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The *L*-fuzzy cover spaces and *L*-fuzzy compact open topology

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Abstract
In this paper, a new concept of L-fuzzy cover spaces regarding fuzzy topological spaces is added. Secondly, the ideas of L-fuzzy compact open topology is established and the number of their interesting properties are studied.

Keywords L-Fuzzy space · L-Fuzzy cover spaces and L-Fuzzy compact open topology

Mathematics Subject Classification 54A40 · 54E55

1 Introduction

Zadeh added the essential standards of fuzzy sets in his classical paper [14]. Fuzzy sets have applications in lots of fields of engineering, social technological know-how, economics, clinical science and many others,. In mathematics, topology furnished the most natural framework for the ideas of fuzzy units to flourish. Bayramov [1] added and advanced the concept of L-fuzzy topological spaces. Additionally fibrewise variations of homotopy concept were studied in [8, 11]. The perception of fuzzy homotopy principle was delivered by way of G. Culvacioglu and M. Citil in [10]. The essential organization of fuzzy topological areas was brought by using Abdul Razak Salleh and Mohammad faucet in [2, 7]. Prompted through [2, 7], fuzzy essential organization in fuzzy topological areas became prolonged to numerous fuzzy structure spaces in [5, 6]. The concept of compact-open topology has a important position in defining function spaces in standard topology. Routaray et al. [9] introduced the concept of fuzzy - structure covering map, fuzzy $\Im^*$-Structure compact open topology.

One of the most crucial ideas in general topology is compactness, which many authors [3, 13] have extended to L-topological space.The objective of this article is to present an original concept of L-fuzzy compact-open topology and contribute some theories and effects relating to this concept.

The rest of paper is organized as follows: Section "Introduction" highlights the importance of fuzzy topology. The "Preliminaries" section emphasizes the fuzzy logic and L-fuzzy topology. The next part discusses the L-fuzzy covering space and some related theorems. The

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section "L-fuzzy compact open topology" details suggested the concept of L-fuzzy compact open topology, L-fuzzy connected set and L-fuzzy path connected space. The contributions of the work are summarized in the "Conclusion" section which highlights the achievements and identifies possible direction for further study.

2 Preliminaries

Definition 2.1 [14] A function from a non-empty set X to a unit interval $I = [0, 1]$ is called a fuzzy set λ. I^X denotes the fuzzy set family as a whole.

Definition 2.2 [8] Let X be a set and τ be a family of fuzzy subsets of X. Then τ is called fuzzy topology on X if satisfies the following conditions:

(i) $0_X, 1_X \in \tau$
(ii) If $\lambda, \mu \in \tau$ then $\lambda \wedge \mu \in \tau$
(iii) If $\lambda_i \in \tau$ for all I then $\vee\lambda_i \in \tau$.

Definition 2.3 [12] For any fuzzy set $A \in F(X)$ and any $\lambda \in [0, 1]$, the λ-cut and strong λ-cut of A are respectively defined as follows: $A_\lambda = \{x \in X : A(x) \geq \lambda\}$, $A_{\langle\lambda\rangle} = \{x \in X : A(x) \geq \lambda\}$, where $A(x) = \mu A(x)$ since $A(x)$ is more convenient than $\mu A(x)$.

Definition 2.4 [12] Let I^τ be set of all monotonic decreasing maps $\lambda : \mathbb{R} \to L$ (where L is completely distributive lattice) satisfying:

(i) $\lambda(t) = 1$ for $t < 0$,
(ii) $\lambda(t) = 0$ for $t > 1$.

For $\lambda, \mu \in I^\Gamma$, we define that $\lambda \equiv \mu$ iff $\lambda(t-) = \mu(t-)$ and $\lambda(t+) = \mu(t+)$ for all $t \in \mathbb{R}$, where $\lambda(t-) = \inf_{s<t}\lambda(s)$ and $\lambda(t+) = \sup_{s>t}\lambda(s)$. Then $\equiv$ is an equivalence relation on I^Γ, $[\lambda]$ denotes the equivalence class of $\lambda \in I^\Gamma$ and the quotient set $I^\Gamma / \equiv$ is called the L-fuzzy unit interval which in symbols is written $I(L)$.

