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A survey on topological structures on fuzzy rough sets

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Abstract
Fuzzy rough set theory gives a mathematical tool for studying unsettled knowledge that is beclouded, inexact, and mutually exclusive. The perception and conclusions of fuzzy rough sets theory are inextricably linked to topological perception. The topological appearance and its applications in fuzzy rough sets theory have been extensively discussed by researchers. The underlying subordinate of topology and classic fuzzy rough sets theory, as well as the expressive work done in this area over the previous years, are highlighted in this research.

Keywords Rough sets · Fuzzy sets · Fuzzy rough sets · Approximation space

Mathematics Subject Classification 54A40 · 06B23 · 03E72

1 Introduction

The theory of fuzzy sets was first introduced by Zadeh [76] in 1965 to study unclear boundaries of sets. After that, many studies attempted to generalize the fuzzy set by using various approaches. In [35], Pawlak introduced the concept of rough sets. This new mathematical tool for data reasoning has benefited machine learning, intelligent systems with insufficient and incomplete information, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields [36–39]. An approximation space is the underlying structure of rough set theory. Lower and upper approximations can be induced based on it. In 1990, Dubois and Prade [12] developed the concepts of rough fuzzy sets and fuzzy rough sets, based on approximations of fuzzy sets by crisp approximation spaces and crisp sets by fuzzy approximation spaces; and pointed out that a rough fuzzy set is a particular instance of a fuzzy rough set. Many researchers have since developed other fuzzy set generalizations of rough approximations (see [10, 12, 33, 46, 52, 53, 57, 58, 66]). The most frequent fuzzy rough set is found by replacing crisp relations on the universe with fuzzy relations and crisp subsets with fuzzy sets. As

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in classical examples, there are two techniques for developing fuzzy rough set theory. One is the constructive technique, which involves constructing lower and upper approximation operators from fuzzy relations. The alternative is the axiomatic method, which requires that the lower and upper approximation operators fulfill the same set of axioms as the created ones. Morsi and Yakout [31] provided axioms for the fuzzy rough set model. Wu et al. [66] proposed a broad framework for studying fuzzy rough sets that include both constructive and axiomatic approaches. Yeung et al. [71] looked into the generalization of fuzzy rough sets and discovered a relation between fuzzy preorders and saturated Kuratowski fuzzy closure operators that met an additional criterion. Over fuzzy lattices, Liu [28] proposed the concept of generalized rough sets. The crisp [37, 38] and fuzzy generalizations [12] of rough sets can be put into one framework using this approach.

Atanassov [2] introduced the idea of intuitionistic fuzzy sets. In Sostak's sense, Coker and Demirci [8] defined intuitionistic fuzzy topological spaces as a generalization of smooth and intuitionistic fuzzy topological spaces. Samanta and Mondal [48] proposed the idea of intuitionistic fuzzy rough sets by combining the concepts of fuzzy rough sets and intuitionistic fuzzy sets. Zhou et al. [80, 81] defined the lower and upper approximations of an intuitionistic fuzzy approximation space. Many researchers [68, 73–75], have investigated several key characteristics of intuitionistic fuzzy approximation operators. Working under the by name "intuitionistic" raised questions about the applicability of the term, particularly when dealing with a complete lattice L. These questions were answered in 2005 by Gutierrez Garcia, and Rodabaugh [15]. They demonstrated that this word is inappropriate for use in mathematics and applications. They came to the conclusion that the term "L-double" should be used in place of the word "intuitionistic". As a generalization of intuitionistic fuzzy topology and L-fuzzy topology, Samanta and Mondal [49] presented the concept of intuitionistic gradation of openness (L-double fuzzy topology).

Later several other generalizations of fuzzy sets were further generalized in the framework of fuzzy rough sets. For example, multi-fuzzy rough sets [63] generalize multi-fuzzy sets [50, 51], hesitant fuzzy rough sets [6, 11, 13, 27, 69] generalize hesistant fuzzy sets [61, 62]. In this paper, we take a survey of topological study on various models of fuzzy rough sets. The paper collects fuzzy rough set models (their definitions and properties) and the topological structures studied on them.

The following is the format of the paper: Sect. 2 collects some basic definitions and properties of topological spaces, rough set theory and fuzzy set theory. Section 3 brings together research on topological theory generated in the framework of fuzzy rough sets, as well as its features. In Sect. 4, we study the topological structures on fuzzy rough sets and fuzzy topologies generated by them. On fuzzy rough sets, there is a relation between fuzzy preorders and fuzzy topologies. In Sect. 5, we collect the work done on intuitionistic fuzzy rough sets, which are derived from intuitionistic fuzzy approximation spaces and study their intuitionistic fuzzy topological spaces. In Sect. 6, we review the ideas of L-double fuzzy rough sets and their topological features, as well as L-double fuzzy topology derived from L-double fuzzy approximation operators. In Sect. 7, we look at some topological theories on multi-fuzzy rough sets. Section 8 discusses hesistant fuzzy topological spaces and its basic characterization in terms of fuzzy rough sets. Finally in last section (Sect. 9), we conclude the survey.

2 Preliminaries and basic results

Throughout this paper, X refers to a non-empty set known as the universal set; Θ refers to an equivalence relation on X; $P(X)$ refers to the power set of X; Λ is an arbitrary index set, and X^c refers to the complement of a set X. The essential concepts, properties, and operations of topology, rough set theory, and fuzzy set theory are collected in this section.

2.1 Topological spaces

Definition 2.1 [21, 53] Let X be a non-empty set. Then a collection τ of subsets of X is called a topology for X, if τ satisfies the following axioms:

(1) $\emptyset$ and X are in τ,

(2) if $\{G_\alpha : \alpha \in \Lambda\}$ is an arbitrary collection of sets in τ, then $\cup\{G_\alpha : \alpha \in \Lambda\}$ is in τ,

(3) if G_1 and G_2 are two sets in τ, then $G_1 \cap G_2$ is in τ.

The pair (X, τ) is called a topological space.

In a topological space (X, τ), the members of τ are called open sets of X. The compliments of open sets are closed sets of X.

Definition 2.2 [53] Let (X, τ) be a topological space and let $A \subseteq X$. Then the interior of A is the union of all open subsets of A and is denoted by $int_\tau(A)$ or A^o, i.e.,

$$int_\tau(A) = A^o = \bigcup\{G \in \tau : G \subseteq A \text{ and G is an open set}\}.$$

Definition 2.3 [53] Let (X, τ) be a topological space and let $A \subseteq X$. Then the intersection of all closed supersets of A is called the closure of A and is denoted by $\overline{A}$ or $cl_\tau(A)$, i.e.,

$$cl_\tau(A) = \overline{A} = \bigcap\{F \subseteq X : A \subseteq F \text{ and F is a closed set in } \tau\}.$$

2.2 Rough set theory

[53]. Let $A \subseteq X$ and Θ be an equivalence relation on X. The lower approximation of the set A with respect to Θ is the set of all objects which can be certainly classified as A with respect to Θ and denoted by $\underline{\Theta}(A)$ is

$$\underline{\Theta}(A) = \{x \in X : [x]_\Theta \subseteq A\},$$

where $[x]_\Theta$ denotes the equivalence class of x with respect to the relation Θ.

The upper approximation of the set X with respect to Θ is the set of all objects which can be possibly classified with respect to Θ and is denoted by $\overline{\Theta}(A)$.

$$\overline{\Theta}(A) = \{x \in X : [x]_\Theta \cap A \neq \emptyset\}.$$

The boundary region of A is denoted as $BN_\Theta(A) = \overline{\Theta}(A) - \underline{\Theta}(A)$. A set A is crisp if the boundary region of A is empty, $i.e$, $BN_\Theta(A) = \emptyset$, otherwise the set A is rough.

Rough sets can be also expressed by a rough membership function, namely

$$\nu_A(x) = \frac{|[x]_\Theta \cap A|}{|[x]_\Theta|}, \quad x \in X.$$

The membership function is a type of fuzzy set, and its value may be defined as the degree of certainty with which x belongs to A.

Properties of approximation operators:
Let $A, B \subseteq X$. Then

(1) $\underline{\Theta}(A) \subseteq A \subseteq \overline{\Theta}(A)$,
(2) $\underline{\Theta}(\emptyset) = \overline{\Theta}(\emptyset) = \emptyset\,;\ \ \underline{\Theta}(X) = \overline{\Theta}(X) = X$,
(3) $\overline{\Theta}(A \cup B) = \overline{\Theta}(A) \cup \overline{\Theta}(B)$,
(4) $\underline{\Theta}(A \cap B) = \underline{\Theta}(A) \cap \underline{\Theta}(B)$,
(5) $\underline{\Theta}(A \cup B) \supseteq \underline{\Theta}(A) \cup \underline{\Theta}(B)$,
(6) $\overline{\Theta}(A \cap B) \subseteq \overline{\Theta}(A) \cap \overline{\Theta}(B)$,
(7) $A \subseteq B \ \Rightarrow \underline{\Theta}(A) \subseteq \underline{\Theta}(B)$ and $\overline{\Theta}(A) \subseteq \overline{\Theta}(B)$,
(8) $\underline{\Theta}(A^c) = (\overline{\Theta}(A))^c$ and $\overline{\Theta}(A^c) = (\underline{\Theta}(A))^c$,
(9) $\underline{\Theta}(\underline{\Theta}(A)) = \underline{\Theta}(A)$ and $\overline{\Theta}(\overline{\Theta}(A)) = \overline{\Theta}(A)$,
(10) $\underline{\Theta}(\overline{\Theta}(A)) = \overline{\Theta}(A)$ and $\overline{\Theta}(\underline{\Theta}(A)) = \underline{\Theta}(A)$,
(11) $\underline{\Theta}(A) \subseteq \underline{\Theta}(\underline{\Theta}(A))$, $\overline{\Theta}(\overline{\Theta}(A)) \subseteq \overline{\Theta}(A)$, $\overline{\Theta}(A) \subseteq \underline{\Theta}(\overline{\Theta}(A))$,
$\overline{\Theta}(\underline{\Theta}(A)) \subseteq \underline{\Theta}(A)$, and $A \subseteq \underline{\Theta}(\overline{\Theta}(A))$.

2.3 Fuzzy set theory

Fuzzy sets were first introduced by Zadeh [76] as an extension of crisp sets (classical sets). A fuzzy set is a technique to express the sets whose boundaries are not well defined. Fuzzy sets also use a several (or infinite) valued membership function, as compared to classical set theory, which uses a two-valued membership function ($i.e.$, an element is either in a set or it is not). For lattice theory, we refer [72].

Definition 2.4 [72] Let X be a universal set. A fuzzy set F in X is characterized by its membership function denoted by ν_F; that is $\nu_F : X \to [0, 1]$ and $\nu_F(x)$ is called membership grade of x in fuzzy set F for each $x \in X$. We write a fuzzy set F as

$$F = \{(x, \nu_F(x)) : x \in X\}.$$

Here, each pair $(x, \nu_F(x))$ is called a singleton.

If $X = \{x_1, x_2, x_3, \cdots, x_n\}$ is a finite set and F is a fuzzy set in X, then we often use the notation

$$F = \sum_{i=1}^{n} \frac{\nu_F(x_i)}{x_i} = \frac{\nu_F(x_1)}{x_1} + \frac{\nu_F(x_2)}{x_2} + \frac{\nu_F(x_3)}{x_3} + \cdots + \frac{\nu_F(x_n)}{x_n},$$

where the term $\dfrac{\nu_F(x_i)}{x_i}$, $\ i = 1, 2, 3, \ldots, n$, signifies that $\nu_F(x_i)$ is the membership grade of x_i in the fuzzy set F and the plus sign represents the union. Fuzzy sets of this type used in the literature, are called ordinary fuzzy sets.

Definition 2.5 [33] Let X be a set and L be a lattice. An L-fuzzy set F in X is defined by a membership function $\nu_F : X \to L$ that assigns a 'grade' or 'degree of membership' $\nu_F(x) \in L$ with each point $x \in X$.
In particular, L could the closed interval $I = [0, 1]$.

Let F and G be fuzzy sets in X. Then

(1) $F = G$ if and only if $\nu_F(x) = \nu_G(x)$, for all $x \in X$,
(2) $F \subseteq G$ if and only if $\nu_F(x) \leq \nu_G(x)$, for all $x \in X$,
(3) $E = F \cup G$ if and only if $\nu_E(x) = \max[\nu_F(x), \nu_G(x)]$, $x \in X$,
(4) $H = F \cap G$ if and only if $\nu_H(x) = \min[\nu_F(x), \nu_G(x)]$, $x \in X$,
(5) the complement of F, denoted by F^c, is defined by
$\nu_{F^c}(x) = 1 - \nu_F(x)$, $x \in X$.

The family of all fuzzy sets on X is represented by $\mathbb{F}(X)$. Let $\alpha \in [0, 1]$. If $\nu_F(x) = \alpha$, for every $x \in X$, then the fuzzy set $F \in \mathbb{F}(X)$ is constant and is written by α_X. Moreover, x_α is said to be a fuzzy point if, for every x in X, $\nu_{x_\alpha}(y) = \alpha$, if $x = y$ and $\nu_{x_\alpha}(y) = 0$, if $y \neq x$, where the point x is called support and α is called its height [72]. Denote by $\mathcal{F}(X) = \{x_\alpha : x \in X,\ \alpha \in (0, 1]\}$.

Definition 2.6 [5, 25] A collection $\eta \subseteq \mathbb{F}(X)$ is said to be a fuzzy topology, if it satisfies the following conditions:

(1) $\alpha_X \in \eta, \quad \alpha \in [0, 1]$,
(2) $F \cap G \in \eta, \quad F, G \in \eta$,
(3) $\bigcup_{i \in \Lambda} F_i \in \eta, \quad \{F_i\}_{i \in \Lambda} \subseteq \eta$.

Further, if η additionally satisfies

(4) $\bigcap_{i \in \Lambda} F_i \in \eta, \quad \{F_i\}_{i \in \Lambda} \subseteq \eta$,

then η is said to be an Alexandrov fuzzy topology on X.

