Afrika Matematika (2024) 35:53
https://doi.org/10.1007/s13370-024-01194-5

Inverse nodal problem with eigenparameter boundary conditions

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Received: 29 December 2023 / Accepted: 1 May 2024 / Published online: 22 May 2024

Abstract
The reconstruction of potential function using nodal parameters is an inverse problem that has been studied in this work. An efficient and highly helpful transformation allowed for the extraction of a reconstruction formula for the problem's potential function by a narrow collection of nodal data only. Additionally, the method's efficacy was shown by a few numerical illustrations.

Keywords Nodal parameters · Sturm–Liouville equation · Prüfer substitution

Mathematics Subject Classification 34A55 · 34L05 · 34L20

1 Introduction

Consider the following differential equation

$$y'' + (\lambda^2 + \mu - q(x))y = 0, \quad 0 \le x \le \pi, \tag{1.1}$$

where $\mu \neq 1$ is any real number, λ is a spectral parameter and $q(x) \in C^1[0, \pi]$ (see [1]). Let $y(x, \lambda)$ be the eigenfunction of (1.1) with the following conditions

$$y(0, \lambda) = 0, \tag{1.2}$$

$$ay^{'}(\pi, \lambda) + \lambda y(\pi, \lambda) = 0, \tag{1.3}$$

for a real number $a \neq 0$. Here, $0 < x_1^{k,\mu} < \cdots < x_{k-1}^{k,\mu} < \pi$ are nodal points and $l_j^{k,\mu} = x_{j+1}^{k,\mu} - x_j^{k,\mu}$ are nodal length of (1.1)–(1.3). The eigenfunction $y_k(x, \lambda_k^\mu)$ corresponding to λ_k^μ has $k+1$ nodal points in $[0, \pi]$ including 0 and π [2].

It is crucial to understand whether or not the spectrum analysis alerts when the spectral parameter is present in both the boundary conditions and the equation. This eigenvalue problem is not of the usual type. Friedman is the rightful owner of the methodology for

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handling such issues [3]. He looked at the operator formula approach to solving Sturm–Liouville problems with boundary conditions that depend on eigenparameters. Issues with the linear eigenvalue parameter boundary conditions have been thoroughly examined due to the significance of use in physics, probability theory, and other fields. Eigenparameter uses in the physical domain Many authors have taken into consideration dependent Sturm–Liouville problems because of their extensive applications in engineering, mathematical physics, and mechanics [4–10]. As known, Theories that expand on the eigenvector and eigenvalue theory of a single square matrix to a far more comprehensive description of the structure of operators in a range of mathematical spaces are collectively referred to as spectral theory. Inverse and direct problems make up the two categories of problems covered in spectral theory. Direct problems have a long history. The history of inverse problems is relatively closer. Inverse problems have an incredible number of applications in physics and engineering. In the inverse problem involving a differential operator, the aim is to obtain the operator by using some information sets. Spectral parameters such as spectrum, spectral parameter and nodal datas are used to create the operator. The most direct result of these theories is the theory in which nodal datas are used. These kinds of problems are called inverse nodal problems and they have a wide application area. Over the years, this type of problems have been handled by many operators as Sturm–Liouville, Dirac etc. with different boundary conditions [11–19].

Recently, a transformation called the Prüfer has been used to further simplify and make the solution of inverse nodal problems more effective. There are many studies using this transformation and with important results [20, 21].

In this study, we will construct the potential function for the given operator while the nodal datas are already available by using the Prüfer transform effectively. In fact, this problem has been solved by another method in [2, 22]. The strength of the method we use will thus be better seen. Later, we will support our theories and obtained results with some numerical presentations.

2 Asymptotics forms of eigenvalues and nodal parameters

$$\begin{aligned} y(x,\lambda) &= s\left(x\right)\sin\left(\lambda\theta\left(x\right)\right), \\ y'(x,\lambda) &= \lambda s\left(x\right)\cos\left(\lambda\theta\left(x\right)\right), \end{aligned} \tag{2.1}$$

or

$$\frac{y(x,\lambda)}{y'(x,\lambda)} = \frac{1}{\lambda}\tan\left(\lambda\theta(x)\right). \tag{2.2}$$

This substitution represents a legitimate change of variables provided s is never zero. After some straightforward computations with (1.1) and (2.1), we obtain the expression

$$\theta'\left(x,\lambda\right) = 1 - \frac{1}{\lambda^2}\left[q(x)-\mu\right]\sin^2\left(\lambda\theta(x)\right), \tag{2.3}$$

which plays an important role throughout our study.