We define an L-fuzzy topology τ on $I(L)$ by taking as a subbase $\{L_{t,R_t:t\in\mathbb{R}}\}$, where we define $L_t([\lambda]) = (\lambda(t-))'$ and $R_t([\lambda]) = (\lambda(t+))'$. The topology τ is called the standart topology on $I(L)$, and the base of τ is $\{L_s \wedge R_t : s, t \in \mathbb{R}\}$.

Definition 2.5 Let $f, g : (X, \tau) \to (Y, \sigma)$ be L-fuzzy continuous maps. We say that f is L-fuzzy homotopic to g if there exists an L-fuzzy continuous map $F : (X, \tau) \times (I(L), \tau) \to (Y, \sigma)$ such that $F(a_\alpha, [\lambda_0]) = f(a_\alpha)$ and $F(a_\alpha, [\lambda_1]) = g(a_\alpha)$ for every L-fuzzy point $a_\alpha \in (X, \tau)$ where $i = 0, 1$.

$$\lambda_i(t) = \begin{cases} 1, & t < i \\ 0, & t > i \end{cases}$$

The map F is called an L-fuzzy homotopy between f and g, and written $F : f \cong_L g$.

Definition 2.6 Let (X, τ) and (Y, σ) be any two L-fuzzy space. Let $p : (X, \tau) \to (Y, \sigma)$ is called a L-fuzzy covering space if and only if

(i) p is L-fuzzy onto.
(ii) For every $a_\alpha \in X$ there exists a neighborhood $a_\alpha \in U$ such that $p^{-1}(u) = \cup S_i$ such that each s_i is L-fuzzy homeomorphic to U.

Each s_i is called a sheet. Each u for which $p^{-1}(u) = \cup s_i$ is said to be L-fuzzy covered. $p^{-1}(a_\alpha)$ is called a L-fuzzy fiber.

Definition 2.7 Let $p : (X, \tau) \to (Y, \sigma)$ be a L-fuzzy covering map and let $f : (\tilde{X}, \tilde{\sigma}) \to (Y, \sigma)$ be a L-fuzzy continuous function. Then a map $\tilde{f} : (\tilde{X}, \tilde{\sigma}) \to (X, \tau)$ is said to be L-fuzzy lift on the map f if $p \circ \tilde{f} = f$.

3 *L*-Fuzzy covering space

Theorem 3.1 *Let $p : (X_1, \tau, x_\lambda) \to (X_2, \sigma, x_{\lambda_2})$ be a L-fuzzy covering space generated by τ and σ. Let $f : (Y, \eta, y_\lambda) \to (X_2, \sigma, x_{\lambda_2})$ be a arbitrary L-fuzzy map. If (Y, η, y_λ) is L-fuzzy connected then $f^{'}$ is unique (if it exists).*

Proof Let $f^{''}$ be another L-fuzzy lifting of the map $pf^{'} = f$, $pf^{''} = f$. Define

$$A = \left\{ y \in Y : f^{'}(y) = f^{''}(y) \right\}$$

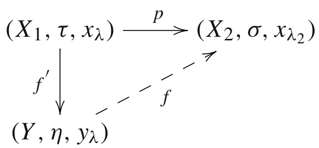
and

$$B = \left\{ y \in Y : f^{'}(y) \neq f^{''}(y) \right\}$$

clearly $Y = A \cup B$ and $A \cap B = \phi$.

For $y \in A$ we have $f^{'}(y) = f^{''}(y)$ and $pf^{'}(y) = f(y)$ implies $pf^{'}(y) \in u$ from this line it is clear that $f^{'}(y) \in p^{-1}(u)$. Again $f^{'}(y) \in S$, $f^{''}(y) \in S$ this implies $y \in f^{'}(S) \cap f^{''}(S) \subset A$.