Definition 2.7 [54] Let (X, η) be a fuzzy topological space. Then

(1) $\overline{\xi}(x_\alpha) = \{F \in \mathbb{F}(X) :$ F is closed remote neighbourhood of $x_\alpha\}$,
(2) $(X, \mathcal{F})$ is called T_{-1}, if for any $x_\alpha, x_\beta \in \mathcal{F}(X)$ and $\beta < \alpha$ there exists $F \in \overline{\xi}(x_\alpha)$ such that $x_\beta \in F$, or, there exists $G \in \overline{\xi}(x_\beta)$ such that $x_\alpha \in G$,
(3) $(X, \mathcal{F})$ is sub-T_0, if for any $x, y \in X$ and $x \neq y$, there exist $\alpha \in (0, 1]$ and $F \in \overline{\xi}(x_\alpha)$ such that $y_\alpha \in F$, or, there exist $\alpha \in (0, 1]$ and $G \in \overline{\xi}(y_\alpha)$ such that $x_\alpha \in G$.

Definition 2.8 [56] A Kuratowski fuzzy closure operator on X is a function $cl : \mathbb{F}(X) \to \mathbb{F}(X)$ such that for every $\alpha \in I,\ F, G \in \mathbb{F}(X)$,

(1) $cl(\alpha_X) = \alpha_X$,
(2) $F \leq cl(F)$,
(3) $cl(F \vee G) = cl(F) \vee cl(G)$ and
(4) $cl(cl(F)) = cl(F)$.

A Kuratowski fuzzy closure operator cl on X is said to be saturated if and only if for every $F_i \in \mathbb{F}(X), i \in \Lambda,\ cl(\bigvee\{F_i : i \in \Lambda\}) = \bigvee\{cl(F_i) : i \in \Lambda\}$.

Every Kuratowski fuzzy closure operator cl on X induces a fuzzy topology on X in which a fuzzy set F is closed if and only if $cl(F) = F$.

2.4 Fuzzy relation

Definition 2.9 [65]

(1) A fuzzy set $\mathcal{F} \in \mathbb{F}(X \times Y)$ is called a fuzzy relation from X to Y.
(2) If $\bigvee_{y \in Y} \mathcal{F}(x, y) = 1$ for every $x \in X$, then $\mathcal{F}$ is said to be serial.
(3) If there exists $y \in Y$ such that $\mathcal{F}(x, y) = 1$ for every $x \in X$, then $\mathcal{F}$ is said to be strongly serial.

If $X = Y$, then $\mathcal{F}$ is said to be fuzzy relation on X. For each fuzzy relation $\mathcal{F}$ on X, a fuzzy relation $\mathcal{F}^{-1}$ is defined as $\mathcal{F}^{-1}(x, y) = \mathcal{F}(y, x)$, for every $x, y \in X$. A fuzzy relation $\mathcal{F}$ is said to be inverse serial if for every $x \in X$ there exists a $y \in X$ such that $\mathcal{F}(y, x) = 1$.

Definition 2.10 [77] Consider the fuzzy relation $\mathcal{F}$ on X.

(1) If $\mathcal{F}(x, x) = 1, \ \ x \in X$, then $\mathcal{F}$ is reflexive.
(2) If $\mathcal{F}(x, y) = \mathcal{F}(y, x), \ \ x, y \in X$, then $\mathcal{F}$ is symmetric.
(3) If $\mathcal{T}(\mathcal{F}(x, y), \mathcal{F}(y, z)) \leq \mathcal{F}(x, z), \ \ x, y, z \in X$, then $\mathcal{F}$ is $\mathcal{T}$-transitive.
(4) If $\mathcal{F}(x, x) \neq 1, \ \ x \in X$, then $\mathcal{F}$ is irreflexive.
(5) If $\mathcal{F}(x, x) = 0, \ \ x \in X$, then $\mathcal{F}$ is antireflexive.
(6) If $\min\{\mathcal{F}(x, y), \mathcal{F}(y, x)\} = 0, \ \ (x, y) \in X \times X$, then $\mathcal{F}$ is asymmetric.
(7) If $\min\{\mathcal{F}(x, y), \mathcal{F}(y, x)\} = 0, \ \ (x, y) \in X \times X$, such that $x \neq y$, then $\mathcal{F}$ is antisymmetric.
(8) If $\max\{\mathcal{F}(x, y), \mathcal{F}(y, z)\} \geq \mathcal{F}(x, z), \ \ x, y, z \in X$, then $\mathcal{F}$ is negatively transitive.
(9) If $\max\{\mathcal{F}(x, y), \mathcal{F}(y, x)\} = 1, \ \ x, y \in X$, then $\mathcal{F}$ is total.
(10) If $\max\{\mathcal{F}(x, y), \mathcal{F}(y, x)\} = 1, \ \ (x, y) \in X \times X$, such that $x \neq y$, then $\mathcal{F}$ is connecting.
(11) If $\mathcal{F}$ is reflexive, transitive and antisymmetric, then $\mathcal{F}$ is a fuzzy partial order relation.

In short, if $\mathcal{T} = \wedge$, then $\mathcal{T}$-transitive is said to be transitive. If a fuzzy relation $\mathcal{F}$ is reflexive and symmetric, it is said to be a fuzzy tolerance relation. Furthermore, $\mathcal{F}$ is said to be $\mathcal{T}$-fuzzy preorder, if it is reflexive and $\mathcal{T}$-transitive. A fuzzy $\mathcal{T}$-equivalence relation is reflexive, symmetric, and $\mathcal{T}$-transitive. Similarly, if a fuzzy relation $\mathcal{F}$ is both reflexive and transitive, then it is known to as a fuzzy preorder. Moreover, a fuzzy relation is a fuzzy equivalence relation, if it is reflexive, symmetric and transitive.

Definition 2.11 [14] A fuzzy relation $\mathcal{F}$ on a set X is said to be fuzzy interval order if

(1) $\max\{\mathcal{F}(x, y), \mathcal{F}(y, x)\} = 1$, for all $x, y \in X$,
(2) $\min\{\mathcal{F}(x, y), \mathcal{F}(z, w)\} \leq \max\{\mathcal{F}(x, w), \mathcal{F}(z, y)\}$, for all $x, y, z, w \in X$.

A fuzzy relation on X is called a fuzzy semiorder if

(3) it is a fuzzy interval order,
(4) $\max\{\mathcal{F}(x, z), \mathcal{F}(z, w)\} \geq \min\{\mathcal{F}(x, y), \mathcal{F}(y, w)\}$, for all $x, y, z, w \in X$.

Definition 2.12 [54] Let $\mathcal{F}$ be a fuzzy relation on X. Then $\mathcal{F}$ is said to as a pseudo constant if there exists $\alpha \in I$ such that for each $x, y \in X$,

$$\mathcal{F}(x, y) = \begin{cases} 1, & if \quad x = y \\ \alpha, & if \quad x \neq y \,. \end{cases}$$

Every pseudo constant fuzzy relation is obviously a fuzzy equivalence relation.

3 Fuzzy rough sets

In 1990, Dubois and Prade [12] were the first to develop fuzzy rough sets by replacing fuzzy binary relations for crisp binary relations. We will study the definition of fuzzy rough sets, as well as some important results, in this section. Firstly, let us define fuzzy rough sets.

Definition 3.1 [24, 45, 48] Let $\mathcal{F}$ denotes a fuzzy relation from X to Y. Then the triple $(X, Y, \mathcal{F})$ is known as a fuzzy approximation space. If $X = Y$ and $\mathcal{F}$ is a fuzzy relation on X, then $(X, \mathcal{F})$ is a fuzzy approximation space whose upper approximation $\overline{\mathcal{F}}$ and lower approximation $\underline{\mathcal{F}}$ are defined as, for any $F \in \mathbb{F}(X)$,

$$\overline{\mathcal{F}}(F)(x) = \bigvee_{y \in X} (\mathcal{F}(x, y) \wedge F(y)), \text{ for all } \ x \in X,$$

$$\underline{\mathcal{F}}(F)(x) = \bigwedge_{y \in X} (1 - \mathcal{F}(x, y) \vee F(y)), \text{ for all } \ x \in X,$$

A set $F \subseteq X$ is said to be definable on $(X, \mathcal{F})$ if $\overline{\mathcal{F}}(F) = \underline{\mathcal{F}}(F)$ otherwise, F is (fuzzy) undefinable, or (fuzzy) rough.
Pawlak rough sets are a simple form of fuzzy rough set, when $\mathcal{F}$ is a crisp equivalence relation and $F \subseteq X$.

3.1 Properties of fuzzy approximation operators

[40]. In a fuzzy approximation space $(X, \mathcal{F})$, the operators $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ satisfies the following conditions for any $F, G \in \mathbb{F}(X)$ and a fuzzy constant α_X :

(1) $\underline{\mathcal{F}}(F) = \left(\overline{\mathcal{F}}(F^c)\right)^c$,
(2) $\overline{\mathcal{F}}(F) = \left(\underline{\mathcal{F}}(F^c)\right)^c$,
(3) $\underline{\mathcal{F}}(X) = X$,
(4) $\overline{\mathcal{F}}(\emptyset) = \emptyset$,
(5) $\underline{\mathcal{F}}(F \cap G) = \underline{\mathcal{F}}(F) \cap \underline{\mathcal{F}}(G)$,
(6) $\overline{\mathcal{F}}(F \cup G) = \overline{\mathcal{F}}(F) \cup \overline{\mathcal{F}}(G)$,
(7) $F \subseteq G \Rightarrow \underline{\mathcal{F}}(F) \subseteq \underline{\mathcal{F}}(G)$,
(8) $F \subseteq G \Rightarrow \overline{\mathcal{F}}(F) \subseteq \overline{\mathcal{F}}(G)$,
(9) $\underline{\mathcal{F}}(F \cup G) \supseteq \underline{\mathcal{F}}(F) \cup \underline{\mathcal{F}}(G)$,
(10) $\overline{\mathcal{F}}(F \cap G) \subseteq \overline{\mathcal{F}}(F) \cap \overline{\mathcal{F}}(G)$,
(11) $\underline{\mathcal{F}}(F \cup \alpha_X) = \underline{\mathcal{F}}(F) \cup \alpha_X$,
(12) $\overline{\mathcal{F}}(F \cap \alpha_X) = \overline{\mathcal{F}}(F) \cap \alpha_X$.
More generally, we have the following properties for any $F_i \in \mathbb{F}(X), \ (i \in \Lambda)$,
(13) $\underline{\mathcal{F}}(\bigcap_{i \in \Lambda} F_i) = \bigcap_{i \in \Lambda} \underline{\mathcal{F}}(F_i)$,
(14) $\overline{\mathcal{F}}(\bigcup_{i \in \Lambda} F_i) = \bigcup_{i \in \Lambda} \overline{\mathcal{F}}(F_i)$,
(15) $\underline{\mathcal{F}}(\bigcup_{i \in \Lambda} F_i) \supseteq \bigcup_{i \in \Lambda} \underline{\mathcal{F}}(F_i)$,
(16) $\overline{\mathcal{F}}(\bigcap_{i \in \Lambda} F_i) \subseteq \bigcap_{i \in \Lambda} \overline{\mathcal{F}}(F_i)$.
If $\mathcal{F}$ is a serial fuzzy relation on X, then
(17) $\underline{\mathcal{F}}(\emptyset) = \emptyset$,
(18) $\overline{\mathcal{F}}(X) = X$,
(19) $\underline{\mathcal{F}}(F) \subseteq \overline{\mathcal{F}}(F)$.
If $\mathcal{F}$ is a inverse serial fuzzy relation on X, then for any $F \in \mathbb{F}(X)$
(20) $\underline{\mathcal{F}}(F) = X \ if \ and \ only \ if \ F = X$,
(21) $\overline{\mathcal{F}}(F) = \emptyset \ if \ and \ only \ if \ F = \emptyset$,
(22) $\underline{\mathcal{F}}(\lambda_x^c) \neq X \ for \ any \ x \in X$,
(23) $\overline{\mathcal{F}}(\lambda_x) \neq \emptyset \ for \ any \ x \in X$,
(24) $\underline{\mathcal{F}}(F) > 0 \Rightarrow F > 0$,
(25) $\overline{\mathcal{F}}(F) < 1 \Rightarrow F < 1$.
If $\mathcal{F}$ is both reflexive and symmetrical fuzzy relation on X, then
(26) $\underline{\mathcal{F}}(\underline{\mathcal{F}}(F)) \subseteq \underline{\mathcal{F}}(F) \subseteq F \subseteq \overline{\mathcal{F}}(F) \subseteq \overline{\mathcal{F}}(\overline{\mathcal{F}}(F))$.

A binary operation $\mathcal{T} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ and $\mathcal{S} : [0, 1] \times [0, 1] \rightarrow [0, 1])$, respectively, are called t-norm and t-conorm on $[0, 1]$, if it is commutative, associative, increasing in every argument and has a unit element of 1 and 0, respectively. The fundamental t-norm and t-conorm are defined as, respectively, $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$, for every $\alpha, \beta \in [0, 1]$.

If the following holds, for every $\{\alpha_i\}_{i\in\Lambda} \subseteq [0, 1]$ and $\beta \in [0, 1]$,

$$\mathcal{T}\left(\bigvee_{i\in\Lambda} \alpha_i, \beta\right) = \bigvee_{i\in\Lambda} \mathcal{T}(\alpha_i, \beta),$$

then a t-norm $\mathcal{T}$ is said to be left-continuous. If a mapping $\mathcal{I} : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies the Boolean implicator boundary conditions (cf. [7]) and is decreasing in the first argument while increasing in the second argument, then it is called a fuzzy implicator on $[0, 1]$. If the following conditions hold for every $\{\alpha_i\}_{i\in\Lambda} \subseteq [0, 1]$ and $\beta \in [0, 1]$,

$\mathcal{I}(\bigvee_{i\in\Lambda} \alpha_i, \beta) = \bigwedge_{i\in\Lambda} \mathcal{I}(\alpha_i, \beta)$ and $\mathcal{I}(\beta, \bigwedge_{i\in\Lambda} \alpha_i) = \bigwedge_{i\in\Lambda} \mathcal{I}(\beta, \alpha_i)$,
then a fuzzy implicator $\mathcal{I}$ is said to be left-continuous in the first argument and right-continuous in the second argument.

