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On characterization of being a generalized Fibonacci *Q*-matrix of linear combinations of two generalized Fibonacci *Q*-matrices

A. Öndül[1] · T. Demirkol[1] · H. Özdemir[1]

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Abstract
It is given a characterization of being a matrix $Q^{(k)}_{g(a_3,b_3)}$ of linear combination of a matrix $Q^{(n)}_{g(a_1,b_1)}$ and a matrix $Q^{(m)}_{g(a_2,b_2)}$, where $a_i, b_i \in \mathbb{R}^*$, $i = 1, 2, 3$, $m, n, k \in \mathbb{Z}$, and $Q^{(l)}_{g(a,b)}$ denotes an (a, b)-generalized Fibonacci Q-matrix with $l \in \mathbb{Z}$. In addition, some examples are presented illustrating the main result. Finally, some applications of the main result obtained are given.

Keywords Fibonacci numbers · Fibonacci Q-matrix · Generalized Fibonacci numbers · Linear combination · Matrix equations

Mathematics Subject Classification 15A15 · 15A16 · 15A24 · 11B39 · 11Y55

1 Introduction and preliminaries

In this section, we remind some fundamental definitions and propositions related to Fibonacci and generalized Fibonacci sequences.

The classical Fibonacci sequence $\{F_n\}$ is defined by the recurrence relation

$$F_{n+1} = F_n + F_{n-1} \quad \text{for all integers} \quad n \geq 1,$$

with $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence with negative subscript is determined by the relation

$$F_{-n} = (-1)^{n+1} F_n \tag{1}$$

for all integers $n \geq 1$, see, for instance, [4].

It is well known that the identity

$$F_a F_b - F_c F_d = (-1)^r (F_{a-r} F_{b-r} - F_{c-r} F_{d-r}) \tag{2}$$

✉ T. Demirkol
tpetik@sakarya.edu.tr

1 Department of Mathematics, Sakarya University, 54187 Serdivan, Sakarya, Turkey

holds for all integers r, where a, b, c and d are integers such that $a + b = c + d$ [1]. In addition, the identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \tag{3}$$

is known as Cassini identity, see, for example, [4].

Now, consider the sequence $\{G_n\}$ defined by the relation

$$G_n = G_{n-1} + G_{n-2}$$

for $n \geq 3$ with the initial conditions $G_1 = a$ and $G_2 = b$, where a and b are arbitrary nonzero real numbers. This sequence is called the generalized Fibonacci sequence (or gibonacci sequence) [5]. The constant $\mu = a^2 + ab - b^2$ produced by a and b in the definition is called the characteristic of generalized Fibonacci sequence.

There is a relation between the Fibonacci sequence and the generalized Fibonacci sequence as follows:

$$G_n = aF_{n-2} + bF_{n-1} \quad \text{for} \quad n \geq 3. \tag{4}$$

Of course, there are many different generalizations of Fibonacci sequences, see for example, [2, 3]. However, throughout the work, we shall use the definition in (4) of the generalized Fibonacci sequence. And also, the identity

$$G_{m+n} = G_m F_{n+1} + G_{m-1} F_n \tag{5}$$

in [4] will be used in the remaining parts of the work.

Fibonacci sequences have amazing application in coding, encryption, and decryption, see, for example, [12, 13]. On the other hand, these types of special sequences occur in many places in nature and art, see, for example, [4, 14]. One such example is that generalized Fibonacci numbers appear in bee colonies in examining the genealogical growth from generation to generation, see for example [4].

The main result in this work includes the main results in [10] and [11].

From now on, nth term of generalized Fibonacci sequence defined as in (4) will be denoted by $G_{(a,b)}^{(n)}$. On the other hand, we know that $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ holds for all integer n, where $Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, which is known as Fibonacci Q-matrix [6, 7]. Hence, we easily get that

$$aQ^{n-2} + bQ^{n-1} = \begin{bmatrix} aF_{n-1} + bF_n & aF_{n-2} + bF_{n-1} \\ aF_{n-2} + bF_{n-1} & aF_{n-3} + bF_{n-2} \end{bmatrix} \tag{6}$$

for all integers n with $a, b \in \mathbb{R}^*$. For the sake of simplicity, we will denote the matrix in (6) by $Q_{g(a,b)}^{(n)}$, and call this matrix as (a, b)-generalized Fibonacci Q-matrix.

2 Being a matrix $Q_{g(a_3,b_3)}^{(k)}$ of linear combinations of a matrix $Q_{g(a_1,b_1)}^{(n)}$ and a matrix $Q_{g(a_2,b_2)}^{(m)}$

Consider the matrix equation

$$c_1 Q_{(a_1,b_1)}^{(n)} + c_2 Q_{(a_2,b_2)}^{(m)} = Q_{(a_3,b_3)}^{(k)}, \tag{7}$$

where c_1, c_2 are unknowns, $m, n, k \in \mathbb{Z}$, and $a_i, b_i \in \mathbb{R}^*$, $i = 1, 2, 3$.