If not let there exists $z \in B$ so $f^{'}(z) \in S$, $f^{''}(z) \in S$ and $f^{'}(z) \neq f^{''}(z)$ implies $pf^{'}(z) \neq pf^{''}(z)$. So we get $f(z) \neq f(z)$ which is a contradiction. So A is a L-fuzzy open set. Similarly B is also L-fuzzy open. Since Y is L-fuzzy connected one of A and B must be empty.



Clearly $f^{'}(x_\lambda) = y_\lambda$ and $f^{''}(x_\lambda) = y_\lambda$. So $x_\lambda \in A$ and $A \neq \phi$, $B = \phi$. This competes the proof. □

Theorem 3.2 *Let $p : (X_1, \tau, x_\lambda) \to (X_2, \sigma, x_{\lambda_2})$ be a L-fuzzy covering space and ω be a L-fuzzy path in (X_1, τ, x_λ), then there exists a unique $\omega^{'} : I \to X_1$ such that $\omega^{'} = \omega$.*

Proof Since I is L-fuzzy connected, $\omega^{'}$ (if it exists) must be unique. Now we shall prove that $\omega^{'}$ exists.

Case-1: Suppose X_2 itself is L-fuzzy structure covered, i.e., $p^{-1}(X_2) = \cup S_i = E$. x_{λ_1} belongs to some sheet S_i. Then p/S_i is a homeomorphism and ψ be the inverse map, i.e., $\psi : X_2 \to S_i$. Clearly ψ exists and L-fuzzy continuous. Since p/S_i is a homeomorphism. Let $\omega : I \to X_2$ then $\psi \circ \omega : I \to S_i$, i.e., E. Let $\omega^{'} = \psi \circ \omega$. So $\omega^{'}$ is a L-fuzzy path in X_2. This is our required $\omega^{'}$ is a L-fuzzy path in S_i.
To show that $p\omega^{'} = \omega$ $p\omega^{'} = (p/S_i)\omega^{'} = (p/S_i)\psi \circ \omega = \omega$ since $(p/S_i)\psi = id$, i.e., ψ is the inverse of p/S_i.

Case-II: For each $x_\lambda \in X_2$ there exists a L-fuzzy neighborhood $x_\lambda \in X_{U_\lambda}$ which is L-fuzzy covered and each $\{\omega^{-1}(U_{x_\lambda})\}$ is a L-fuzzy open set. Thus the collection of these L-fuzzy open sets will be a L-fuzzy covering I. Since I is L-fuzzy compact it is possible to choose a L-fuzzy finite covering

$$0 = t_0 < t_1 < t_2 \cdots < t_n = 1$$

such that $[t_i, t_{i+1}] \subset \omega^{-1}(U_x)$ for some x and for all i. $\omega[t_i, t_{i+1}] \subset \omega^{-1}(U_x)$ for some x and for all i.

There exists $\omega_1^{'} : [t_0, t_1] \to X_1$ such that $p\omega_1^{'} = \omega/[t_0, t_1]$ similarly there exists $\omega_2^{'} : [t_1, t_2] \to X_1$ such that $p\omega_2^{'} = \omega/[t_1, t_2]$

Define $\omega^{'}(t) = \begin{cases} \omega_1^{'}(t) & t_0 \le t \le t_1 \\ \omega_2^{'}(t) & t_1 \le t \le t_2 \\ \vdots & \\ \omega_n^{'}(t) & t_{n-1} \le t \le t_n \end{cases}$ □

Theorem 3.3 *Let $f : (Y, \tau) \to (X, \sigma)$ admit a L-fuzzy lifting $f^{'}$ generated by τ and σ; then any L-fuzzy homotopy $F : (Y, \tau) \times (I(L), \tau) \to (X, \sigma)$ with $F(a_\alpha, [\lambda_0]) = f(a_\alpha)$ can be L-fuzzy lifted to a L-fuzzy homotopy $F^{'} : (Y, \tau) \times (I(L), \tau) \to E$ with $F^{'}(a_\alpha, [\lambda_0]) = f^{'}(y)$*

Proof Given that $F(a_\alpha, [\lambda_0]) = f(a_\alpha)$. Theorem states that there exists $F^{'}$ such that $F^{'}(a_\alpha, [\lambda_0]) = f^{'}(a_\alpha)$.