Theorem 2.1 *The eigenvalues of the problem (1.1)–(1.3) have the following asymptotic expressions*

$$\lambda_k^{\mu} = k - \frac{arctana}{\pi} - \frac{\mu}{2k - \frac{2arctana}{\pi}} + \frac{1}{2k\pi - 2arctana}\int_0^{\pi} q(x)dx + O\left(\frac{1}{k^2}\right), \tag{2.4}$$

as $k \to \infty$.

Proof Let us take $\lambda = \lambda_k^\mu$ as an eigenvalue of the problem (1.1)–(1.3), according to the right endpoint condition, then

$$ay'(\pi, \lambda) + \lambda_k^\mu y(\pi, \lambda) = 0,$$

or equivalently

$$-a = \tan\left(\lambda_k^\mu \theta(\pi, \lambda_k^\mu)\right).$$

It follows that

$$\frac{k\pi - arctana}{\lambda_k^\mu} = \theta(\pi, \lambda_k^\mu).$$

On the other hand, integration of the phase function $\theta'\left(x, \lambda_k^\mu\right)$ in (2.3) with respect to x on $[0, \pi]$ yields

$$\theta(\pi, \lambda_k^\mu) = \pi - \frac{1}{(\lambda_k^\mu)^2}\int_0^\pi (q(x) - \mu)\sin^2\left(\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx.$$

Then, we have

$$\frac{k\pi - arctana}{\lambda_k^\mu} = \pi - \frac{1}{(\lambda_k^\mu)^2}\int_0^\pi q(x)\sin^2\left(\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx + \frac{\mu}{(\lambda_k^\mu)^2}\int_0^\pi \sin^2\left(\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx.$$

Using the identity

$$1 - 2\sin^2\left(\lambda_k^\mu \theta(t)\right) = \cos\left(2\lambda_k^\mu \theta(t)\right) = \frac{1}{2\lambda_k^\mu \theta'(t)}\frac{d}{dt}\left[\sin(2\lambda_k^\mu \theta(t)\right],$$

we get

$$\frac{k\pi - arctana}{\lambda_k^\mu} = \pi + \frac{\mu\pi}{2(\lambda_k^\mu)^2} - \frac{1}{2(\lambda_k^\mu)^2}\int_0^\pi q(x)dx$$
$$+ \frac{1}{(\lambda_k^\mu)^2}\int_0^\pi \frac{q(x) - \mu}{4(\lambda_k^\mu)\theta'}\frac{d}{dx}\left[\sin(2\lambda_k^\mu \theta(x, \lambda_k^\mu))\right] dx.$$

If the second term from the right above is written asymptotically, we get

$$\int_0^\pi \frac{q(x) - \mu}{2\lambda_k^\mu \theta'}\frac{d}{dx}\left[\sin(2\lambda_k^\mu \theta(x, \lambda_k^\mu))\right] dx = O\left(\frac{1}{\lambda_k^\mu}\right),$$

by using integration by parts method where $|k|$ is sufficiently large. After recollecting all terms above, since

$$\lim_{k\to\infty}\frac{\mu}{2k - \frac{2\arctan\alpha}{\pi}} + \lim_{k\to\infty}\frac{1}{2k\pi - 2\arctan\alpha}$$

exists as a finite real number, and $O\left(\frac{1}{k^2}\right) \to 0$, as $k \to \infty$ it can be written as $\lambda_k^\mu \cong k - \frac{\arctan\alpha}{\pi} + o(1)$ by the definition of little o. It yields

$$\lambda_k^\mu = k - \frac{arctana}{\pi} - \frac{\mu}{2k - \frac{2arctana}{\pi}} + \frac{1}{2k\pi - 2arctana}\int_0^\pi q(x)dx + O\left(\frac{1}{k^2}\right),$$

for $k \to \infty$. It completes the proof.

Remark 2.2 For $q(x) \geq \mu$, the eigenvalues $\{\lambda_k^\mu\}$of the problem (1.1)–(1.3) are all real for sufficiently large $|k|$.