A negator $\mathcal{N} : [0, 1] \rightarrow [0, 1]$ is a decreasing mapping that satisfies the conditions $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. Further $\mathcal{N}$ is said to be involutive if the $\mathcal{N}(\mathcal{N}(x)) = x$, for every x in $[0, 1]$. An implicator $\mathcal{I}$ satisfies $\mathcal{I}(0, 0) = 1, \ \mathcal{I}(1, x) = x$, for every $x \in [0, 1]$. If $\mathcal{T}$ is a t-norm, the mapping $\mathcal{I}_\mathcal{T}$ defined by, for every x and y in $[0, 1]$, $\mathcal{I}_\mathcal{T}(x, y) = \sup\{\lambda : \lambda \in [0, 1] \text{ and } \mathcal{T}(x, \lambda) \leq y\}$ is an implicator (of $\mathcal{T}$), also known as the residual implicator. If $\mathcal{T}$ is a t-norm and $\mathcal{N}$ is an involutive negator, then the mapping $\mathcal{I}_\mathcal{T}$ given by for every x, y in $[0, 1]$, $\mathcal{I}_\mathcal{T}, \mathcal{N}(x, y) = \mathcal{N}(\mathcal{T}(x, \mathcal{N}(y)))$ is an implicator, which is called the S-implicator induced by $\mathcal{T}$ and $\mathcal{N}$.

Theorem 3.1 [40] *Let $\mathcal{L}$ and $\mathcal{M}$ be two unary operators on $\mathbb{F}(X)$. If the following conditions are satisfied by $\mathcal{L}$ and $\mathcal{M}$ for any $F, F_i \in \mathbb{F}(X), \ i \in \Lambda$ and $\alpha_X \in [0, 1]$.*

(1) $\mathcal{L}(F) = (\mathcal{M}(F^c))^c$,
(2) $\mathcal{M}(F) = (\mathcal{L}(F^c))^c$,
(3) $\mathcal{L}(F \cup \alpha_X) = \mathcal{L}(F) \cup \alpha_X$,
(4) $\mathcal{M}(F \cap \alpha_X) = \mathcal{M}(F) \cap \alpha_X$,
(5) $\mathcal{L}\left(\bigcap_{i\in\Lambda} F_i\right) = \bigcap_{i\in\Lambda} \mathcal{L}(F_i)$,
(6) $\mathcal{M}\left(\bigcup_{i\in\Lambda} F_i\right) = \bigcup \mathcal{M}(F_i)$,

then there exists a fuzzy binary relation $\mathcal{F}$ on X such that $\mathcal{L} = \underline{\mathcal{F}}, \ \ \mathcal{M} = \overline{\mathcal{F}}$.

Proposition 3.1 [3] *If $\mathcal{I}$ is a residual implicator of left-continuous t-norm $\mathcal{T}$, then for any fuzzy set F in X,*

$$\underline{\mathcal{F}}(\overline{\mathcal{F}}(F)) \subseteq F \subseteq \overline{\mathcal{F}}(\underline{\mathcal{F}}(F)).$$

In general, Proposition 3.1 does not holds for other t-norms and implicators that do not satisfy the properties

$$\mathcal{T}(x, \mathcal{I}(x, y)) \leq y \ \text{ and } y \leq \mathcal{I}(x, \mathcal{T}(x, y)).$$

For example, let $X = \{\alpha, \beta\}$ be a finite set and $\mathcal{F}$ be a fuzzy relation on X. Define a fuzzy set F in X by $F(\alpha) = 1$ and $F(\beta) = 0.8$. Moreover, let $\mathcal{T} = \mathcal{T}_M$ and $\mathcal{I} = \mathcal{I}_{\mathcal{I}_M}, \mathcal{N}_s$ be its S-implicator. Then $\overline{\mathcal{F}}(F)(\alpha) = 1$ and $\overline{\mathcal{F}}(F)(\beta) = 0.8$. Hence

$$\underline{\mathcal{F}}(\overline{\mathcal{F}}(F))(\alpha) = \min\left(\max(0, 1), \max(0.8, 0.8)\right) = 0.8.$$

Thus $F \nsubseteq \underline{\mathcal{F}}(\overline{\mathcal{F}}(F))$. From all of the above, we obtain

$$\underline{\mathcal{F}}(\underline{\mathcal{F}}(F)) \subseteq \underline{\mathcal{F}}(F) \subseteq \underline{\mathcal{F}}(\overline{\mathcal{F}}(F)) \subseteq F \subseteq \overline{\mathcal{F}}(\underline{\mathcal{F}}(F)) \subseteq \overline{\mathcal{F}}(F) \subseteq \overline{\mathcal{F}}(\overline{\mathcal{F}}(F)),$$

If $\mathcal{I}$ is residual implicator of left-continuous $\mathcal{T}$, then the above inequality is valid for any reflexive and symmetric fuzzy relation $\mathcal{F}$.

Proposition 3.2 [4, 7] *If $\mathcal{T}$ is a left-continuous t-norm, then for every fuzzy set F in X, $\overline{\mathcal{F}}(\overline{\mathcal{F}}(F)) = \overline{(\mathcal{F} \circ \mathcal{F})}(F)$. Further if $\mathcal{I}$ is left-continuous in its first component and right-continuous in its second component and if $\mathcal{T}$ and $\mathcal{I}$ satisfy the shunting principle,*

$$\mathcal{I}(\mathcal{T}(x, y), z) = \mathcal{I}(x, \mathcal{I}(y, z)),$$

then for every fuzzy set F in X,

$$\underline{\mathcal{F}}(\underline{\mathcal{F}}(F)) = \underline{(\mathcal{F} \circ \mathcal{F})}(F).$$

Proposition 3.3 [3, 4, 7, 46] *If $\mathcal{F}$ is a fuzzy $\mathcal{T}$-equivalence relation in X, where $\mathcal{T}$ is a left-continuous t-norm and its residual implicator, then for every fuzzy set F in X,*

$$\underline{\mathcal{F}}(\overline{\mathcal{F}}(F)) = \underline{\mathcal{F}}(F) \ \text{ and } \ \overline{\mathcal{F}}(\underline{\mathcal{F}}(F)) = \overline{\mathcal{F}}(F).$$

It means that by using a fuzzy $\mathcal{T}$-equivalence relation to model approximate equality, we obtain maximal reduction or expansion in one phase, regardless of the approximations. When $\mathcal{F}$ is not $\mathcal{T}$- transitive and the universe X is finite, it is known that the $\mathcal{T}$-transitive closure of $\mathcal{F}$, and is denoted by $\mathcal{F}^{|X-1|}$ (assuming $|X| \geq 2$) [32]. Hence

$$\mathcal{F}^{|X-1|} \circ \mathcal{F}^{|X-1|} = \mathcal{F}^{|X-1|}.$$

In other words, with the lower and upper approximations, maximal reduction and expansion will be reached in at most $|X-1|$ steps, while with the tight lower and the loose upper approximation, it can take at most $\lceil |X-1|/2 \rceil$ steps.

4 Topological structures on fuzzy rough sets

Qin and Pei [45] examined topological structures of fuzzy rough sets or fuzzy approximation spaces and pointed out that the set of all reflexive, transitive fuzzy relations and the set of all fuzzy topologies satisfying the (TC) axiom have a one-to-one correspondence. Lai and Zhang [25], found a strong link between fuzzy preorders and saturated fuzzy topologies that satisfy an additional but differently formulated condition. Throughout this section, let $(X, \mathcal{F})$ be a fuzzy approximation, where X is a non-empty finite set (see [12]), unless stated otherwise.

4.1 Fuzzy topologies from fuzzy approximation spaces

The topological structures of fuzzy approximation spaces are discussed in this subsection. For every $\mathcal{F} \in \mathbb{F}(X \times X)$, let

$$\begin{aligned}
\eta_{\mathcal{F}} &= \{F \in \mathbb{F}(X) : F = \underline{\mathcal{F}}(F)\},\\
\theta_{\mathcal{F}} &= \{\underline{\mathcal{F}}(F) : F \in \mathbb{F}(X)\},\\
s_{\mathcal{F}} &= \bigwedge_{x,y \in X, x \neq y} \mathcal{F}(x, y),\\
t_{\mathcal{F}} &= \bigvee_{x,y \in X, x \neq y} \mathcal{F}(x, y).
\end{aligned}$$

If $\mathcal{F}$ is a fuzzy preorder, then $\theta_{\mathcal{F}} = \eta_{\mathcal{F}}$.
Also, $\eta_{\mathcal{F}}$ is the Alexandrov fuzzy topology induced by $(X, \mathcal{F})$, if $(X, \mathcal{F})$ is a reflexive fuzzy approximation space $(X, \mathcal{F})$. This topology satisfies the following properties [54]:

1. $(X, \eta_{\mathcal{F}})$ is not connected,
2. $(X, \eta_{\mathcal{F}})$ is T_{-1},
3. (a) if $(X, \eta_{\mathcal{F}})$ is sub-T_0, then for any $x, y \in X$ such as $x \neq y$, $\mathcal{F}(x, y) \wedge \mathcal{F}(y, x) < 1$, (b) if $t_{\mathcal{F}} < 1$, then $(X, \eta_{\mathcal{F}})$ is sub-T_0.

Proposition 4.1 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two reflexive fuzzy relations on X. Then*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \eta_{\mathcal{F}_2} \subseteq \eta_{\mathcal{F}_1}$,
(2) $\eta_{\mathcal{F}_1} \subseteq \eta_{\mathcal{F}_2} \Rightarrow \bigvee_{y \in X-\{x\}} \mathcal{F}_2(x, y) \leq \bigvee_{y \in X-\{x\}} \mathcal{F}_1(x, y), \quad x \in X$,
(3) *$\mathcal{F}_1 \cup \mathcal{F}_2$ is reflexive,*
(4) *$\eta_{\mathcal{F}_1 \cup \mathcal{F}_2}$ is an Alexandrov fuzzy topology on X,*
(5) $\eta_{\mathcal{F}_1 \cup \mathcal{F}_2} = \eta_{\mathcal{F}_1} \cap \eta_{\mathcal{F}_2}$.

Theorem 4.1 [45] *Let $(X, \mathcal{F})$ be a fuzzy preorder approximation space. Then*

(1) *$\theta_{\mathcal{F}}$ is a fuzzy topology on X,*
(2) *the interior operator of $\theta_{\mathcal{F}}$ is $\underline{\mathcal{F}}$,*
(3) *the closure operator of $\theta_{\mathcal{F}}$ is $\overline{\mathcal{F}}$.*

If $\mathcal{F}$ be a fuzzy relation on X, then

(4) $\mathcal{F}$ is transitive emplies $\theta_{\mathcal{F}} \subseteq \eta_{\mathcal{F}}$,
(5) an Alexandrov fuzzy topology on X is $\eta_{\mathcal{F}}$,
(6) if $\mathcal{F}$ is reflexive, then for each $F \in \mathbb{F}(X)$, $int_{\eta_{\mathcal{F}}}(F) \subseteq \underline{\mathcal{F}}(F) \subseteq F \subseteq \overline{\mathcal{F}}(F) \subseteq cl_{\eta_{\mathcal{F}}}(F)$,
(7) $F \in (\eta_{\mathcal{F}})^c$ if and only if $\overline{\mathcal{F}}(F) \subseteq F$,
(9) for every $\alpha \in I$, $\alpha_X \in (\eta_{\mathcal{F}})^c$.

4.2 Fuzzy approximation spaces from fuzzy topologies

A fuzzy topology is generated by a reflexive fuzzy approximation space, as we saw in Sect. 4.1. In this subsection, we look at the opposite problem: how can a fuzzy topology be linked to a fuzzy approximation space that provides the specified fuzzy topology.

Let η be the fuzzy topology on X. Define the fuzzy relation $\mathcal{F}_\eta$ on X by $\mathcal{F}_\eta(x, y) = cl_\eta(y)(x),\ (x, y) \in X \times X$. Then $\mathcal{F}_\eta$ is called the fuzzy relation induced by η on X and $(X, \mathcal{F}_\eta)$ is called the fuzzy approximation space induced by η on X.

Theorem 4.2 [54] *Let $\mathcal{F}$ be a reflexive fuzzy relation, $\eta_{\mathcal{F}}$ be the fuzzy topology generated by $\mathcal{F}$ on X, and $\mathcal{F}_{\eta_{\mathcal{F}}}$ be the fuzzy relation generated by $\eta_{\mathcal{F}}$ on X. If $\mathcal{F}$ is transitive, then $\mathcal{F}_{\eta_{\mathcal{F}}} = \mathcal{F}$.*

For a fuzzy topology η on X, the following condition is said to be the (CC) axiom: for any $\alpha \in I$ and $F \in \mathbb{F}(X)$,

$$(CC) \qquad cl_\eta(\alpha_X F) = \alpha_X cl_\eta(F),$$

where cl_η is the fuzzy closure operator on X.
Assume that η is a fuzzy topology on X. If η satisfies the (CC) axiom, then

(1) *the closure fuzzy operator of η is $\mathcal{F}_\eta$,*
(2) *η is an Alexandrov fuzzy topology on X.*

Further, $\eta_{\mathcal{F}}$ satisfies the (CC) axiom if $\mathcal{F}$ is a preorder fuzzy relation on X. Finally, we have the following theorem given by Tang et al. [54].

Theorem 4.3 *Let η be a fuzzy topology on X, $\mathcal{F}_\eta$ be the fuzzy relation generated by η on X, and $\eta_{\mathcal{F}_\eta}$ be the fuzzy topology generated by $\mathcal{F}_\eta$ on X. Then*

$$\eta_{\mathcal{F}_\eta} = \eta \ \textit{if and only if} \ \ \eta \ \textit{follows} \ \ (CC) \ axiom.$$

Theorem 4.4 *Consider a fuzzy topological space (X, η). Then (X, η) is a fuzzy approximation space provided any one of the following conditions is satisfied:*

(1) *η satisfies (CC) axiom,*
(2) *$F \in \mathbb{F}(X)$ and for any $\alpha \in I$, $int_\eta(\alpha_X \cup F) = \alpha_X \cup int_\eta(F)$,*
(3) *there exists a preorder fuzzy relation $\mathcal{F}$ on X such that $\overline{\mathcal{F}}$ is the closure operator of η,*
(4) *there exists a preorder fuzzy relation $\mathcal{F}$ on X such that $\underline{\mathcal{F}}$ is the interior operator of η,*
(5) *the closure operator of η is $\overline{\mathcal{F}_\eta}$,*
(6) *the interior operator of η is $\underline{\mathcal{F}_\eta}$.*

Moreover, we have the following result that a fuzzy topology generates a fuzzy approximation space.