Case-1: If the whole space X is evenly L-fuzzy covered then $p^{-1}(X) = \cup S_i = E$. p/S_i defines a L-fuzzy homeomorphism from S_i to X. Thus $S_i \cong X$. Hence there exists inverse map ψ such that $F^{'} = \psi \circ F$ where $F^{'} : Y \times I \to E$. Clearly F, ψ and $F^{'}$ are L-fuzzy continuous. $F^{'}(a_\alpha, [\lambda_0]) = \psi F(a_\alpha, [\lambda_0]) = \psi f(a_\alpha) = f^{'}(a_\alpha)$.

Case-II: Let $a_\alpha \in Y$ and $\lambda_t \in I$, then $F(\lambda_t, a_\alpha) \in X$. Since E is L-fuzzy covering space there exists a L-fuzzy neighborhood $U_{F(a_\alpha,\lambda_t)}$ which is evenly covered $\{F^{-1}(U_{F(a_\alpha,\lambda_t)})\}$ where a_α is fixed and $\lambda_t \in I$.Clearly $\{a_\alpha\} \otimes I$is L-fuzzy compact set and $F^{-1}(U_{F(a_\alpha,\lambda_t)})$ is a L-fuzzy covering set of $\{a_\alpha\} \otimes I$. Now consider $0 = \lambda_{t_0} < \lambda_{t_1} < \lambda_{t_2} < \cdots \lambda_{t_n} = 1$ and we get $\{a_\alpha\} \times [\lambda_{t_i}, \lambda_{t_{i+1}}] \subset F^{-1}(U_{F(a_\alpha,\lambda_t)})$. But $F^{-1}(U_{F(a_\alpha,\lambda_t)})$ is a L-fuzzy open set containing (a_α, λ_t). Choose a neighbourhood N_{a_α} of a_α such that $F(N_1 \times [\lambda_{t_i}, \lambda_{t_{i+1}}]) \subset U_{F(a_\alpha,\lambda_t)}$. Thus $U_{F(a_\alpha,\lambda_t)}$ is evenly L-fuzzy covered. □

Corollary 3.4 *Let (X, τ) and (Y, σ) be two L-fuzzy structure and $(I(L), \tau)$ be L-fuzzy space introduced by τ. Let $\alpha, \beta : I(L) \to X$ and $p : Y \to X$.If $\alpha \simeq_L \beta$ then $\alpha^{'} \simeq_L \beta^{'}$.*

Proof Let $\alpha : (I(L), \tau) \to (X, \tau)$ be any $L-$fuzzy continuous function. Define a function $F : (I(L), \tau) \times (I(L), \tau) \to (X, \tau)$ such that $F(a_\alpha, [\lambda_0]) = \alpha(a_\alpha)$ and $F(a_\alpha, [\lambda_1]) = \beta(a_\alpha)$. Then there exists $F^{'}$ such that $F^{'}(a_\alpha, [\lambda_0]) = \alpha^{'}(a_\alpha)$, $pF^{'}(a_\alpha, [\lambda_1]) = F(a_\alpha, [\lambda_1])$ and $pF^{'}(a_\alpha, [\lambda_1]) = \beta(a_\alpha)$. So$F^{'}(a_\alpha, [\lambda_1])$ is a lifting of β. $\beta^{'}$ is also lifting of β. Since $I(L)$ is L-connected both are equal.$\beta^{'}(a_\alpha) = F^{'}(a_\alpha, [\lambda_1])$. Clearly we have $\alpha^{'} \simeq_L \beta^{'}$. □