Proof Let λ be an eigenvalue of the problem (1.1)–(1.3) corresponding to the eigenfunction $y(x)$. Multiplying both sides of the Eq. (1.1) $\overline{y}(x)$ and integrating the obtained result $[0, \pi]$ with respect to x yields

$$-\int_0^\pi y''(x)\overline{y}(x)dx + \int_0^\pi (q(x) - \mu)\,|y(x)|^2\,dx = \lambda^2 \int_0^\pi |y(x)|^2\,dx.$$

By applying integration by parts method, we get

$$y'(0)\overline{y}(0) - y'(\pi)\overline{y}(\pi) + \int_0^\pi \left[\left|y'(x)\right|^2 + (q(x) - \mu)\,|y(x)|^2\right] dx = \lambda^2 \int_0^\pi |y(x)|^2\,dx.$$

On the other hands, we conclude that

$$y'(0)\overline{y}(0) - y'(\pi)\overline{y}(\pi) = \frac{\lambda |y(\pi)|^2}{\alpha},$$

by (1.2) and (1.3). Considering last relations together, it yields

$$A\lambda^2 - B\lambda - C = 0,$$

where $A = \int_0^\pi |y(x)|^2\,dx$, $B = \frac{|y(\pi)|^2}{\alpha}$, $C = \int_0^\pi \left[\left|y'(x)\right|^2 + (q(x) - \mu)\,|y(x)|^2\right] dx$. Since

$$\lambda = \frac{B \pm \sqrt{B^2 + 4AC}}{2A},$$

and $A, C > 0$ if $q(x) \geq \mu$ on $[0, \pi]$, λ is real. It completes the proof.

Theorem 2.3 *Asymptotic formulae of the nodal points for the problem* (1.1)–(1.3) *are as follows*

$$x_j^{k,\mu} = \frac{j\pi^2}{(k\pi - arctana)\,A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2\,A_k} \int_0^{x_j^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right), \qquad (2.5)$$

where $k \neq \frac{arctana}{\pi}$, $A_k = \frac{\mu}{2(\lambda_k^\mu)^2} + 1$, *as* $k \to \infty$.

Proof By integrating $\theta'(x, \lambda)$ on $[0, x_j^{k,\mu}]$ and using the fact $\theta(x_j^{k,\mu}) = \dfrac{j\pi}{\lambda_k^\mu}$, we may easily conclude that

$$\frac{j\pi}{\lambda_k^\mu} = A_k x_j^{k,\mu} - \frac{1}{\left(\lambda_k^\mu\right)^2} \int_0^{x_j^{k,\mu}} q(x)\sin^2\left(\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx - \frac{\mu}{2\left(\lambda_k^\mu\right)^2} \int_0^{x_j^{k,\mu}} \cos\left(2\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx,$$

or

$$x_j^{k,\mu} = \frac{j\pi}{\lambda_k^\mu A_k} + \frac{1}{\left(\lambda_k^\mu\right)^2 A_k} \int_0^{x_j^{k,\mu}} q(x) \sin^2\left(\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx$$
$$+ \frac{\mu}{2\left(\lambda_k^\mu\right)^2 A_k} \int_0^{x_j^{k,\mu}} \cos\left(2\lambda_k^\mu \theta(x, \lambda_k^\mu)\right) dx. \tag{2.6}$$

Equation (2.6) may be also written as

$$x_j^{k,\mu} = \frac{j\pi}{\lambda_k^\mu A_k} + \frac{1}{2\left(\lambda_k^\mu\right)^2 A_k} \int_0^{x_j^{k,\mu}} q(x)dx - \frac{1}{2\left(\lambda_k^\mu\right)^2 A_k} \int_0^{x_j^{k,\mu}} (q(x) - \mu)\cos(2\lambda_k^\mu \theta(x, \lambda_k^\mu))dx. \tag{2.7}$$

The last term is attempted to add on on right side asymptotically. Then by integration by parts;

$$\frac{1}{2\left(\lambda_k^\mu\right)^2 A_k} \int_0^{x_j^{k,\mu}} (q(x) - \mu)\cos(2\lambda_k^\mu \theta(x))dx = O\left(\frac{1}{\left(\lambda_k^\mu\right)^3}\right).$$

Therefore, If the above asymptotic expression is used in (2.7) for $k \to \infty$, and the relation $\lambda_k^\mu \cong k - \frac{arctana}{\pi} + o(1)$, it yields

$$x_j^{k,\mu} = \frac{j\pi^2}{(k\pi - arctana)\, A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2 A_k} \int_0^{x_j^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right),$$

which completes the proof. it follows that the set of all nodal points is dense in $(0, \pi)$.

