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A study of bi-bases of ternary semigroups

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Abstract
In this paper, we introduce the bi-bases of a ternary semigroup. The results of this paper are based on the bi-ideals generated by a non-empty subset of a ternary semigroup. Moreover, we define the quasi-order relation of a ternary semigroup and study some of their interesting properties.

Keywords Ternary semigroup · Bi-bases · Bi-ideal

Mathematics Subject Classification 20M12 · 20N99

1 Introduction and preliminaries

The idea of investigation of n-ary algebras, i.e. the sets with one n-ary operation, was given by Kasner [3]. In particular, n-ary semigroups are known as ternary semigroups for n = 3 with one associative operation [6]. Kerner [4] expressed many applications of ternary structures in physics. The concept of ideal in ternary semigroup was given by Sioson [7]. He also defined regular ternary semigroups. The properties of quasi-ideals and bi-ideals in ternary semigroups were studied by Dixit and Dewan [1].

Tamura [8] introduced the notion of a (right)left base of semigroup. Later, Fabrici described a semigroup structure containing one-sided bases [2]. Thongkam and Changphas [9] introduced the notion of left bases and right bases of a ternary semigroup. Kumoon and Changphas [5] introduced the concept of bi-bases in the semigroups and discussed some interesting results.

To start with, we need the following.

Definition 1.1 [6] A non-empty set S is called a ternary semigroup if there exists a ternary operation $S \times S \times S \rightarrow S$, written as$(x_1, x_2, x_3) \rightarrow [x_1 x_2 x_3]$, satisfying the following

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identity for any $x_1, x_2, x_3, x_4, x_5 \in S$,

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]].$$

For non-empty subsets A, B and C of a ternary semigroup S,

$$[ABC] := \{[abc] : a \in A, b \in B \text{ and } c \in C\}.$$

If $A = \{a\}$, then we write $[\{a\}BC]$ as $[aBC]$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $[AbC]$ and $[ABc]$, respectively. Throughout the paper, we denote $[x_1x_2x_3]$ by $x_1x_2x_3$ and $[ABC]$ as ABC.

Definition 1.2 [7] A non-empty subset B of a ternary semigroup S is called a ternary subsemigroup of S, if $BBB \subseteq B$.

Definition 1.3 [1] A non-empty subset B of a ternary semigroup S is called a bi-ideal of S if $BSBSB \subseteq B$.

Proposition 1.1 [1] *Let B be a non-empty subset of a ternary semigroup S without identity. Then $B \cup [BBB] \cup [BSBSB]$ is the smallest bi-ideal of S containing B.*

Remark 1.1 In this paper, smallest bi-ideal of S containing B is denoted by $(B)_b$.

2 Main results

Definition 2.1 Let S be a ternary semigroup. A non-empty subset B of S is called a bi-base of S if it satisfies the following conditions:

(1) $S = (B)_b$ (i.e. $S = B \cup BBB \cup BSBSB$);
(2) If A is a subset of B such that $S = (A)_b$, then $A = B$.

Example 2.1 Let $S = \{a, b, c, d\}$ with $xyz = (x \circ y) \circ z$, for all $x, y, z \in S$ and a ternary operation, $'\circ'$ given by the following table:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	b
d	a	a	b	b

Then S is a ternary semigroup. Let $B = \{b, c, d\}$, then clearly $(B)_b = B \cup BBB \cup BSBSB = S$ and there is no proper subset A of B such that $S = (A)_b$. This shows that B is a bi-base of S.

Theorem 2.1 *If B is a bi-base of a ternary semigroup S and $a, b \in B$ such that $a \in bbb \cup bSbSb$, then $a = b$.*

Proof Let B be a bi-base of a ternary semigroup S and $a, b \in B$ such that $a \in bbb \cup bSbSb$, and suppose that $a \neq b$. Consider $A = B \backslash \{a\}$, then $A \subseteq B$. Since $a \neq b$, therefore $b \in A$. Clearly, $(A)_b \subseteq S$. Let $x \in S$, then by $S = (B)_b$, we have $x \in B \cup BBB \cup BSBSB$. Now, three cases arise:

Case 1: For an element $x \in B$, we have two subcases

Subcase 1.1: If $x \neq a$, then $x \in B\backslash\{a\} = A \subseteq (A)_b$.

Subcase 1.2: If $x = a$, then by assumption, we have $x = a \in bbb \cup bSbSb \subseteq AAA \cup ASASA \subseteq (A)_b$.