Corollary 3.5 *Let (X, τ) and (Y, σ) be any two L-fuzzy path connected spaces and $p : (X, \tau) \to (Y, \sigma)$ be a L-fuzzy continuous function. Let x_λ be a fuzzy point in (X, τ) and $p_* : \Pi_1(X, x_\lambda) \to \Pi_1(Y, p(x_\lambda))$. Then p_* induces a L-fuzzy monomorphism.*

Proof For each $[\alpha], [\beta] \in \Pi_1(X, x_\lambda)$,

$$\begin{aligned} p_*([\alpha] \circ [\beta]) &= p_*[\alpha * \beta] \\ &= [p \circ (\alpha * \beta)] \\ &= [(p \circ \alpha) * (p \circ \beta)] \\ &= [p \circ \alpha] * [p \circ \beta] \\ &== p_*([\alpha]) \circ p_*([\beta]). \end{aligned}$$

Thus p_* is a L-fuzzy homomorphism. Let $[\alpha] \in \Pi_1(X, x_\lambda)$ then $p_*[\alpha] = [\gamma]$ (where $\gamma : I \to Y$) this implies $[p \circ \alpha] \simeq [\gamma]$. So we get $p \circ \alpha \simeq_L \gamma$. Again γ has a lifting in X. Suppose δ is its lifting, i.e., $p\delta = \gamma$. $p\delta(x_t) = \gamma(x_t) = p(x_t)$. Let $(x_t) = y_t$. Now we get $\delta(x_t) \in p^{-1}(y_t)$. But $p^{-1}(y_t)$ is a fuzzy discrete set of points in X. $\delta(I) \subset p^{-1}(y_t)$, $\delta(I)$ is L-fuzzy connected, hence it must a singleton. Since the points are L-fuzzy discrete $\delta(I) = x_\lambda.\delta$ is the L-fuzzy lifting of γ and α is the L-fuzzy lifting of $p \circ \alpha$. Therefore $\alpha \simeq_L \delta$ and $[\alpha]$ is the identity class. □

4 *L*-Fuzzy compact open topology

Let (X, τ) and (Y, σ) be any two L-fuzzy spaces generated by τ and σ.

Let

$$Y^X = \{f : (X, \tau) \to (Y, \sigma) |\ \ f \text{ is } L\text{-fuzzy continuous function.}\}$$

We give this class Y^X a topology called the L-fuzzy compact open topology as follows:

Let

$$\begin{aligned} &\kappa : \{K : I \to X : K \text{ is } L\text{-fuzzy compact in } X\}. \\ &\eta = \{U : I \to Y \text{ such that } U \text{ is } L\text{-fuzzy open in } Y\}. \end{aligned}$$

For any $K \in \kappa$ and $U \in \eta$, let

$$W(K, U) = \left\{\omega \in Y^X : \omega(K) \subseteq U\right\}.$$

The collection $\{W(K, U) : K \in \kappa, U \in \eta\}$ can be as a fuzzy subbase to generate a L-fuzzy topology on the class Y^X, called the L-fuzzy compact-open topology. The class Y^X with this topology is called L-fuzzy compact-open topological space. Unless otherwise stated, Y^X will always have the L-fuzzy compact-open topology.