Remark 2.4 Let a dense subset X of the nodal points be given. Then, $x_{j_k(x)}^{k,\mu} \to x$ for $k \to \infty$ where $j = j_k(x) = \max\left\{j : x_j^{k,\mu} < x\right\}$.

Theorem 2.5 *The asymptotic formulae of the nodal lengths for the problem (1.1)–(1.2) satisfy the below asymptotic expression*

$$l_j^{k,\mu} = \frac{\pi^2}{(k\pi - arctana)\, A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2 A_k} \int_{x_j^{k,\mu}}^{x_{j+1}^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right), \tag{2.8}$$

as $k \to \infty$.

Proof By the notion of nodal length distance between two consecutive nodal points- for

$$x_{j+1}^{k,\mu} = \frac{(j+1)\,\pi^2}{(k\pi - arctana)\,A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2\,A_k}\int_0^{x_{j+1}^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right)$$

$$x_{j}^{k,\mu} = \frac{j\pi^2}{(k\pi - arctana)\,A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2\,A_k}\int_0^{x_{j}^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right)$$

we obtain

$$l_j^{k,\mu} = \frac{\pi^2}{(k\pi - arctana)\,A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2\,A_k}\int_{x_j^{k,\mu}}^{x_{j+1}^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right).$$

3 Main results

This section is devoted to an asymptotic expresison for $q(x)$ for the problem (1.1)–(1.3) by using nodal parameters. Actually, this method has been used in many problems. However, the spectral parameter μ included in this equation and λ in boundary conditions distinguish the proof from other consequences.

Theorem 3.1 *Let $q \in C^1[0,\pi]$ be a function defined on interval $0 \le x \le \pi$. Then*

$$q(x) = \lim_{k\to\infty} 2\,(k\pi - arctana)\left(\frac{(k\pi - arctana)\,A_k}{\pi^2} - \frac{1}{l_j^{k,\mu}}\right), \tag{3.1}$$

for almost every $x \in (0,\pi)$ and $j = j_k(x)$.

Proof By making some straightforward computations in (2.8), we get

$$l_j^{k,\mu} = \frac{\pi^2}{(k\pi - arctana)\,A_k} + \frac{\pi^2}{2\,(k\pi - arctana)^2\,A_k}\int_{x_j^{k,\mu}}^{x_{j+1}^{k,\mu}} q(x)dx + O\left(\frac{1}{k^3}\right)$$

and applying the mean value theorem for integration, with fixed k, there exists a number $z \in \left(x_j^{k,\mu}, x_{j+1}^{k,\mu}\right)$, and the following is obtained

$$l_j^{k,\mu} q(z) = 2\left(l_j^{k,\mu} - \frac{\pi^2}{(k\pi - arctana)\,A_k}\right)\frac{(k\pi - arctana)^2\,A_k}{\pi^2} + O\left(\frac{1}{k^3}\right).$$

For sufficiently large values of k, we get

$$q(x) = \lim_{k\to\infty} 2\,(k\pi - arctana)\left(\frac{(k\pi - arctana)\,A_k}{\pi^2} - \frac{1}{l_j^{k,\mu}}\right).$$

This completes the proof. Now, let's give some numerical conclusions to embody the eigenvalues and nodal parameter concepts for the problem (1.1)–(1.3). Let us consider (1.1)–(1.3)

Table 1 The eigenvalues λ_k^{μ}; $k = \overline{1, 10}$ of Sturm–Liouville problem for $a = 1$, $\mu = 2$ and $q(x) = x^2$

λ_1^2	λ_2^2	λ_3^2	λ_4^2	λ_5^2	λ_6^2	λ_7^2	λ_8^2	λ_9^2	λ_{10}^2
1.60991	2.11853	2.98452	3.92198	4.88578	5.86216	6.84555	7.83322	8.82371	9.81615

Table 2 The nodal points $x_j^{k,\mu}$; $j, k = \overline{1, 10}$ of Sturm–Liouville problem for $a = 1$, $\mu = 2$ and $q(x) = x^2$