Theorem 4.5 *Let the fuzzy topological space be defined by (X, η). If there exists a reflexive fuzzy relation $\mathcal{F}$, then (X, η) is a fuzzy approximation space such that*

$$\eta = \bigcup \left\{ F \in \mathbb{F}(X) : \textit{for every} \ \ (x, y) \in \mathcal{F}, \ F^c(x) \wedge F^c(y) \geq \mathcal{F}(x, y) \right\}.$$

4.3 Relation between fuzzy preorders and fuzzy topologies

The family of all saturated fuzzy topologies on X satisfying the specific extra condition is in close relationship with the family of fuzzy preorders on X (see [25, 45, 71]). In fact, the fuzzy topologies involved in [25, 45, 71], are either the same or closely related. In [56], it is shown that the observation in [25, 45, 71] regarding the relationship are actually equivalent. In this subsection, let X be a non-empty set.

Definition 4.1 (1) Let $(X, \mathcal{F})$ be a fuzzy preordered set. Then $F \in \mathbb{F}(X)$ is called an upper set of $(X, \mathcal{F})$ if $F(x) \wedge \mathcal{F}(x, y) \leq F(y)$ for every $x, y \in X$.

(2) For $\beta \in I, \ \ F \in \mathbb{F}(X), x \in X$,
$$(\beta \to F)(x) = \beta \to F(x) = \begin{cases} 1 \qquad , & if \ \beta \leq F(x), \\ F(x), & if \ \beta > F(x). \end{cases}$$

Proposition 4.2 [71] *The set of all fuzzy preorders on X and the set of all saturated fuzzy topologies η on X are in one-to-one correspondence such that for every $\beta \in I$ and for every $F \in \eta, \ \beta \to F \in \eta$.*

Proposition 4.3 *Let $(X, \mathcal{F})$ be a fuzzy preordered set and $F \in \mathbb{F}(X)$. Then F is an upper set of $(X, \mathcal{F})$ if and only if $F = \overline{\mathcal{F}}(F)$.*

A fuzzy topology η on X is said to satisfy:

(1) The (TC) axiom if, for every $x, y \in X$, whenever there exists some $F \in \eta$ such that $F(x) > F(y)$, then there also exists some $F_* \in \eta$ such that $F_*(y) = 0$ and $F_*(x) \geq F(x)$.
(2) The (TC^*) axiom if, for every $x, y \in X$, whenever there exists some $F \in \eta$ such that $F(x) < F(y)$, then there also exits some $F^* \in \eta$ such that $F^*(y) = 1$ and $F^*(x) \leq F(x)$.

The set of all fuzzy preorders on X and the set of saturated fuzzy topologies η on X that satisfies the (TC) axiom or (TC^*) axiom are shown to have a one-to-one relationship in [18, 23, 70]. Moreover, a fuzzy topological space (X, η) satisfies the (TC^*) axiom or (TC) $axiom$ if and only if $\beta \to F \in \eta$, $((\beta \to F^c)^c \in \eta)$, for every $\beta \in I$ and for every $F \in \eta$.

Proposition 4.4 *Let cl be a Kuratowski fuzzy closure operator on X and $I = [0, 1]$. Then there is a fuzzy preorder $\mathcal{F}$ on X such that $cl = \overline{\mathcal{F}}$ if and only if*

(1) $cl(\bigvee_{i\in I} F_i) = \bigvee_{i\in I} cl(F_i)$, $F_i \in \mathbb{F}(X)$,
(2) $cl(\beta \bigwedge F) = \beta \bigwedge cl(F)$, *for every* $F \in \mathbb{F}(X)$, $\beta \in I$, *and*
(3) *for every* $\beta \in I$ *and for every* $F \in \mathbb{F}(X)$, $cl(\beta \to F) = \beta \to cl(F)$ *if and only if* $cl(\beta \bigwedge F) = \beta \bigwedge cl(F)$.

Proposition 4.5 *The set of all fuzzy preorders on X has a one-to-one correspondence with the collection of all Kuratowski saturated fuzzy closure operators cl on X such that for every $\beta \in I$ and for every $F \in \mathbb{F}(X)$, $cl(\beta \wedge F) = \beta \wedge cl(F)$.*

Remark 4.1 Most of the results studied for fuzzy rough sets have been generalized by different researchers for L-fuzzy rough sets in [30, 34, 44, 64].

5 Intuitionistic fuzzy rough sets

All the elements of a set have a membership value in the fuzzy set theory, and the non-membership values are ignored. However, because of the presence of doubt, this is not true in many real-life problems. In fuzzy set theory, if $\nu_F(x)$ is the degree of membership of an element x in F, then the degree of non-membership of x is determined using the mathematical formula $(1 - \nu_F(x))$ assuming that deterministic component governs the whole degree of membership and the indeterministic part has no contribution in the degree of membership. In [2], the intuitionistic fuzzy set theory was introduced because the fuzzy set theory is not always applicable in real life. The intuitionistic fuzzy set theory reduces to the fuzzy set theory if the indeterministic element is zero. It shows that the intuitionistic fuzzy set model is an expanded version of the fuzzy set model. Therefore, a better model than a fuzzy rough set on two universal sets is an intuitionistic fuzzy rough set on two universal sets. We define the basic concepts leading to an intuitionistic fuzzy rough set on two universal sets. We denote ν for membership and ϑ for non-membership functions associated with an intuitionistic fuzzy rough set on two universal sets.

Definition 5.1 [2] An intuitionistic fuzzy (IF, in short) set F in X is an object of the form

$$F = \{< x, \nu_F(x), \vartheta_F(x) >: x \in X\},$$

where $\nu_F : X \to [0, 1]$ and $\vartheta_F : X \to [0, 1]$ satisfy $0 \leq \nu_F(x) + \vartheta_F(x) \leq 1$ for every $x \in X$,and $\nu_F(x)$ and $\vartheta_F(x)$ are, respectively, used to define the degree of membership and the degree of non-membership of the element x to F. The collection of all IF subsets of X is denoted by $\mathbf{IF}(X)$.The complement of an IF set F is given by $F^c = \{< x, \vartheta_F(x), \nu_F(x) >: x \in X\}$.

Clearly, every fuzzy set has a structure $\{< x, \nu_F(x), 1 - \nu_F(x) > x \in X\}$ and is an IF set. Every IF set of the form $\{< x, 1, 0 >: x \in X\}$ is still considered a crisp set F and if $y \notin F$, then $\nu_F(y) = 0$ and $\vartheta_F(y) = 1$.
Let $F, G \in \mathbf{IF}(X)$. Some fundamental operations on $\mathbf{IF}(X)$ are as follows:

(1) $F \subseteq G$ if and only if $\nu_F(x) \leq \nu_G(x)$ and $\vartheta_F(x) \geq \vartheta_G(x)$ for all $x \in X$,
(2) $F \supseteq G$ if and only if $G \subseteq F$,
(3) $F = G$ if and only if $F \subseteq G$, and $G \subseteq F$,
(4) $F \cap G = \{< x, \min(\nu_F(x), \nu_G(x)), \max(\vartheta_F(x), \vartheta_G(x)) >: x \in X\}$,
(5) $\bigcap_{i\in\Lambda} F_i = \{< x, \bigwedge_{i\in\Lambda} \nu_{F_i}(x), \bigvee_{i\in\Lambda} \vartheta_{F_i}(x) >: x \in X\},\ F_i \in \mathbf{IF}(X), i \in \Lambda$,
(6) $F \cup G = \{< x, \max(\nu_F(x), \nu_G(x)), \min(\vartheta_F(x), \vartheta_G(x)) >: x \in X\}$,
(7) $\bigcup_{i\in\Lambda} F_i = \{< x, \bigvee_{i\in\Lambda} \nu_{F_i}(x), \bigwedge_{i\in\Lambda} \vartheta_{F_i}(x) >: x \in X\},\ F_i \in \mathbf{IF}(X), i \in \Lambda$.

A constant IF set $(\alpha, \beta)_X = \{< x, \alpha, \beta >: x \in X\}$, where $0 \leq \alpha, \beta \leq 1$. A special IF set (IF singleton set) 1_y for $y \in X$, is defined as follows:

$$\nu_{1_y}(x) = \begin{cases} 1, \, if & x = y \\ 0, \, if & x \neq y \end{cases} \qquad \vartheta_{1_y}(x) = \begin{cases} 0, \, if & x = y \\ 1, \, if & x \neq y \end{cases}$$

$$\nu_{1_{X-\{y\}}}(x) = \begin{cases} 0, \, if & x = y \\ 1, \, if & x \neq y \end{cases} \qquad \vartheta_{1_{X-\{y\}}}(x) = \begin{cases} 1, \, if & x = y \\ 0, \, if & x \neq y \end{cases}$$

The IF universe set is $1^c_X = X = \{< x, 1, 0 >: x \in X\}$ and the IF empty set is $0^c_X = \emptyset = \{< x, 1, 0 >: x \in X\}$. An IF relation on X is an IF subset of $X \times X$ that is,

$$\mathcal{F} = \{< (x, y), \nu_{\mathcal{F}}(x, y), \vartheta_{\mathcal{F}}(x, y) >: x, y \in X\},$$

where

$$\nu_{\mathcal{F}} : X \times X \to [0, 1], \quad \vartheta_{\mathcal{F}} : X \times X \to [0, 1]$$

satisfies $0 \leq \nu_{\mathcal{F}}(x, y) + \vartheta_{\mathcal{F}}(x, y) \leq 1$ for every $(x, y) \in X \times X$. We denote by $IF\mathcal{F}(X \times X)$, the collection of all IF relations on X. For other properties of IF sets, we refer [79].

Definition 5.2 [9] An IF topology on a non-empty set X is a family ψ of IF sets in X satisfying the following axioms:

(1) $0^c_X, 1^c_X \in \psi$,
(2) $F \cap G \in \psi$, for any $F, G \in \psi$,
(3) $\bigcup_{i\in\Lambda} F_i \in \psi$, for an arbitrary family $\{F_i : i \in \Lambda\} \subseteq \psi$.

5.1 Intuitionistic fuzzy rough sets induced from intuitionistic fuzzy approximation spaces

In this subsection, we study the rough set approximations of IF sets with respect to a IF approximation space and features of IF rough approximation operators.

Definition 5.3 [1, 80] If $\mathcal{F} \in IF\mathcal{F}(X \times X)$, then the pair $(X, \mathcal{F})$ is called an IF approximation space. For $F \in \mathbf{IF}(X)$, the upper and lower approximations of F with respect to $(X, \mathcal{F})$ are given by $\overline{\mathcal{F}}(F)$ and $\underline{\mathcal{F}}(F)$, and are defined as follows:

$$\overline{\mathcal{F}}(F) = \{< x, \nu_{\overline{\mathcal{F}}(F)}(x), \vartheta_{\overline{\mathcal{F}}(F)}(x) >: x \in X\},$$
$$\underline{\mathcal{F}}(F) = \{< x, \nu_{\underline{\mathcal{F}}(F)}(x), \vartheta_{\underline{\mathcal{F}}(F)}(x) >: x \in X\},$$

where

$$\nu_{\overline{\mathcal{F}}(F)}(x) = \bigvee_{y\in X} [\nu_{\mathcal{F}}(x, y) \wedge \nu_F(y)], \quad \vartheta_{\overline{\mathcal{F}}(F)}(x) = \bigwedge_{y\in X} [\vartheta_{\mathcal{F}}(x, y) \vee \vartheta_F(y)]$$
$$\nu_{\underline{\mathcal{F}}(F)}(x) = \bigwedge_{y\in X} [\vartheta_{\mathcal{F}}(x, y) \vee \nu_F(y)], \quad \vartheta_{\underline{\mathcal{F}}(F)}(x) = \bigvee_{y\in X} [\nu_{\mathcal{F}}(x, y) \wedge \vartheta_F(y)].$$

The pair $(\underline{\mathcal{F}}(F), \overline{\mathcal{F}}(F))$ is said to be an IF rough set of F with respect to $(X, \mathcal{F})$, and $\overline{\mathcal{F}}, \underline{\mathcal{F}} : \mathbf{IF}(X) \to \mathbf{IF}(X)$ are respectively called as upper and lower IF rough approximation operators.

If $\mathcal{F}$ is a crisp binary relation on X, then $(X, \mathcal{F})$ is a crisp approximation space, the IF rough approximation operators are induced from a crisp approximation space, that is, for all $F \in \mathbf{IF}(X)$,

$$\overline{\mathcal{F}}(F) = \left\{< x, \nu_{\overline{\mathcal{F}}(F)}(x), \vartheta_{\overline{\mathcal{F}}(F)}(x) >: x \in X\right\},$$
$$\underline{\mathcal{F}}(F) = \left\{< x, \nu_{\underline{\mathcal{F}}(F)}(x), \vartheta_{\underline{\mathcal{F}}(F)}(x) >: x \in X\right\},$$

where

$$\nu_{\overline{\mathcal{F}}(F)}(x) = \bigvee_{y \in x\mathcal{F}} \nu_F(y), \quad \vartheta_{\overline{\mathcal{F}}(F)}(x) = \bigwedge_{y \in x\mathcal{F}} \vartheta_F(y),$$
$$\nu_{\underline{\mathcal{F}}(F)}(x) = \bigwedge_{y \in x\mathcal{F}} \nu_F(y), \quad \vartheta_{\underline{\mathcal{F}}(F)}(x) = \bigvee_{y \in x\mathcal{F}} \vartheta_F(y).$$

An IF rough set degenerates to a fuzzy rough set if $\mathcal{F}$ is a fuzzy relation on X and F is a fuzzy set [67].

Definition 5.4 Let $F \in \mathbf{IF}(X)$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the (α, β)-level cut set of F, given by F_α^β, is defined as the following:

$$F_\alpha^\beta = \{x \in X : \nu_F(x) \geq \alpha, \ \vartheta_F(x) \leq \beta\}.$$

5.2 Relation between intuitionistic fuzzy approximation spaces and intuitionistic fuzzy topological space

In this section, we investigate the relation between IF rough set approximations and IF topologies by generalizing IF rough set theory inside the framework of IF topological spaces (see [80]). Let $\mathcal{F}$ represents an IF reflexive and transitive binary relation on X, and $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ represent the IF rough approximation operators defined in Definition 5.3.