Theorem 4.1 *Let (X, τ) and (Y, σ) be two L-fuzzy compact space. Let a_α be any L-fuzzy point in X and N be a L-fuzzy open set in the L-fuzzy product space $X \times Y$ containing $a_\alpha \times Y$. Then there exists some L-fuzzy neighborhood W of a_α in X such that $a_\alpha \times X \subseteq W \times Y \subseteq N$.*

Proof It is clear that $x_t \times Y$ is L-fuzzy homeomorphic to Y and hence $x_t \times Y$ is L-fuzzy compact. We cover $\{x_t\} \times Y$ by the basis elements $\{U \times V\}$ (for the L-fuzzy topology of $X \times Y$) lying in N. Since $\{x_t\} \times Y$ is L-fuzzy compact, $\{U \times V\}$ has a finite subcover, say, a finite number of L-fuzzy basis elements $U_1 \times V_1, \cdots U_n \times V_n$. Without loss of generality we assume that $x_t \in U_i$ for each $i = 1, 2 \cdots n$; since otherwise the basis elements would be superfluous. Let $W = \bigcap_{i=1}^n U_i$. Clearly W is L-fuzzy open and $x_t \in W$. We show that

$$W \times Y \subseteq \bigcup_{i=1}^n (U_i \times V_i)$$

Let (x_r, y_s) be any L-fuzzy point in $W \times Y$. We consider the L-fuzzy point (x_t, y_s). Now $(x_t, y_s) \in U_i \times V_i$ for some i; thus $y_s \in V_i$. But $x_r \in U_j$ for every $j = 1, 2 \cdots n$ (because $x_r \in W$). Therefore $(x_r, y_s) \in U_i \times V_i$, as desired. But $U_i \times V_i \subseteq N$for all $i = 1, 2, \cdots, n$; and $W \times Y \subseteq \bigcup_{i=1}^{n}(U_i \times V_i)$. Therefore $W \times Y \subseteq N$. □

Theorem 4.2 *Let Y be a L-fuzzy connected and L-fuzzy locally path connected space. $P(Y, y_0)$ denote the set of all L-fuzzy paths whose initial point is y_0. Then $\phi : P(Y, y_0) \to Y$ is L-fuzzy continuous, L-fuzzy onto and L-fuzzy open map.*

Proof We have

$$W(K, U) = \left\{\omega \in Y^X : \omega(K) \subseteq U\right\}.$$

So clearly $\{\omega : \omega(1) \subset U\}$ be L-fuzzy open set in the L-fuzzy compact open topology. Let $\phi : P(Y, y_0) \to Y$ and consider u as L-fuzzy open set in Y. So

$$\phi^{-1}(u) = \{\omega : \phi(\omega) \in u\} = \{\omega : \omega(1) \in u\}$$

is L-fuzzy open. This implies ϕ is L-fuzzy continuous map.

Let $\omega \in P(Y, y_0)$ and $W = \bigcap_{i=1}^{n} W(K_i, U_i)$ where K_i be L-fuzzy compact and U_i be L-fuzzy open set. Arrange the K_i according to decreasing end points. Choose $j \le n$ such that

$$1 \in K_1 \cap K_2 \cap \cdots K_j$$

and

$$1 \notin K_{j+1} \cap K_{J+2} \cap \cdots K_n$$

Again $\omega \in W(K_i, U_i)$ then $\omega(K_i) \subset U_i$ for all $i = 1, 2, \cdots n$ and $i = 1, 2, \cdots j$. From this it is clear that $\omega(1) \in \bigcap_{i=1}^{n} U_i$. Choose L-fuzzy connected neighborhood of V of $\omega(1)$ such that

$$V \subset \cap_{i=1}^{n} U_i.$$

Choose $t' \in (0, 1)$ such that

$$[t', 1] \cap [K_{j+1} \cup \ . \ . \ . \ \cup K_n] = \phi$$

such that $\omega[t', 1] \subset V$.