$x_j^{k,2}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
$j = 1$	155.107	1.65345	1.04993	0.792148	0.636624	0.531651	0.456036	0.39904	0.354582	0.318961
$j = 2$	32339.8	5.92077	2.27271	1.62025	1.2846	1.06779	0.914129	0.799126	0.709741	0.638261
$j = 3$	1.02247×10^6	29.0019	4.05344	2.53501	1.95723	1.61328	1.37643	1.20133	1.06606	0.958242
$j = 4$	1.2678×10^7	152.653	7.37327	3.6208	2.67191	2.17375	1.84526	1.60679	1.42415	1.27925
$j = 5$	9.13303×10^7	698.032	14.6071	5.02349	3.4528	2.75602	2.32326	2.0167	1.78464	1.60165
$j = 6$	4.62629×10^8	2695.1	31.1206	6.99329	4.33423	3.36863	2.81343	2.43241	2.14821	1.92582
$j = 7$	1.83189×10^9	8956.62	68.2637	9.95072	5.36516	4.0224	3.31931	2.85537	2.51556	2.25214
$j = 8$	6.04873×10^9	26246.1	148.646	14.586	6.61508	4.73118	3.84508	3.28725	2.88749	2.58104
$j = 9$	1.73726×10^{10}	69254.7	314.927	22.0059	8.18192	5.51272	4.39574	3.72991	3.26485	2.91296
$j = 10$	4.46814×10^{10}	167372	643.798	33.9456	10.2023	6.38968	4.97727	4.1855	3.64857	3.24836

for some constant values $a = 1$, $\mu = 2$. Tables 1, 2, and 3 show the behavior of nodal lengths, nodal positions, and eigenvalues, when $q(x) = x^2$, respectively.

Table 1 shows that the sequence of eigenvalues increases as n expands. This aligns with the overall theoretical framework.

Table 2 shows that the nodal points oscillate within the studied range as the value of k rises. As a result, the issue is steady and the conclusions are clear and precise.

The numerical findings in this particular situation under discussion demonstrate the applicability of the found formulas for fundamental theorems.

4 Conclusion

The Prüfer transform is used in this study to solve the inverse nodal problem for the Sturm–Liouville equation, which incorporates parameters in both the independent and boundary conditions. In the qualitative theory of second order Sturm–Liouville differential equations, this transformation is a helpful tool. The advantages of this will become clear eventually, but to give you a rough idea, it makes zero counting incredibly effective. This transformation is a very efficient procedure, as demonstrated by a few numerical results.

Table 3 The nodal lengths $l_j^{k,\mu}$; $j, k = \overline{1, 10}$ of Sturm–Liouville problem for $a = 1$, $\mu = 2$ and $q(x) = x^2$

$l_j^{k,2}$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$	$k = 9$	$k = 10$
$j = 1$	7.23151×10^{12}	10.5043	1.23675	0.828422	0.647995	0.536142	0.458093	0.400086	0.355159	0.3193
$j = 2$	2.28535×10^{17}	1077.89	2.1141	0.920574	0.672923	0.545519	0.462301	0.402207	0.356322	0.319981
$j = 3$	4.35448×10^{20}	157235	7.64992	1.13359	0.716877	0.56066	0.46886	0.405459	0.35809	0.321011
$j = 4$	1.62442×10^{23}	1.49787×10^{7}	54.8393	1.6691	0.791402	0.583144	0.478106	0.409935	0.360494	0.322402
$j = 5$	2.10068×10^{25}	8.5611×10^{8}	536.469	3.18195	0.920215	0.615619	0.490543	0.415766	0.363577	0.324169
$j = 6$	1.29318×10^{27}	3.11065×10^{10}	5706.8	7.94522	1.15246	0.662462	0.506921	0.423136	0.367393	0.326333
$j = 7$	4.60018×10^{28}	7.72676×10^{11}	58775.6	24.3554	1.59218	0.730878	0.528319	0.432291	0.372016	0.328919
$j = 8$	1.0737×10^{30}	1.39783×10^{13}	553802	84.8434	2.46581	0.832796	0.556298	0.443561	0.377539	0.331959
$j = 9$	1.7951×10^{31}	1.9389×10^{14}	4.66831×10^{6}	317.491	4.28047	0.988165	0.593108	0.457384	0.38408	0.335492
$j = 10$	2.2898×10^{32}	2.14875×10^{15}	$3.50417\text{x}\ 10^{7}$	1228.61	8.20257	1.23075	0.641995	0.474335	0.391784	0.339564

Acknowledgements Not applicable.

Author Contributions Theoretical part has written by the second author and numerical part has written by first author.

Funding Not applicable.

Data availability Data usage is not applicable to this article as no data were created or analysed in this study. The study is directly on inverse nodal problem and reconstructed of potential function.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Ethics approval Not applicable.

Consent to participate Not applicable.

Consent for publication Not applicable.

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