Case 2: If $x \in BBB$. Then for $x = b_1b_2b_3$, for some $b_1, b_2, b_3 \in B$, we have eight subcases

Subcase 2.1: If $b_1 = a = b_2 = b_3$. Then, by assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \\ &\in (bbb \cup bSbSb)(bbb \cup bSbSb)(bbb \cup bSbSb) \\ &\subseteq (AAA \cup ASASA)(AAA \cup ASASA)(AAA \cup ASASA) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2: If $b_1 \neq a$, $b_2 = a$, $b_3 = a$. Then, by assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \\ &\in (B\backslash\{a\})(bbb \cup bSbSb)(bbb \cup bSbSb) \\ &= (B\backslash\{a\}bbb \cup B\backslash\{a\}bSbSb)(bbb \cup bSbSb) \\ &= B\backslash\{a\}bbbbbb \cup B\backslash\{a\}bbbbSbSb \cup B\backslash\{a\}bSbSbbbb \cup B\backslash\{a\}bSbSbbbSbSb \\ &\subseteq AAAAAAA \cup AAAASASA \cup AASASAAAA \cup AASASAASASA \\ &\subseteq ASASA \subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 2.3 for $b_2 \neq a$, $b_1 = a$, $b_3 = a$ and subcase 2.4 for $b_3 \neq a$, $b_1 = a$, $b_2 = a$.

Subcase 2.5: If $b_1 \neq a$, $b_2 \neq a$, $b_3 = a$. Then, by assumption and $A = B\backslash \{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \\ &\in (B\backslash\{a\})(B\backslash\{a\})(bbb \cup bSbSb) \\ &= B\backslash\{a\}B\backslash\{a\}bbb \cup B\backslash\{a\}B\backslash\{a\}bSbSb \\ &\subseteq AAAAA \cup AAASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 2.6 for $b_2 \neq a$, $b_3 \neq a$, $b_1 = a$ and subcase 2.7 for $b_1 \neq a$, $b_3 \neq a$, $b_2 = a$.

Subcase 2.8: If $b_1 \neq a$, $b_2 \neq a$ and $b_3 \neq a$. By assumption and for $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \\ &\in (B\backslash\{a\})(B\backslash\{a\})(B\backslash\{a\}) \\ &\subseteq AAA \\ &\subseteq (A)_b. \end{aligned}$$

Case 3: If $x \in BSBSB$. Then $x = b_1s_1b_2s_2b_3$, for some $b_1, b_2, b_3 \in B$ and $s_1, s_2 \in S$. Again, we have eight subcases.

Subcase 3.1: If $b_1 = a = b_2 = b_3$. By assumption

$$\begin{aligned} x &= b_1s_1b_2s_2b_3 \\ &\in (bbb \cup bSbSb)S(bbb \cup bSbSb)S(bbb \cup bSbSb) \\ &\subseteq (AAA \cup ASASA)S(AAA \cup ASASA)S(AAA \cup ASASA) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: If $b_1 \neq a$, $b_2 = a$, $b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 s_1 b_2 s_2 b_3 \\ &\in (B\backslash\{a\})S(bbb \cup bSbSb)S(bbb \cup bSbSb) \\ &= (B\backslash\{a\}Sbbb \cup B\backslash\{a\}SbSbSb)S(bbb \cup bSbSb) \\ &= B\backslash\{a\}SbbbSbbb \cup B\backslash\{a\}SbbbSbSbSb \cup B\backslash\{a\}SbSbSbSbbb \cup B\backslash\{a\}SbSbSbSbSbSb \\ &\subseteq ASAAASAAA \cup AAAASASA \cup ASASASAAAA \cup ASASASASASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 3.3 for $b_2 \neq a$, $b_1 = a$, $b_3 = a$ and the subcase 3.4 for $b_3 \neq a$,$b_1 = a$, $b_2 = a$.
Subcase 3.5: If $b_1 \neq a$,$b_2 \neq a$,$b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 s_1 b_2 s_2 b_3 \\ &\in (B\backslash\{a\})S(B\backslash\{a\})S(bbb \cup bSbSb) \\ &= B\backslash\{a\}SB\backslash\{a\}Sbbb \cup B\backslash\{a\}SB\backslash\{a\}SbSbSb \\ &\subseteq ASASAAA \cup ASASASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 3.6 for $b_2 \neq a$, $b_3 \neq a$, $b_1 = a$ and subcase 3.7 for $b_1 \neq a$, $b_3 \neq a$, $b_2 = a$.
Subcase 3.8: If $b_1 \neq a$, $b_2 \neq a$ and $b_3 \neq a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 s_1 b_2 s_2 b_3 \\ &\in (B\backslash\{a\})S(B\backslash\{a\})S(B\backslash\{a\}) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Thus in all cases $x \in (A)_b$, it implies $(A)_b = S$, which is a contradiction, as B is a bi-base of S. Hence, we have $a = b$ as required. □