Theorem 5.1 *Let* $F_i \in \mathbf{IF}(X), \ \ i \in \Lambda$. *Then*

$$\underline{\mathcal{F}}\left(\bigcup_{i \in \Lambda} \underline{\mathcal{F}}(F_i)\right) = \bigcup_{i \in \Lambda} \underline{\mathcal{F}}(F_i).$$

In the following theorems, Zhou et al. [80] proved that an IF reflexive and transitive approximation space can generate an IF topological space, in which the IF topology is formed by the family of all lower approximations of IF sets concerning the IF approximation space. In addition, the interior and closure operators of the IF topological space are the lower and upper IF rough approximation operators, respectively.

Theorem 5.2 *Let* $(X, \mathcal{F})$ *be an* IF *reflexive and transitive approximation space. Then*

(1) $\psi_{\mathcal{F}} = \{\underline{\mathcal{F}}(F) : F \in \mathbf{IF}(X)\}$ *is an* IF *topology on* X.
(2) *Further, for every* $F \in \mathbf{IF}(X)$,

(i) $\underline{\mathcal{F}}(F) = int(F) = \bigcup\{\underline{\mathcal{F}}(G) : \underline{\mathcal{F}}(G) \subseteq F, G \in \mathbf{IF}(X)\}$

(ii) $\overline{\mathcal{F}}(F) = cl(F) = \bigcap\{(\underline{\mathcal{F}}(G))^c : (\underline{\mathcal{F}}(G))^c \supseteq F, G \in \mathbf{IF}(X)\}$
$= \bigcap\{\overline{\mathcal{F}}(G) : \overline{\mathcal{F}}(G) \supseteq F, G \in \mathbf{IF}(X)\}.$

The following theorem proves that the generating IF topology can also represent an IF reflexive and transitive relation.

Theorem 5.3 *Let $(X, \mathcal{F})$ denotes an IF reflexive and transitive approximation space, and (X, ψ) denotes the IF topological space generated by $(X, \mathcal{F})$. Then*

$$\nu_{\mathcal{F}}(x, y) = \bigwedge_{G \in (y)_\psi} \nu_G(x), \ \ \vartheta_{\mathcal{F}}(x, y) = \bigvee_{G \in (y)_\psi} \vartheta_G(x), \ \ x, y \in X,$$

where $(y)_\psi = \{G \in \mathbf{IF}(X) : G^c \in \psi_{\mathcal{F}}, \ \nu_G(y) = 1, \ \vartheta_G(y) = 0\}.$

An IF reflexive and transitive approximation space gives an IF topological space. Conversely, we will look at the conditions under which an IF topological space and an IF approximation space can be connected to produce the same IF topological space. In [80], the authors provide the necessary and sufficient conditions to satisfy the requirement. For this purpose, we consider the fuzzy topology defined in [5].

Definition 5.5 [5] A fuzzy topology on a set X is a family η of fuzzy sets in X that satisfies the following conditions:

(1) $\emptyset, X \in \eta$,
(2) if $F, G \in \eta$, then $F \cap G \in \eta$,
(3) if $F_i \in \eta, i \in \Lambda$, then $\bigcup_{i \in \Lambda} F_i \in \eta$.

Lemma 5.1 [9] *Let (X, ψ) be an IF topological space. Then*

(1) $\eta_1 = \{\nu_{\mathcal{G}} : \mathcal{G} \in \psi\}$ *is a fuzzy topology on X in Chang's sense,*
(2) $\eta_2 = \{1 - \vartheta_{\mathcal{G}} : \mathcal{G} \in \psi\}$ *is a fuzzy topology on X in Chang's sense and* $\eta_2^* = \{1 - \vartheta_{\mathcal{G}} : \mathcal{G} \in \psi\}$ *is the family of all fuzzy closed sets of η_2.*

Let (X, ψ) be an IF topological space and $\mathbf{C}, \mathbf{I} : \mathbf{IF}(X) \to \mathbf{IF}(X)$ be the IF closure and IF interior operators, respectively. Then $\mathbf{C}$ may be characterised by a pair of operators $\mathbf{C} = (\mathbf{C}_\nu, \mathbf{C}_\vartheta)$, where $\mathbf{C}_\nu$ represents the fuzzy closure operator generated by η_2, and $\mathbf{C}_\vartheta$ represents the fuzzy interior operator generated by η_1. Moreover, the pair of operators $\mathbf{I} = (\mathbf{I}_\nu, \mathbf{I}_\vartheta)$ characterise to $\mathbf{I}$, where $\mathbf{I}_\nu$ represents the fuzzy interior operator generated by η_1 and $\mathbf{I}_\vartheta$ represents the fuzzy closure operator generated by η_2. In fact $\mathbf{C}_\nu, \mathbf{C}_\vartheta, \mathbf{I}_\nu, \mathbf{I}_\vartheta : \mathbb{F}(X) \to \mathbb{F}(X)$. For $F \in \mathbf{IF}(X)$, $\mathbf{C}(F) = (\mathbf{C}_\nu(\nu), \mathbf{C}_\vartheta(\vartheta))$ such that $\nu_{\mathbf{C}(F)} = \mathbf{C}_\nu(\nu)$ and $\vartheta_{\mathbf{C}(F)} = \mathbf{C}_\vartheta(\vartheta)$, $\mathbf{I}(F) = (\mathbf{I}_\nu(\nu), \mathbf{I}_\vartheta(\vartheta))$ such that $\nu_{\mathbf{I}(F)} = \mathbf{I}_\nu(\nu)$ and $\vartheta_{\mathbf{I}(F)} = \mathbf{I}_\vartheta(\vartheta)$.

Imposing some sufficient and necessary conditions, an IF interior (closure, respectively) operator derived from an IF topological space can be associated with an IF reflexive and transitive relation such that the induced lower (upper, respectively) IF rough approximation operator is just the IF interior (closure, respectively) operator.

Theorem 5.4 *Let (X, ψ) be an IF topological space and $\mathbf{C}, \mathbf{I} : \mathbf{IF}(X) \to \mathbf{IF}(X)$ be the IF closure operator and the IF interior operator, respectively. Then there exists an IF reflexive and transitive relation $\mathcal{F}$ on X such that $\overline{\mathcal{F}}(F) = \mathbf{C}(F)$ and $\underline{\mathcal{F}}(F) = \mathbf{I}(F)$, for every $F \in \mathbf{IF}(X)$, if and only if $\mathbf{C}$ satisfies (1) and (2), or equivalently, $\mathbf{I}$ satisfies the following conditions (3) and (4) :*

(1) $\mathbf{C}\left(\bigcup_{i\in\Lambda} F_i\right) = \bigcup_{i\in\Lambda} \mathbf{C}(F_i)$, *i.e.*, $\mathbf{C}_\nu\left(\nu_{\bigcup_{i\in\Lambda} F_i}\right) = \mathbf{C}_\nu\left(\bigcup_{i\in\Lambda} \nu_{F_i}\right) = \bigcup_{i\in\Lambda} \mathbf{C}_\nu(\nu_{F_i})$,

$$\mathbf{C}_\vartheta\left(\vartheta_{\bigcup_{i\in\Lambda} F_i}\right) = \mathbf{C}_\vartheta\left(\bigcap_{i\in\Lambda} \vartheta_{F_i}\right) = \bigcap_{i\in\Lambda} \mathbf{C}_\vartheta(\vartheta_{F_i}), \quad F \in \mathbf{IF}(X).$$

(2) $\mathbf{C}(F \cap (\alpha, \beta)_X) = \mathbf{C}(F) \cap (\alpha, \beta)_X$, *i.e.*, $\mathbf{C}_\nu(\nu_{F_X} \cap \alpha_X) = \mathbf{C}_\nu(\nu_F) \cap \alpha_X$,

$$\mathbf{C}_\vartheta(\vartheta_{F_X} \cup \beta_X) = \mathbf{C}_\vartheta(\vartheta_F) \cup \beta_X, \quad F \in \mathbf{IF}(X),\ \alpha, \beta \in [0, 1] \text{ with } \alpha + \beta \le 1.$$

(3) $\mathbf{I}\left(\bigcap_{i\in\Lambda} F_i\right) = \bigcap_{i\in\Lambda} \mathbf{I}(F_i)$, *i.e.*, $\mathbf{I}_\nu\left(\nu_{\bigcap_{i\in\Lambda} F_i}\right) = \mathbf{I}_\nu\left(\bigcap_{i\in\Lambda} \nu_{F_i}\right) = \bigcap_{i\in\Lambda} \mathbf{I}_\nu(\nu_{F_i})$,

$$\mathbf{I}_\vartheta\left(\vartheta_{\bigcap_{i\in\Lambda} F_i}\right) = \mathbf{I}_\vartheta\left(\bigcup_{i\in\Lambda} \vartheta_{F_i}\right) = \bigcup_{i\in\Lambda} \mathbf{I}_\vartheta(\vartheta_{F_i}), \quad F \in \mathbf{IF}(X).$$

(4) $\mathbf{I}(F \cup (\alpha, \beta)_X) = \mathbf{I}(F) \cup (\alpha, \beta)_X$, *i.e.*, $\mathbf{I}_\nu(\nu_{F_X} \cup \alpha_X) = \mathbf{I}_\nu(\nu_F) \cup \alpha_X$,

$$\mathbf{I}_\vartheta(\vartheta_{F_X} \cap \beta_X) = \mathbf{I}_\vartheta(\vartheta_F) \cap \beta_X, \quad F \in \mathbf{IF}(X),\ \alpha, \beta \in [0, 1] \text{ with } \alpha + \beta \le 1.$$

Definition 5.6 Let (X, ψ) be an IF topological space and $\mathbf{C}, \mathbf{I} : \mathbf{IF}(X) \to \mathbf{IF}(X)$ be the IF closure operator and the IF interior operator, respectively. Suppose that $\mathbf{C}$ satisfies the conditions, (1) and (2) or $\mathbf{I}$, satisfies the conditions (3) and (4). Then we call (X, ψ) an IF rough topological space.

Finally, we have the following characterization (see [80]).

Theorem 5.5 *Any universe of discourse has a one-to-one correspondence between the set of all IF reflexive and transitive approximation spaces and the set of all IF rough topological spaces, with the lower and upper IF rough approximation operators as the IF interior and closure operators, respectively.*

In [16], Ghanim defined the fuzzy pseudo-closure operator of a fuzzy topological space, which results in different fuzzy topological properties. Further, the author described the IF pseudo closure operator induced by an IF topological space and investigated its properties. He also studied the IF pseudo-closure operators within the framework of IF approximations. A decomposition theorem for IF sets is stated as follows.

Theorem 5.6 *Let X be an arbitrary non-empty universe of discourse and $F \in \mathbf{IF}(X)$. Then*

$$F = \bigcup_{(\alpha,\beta)\in\mathcal{E}^2} \left(F_\alpha^\beta \cap (\alpha, \beta)_X\right),$$

where $\mathcal{E}^2 = \{(\alpha, \beta) : \alpha, \beta \in [0, 1], \alpha + \beta \le 1\}$ and $(\alpha, \beta)_X$ is the constant set in I^2.

The upper and lower IF rough approximation operators can be recreated by using following processes of approximations.

Corollary 5.1 *Let $(X, \mathcal{F})$ be an IF approximation space and $F \in \mathbf{IF}(X)$. Then*

$$\overline{\mathcal{F}}(F) = \bigcup_{(\alpha,\beta)\in\mathcal{E}^2} \overline{\mathcal{F}_\alpha^\beta}\left(F_\alpha^\beta \cap (\alpha, \beta)_X\right) = \bigcup_{(\alpha,\beta)\in\mathcal{E}^2} \overline{\mathcal{F}_\alpha^\beta}\left(F_{\alpha+}^{\beta+} \cap (\alpha, \beta)_X\right),$$

$$\underline{\mathcal{F}}(F) = \bigcap_{(\alpha,\beta)\in\mathcal{E}^2} \underline{\mathcal{F}_\alpha^\beta}\left(F_{\beta+}^{\alpha+} \cup (\alpha, \beta)_X\right) = \bigcap_{(\alpha,\beta)\in\mathcal{E}^2} \underline{\mathcal{F}_\alpha^\beta}\left(F_\beta^\alpha \cup (\beta, \alpha)_X\right).$$

Definition 5.7 Let (X, ψ) be an IF topological space, and cl_ψ be the IF closure operator given by ψ. Define

$$\mathbb{S}_\psi(F) = \bigcup_{(\alpha,\beta)\in\mathcal{E}^2} cl_\psi\left(F_\alpha^\beta \cap (\alpha,\beta)_X\right), \quad F \in \mathbf{IF}(X).$$

Then the function $\mathbb{S}_\psi : \mathbf{IF}(X) \to \mathbf{IF}(X)$ is said to be IF pseudo-closure operator induced by ψ.

Theorem 5.7 *An IF pseudo-closure operator, defined above, satisfies following conditions, for every $F, G \in \mathbf{IF}(X)$*

(1) $\mathbb{S}_\psi(\emptyset) = \emptyset$,
(2) $F \subseteq \mathbb{S}_\psi(F)$,
(3) $\mathbb{S}_\psi(F \cup G) \supseteq \mathbb{S}_\psi(F) \cup S_\psi(G),\ \mathbb{S}_\psi(F \cap G) \subseteq \mathbb{S}_\psi(F) \cap \mathbb{S}_\psi(G)$,
(4) *if F is an IF closed set, then $\mathbb{S}_\psi(F) = F$ (i.e., $\mathbb{S}_\psi cl_\psi(F) = cl_\psi(F)$),*
(5) $\mathbb{S}_\psi^n(F) \subseteq cl_\psi(F)$ *for every positive integer n, i.e., $cl_\psi \mathbb{S}_\psi(F) = cl_\psi(F),\ F \in \mathbf{IF}(X)$,*
(6) $\mathbb{S}_\psi$ *coincides with cl_ψ as operators from $\mathsf{E}(X)$ to $\mathbf{IF}(X)$, where*

$$\mathsf{E}(X) = \left\{\mathsf{N} \cap (\kappa, l)_X : \mathsf{N} \in P(X), (\kappa, l) \in \mathcal{E}^2\right\}.$$

Corollary 5.2 *Let $\mathcal{F}$ be a crisp reflexive and transitive relation on X. Then $\psi_{\mathcal{F}}^* = \{\underline{\mathcal{F}}(F) : F \in \mathbf{IF}(X)\}$ is an IF topology on X. In addition, $\overline{\mathcal{F}}$ is the pseudo-closure operator generated by $\psi_{\mathcal{F}}^*$ and $\overline{\mathcal{F}}(F) = \bigcup_{(\alpha,\beta)\in\mathcal{E}^2} \overline{\mathcal{F}}(F_\alpha^\beta \cap (\alpha,\beta)_X)$.*

Theorem 5.8 *Let $\psi_{\mathcal{F}}$ be the IF topology induced by an IF reflexive and transitive approximation space $(X, \mathcal{F})$. Then*

$$\mathbb{S}_{\psi_{\mathcal{F}}}(\mathsf{N} \cap (\kappa, l)_X) = \overline{\mathcal{F}}(\mathsf{N} \cap (\kappa, l)_X), \quad \textit{for every } \mathsf{N} \cap (\kappa, l)_X \in \mathsf{E}(X).$$

6 L-Double fuzzy rough sets

In [26], El-Latif et al. introduced a new idea of L-double fuzzy rough sets, where $(L, \wedge, \vee, c, 0_L, 1_L)$ is a fuzzy lattice, i.e., a completely distributive lattice with an order reversing involution $c : L \to L$, where, 0_L and 1_L denote the lowest and greatest elements of the lattice L, respectively. They studied the constructive and axiomatic approaches of L-double fuzzy rough sets. They defined L-double fuzzy rough sets by using Goguen [17] L-fuzzy sets, and L-double fuzzy topology generated by L-double approximation operators given by Samanta and Mondal [49].