We claim that $V \subset \phi(W)$. Let $y^{'} \in V$ to show that $y^{'} \in \phi(W)$, i.e., show that there exists a L-fuzzy path in W whose end point is $y^{'}$. Define a L-fuzzy path $\omega^{'}$ from $\omega(t^{'})$ to $y^{'}$. Now define a L-fuzzy path $\bar{\omega} : I \to Y$

$$\bar{\omega}(t) = \begin{cases} \omega(t), & 0 \le t \le t' \\ \omega'\left(\frac{t-t'}{1-t'}\right), & t' \le t \le 1 \end{cases}$$

For $i = j+1, j+2, \cdots, n$, $\bar{\omega}(K_i) = \omega(K_i) \subset U_i$.

$$\therefore \bar{\omega} \in W(K_i, U_i).$$

For $i = 1, 2, \cdots, j$,

$$\begin{aligned} \bar{\omega}(K_i) &= \bar{\omega}[K_i \cap [0, t']] \cup \bar{\omega}[K_i \cap [t', 1]] \\ &\subset \omega(K_i) \cup \omega'(I) \\ &\subset U_i \cup V = U_i \end{aligned}$$

We get $\bar{\omega}(K_i) \subset U_i$, $i = 1, 2, \cdots$. Let $\bar{\omega} \in W(K_i, U_i)$, so $\bar{\omega} \in \cap_{i=1}^{n} W(K_i, U_i) = W$. This implies $\bar{\omega} \in W$. Since $\phi(\bar{\omega}) = \bar{\omega}(1) = Y'$ we get $V \subset \phi(\omega) \Rightarrow \phi$ is L-fuzzy open.

Now we have to show that ϕ is onto. Clearly ϕ is L-fuzzy path connected as ϕ is L-fuzzy connected and L-fuzzy locally path connected. Let $y \in Y$, then $\exists\ \omega$ from y_0 to y imply $\omega \in P(Y, y_0)$ and $\phi(\omega) =$ end point of $\omega = y_1$. This implies ϕ is onto. □

5 Conclusion and future work

In this work, the concept of L-fuzzy compact open topology has been introduced along with some basic theories. This work will lay the foundation for further research on L-fuzzy compact open topology. We hope to build a concept of fuzzy higher homotopy groups and fuzzy universal covering space using this concept. Further, this topic can be expanded in fuzzy category theory in future.

References

1. Bayramov, S.: The Čech Homology Theory in the Category of Šostak Fuzzy Topological Spaces. Int. J. Contemp. Math. Sci. **5**(9), 433–488 (2010)
2. Culvacioglu, G., Citil, M.: On fuzzy homotopy theory. Adv. Stud. Contemp. Math. (Kyungshang) **12**(1), 163–166 (2006)
3. Chang, C.L.: Fuzzy topological spaces. J. Math. Anal. Appl. **24**, 182–190 (1968)
4. Flori, C.: A topos formulation of history quantum theory. J. Math. Phys. **51**, 1–31 (2010)
5. El-Ghoul, M., Attiya, H.I.: The dynamical fuzzy topological space and its folding. J. Fuzzy Math. **12**(3), 685–693 (2004)
6. Massey, W.S.: Algebraic Topology - An Introduction. Harcourt, Brace and World, New York (1967)
7. Munkres, J.: Topology, 2nd edition, Pearson Prentice Hall, New Jersey
8. Palaniappan, N.: Fuzzy Topology, Alpha Science International, Pangbourne England
9. Routaray, M., Sahu, P.K., Naik, S.: Study on Fuzzy $\Im^*$-structure compact-open topology. Int J Fuzzy Logic Intell Syst **23**(3), 311–317 (2023)
10. Salleh, A.R., Md, A.O.: Tap?: The fundamental groupoid of fuzzy topological spaces. Sains Malaysina **16**(4), 447–454 (1987)
11. Salleh, A.R., Tap, A.O.M.: The fundamental group of fuzzy topological spaces. Sains Malaysina **15**(4), 397–407 (1986)
12. Salleh, A.R.: The homotopy property of the induced homomorphisms on homology groups of fuzzy topological spaces. Fuzzy Sets Syst. **56**, 111–116 (1993)
13. Shi, F.G.: A new definition of fuzzy compactness. Fuzzy Sets Syst. **158**, 1486–1495 (2007)
14. Zadeh, L.A.: Fuzzy Sets. Inf. Control **8**, 338–353 (1965)