(j+1)}}(\mathfrak{p}\xi+\mathfrak{p}u(i))\bar{\chi}_k(\xi)\chi_l(\xi)d\xi \\
&= 0.
\end{aligned}
$$

□

3.2 Construction of wavelet packets

Using following proposition and theory of convolution of Fourier transform, we construct orthogonal wavelet packets in $H^s(\mathbb{K})$ at j^{th} level in the other form.

Proposition 3.8 *Consider the functions $\{w_n : n \geq 0\}$ the wavelet packet corresponding to the MRA $\{V_j : j \in \mathbb{Z}\}$ in $L^2(\mathbb{K})$ $j \in \mathbb{Z}$ (for more detail see Ref. [4]). Then*

$$\langle w_{j,k,m}, w_{j,l,n}\rangle_{L^2(\mathbb{K})} = \delta_{m,n}\delta_{k,l}, \tag{3.4}$$

where $w_{j,k,n}(.) = q^{\frac{j}{2}} w_n(\mathfrak{p}^{-j}. - u(k))$, $k \in \mathbb{N}_0$ and $j \in \mathbb{Z}$.

Theorem 3.9 *Suppose $\rho(.) = \gamma^{-\frac{s}{2}}(.)$ and $w_{j,k,n}(.)$ as in above proposition. Then*

$$\langle w^{(j)}_{j,k,n}, w^{(j)}_{j,l,n}\rangle_{H^s(\mathbb{K})} = \delta_{k,l},$$

where $w^{(j)}_{j,k,n}(.) = \rho(.) * w_{j,k,n}(.)$ and $*$ denotes convolution of two functions.

Proof By using the convolution theorem, we have

$$
\begin{aligned}
\langle w^{(j)}_{j,k,n}, w^{(j)}_{j,l,n}\rangle_{H^s(\mathbb{K})} &= \int_{\mathbb{K}} \hat{\gamma}^s(\xi)\hat{\gamma}^{-\frac{s}{2}}(\xi)\hat{w}_{j,k,n}(\xi)\hat{\gamma}^{-\frac{s}{2}}(\xi)\overline{\hat{w}_{j,l,n}(\xi)}d\xi \\
&= \int_{\mathbb{K}} \hat{w}_{j,k,n}(\xi)\overline{\hat{w}_{j,l,n}(\xi)}d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{K}} w_{j,k,n}(x)\overline{w_{j,l,n}(x)}dx \\
&= \delta_{k,l}.
\end{aligned}
$$

□

Example 3.10 In this presented example, we have constructed the orthogonal wavelet packets at jth level by using the above theorem. For this, we need the orthogonal wavelet packets in $L^2(\mathbb{K})$.
We recall the MRA which is given by Jiang et. al. [5], they considered the scaling function $\phi(x) = \eta_{\mathfrak{D}}(x)$, where $\eta_{\mathfrak{D}}$ is a characteristic function on $\mathfrak{D}$. The low-pass filter $m_0(\xi)$ of the MRA is given by the formula:

$$
m_0(\xi) = \frac{1}{q}\sum_{k=0}^{q-1}\overline{\chi_k(\xi)} = \begin{cases} 1, & if\ |\xi| \le q^{-1}, \\ 0, & if\ q^{-1} < |\xi| \le 1, \end{cases}
$$

and high pass filters are given by

$$
m_t(\xi) = \frac{1}{\sqrt{2q}}(\overline{\chi_t(\xi)} - \overline{\chi_{t-1}(\xi)}),\ \ t = 1, 2, ..., q-1,
$$

the associated basic wavelet functions are $\psi_t(x) = \sqrt{\frac{q}{2}}[\eta_{\mathfrak{D}}(\mathfrak{p}^{-1}x - u(t)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-1}x - u(t-1))]$.
Then, the corresponding orthogonal wavelet packets are given by [4]

$$
\begin{aligned}
w_0(x) &= \eta_{\mathfrak{D}}(x), \\
w_t(x) &= \sqrt{\frac{q}{2}}[\eta_{\mathfrak{D}}(\mathfrak{p}^{-1}x - u(t)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-1}x - u(t-1))],\ \text{for}\ 1 \le t \le q-1, \\
w_q(x) &= \sqrt{\frac{q}{2}}\sum_{k=0}^{q-1}[\eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - u(1) - \mathfrak{p}^{-1}u(k)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - u(0) - \mathfrak{p}^{-1}u(k))], \\
w_{q+1}(x) &= \frac{q}{2}[\eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(1) - u(1)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(1)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - u(1)) \\
&\quad + \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x)], \\
w_{q+2}(x) &= \frac{q}{2}[\eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(2) - u(1)) - \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(2)) \\
&\quad - \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(1) - u(1)) + \eta_{\mathfrak{D}}(\mathfrak{p}^{-2}x - \mathfrak{p}^{-1}u(1))],
\end{aligned}
$$

.
.
.

Now, by using Theorem 3.9, we get the wavelet packets in $H^s(\mathbb{K})$ at jth level (Table 1)

Table 1 Basis functions for the wavelet packet decomposition at jth level in $H^s(\mathbb{K})$

Space	Generator of wavelet packets	Formula
$W_j^{0,0}$	$w_0^{(j)}(x) = w_0(\mathfrak{p}^{-j}x) * \gamma^{-\frac{s}{2}}(x)$	$span\{q^{\frac{j}{2}} w_0^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\}$
$W_j^{0,t}$	$w_t^{(j)}(x) = w_t(\mathfrak{p}^{-j}x) * \gamma^{-\frac{s}{2}}(x)$	$span\{q^{\frac{j}{2}} w_t^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\}$
$W_j^{1,0}$	$w_q^{(j)}(x) = w_q(\mathfrak{p}^{-j}x) * \gamma^{-\frac{s}{2}}(x)$	$span\{q^{\frac{j}{2}} w_q^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\}$
$W_j^{1,1}$	$w_{q+1}^{(j)}(x) = w_{q+1}(\mathfrak{p}^{-j}x) * \gamma^{-\frac{s}{2}}(x)$	$span\{q^{\frac{j}{2}} w_{q+1}^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\}$
$W_j^{1,2}$	$w_{q+2}^{(j)}(x) = w_{q+2}(\mathfrak{p}^{-j}x) * \gamma^{-\frac{s}{2}}(x)$	$span\{q^{\frac{j}{2}} w_{q+2}^{(j)}(\mathfrak{p}^{-j}. - u(k)) : k \in \mathbb{Z}\}$
.	.	.
.	.	.
.	.	.

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