Theorem 2.2 *Let B be a bi-base of a ternary semigroup S and $a, b, c, d \in B$ such that $a \in bcd \cup bScSd$, then $a = b$ or $a = c$ or $a = d$.*

Proof Assume that $a \in bcd \cup bScSd$ and if possible $a \neq b$, $a \neq c$ and $a \neq d$. Consider $A = B\backslash\{a\}$, then $A \subseteq B$. Since $a \neq b$, $a \neq c$ and $a \neq d$, we have $b, c, d \in A$. Therefore $(A)_b \subseteq S$. Let $x \in S$. Then, by $S = (B)_b$, we have $x \in B \cup BBB \cup BSBSB$. Now, three cases arise:

Case 1: For $x \in B$, we have two subcases
Subcase 1.1: If $x \neq a$, then by assumption $x \in B\backslash\{a\} = A \subseteq (A)_b$.
Subcase 1.2: If $x = a$, then by hypothesis $x = a \in bcd \cup bScSd \subseteq AAA \cup ASASA \subseteq (A)_b$.

Case 2: If $x \in BBB$. Then $x = b_1 b_2 b_3$, for some $b_1, b_2, b_3 \in B$. Now, we have eight subcases
Subcase 2.1: If $b_1 = a = b_2 = b_3$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 b_2 b_3 \\ &\in (bcd \cup bScSd)(bcd \cup bScSd)(bcd \cup bScSd) \\ &\subseteq (AAA \cup ASASA)(AAA \cup ASASA)(AAA \cup ASASA) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 2.2: If $b_1 \neq a, b_2 = a, b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \in (B\backslash\{a\})(bcd \cup bScSd)(bcd \cup bScSd) \\ &= (B\backslash\{a\}bcd \cup B\backslash\{a\}bScSd)(bcd \cup bScSd) \\ &= B\backslash\{a\}bcdbcd \cup B\backslash\{a\}bcdbScSd \cup B\backslash\{a\}bScSdbcd \cup B\backslash\{a\}bScSdbScSd \\ &\subseteq AAAAAAA \cup AAAASASA \cup AASASAAAA \cup AASASAASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 2.3 for $b_2 \neq a, b_1 = a, b_3 = a$ and subcase 2.4 for $b_3 \neq a, b_1 = a, b_2 = a$.

Subcase 2.5: If $b_1 \neq a, b_2 \neq a, b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \in (B\backslash\{a\})(B\backslash\{a\})(bcd \cup bScSd) \\ &= B\backslash\{a\}B\backslash\{a\}bcd \cup B\backslash\{a\}B\backslash\{a\}bScSd \\ &\subseteq AAAAA \cup AAASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 2.6 for $b_2 \neq a, b_3 \neq a, b_1 = a$ and the subcase 2.7 for $b_1 \neq a, b_3 \neq a, b_2 = a$.

Subcase 2.8: If $b_1 \neq a, b_2 \neq a$ and $b_3 \neq a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1b_2b_3 \\ &\in (B\backslash\{a\})(B\backslash\{a\})(B\backslash\{a\}) \\ &\subseteq AAA \\ &\subseteq (A)_b. \end{aligned}$$

Case 3: If $x \in BSBSB$. Then $x = b_1s_1b_2s_2b_3$, for some $b_1, b_2, b_3 \in B$ and $s_1, s_2 \in S$. Again, we have eight subcases