Definition 6.1 Let X and Y be two arbitrary sets. The pair $(\mathcal{F}, \mathcal{F}^*)$ of maps $\mathcal{F}, \mathcal{F}^* : X \times Y \to L$ is said to be L-double fuzzy relation on $X \times Y$ if $\mathcal{F}(x, y) \leq (\mathcal{F}^*(x, y))^c$, for every $(x, y) \in X \times Y$. Moreover, $\mathcal{F}(x, y)$ (respectively, $\mathcal{F}^*(x, y)$), referred to as the degree of relation (respectively, non-relation) between x and y.

An L-double fuzzy relation $(\mathcal{F}, \mathcal{F}^*)$ on X is called:

(1) L-double fuzzy reflexive, if $\mathcal{F}(x, x) = 1_L$ and $\mathcal{F}^*(x, x) = 0_L,\ x \in X$,
(2) L-double fuzzy symmetric, if $\mathcal{F}(x, y) = \mathcal{F}(y, x)$ and $\mathcal{F}^*(x, y) = \mathcal{F}^*(y, x),\ x, y \in X$,
(3) L-double fuzzy transitive, if for every $x, y, z \in X$,
$\mathcal{F}(x, z) \geq \bigvee_{y\in X}(\mathcal{F}(x, y) \wedge \mathcal{F}(y, z))$ and $\mathcal{F}^*(x, z) \leq \bigwedge(\mathcal{F}^*(x, y) \vee \mathcal{F}^*(y, z))$,

(4) L-double fuzzy serial, if for every $x \in X$, there exists $y \in X$ such that $\mathcal{F}(x, y) = 1_L$ and $\mathcal{F}^*(x, y) = 0_L$.

An L-double fuzzy relation $(\mathcal{F}, \mathcal{F}^*)$ on X is called an L-double fuzzy equivalence relation if it is L-double fuzzy reflexive, L-double fuzzy symmetric and L-double fuzzy transitive. The triplet $(X, \mathcal{F}, \mathcal{F}^*)$ is called an L-double fuzzy approximation space.

For example, let $X = \{l, m, n\}$ and $L = [0, 1]$. Define $\mathcal{F}, \mathcal{F}^* : X \times X \to L$ as follows:

$$\mathcal{F} = \begin{pmatrix} 1 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}, \quad \mathcal{F}^* = \begin{pmatrix} 0 & 0 & 0.4 \\ 0 & 0 & 0.4 \\ 0.4 & 0.4 & 0 \end{pmatrix}.$$

Then, $(\mathcal{F}, \mathcal{F}^*)$ is an L-double fuzzy reflexive, symmetric, transitive and serial relation.

Definition 6.2 Let L be a fuzzy lattice and $(\mathcal{F}, \mathcal{F}^*)$ be an L-double fuzzy relation on X. For every L-fuzzy set F on X, the pairs $((\underline{\mathcal{F}}(F), \underline{\mathcal{F}^*}(F)), (\overline{\mathcal{F}}(F), \overline{\mathcal{F}^*}(F)))$ of maps $\underline{\mathcal{F}}(F), \underline{\mathcal{F}^*}(F), \overline{\mathcal{F}}(F), \overline{\mathcal{F}^*}(F) : X \to L$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation of the L-fuzzy set F, respectively, where

$$\underline{\mathcal{F}}(F) = \bigwedge_{y \in X} ((\mathcal{F}(x, y))^c \vee F(y)), \;\; \underline{\mathcal{F}^*}(F) = \bigvee_{y \in X} ((\mathcal{F}^*(x, y))^c \wedge F^c(y)), \;\; x \in X,$$

and

$$\overline{\mathcal{F}}(F) = \bigvee_{y \in X} ((\mathcal{F}(x, y)) \wedge F(y)), \;\; \overline{\mathcal{F}^*}(F) = \bigwedge_{y \in X} ((\mathcal{F}^*(x, y)) \vee F^c(y)), \;\; x \in X.$$

The quaternary $(\underline{\mathcal{F}}(F), \underline{\mathcal{F}^*}(F), \overline{\mathcal{F}}(F), \overline{\mathcal{F}^*}(F))$ is called an L-double fuzzy rough set of F. The pairs $(\underline{\mathcal{F}}, \underline{\mathcal{F}^*}), (\overline{\mathcal{F}}, \overline{\mathcal{F}^*})$ of operators $\underline{\mathcal{F}}, \underline{\mathcal{F}^*}, \overline{\mathcal{F}}, \overline{\mathcal{F}^*} : L^X \to L^X$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation operators, respectively, and the triplets $(X, \underline{\mathcal{F}}, \underline{\mathcal{F}^*}), (X, \overline{\mathcal{F}}, \overline{\mathcal{F}^*})$ are called L-double fuzzy lower approximation and L-double fuzzy upper approximation spaces, respectively.

Remark 6.1 Let a mapping $\mathcal{F} : X \times X \to L$ be an L-fuzzy relation on X and $(\underline{\mathcal{F}}(F), \overline{\mathcal{F}}(F))$ be a extended rough set of $F \in L^X$. Define a map $\mathcal{F}^* : X \times X \to L$ by, $\mathcal{F}^*(x, y) = (\mathcal{F}(x, y))^c, \;\; (x, y) \in X \times X$. Define L-fuzzy sets $\underline{\mathcal{F}^*}(F), \overline{\mathcal{F}^*}(F) : X \to L$ as: $(\underline{\mathcal{F}^*}(F))(x) = (\underline{\mathcal{F}}(F))^c(x)$ and $(\overline{\mathcal{F}^*}(F))(x) = (\overline{\mathcal{F}}(F))^c(x), \;\; x \in X$.
Then $(\mathcal{F}, \mathcal{F}^*)$ is an L-double fuzzy relation on X and $(\underline{\mathcal{F}}(F), \underline{\mathcal{F}^*}(F), \overline{\mathcal{F}}(F), \overline{\mathcal{F}^*}(F))$ is an L-double fuzzy rough set of F. Therefore, an L-double fuzzy rough set is a generalization of generalized rough set.

Properties of L-double fuzzy rough sets: Let L be a fuzzy lattice, and $(\mathcal{F}, \mathcal{F}^*)$ be an L-double fuzzy relation on X, $F \in L^X$. Then,

(1) $\overline{\mathcal{F}}(F) \leq (\overline{\mathcal{F}^*}(F))^c$ and $\underline{\mathcal{F}}(F) \geq (\underline{\mathcal{F}^*}(F))^c, \;\; F \in L^X$,
(2) $\underline{\mathcal{F}} 1_X = 1_X$ and $\underline{\mathcal{F}^*} 1_X = 0_X$,
(3) $\overline{\mathcal{F}} 0_X = 0_X$ and $\overline{\mathcal{F}^*} 0_X = 1_X$,
(4) $\underline{\mathcal{F}}(\bigwedge_{i \in \Lambda} F_i) = \bigwedge_{i \in \Lambda} \underline{\mathcal{F}}(F_i)$ and $\underline{\mathcal{F}^*}(\bigwedge_{i \in \Lambda} F_i) = \bigvee_{i \in \Lambda} \underline{\mathcal{F}^*}(F_i)$, for each family $\{F_i : i \in \Lambda\} \subseteq L^X$,
(5) $\overline{\mathcal{F}}(\bigvee_{i \in \Lambda} F_i) = \bigvee_{i \in \Lambda} \overline{\mathcal{F}}(F_i)$ and $\overline{\mathcal{F}^*}(\bigvee_{i \in \Lambda} F_i) = \bigwedge_{i \in \Lambda} \overline{\mathcal{F}^*}(F_i)$, for each family $\{F_i : i \in \Lambda\} \subseteq L^X$,
(6) if $F \leq G$, then $\underline{\mathcal{F}}(F) \leq \underline{\mathcal{F}}(G)$ and $\underline{\mathcal{F}^*}(F) \geq \underline{\mathcal{F}^*}(G)$,
(7) if $F \leq G$, then $\overline{\mathcal{F}}(F) \leq \overline{\mathcal{F}}(G)$ and $\overline{\mathcal{F}^*}(F) \geq \overline{\mathcal{F}^*}(G)$,

(8) $\underline{\mathcal{F}}(F) \vee \underline{\mathcal{F}}(G) \leq \underline{\mathcal{F}}(F \vee G)$ and $\underline{\mathcal{F}^*}(F) \wedge \underline{\mathcal{F}^*}(G) \geq \underline{\mathcal{F}^*}(F \vee G),\ F, G \in L^X$,
(9) $\overline{\mathcal{F}}(F \wedge G) \leq \overline{\mathcal{F}}(F) \wedge \overline{\mathcal{F}}(G)$ and $\overline{\mathcal{F}^*}(F \wedge G) \geq \overline{\mathcal{F}^*}(F) \vee \overline{\mathcal{F}^*}(G),\ F, G \in L^X$,
(10) $\overline{\mathcal{F}}(F^c) = (\underline{\mathcal{F}}(F))^c$ and $\overline{\mathcal{F}^*}(F^c) = (\underline{\mathcal{F}^*}(F))^c$,
(11) $\underline{\mathcal{F}}(F^c) = (\overline{\mathcal{F}}(F))^c$ and $\underline{\mathcal{F}^*}(F^c) = (\overline{\mathcal{F}^*}(F))^c$.

6.1 L-Double fuzzy topology from L-double fuzzy approximation operators

In this section, we will show that an L-double fuzzy upper (respectively lower) approximation operator on X induces an Alexandrov L-double fuzzy topology on X (see [26]). First, let us come to the definition of L-double fuzzy topology on X.

Definition 6.3 [49] The pair (η, η^*) of maps $\eta, \eta^* : L^X \to L^X$ is called an L-double fuzzy topology on X if it satisfies the following conditions:

(1) $\eta(F) \leq (\eta^*(F))^c, \quad F \in L^X$,
(2) $\eta(0_X) = \eta(1_X) = 1_L,\ \eta^*(0_X) = \eta^*(1_X) = 0_L$,
(3) $\eta(F \wedge G) \geq \eta(F) \wedge \eta(G)$ and $\eta^*(F \wedge G) \leq \eta^*(F) \vee \eta^*(G),\ \ F, G \in L^X$,
(4) $\eta(\bigvee_{i\in\Lambda} F_i) \geq \bigwedge_{i\in\Lambda} \eta(F_i)$ and $\eta^*(\bigvee_{i\in\Lambda} F_i) \leq \bigvee_{i\in\Lambda} \eta^*(F_i)$, for any family $\{F_i : i \in \Lambda\} \subseteq L^X$.
The triplet (X, η, η^*) is called an L-double fuzzy topological space. An L-double fuzzy topology (η, η^*) is called an Alexandrov if it satisfies:
(5) $\eta(\bigwedge_{i\in\Lambda} F_i) \geq \bigwedge_{i\in\Lambda} \eta(F_i)$ and $\eta^*(\bigwedge_{i\in\Lambda} F_i) \leq \bigvee_{i\in\Lambda} \eta^*(F_i)$ for any family $\{F_i : i \in \Lambda\} \subseteq L^X$.

For example: Let $X = \{l, m, n\}$ and $L = I$. Define $G \in L^X$ as follows:

$$G(l) = 0.2, \quad G(m) = 0.4, \quad G(n) = 0.7.$$

Define $\eta, \eta^* : L^X \to L$ as follows:

$$\eta(F) = \begin{cases} 1_L, & if\ F \in \{0_X, 1_X\} \\ 0.5, & if\ F = G \\ 0_L, & if\ otherwise, \end{cases} \qquad \eta^*(F) = \begin{cases} 0_L, & if\ F \in \{0_X, 1_X\} \\ 0.3, & if\ F = G \\ 1_L, & if\ otherwise. \end{cases}$$

Then (η, η^*) is an L-double fuzzy topology on X.

Definition 6.4 [49] Let $f : (X, \eta_1, \eta_1^*) \to (Y, \eta_2, \eta_2^*)$ be a map between L-double fuzzy topological spaces (X, η_1, η_1^*) and (Y, η_2, η_2^*). Then f is said to be continuous if for every $F \in L^Y, \eta_1(f^{\leftarrow}(F)) \geq \eta_2(F)$ and $\eta_1^*(f^{\leftarrow}(F)) \leq \eta_2^*(F)$, where $f^{\leftarrow}(F)(x) = F(f(x)),\ \ F \in L^Y,\ \ x \in X$.

Theorem 6.1 [26] *Let $(X, \overline{\mathcal{F}}, \overline{\mathcal{F}}^*)$ be an L-double fuzzy upper approximation space with $\overline{\mathcal{F}}(F) \geq (\overline{\mathcal{F}}^*(F))^c$, for every $F \in L^X$. Define $\eta_{\overline{\mathcal{F}}}, \eta^*_{\overline{\mathcal{F}}^*} : L^X \to L$ as follows: for every $F \in L^X$*

$$\eta_{\overline{\mathcal{F}}}(F) = \bigwedge_{x\in X} \left((\overline{\mathcal{F}}(F))^c(x) \vee F(x)\right),$$
$$\eta^*_{\overline{\mathcal{F}}^*}(F) = \bigvee_{x\in X} \left((\overline{\mathcal{F}}^*(F))^c(x) \wedge F^c(x)\right).$$

Then $(\eta_{\overline{\mathcal{F}}}, \eta^*_{\overline{\mathcal{F}}^*})$ is an Alexandrov L-double fuzzy topology on X.