Subcase 3.1: If $b_1 = a = b_2 = b_3$. By assumption, we have

$$\begin{aligned} x &= b_1s_1b_2s_2b_3 \\ &\in (bcd \cup bScSd)S(bcd \cup bScSd)S(bcd \cup bScSd) \\ &\subseteq (AAA \cup ASASA)S(AAA \cup ASASA)S(AAA \cup ASASA) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Subcase 3.2: If $b_1 \neq a, b_2 = a, b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1s_1b_2s_2b_3 \\ &\in (B\backslash\{a\})S(bcd \cup bScSd)S(bcd \cup bScSd) \\ &= (B\backslash\{a\}Sbcd \cup B\backslash\{a\}SbScSd)S(bcd \cup bScSd) \\ &= B\backslash\{a\}SbcdSbcd \cup B\backslash\{a\}SbcdSbScSd \cup B\backslash\{a\}SbScSdSbcd \cup B\backslash\{a\}SbScSdSbScSd \\ &\subseteq ASAAASAAA \cup AAAASASA \cup ASASASAAAA \cup ASASASASASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 3.3 for $b_2 \neq a, b_1 = a, b_3 = a$ and subcase 3.4 for $b_3 \neq a, b_1 = a, b_2 = a$.

Subcase 3.5: If $b_1 \neq a, b_2 \neq a, b_3 = a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 s_1 b_2 s_2 b_3 \in (B\backslash\{a\})S(B\backslash\{a\})S(bcd \cup bScSd) \\ &= B\backslash\{a\}SB\backslash\{a\}Sbcd \cup B\backslash\{a\}SB\backslash\{a\}SbScSd \\ &\subseteq ASASAAA \cup ASASASASA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Similarly, we can prove the subcase 3.6 for $b_2 \neq a, b_3 \neq a, b_1 = a$ and subcase 3.7 for $b_1 \neq a, b_3 \neq a, b_2 = a$.

Subcase 3.8: If $b_1 \neq a, b_2 \neq a$ and $b_3 \neq a$. By assumption and $A = B\backslash\{a\}$, we have

$$\begin{aligned} x &= b_1 s_1 b_2 s_2 b_3 \\ &\in (B\backslash\{a\})S(B\backslash\{a\})S(B\backslash\{a\}) \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Thus in all cases $x \in (A)_b$. It follows that $(A)_b = S$, which is a contradiction as B is a bi-base S. Hence, $a = b$ or $a = c$ or $a = d$. □

Definition 2.2 Let S be a ternary semigroup.Then a quasi-order on S is defined as $a \leqslant_b b \Leftrightarrow (a)_b \subseteq (b)_b$ for any $a, b \in S$.

Theorem 2.3 *Let B be a bi-base of a ternary semigroup S. Then $a \leqslant_b b$, $a \leqslant_b c$ and $b \leqslant_b c$ if and only if $a = b = c$ for any $a, b, c \in S$.*

Proof It is straightforward. □

Theorem 2.4 *Let B be a bi-base of a ternary semigroup S such that $a, b, c, d \in B$ and $s \in S$, then following statements are true:*

(1) *If $a \in bcd \cup bcdbcdbcd \cup bcdSbcdSbcd$, then $a = b$ or $a = c$ or $a = d$.*
(2) *If $a \in bscsd \cup bscsdsbscsdsbscsd \cup bscsdSbscsdSbscd$, then $a = b$ or $a = c$ or $a = d$.*

Proof (1) Assume that $a \in bcd \cup bcdbcdbcd \cup bcdSbcdSbcd$ and suppose that $a \neq b$, $a \neq c$ and $a \neq d$. Let $A = B\backslash\{a\}$. Then $A \subseteq B$. Since $a \neq b, a \neq c$ and $a \neq d$, we have $b, c, d \in A$. We show that $B \subseteq (A)_b$. Let $x \in B$ and if $x \neq a$, then $x \in A$, and so $x \in (A)_b$. If $x = a$ then by assumption, we have

$$\begin{aligned} x &\in bcd \cup bcdbcdbcd \cup bcdSbcdSbcd \\ &\subseteq AAA \cup AAAAAAAAA \cup AAASAAASAAA \\ &\subseteq ASASA \\ &\subseteq (A)_b. \end{aligned}$$

Thus $B \subseteq (A)_b$. This implies that $(B)_b \subseteq (A)_b$. Since B is a bi-base of S, therefore $S = (B)_b \subseteq (A)_b \subseteq S$. It implies $S = (A)_b$, which is a contradiction. Hence $a = b$ or $a = c$ or $a = d$.

(2) The proof is similar to (1). □

Theorem 2.5 *Let B be a bi-base of a ternary semigroup S. Then the following statements are true:*

(1) *For any $a, b, c, d \in B$ if $a \neq b$, $a \neq c$ and $a \neq d$ then $a \nleqslant_b bcd$.*

(2) *For any* $a, b, c, d \in B$ *and* $s \in S$, *if* $a \neq b$, $a \neq c$ *and* $a \neq d$, *then* $a \nleqslant_b bscsd$.