Theorem 6.2 [26] *Let $(X, \underline{\mathcal{F}}, \underline{\mathcal{F}}^*)$ be an L-double fuzzy lower approximation space with $\underline{\mathcal{F}}(F) \leq (\underline{\mathcal{F}}^*(F))^c$, for every $F \in L^X$. Define $\eta_{\underline{\mathcal{F}}}, \eta^*_{\underline{\mathcal{F}}^*} : L^X \to L$ as follows: for every $F \in L^X$*

$$\eta_{\underline{\mathcal{F}}}(F) = \bigwedge_{x \in X} \left((\underline{\mathcal{F}}(F))(x) \vee F^c(x)\right),$$

$$\eta^*_{\underline{\mathcal{F}}^*}(F) = \bigvee_{x \in X} \left((\underline{\mathcal{F}}^*(F))(x) \wedge F(x)\right).$$

Then $(\eta_{\underline{\mathcal{F}}}, \eta^*_{\underline{\mathcal{F}}^*})$ is an Alexandrov L-double fuzzy topology on X.

Definition 6.5 Let $(X, \mathcal{F}_1, \mathcal{F}_1^*)$ and $(Y, \mathcal{F}_2, \mathcal{F}_2^*)$ be two L-double fuzzy approximation spaces.

(1) The map $f : (X, \mathcal{F}_1, \mathcal{F}_1^*) \to (Y, \mathcal{F}_2, \mathcal{F}_2^*)$ is called $\mathcal{F}$-map if $\mathcal{F}_1(x, y) \leq \mathcal{F}_2(f(x), f(y))$ and $\mathcal{F}_1^*(x, y) \geq \mathcal{F}_2^*(f(x), f(y))$, for every $(x, y) \in X \times Y$,
(2) The map $f : (X, \overline{\mathcal{F}_1}, \overline{\mathcal{F}_1^*}) \to (Y, \overline{\mathcal{F}_2}, \overline{\mathcal{F}_2^*})$ is called $\overline{\mathcal{F}}$-map if $\overline{\mathcal{F}_1}(f^{\leftarrow}(F)) \leq f^{\leftarrow}(\overline{\mathcal{F}_2}(F))$ and $\overline{\mathcal{F}_1^*}(f^{\leftarrow}(F)) \geq f^{\leftarrow}(\overline{\mathcal{F}_2^*}(F)$, for every $F \in L^Y$,

If the map $f : (X, \mathcal{F}_1, \mathcal{F}_1^*) \to (Y, \mathcal{F}_2, \mathcal{F}_2^*)$ is a $\mathcal{F}$-map, then $f : (X, \overline{\mathcal{F}_1}, \overline{\mathcal{F}_1^*}) \to (Y, \overline{\mathcal{F}_2}, \overline{\mathcal{F}_2^*})$ is a $\overline{\mathcal{F}}$-map. If the map $f : (X, \overline{\mathcal{F}_1}, \overline{\mathcal{F}_1^*}) \to (Y, \overline{\mathcal{F}_2}, \overline{\mathcal{F}_2^*})$ is $\overline{\mathcal{F}}$-map, then $(X, \eta_{\overline{\mathcal{F}_1}}, \eta^*_{\overline{\mathcal{F}_1^*}}) \to (X, \eta_{\overline{\mathcal{F}_2}}, \eta^*_{\overline{\mathcal{F}_2^*}})$ is continuous.

7 Topology on multi-fuzzy rough sets

Sebastian and Ramakrishnan [50, 51] introduced multi-fuzzy sets as a generalization of fuzzy sets. Ordered sequences of membership functions define multi-fuzzy sets. The multi-fuzzy sets theory provides a foundation for image processing, taste recognition, and pattern recognition problems. Multi-fuzzy sets make it simple to make decisions that involve more than one variable. Multi-fuzzy rough sets [63] are specific hybrid models in the literature. We collect some definitions and important results of multi-fuzzy rough sets in this section. A detailed study on multi-fuzzy topological spaces can be found in [50].

Definition 7.1 [50] Let X be a non-empty set, $\mathbb{N}$ be the collection of all natural numbers, and $\{L_i : i \in \mathbb{N}\}$ be a family of complete lattices. A multi-fuzzy set F in X is a collection of ordered sequences

$$F = \left\{ < x, \nu_F^1(x), \nu_F^2(x), \nu_F^3(x), \ldots, \nu_F^i(x), \ldots >: x \in X \right\}$$

where $\nu_F^i(x) \in L_i^X \quad (i.e., \quad \nu_F^i : X \to L_i)$ for $i \in \mathbb{N}$.

Remark 7.1 The dimension of F is defined as the number of terms in the sequences of membership functions that include only k-terms (a finite number of terms). The set of all multi-fuzzy sets in X with the value domain $\prod_{i \in \Lambda} M_i$ where each M_i is a complete lattice, is denoted by $\prod_{i \in \Lambda} M_i^X$ and is called multi-fuzzy space. Let $L_i = [0, 1]$ for $i \in \mathbb{N}$. Then the set of all multi-fuzzy sets in X, is denoted by $\mathbf{MF}(X)$.

Definition 7.2 [51] A multi-fuzzy topology on X is a subset θ of $\prod_{i \in \Lambda} M_i^X$, where each M_i is a complete lattice, that satisfies the following conditions:

(1) $0_X, 1_X \in \theta$,
(2) $F \cap G \in \theta$, for every $F, G \in \theta$,
(3) $\bigvee_{i\in\Lambda} H_i \in \theta$, for every $H \subset \theta$, that is, the arbitrary union of multi-fuzzy sets of θ is in θ.

A multi-fuzzy topological space is defined as an ordered triple $(X, \prod_{i\in\Lambda} M_i^X, \theta)$. Multi-fuzzy sets in θ are also known as θ-open multi-fuzzy sets in X, or open multi-fuzzy sets in X. If $\alpha_X \in \theta$ for every constant multi-fuzzy set, α_X in X, then θ is termed as a strong multi-fuzzy topology.

Definition 7.3 Let $\mathcal{T}$ be a continuous t-norm on [0, 1] and $\mathcal{I}$ be an implicator on [0, 1]. For a multi-fuzzy approximation space $(X, \mathcal{F})$ and any multi-fuzzy set $F \in \mathbf{MF}(X)$, the $\mathcal{T}$-upper and $\mathcal{I}$-lower multi-fuzzy rough approximation of F, denoted as $\overline{\mathcal{F}}^{\mathcal{T}}(F)$ and $\underline{\mathcal{F}_{\mathcal{I}}}(F)$ respectively, with respect to the approximation space $(X, \mathcal{F})$ are multi-fuzzy sets of X with defined membership functions respectively,

$$\nu^i_{\overline{\mathcal{F}}^{\mathcal{T}}(F)}(x) = \bigvee_{y\in X} \mathcal{T}(\nu^i_{\mathcal{F}}(x, y), \nu^i_F(y)),\ x \in X,\ i \in \mathbb{N},$$

$$\nu^i_{\underline{\mathcal{F}_{\mathcal{I}}}(F)}(x) = \bigwedge_{y\in X} \mathcal{I}(\nu^i_{\mathcal{F}}(x, y), \nu^i_F(y)),\ x \in X,\ i \in \mathbb{N}.$$

The operators $\overline{\mathcal{F}}^{\mathcal{T}}$ and $\underline{\mathcal{F}_{\mathcal{I}}}$ on $\mathbf{MF}(X)$ are given to as $\mathcal{T}$-upper and $\mathcal{I}$-lower multi-fuzzy rough approximation operators of $(X, \mathcal{F})$ respectively and the pair $(\overline{\mathcal{F}}^{\mathcal{T}}, \underline{\mathcal{F}_{\mathcal{I}}})$ is called the $(\mathcal{I}, \mathcal{T})$ multi-fuzzy rough set of F.

A fuzzy implication $\mathcal{I}$ is said to satisfy:

- the exchange principle, if $\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(y, \mathcal{I}(x, z)),\quad x, y, z \in [0, 1]\quad (EP)$.
- the left neutrality property, if $\mathcal{I}(1, x) = x,\quad x \in [0, 1]\ \ (NP)$.

Properties of $\mathcal{I}$-lower multi-fuzzy rough approximation operators:
Let $(X, \mathcal{F})$ be a multi-fuzzy approximation space. Then the $\mathcal{F}$-lower multi-fuzzy rough approximation operator $\underline{\mathcal{F}_{\mathcal{I}}}$ has the following properties:
For every $F, G \in \mathbf{MF}(X),\ F_i \in \mathbf{MF}(X)$ where $i \in \Lambda,\ M \subseteq X,\ (x, y) \in X \times X$ and all $\alpha \in [0, 1],\ i \in \mathbb{N}$

(1) $\underline{\mathcal{F}_{\mathcal{I}}}(\bigcap_{i\in\Lambda} F_i) = \bigcap_{i\in\Lambda} \underline{\mathcal{F}_{\mathcal{I}}}(F_i)$,
(2) $\underline{\mathcal{F}_{\mathcal{I}}}(\alpha_X) \supseteq \alpha_X$, if $\mathcal{I}$ is an NP implicator,
(3) $\underline{\mathcal{F}_{\mathcal{I}}}(\bigcup_{i\in\Lambda} F_i) \supseteq \bigcup_{i\in\Lambda} \underline{\mathcal{F}_{\mathcal{I}}}(F_i)$,
(4) $F \subseteq G \Rightarrow \underline{\mathcal{F}_{\mathcal{I}}}(F) \subseteq \underline{\mathcal{F}_{\mathcal{I}}}(G)$.

Consider the fuzzy continuous implicator $\mathcal{I}$ that satisfies the law of importation (i.e., if $\mathcal{I}(x, \mathcal{I}(y, z)) = \mathcal{I}(\mathcal{T}(x, y), z),\ \ x, y, z \in [0, 1]$ where $\mathcal{T}$ is a t-norm) and $\mathcal{F}$ is a multi-fuzzy relation that is both reflexive and transitive. Then we have the following results.

Lemma 7.1 *For every $F_i \in \boldsymbol{MF}(X), i \in \Lambda$ (Λ is an index set),*

$$\underline{\mathcal{F}_{\mathcal{I}}}\left(\bigcup_{i\in\Lambda} \underline{\mathcal{F}_{\mathcal{I}}}(F_i)\right) = \bigcup_{i\in\Lambda} \underline{\mathcal{F}_{\mathcal{I}}}(F_i).$$

Theorem 7.1 *A multi-fuzzy topology is defined as the collection of all $\mathcal{I}$-lower approximations of multi-fuzzy sets on X,*
i.e., $T_{\boldsymbol{MFR}} = \{\underline{\mathcal{F}_{\mathcal{I}}}(F) : F \in \boldsymbol{MF}(X)\}$ is a strong multi-fuzzy topology on X.

8 Hesitant fuzzy rough sets

Torra and Narukawa introduced the concept of an hesitant fuzzy (HF) sets in [61, 62] as another fuzzy set generalization. This section will study that the HF set allows an element to belong to a set with various possible values between 0 and 1 and how useful it is for determining membership degree, especially when there are multiple values on it. The two tools for solving problems are rough set theory and HF set theory. Despite the absence of a clear relationship between the two ideas, Yang et al. [69] take a significant step forward. Introducing the concept of HF rough sets, they proposed an axiomatic approach of the model. In this section, X is a non-empty and finite universe of discourse.

Definition 8.1 [61, 62] An HF set F on X is defined in terms of a function $h_F(x)$ that generates a subset of [0, 1], when applied to X, i.e.,

$$F = \{< x, h_F(x) >: x \in X\},$$

where $h_F(x)$ is a finite set of distinct values in [0, 1], indicating the different degrees of membership of the element $x \in X$ to F.

We call $h_F(x)$ as an HF element for convenience. Further, $\mathbf{HF(X)}$ denotes the set of all HF sets on X, and is known as the HF power set of X. Moreover, an HF set F is a fuzzy set is just one element in $h_F(x)$. As a particular case, HF sets include fuzzy sets in this situation.

Now, we define the concept of an hesitant fuzzy relation [69].

Definition 8.2 An hesitant fuzzy (HF) relation $\mathcal{H}$ on X is an HF subset of $X \times X$, i.e, $\mathcal{H} = \{< (x, y), h_{\mathcal{H}}(x, y) >: (x, y) \in X \times X\}$, where $h_{\mathcal{H}}(x, y)$ is a set of some different values in [0, 1], denoting the possible membership degrees of the relationship between x and y.

Yang et al. [69] also presented several special HF relations as follows:

Definition 8.3 Let $\mathcal{H}$ be an HF relation on X. Then

(1) $\mathcal{H}$ is said to be serial if for any $x \in X$, there exists $y \in X$ such that $h_{\mathcal{H}}(x, y) = \{1\}$,
(2) $\mathcal{H}$ is said to be reflexive if $h_{\mathcal{H}}(x, x) = \{1\}$, $x \in X$,
(3) $\mathcal{H}$ is said to be symmetric if $h_{\mathcal{H}}(x, y) = h_{\mathcal{H}}(y, x)$, $(x, y) \in X \times X$,
(4) $\mathcal{H}$ is said to be transitive if $h^{\rho(l)}_{\mathcal{H}}(x, y) \wedge h^{\rho(l)}_{\mathcal{H}}(y, z) \leq h^{\rho(l)}_{\mathcal{H}}(x, z)$, $l = 1, 2, \ldots, m$, where $h^{\rho(l)}_{\mathcal{H}}(x, y)$ is the l^{th} largest value in $h_{\mathcal{H}}(x, y)$ and $m = \max\{m(h_{\mathcal{H}}(x, y)), m(h_{\mathcal{H}}(y, z)), m(h_{\mathcal{H}}(x, z))\}$, $m(h_{\mathcal{H}}(x, y))$ is the number of values in $(h_{\mathcal{H}}(x, y))$, $x, y \in X$.

Definition 8.4 The pair $(X, \mathcal{H})$ is called a HF approximation space if $\mathcal{H}$ is an HF relation on X. For every $\mathcal{H} \in \mathbf{HF(X)}$, the lower and upper approximations of $\mathcal{H}$ with respect to $(X, \mathcal{H})$, denoted by $\underline{\mathcal{H}}(F)$ and $\overline{\mathcal{H}}(F)$, respectively, are two HF sets and are defined as follows:

$$\underline{\mathcal{H}}(F) = \{< x, h_{\underline{\mathcal{H}}(F)}(x) >: x \in X\},$$
$$\overline{\mathcal{H}}(F) = \{< x, h_{\overline{\mathcal{H}}(F)}(x) >: x \in X\},$$

where

$$h_{\underline{\mathcal{H}}(F)}(x) = \left\{\bigwedge_{y \in X} h^{\rho(l)}_{\mathcal{H}^c}(x, y) \vee h^{\rho(l)}_F(y) : l = 1, 2, \ldots, m_x\right\},$$
$$h_{\overline{\mathcal{H}}(F)}(x) = \left\{\bigvee_{y \in X} h^{\rho(l)}_{\mathcal{H}}(x, y) \wedge h^{\rho(l)}_F(y) : l = 1, 2, \ldots, m_x\right\},$$

where $m_x = \max_{y \in X}\{m(h_{\mathcal{H}}(x, y)), m(h_F(y))\}$ and $h_{\mathcal{H}^c}(x) = \{1 - h_F^{\rho(l)}(x) : l = 1, 2, \ldots, m\}$.
The pair $(\underline{\mathcal{H}}(F), \overline{\mathcal{H}}(F))$ is said to be HF rough set of F with respect to $(X, \mathcal{H})$, and $\underline{\mathcal{H}}, \overline{\mathcal{H}} : \mathbf{HF(X)} \to \mathbf{HF(X)}$ are referred to as lower and upper HF rough approximation operators, respectively,

Theorem 8.1 *Let $(X, \mathcal{H})$ be an HF approximation space, and $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ the HF approximation operators induced from $(X, \mathcal{H})$. Then, for every $F \in \mathbf{HF(X)}$, $(x, y) \in X \times X$,*

(1) *$\mathcal{H}$ is reflexive iff $\underline{\mathcal{H}}(F) \subseteq F$ iff $F \subseteq \overline{\mathcal{H}}(F)$,*
(2) *$\mathcal{H}$ is symmetric iff $h_{\underline{\mathcal{H}}(1_{X-\{x\}})}(y) = h_{\underline{\mathcal{H}}(1_{X-\{y\}})}(x)$ iff $h_{\overline{\mathcal{H}}(1_x)}(y) = h_{\overline{\mathcal{H}}(1_y)}(x)$,*
(3) *$\mathcal{H}$ is transitive iff $\underline{\mathcal{H}}(F) \subseteq \underline{\mathcal{H}}(\underline{\mathcal{H}}(F))$ iff $\overline{\mathcal{H}}(\overline{\mathcal{H}}(F)) \subseteq \overline{\mathcal{H}}(F)$.*

8.1 Hesitant fuzzy topological spaces

In this subsection, we define some fundamental concepts related to HF topological spaces in the sense of [29].

Definition 8.5 (cf. [29]) Let X be a non-empty set, an HF topology in the sense of Lowen's is a family Ω of HF sets that satisfies the following properties:

(1) $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega$, for every $\alpha_j \in [0, 1]$, $j = 1, 2, \ldots, n$,
(2) $F \cap G \in \Omega$, for any $F, G \in \Omega$,
(3) $\bigcup_{i \in \Lambda} F_i \in \Omega$, for any $F_i \in \Omega,\ i \in \Lambda$.

The pair (X, Ω) is said to be an HF topological space and each HF set F in Ω is said to be an HF open set in (X, Ω). An HF closed set in (X, Ω) is the complement of an HF open set in (X, Ω).

If $\emptyset, X \in \Omega$, replaces the condition (1) of Definition 8.5, then Ω is a HF topology according to [5]. In Lowen's sense, an HF topology must obviously be an HF topology in Chang's sense. We will use HF topology in Lowen's sense.

Definition 8.6 Let (X, Ω) be an HF topological space. The HF interior and HF closure of F are defined as follows for every $F \in \mathbf{HF(X)}$, respectively:
$int(F) = \bigcap\{H : H \in \Omega\ \ and\ \ H \subseteq F\}$,
$cl(F) = \bigcup\{K : K^c \in \Omega\ \ and\ \ F \subseteq K\}$,
and $int\ \ and\ \ cl : \mathbf{HF(X)} \to \mathbf{HF(X)}$ are, respectively, called the HF interior operator and the HF closure operator of Ω.

Theorem 8.2 *Let (X, Ω) be an HF topological space. Then for every $F \in \mathbf{HF(X)}$,*

(1) *F is an HF open set in (X, Ω) if and only if $int(F) = F$,*
(2) *F is an HF closed set in (X, Ω) if and only if $cl(F) = F$.*

Theorem 8.3 *Let (X, Ω) be an HF topological space. Then the following properties hold: for any $F, G \in \mathbf{HF(X)}$ and $\alpha_j \in [0, 1]$, $j = 1, 2, \ldots, n$,*

(1) $(int(F))^c = cl(F^c)$,
(2) $(cl(F))^c = int(F^c)$,
(3) $int(\alpha_1, \alpha_2, \ldots, \alpha_n)_X = (\alpha_1, \alpha_2, \ldots, \alpha_n)_X$,
(4) $cl(\alpha_1, \alpha_2, \ldots, \alpha_n)_X = (\alpha_1, \alpha_2, \ldots, \alpha_n)_X$,

(5) $int(F) \subseteq F$,
(6) $F \subseteq cl(F)$,
(7) $int(int(F)) = int(F)$,
(8) $cl(cl(F)) = cl(F)$,
(9) $int(F \cap G) = int(F) \cap int(G)$,
(10) $cl(F \cup G) = cl(F) \cup cl(G)$.

Theorem 8.4 (1) *If an HF operator $int : \mathbf{HF(X)} \rightarrow \mathbf{HF(X)}$ satisfies properties* (3), (5), (7), (9), *then there exists an HF topology Ω_{int} on X such that $int_{\Omega_{int}} = int$,*
(2) *If an HF operator $cl : \mathbf{HF(X)} \rightarrow \mathbf{HF(X)}$ satisfies properties* (4), (6), (8), (10), *then there exists an HF topology Ω_{cl} on X such that $cl_{\Omega_{cl}} = cl$.*

Next, we study the transformation from HF approximation spaces to HF topological spaces and vice-versa.

8.2 From HF approximation spaces to HF topological spaces

This section defines the relationship between HF rough approximation spaces and HF topological spaces by generalizing HF rough set theory in the framework of HF topological spaces.

Assume that X is a non-empty and finite universe of discourse, $\mathcal{H}$ is an HF relation on X, and $\underline{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are the HF rough approximation operators.

Denote
$\Omega_{\mathcal{H}} = \{F \in \mathbf{HF(X)} : \underline{\mathcal{H}}(F) = F\}$.

Proposition 8.1 *If $\mathcal{H}$ is an HF reflexive and transitive relation on X, and $F_i \in \mathbf{HF(X)}$ for every $i \in \Lambda$, then*

$$\underline{\mathcal{H}}\left(\bigcup_{i \in \Lambda} \underline{\mathcal{H}}(F_i)\right) = \bigcup_{i \in \Lambda} \underline{\mathcal{H}}(F_i).$$

Theorem 8.5 *If $\mathcal{H}$ is an HF reflexive and transitive relation on X, then $\Omega_{\mathcal{H}}$ is an HF topology on X.*

Theorem 8.6 *If $\mathcal{H}$ is an HF reflexive and transitive relation on X, then $\{\underline{\mathcal{H}}(F) : F \in \mathbf{HF(X)}\}$ is an HF topology on X.*

Theorem 8.7 *Let $(X, \Omega_{\mathcal{H}})$ be the HF topological space induced from an HF reflexive and transitive approximation space $(X, \mathcal{H})$, i.e., $\Omega_{\mathcal{H}} = \{\underline{\mathcal{H}}(F) : F \in \mathbf{HF(X)}\}$. Then, for any $F \in \mathbf{HF(X)}$,*

(1) $\underline{\mathcal{H}}(F) = int_{\Omega_{\mathcal{H}}}(F) = \bigcup\{\underline{\mathcal{H}}(G) : \underline{\mathcal{H}}(G) \subseteq F, G \in \mathbf{HF(X)}\}$,
(2) $\overline{\mathcal{H}}(F) = cl_{\Omega_{\mathcal{H}}}(F)$
$= \bigcap\{(\underline{\mathcal{H}}(G))^c : (\underline{\mathcal{H}}(G))^c \supseteq F, G \in \mathbf{HF(X)}\}$
$= \bigcap\{\overline{\mathcal{H}}(G) : \overline{\mathcal{H}}(G) \supseteq F, G \in \mathbf{HF(X)}\}$.

Theorem 8.8 *Let $(X, \mathcal{H})$ be an HF reflexive and transitive approximation space and $(X, \Omega_{\mathcal{H}})$ be the HF topological space induced by $(X, \mathcal{H})$. Then*

$$h_{\mathcal{H}}(x, y) = \wedge_{G \in (y)_{\Omega_{\mathcal{H}}}} h_G(x),$$

where

$$(y)_{\Omega_{\mathcal{H}}} = \{G \in \mathbf{HF(X)} : G^c \in \Omega_{\mathcal{H}}, h_G(y) = \{1\}\}.$$

8.3 From HF topological spaces to HF approximation spaces

As seen from the Sect. 8.2, an HF topological space with HF interior and closure operators that are the lower and upper approximation operators of the provided HF approximation space can be generated from a HF reflexive and transitive approximation space. This subsection assumes the reverse problem: how can a HF topological space be associated with a HF approximation space and generate the same HF topological space.

Theorem 8.9 *Let (X, Ω) be an HF topological space, and $int,\ cl : \mathbf{HF(X)} \to \mathbf{HF(X)}$ be its HF interior operator and HF closure operator, respectively. Then for any $F \in \mathbf{HF(X)}$, there exists a HF reflexive and transitive relation $\Omega_{\mathcal{H}}(F) = int(F)$ and $\Omega_{\mathcal{H}}(F) = cl(F)$ if and only if int satisfies axioms (1) and (2), or equivalently, cl satisfies axioms (3) and (4): for every $F, G \in \mathbf{HF(X)}$, $(\alpha_1, \alpha_2, \ldots, \alpha_n)_X \in P([0, 1])$ (the power set of the closed interval $[0, 1]$),*

(1) $int\,(F \cup (\alpha_1, \alpha_2, \ldots, \alpha_n)_X) = int(F) \cup (\alpha_1, \alpha_2, \ldots, \alpha_n)_X$,
(2) $int(F \cap G) = int(F) \cap int(G)$,
(3) $cl\,(F \cap (\alpha_1, \alpha_2, \ldots, \alpha_n)_X) = cl(F) \cap (\alpha_1, \alpha_2, \ldots, \alpha_n)_X$,
(4) $cl(F \cup G) = cl(F) \cup cl(G)$.

Definition 8.7 Let (X, Ω) be an HF topological space and $int,\ cl : \mathbf{HF(X)} \to \mathbf{HF(X)}$ the induced HF interior and closure operators, respectively. We call (X, Ω) an HF rough topological space if int satisfies the conditions (1) and (2), or if cl satisfies the conditions (3) and (4), respectively.

Let $\mathfrak{R}$ be the collection of all HF reflexive and transitive relations on X, and $\mathfrak{T}$ the collection of all HF rough topological spaces.

Theorem 8.10 (1) *If $\mathcal{H} \in \mathfrak{R}$, $\Omega_{\mathcal{H}}$ is defined by $\Omega_{\mathcal{H}} = \{F \in \mathbf{HF(X)} : \underline{\mathcal{H}}(F) = F\}$ and $\mathcal{H}_{\Omega_{\mathcal{H}}}$ by $h_{\mathcal{H}_{\Omega}}(x, y) = h_{cl(1_y)}(x)$, $(x, y) \in X \times X$, then $\mathcal{H}_{\Omega_{\mathcal{H}}} = \mathcal{H}$.*
(2) *If $\Omega \in \mathfrak{T}$, $\mathcal{H}_{\Omega}$ is defined by $h_{\mathcal{H}_{\Omega}}(x, y) = h_{cl(1_y)}(x)$, $(x, y) \in X \times X$ and $\Omega_{\mathcal{H}_{\Omega}}$ by $\Omega_{\mathcal{H}} = \{F \in \mathbf{HF(X)} : \underline{\mathcal{H}}(F) = F\}$, then $\Omega_{\mathcal{H}_{\Omega}} = \Omega$.*
(3) *There exists a one-one correspondence between $\mathfrak{R}$ and $\mathfrak{T}$.*

9 Conclusion

Topology provides a useful theoretical framework for the study of fuzzy sets and rough sets. In fuzzy rough set theory, approximation operators are fuzzy topological operators. Therefore, a conjoint analysis of fuzzy rough set theory and topology is necessary, which many researchers have done since the area's inception. There are various generalizations of fuzzy sets which have further been generalized in the framework of rough sets. All these generalized structures induce generalized fuzzy topologies and vice-versa. This survey focuses on topological structures on fuzzy rough sets, L-fuzzy rough sets, intuitionistic fuzzy rough sets, multi-fuzzy rough sets, and hesitant fuzzy rough sets.

There are still some unsolved problems/areas that can be discussed in future research on the topological properties of fuzzy rough sets. Some of them are as follows:

(1) Proximity structures [19, 21, 22, 55] and near sets [41–43] in combination with rough sets have various applications in the field of image analysis, pattern recognition, and many other fields [52, 53, 57–60]. The topological study of proximity structures and near sets in the framework of fuzzy rough sets is still unexplored.

(2) Various extension problems of fuzzy topology on topological structures of several fuzzy approximation spaces are still open to get attention from researchers.
(3) The area of topological groups can also be explored in the framework of fuzzy approximation spaces.
(4) In 2019, Zhao et al. [78] introduced the concept of hesitant neutrosophic rough approximation operator over two universes and applied them to handle a decision making problem in medical diagnosis. The topological study on hesitant neutrosophic rough sets is still missing in the literature.

This survey provides a cutting-edge and comprehensive reference for topological structures on fuzzy rough sets and its various generalizations to date.

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