Proof (1) For any $a, b, c, d \in B$, let $a \neq b$, $a \neq c$ and $a \neq d$. Suppose that $a \leqslant_b bcd$, we have

$$\begin{aligned} a &\subseteq (a)_b \\ &\subseteq (bcd)_b \\ &= bcd \cup bcdbcdbcd \cup bcdSbcdSbcd. \end{aligned}$$

By Theorem 2.4(1), it follows that $a = b$ or $a = c$ or $a = d$. This contradict the assumption. Hence $a \nleqslant_b bcd$.
(2) The proof is similar to (1). □

Theorem 2.6 *A non-empty subset* B *of a ternary semigroup* S *is a bi-base of* S *if and only if it satisfies the following conditions*

(1) *For any* $x \in S$,

   (a) *there exists* $b \in B$ *such that* $x \leqslant_b b$,
   (b) *there exists* $b_1, b_2, b_3 \in B$ *such that* $x \leqslant_b b_1b_2b_3$,
   (c) *there exists* $b_1, b_2, b_3 \in B$, $s \in S$ *such that* $x \leqslant_b b_1sb_2sb_3$.

(2) *For any* $a, b, c, d \in B$, *let* $a \neq b$, $a \neq c$ *and* $a \neq d$, *then* $a \nleqslant_b bcd$.
(3) *For any* $a, b, c, d \in B$ *and* $s \in S$, *let* $a \neq b$, $a \neq c$ *and* $a \neq d$, *then* $a \nleqslant_b bscsd$.

Proof Suppose that B is a bi-base of S, then $S = (B)_b$. To prove (1), let $x \in S$, it implies $x \in B \cup BBB \cup BSBSB$. Now, three cases arise:

Case 1: If $x \in B$. Then, $x = b$, for some $b \in B$. This implies $(x)_b \subseteq (b)_b$. Hence $x \leqslant_b b$.

Case 2: If $x \in BBB$, then $x = b_1b_2b_3$, for some $b_1, b_2, b_3 \in B$. It implies $(x)_b \subseteq (b_1b_2b_3)_b$. Hence, $x \leqslant_b b_1b_2b_3$.

Case 3: If $x \in BSBSB$, then $x = b_1sb_2sb_3$ for some $b_1, b_2, b_3 \in B$ and $s \in S$. It implies $(x)_b \subseteq (b_1sb_2sb_3)_b$. Hence $x \leqslant_b b_1sb_2sb_3$. Proofs of (2) and (3) are similar to the Theorem 2.5.
Conversely, suppose that (1), (2) and (3) holds. Then, we have to prove that B is a bi-base of S. Clearly $(B)_b \subseteq S$ and by (1) $S \subseteq (B)_b$ and so $S = (B)_b$. Now, it remains to show that B is minimal subset of S. Suppose that $S = (A)_b$ for some $A \subseteq B$. Since $A \subseteq B$, there exists $b \in B \backslash A$. Since $b \in B \subseteq S = (A)_b$ and $b \notin A$, it implies $b \in AAA \cup ASASA$. Now, two cases arise:

Case 1: If $b \in AAA$, then $b = a_1a_2a_3$, for some $a_1, a_2, a_3 \in A$. As $b \notin A$ so, $b \neq a_1$, $b \neq a_2$ and $b \neq a_3$. It implies $a_1, a_2, a_3 \in A$. Since $b = a_1a_2a_3$, so $(b)_b \subseteq (a_1a_2a_3)_b$. It follows that $b \leqslant_b a_1a_2a_3$ which contradict (2). Hence B is minimal subset of S such that $(B)_b = S$.

Case 2: If $b \in ASASA$, then $b = a_1sa_2sa_3$, for some $a_1, a_2, a_3 \in A$ and $s \in S$. As $b \notin A$ so, $b \neq a_1$, $b \neq a_2$ and $b \neq a_3$. It implies $a_1, a_2, a_3 \in A$. Since $b = a_1sa_2sa_3$, therefore $(b)_b \subseteq (a_1sa_2sa_3)_b$.
Thus, $b \leqslant_b a_1sa_2sa_3$ which contradict (3). Thus there exists no proper subset A of B such that $(A)_b = S$. Hence, B is bi-base of S. □

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