The matrix equation (7) is equivalent to the system of linear equations

$$\begin{aligned} c_1(a_1F_{n-2}+b_1F_{n-1})+c_2(a_2F_{m-2}+b_2F_{m-1}) &= a_3F_{k-2}+b_3F_{k-1} \\ c_1(a_1F_{n-3}+b_1F_{n-2})+c_2(a_2F_{m-3}+b_2F_{m-2}) &= a_3F_{k-3}+b_3F_{k-2}. \end{aligned} \tag{8}$$

It is seen that the determinant of coefficient matrix of the system (8) is

$$\begin{vmatrix} a_1F_{n-2}+b_1F_{n-1} & a_2F_{m-2}+b_2F_{m-1} \\ a_1F_{n-3}+b_1F_{n-2} & a_2F_{m-3}+b_2F_{m-2} \end{vmatrix} = (-1)^n(-a_1G^{(m-n+2)}_{(a_2,b_2)}+b_1G^{(m-n+1)}_{(a_2,b_2)}) \tag{9}$$

taking the identity (2) into account. In case $a_1G^{(m-n+2)}_{(a_2,b_2)} = b_1G^{(m-n+1)}_{(a_2,b_2)}$, the determinant (9) is zero, and nonzero, otherwise. First, suppose that the determinant is nonzero. In this case, it is clear that the matrix equation (7) has unique solution such that

$$c_1 = \frac{(-1)^{k-n}(-a_3G^{(m-k+2)}_{(a_2,b_2)}+b_3G^{(m-k+1)}_{(a_2,b_2)})}{-a_1G^{(m-n+2)}_{(a_2,b_2)}+b_1G^{(m-n+1)}_{(a_2,b_2)}} \text{ and } c_2 = \frac{-a_1G^{(k-n+2)}_{(a_3,b_3)}+b_1G^{(k-n+1)}_{(a_3,b_3)}}{-a_1G^{(m-n+2)}_{(a_2,b_2)}+b_1G^{(m-n+1)}_{(a_2,b_2)}}.$$

Next, consider the case where the determinant is zero. Writing $a_1 = \frac{b_1G^{(m-n+1)}_{(a_2,b_2)}}{G^{(m-n+2)}_{(a_2,b_2)}}$ in the augmented matrix of the system of equations (8) under the condition $G^{(m-n+2)}_{(a_2,b_2)} \neq 0$, and then rearranging the entries of the matrix in view of (5) lead to the augmented matrix

$$\left[\begin{array}{cc|c} \frac{b_1G^{(m)}_{(a_2,b_2)}}{G^{(m-n+2)}_{(a_2,b_2)}} & a_2F_{m-2}+b_2F_{m-1} & a_3F_{k-2}+b_3F_{k-1} \\ \frac{b_1G^{(m-1)}_{(a_2,b_2)}}{G^{(m-n+2)}_{(a_2,b_2)}} & a_2F_{m-3}+b_2F_{m-2} & a_3F_{k-3}+b_3F_{k-2} \end{array}\right]. \tag{10}$$

If the first row of this matrix is multiplied by $-\frac{G^{(m-1)}_{(a_2,b_2)}}{G^{(m)}_{(a_2,b_2)}}$ with $G^{(m)}_{(a_2,b_2)} \neq 0$, and then added this row to the second row, and finally the last entry of the second row is rearranged, then it is obtained the augmented matrix

$$\left[\begin{array}{cc|c} \frac{b_1G^{(m)}_{(a_2,b_2)}}{G^{(m-n+2)}_{(a_2,b_2)}} & G^{(m)}_{(a_2,b_2)} & G^{(k)}_{(a_3,b_3)} \\ 0 & 0 & \frac{(-1)^k(a_3G^{(m-k+2)}_{(a_2,b_2)}-b_3G^{(m-k+1)}_{(a_2,b_2)})}{G^{(m)}_{(a_2,b_2)}} \end{array}\right] \tag{11}$$

which is equivalent to (10). So, the system of linear equations corresponding to the matrix (11) has no solution in case $a_3G^{(m-k+2)}_{(a_3,b_3)} - b_3G^{(m-k+1)}_{(a_3,b_3)} \neq 0$, otherwise, there are finitely many solutions. Now, suppose that $a_3G^{(m-k+2)}_{(a_2,b_2)} = b_3G^{(m-k+1)}_{(a_2,b_2)}$.

If we first take $a_3 = \frac{b_3G^{(m-k+1)}_{(a_2,b_2)}}{G^{(m-k+2)}_{(a_2,b_2)}}$ with $G^{(m-k+2)}_{(a_2,b_2)} \neq 0$ in the equation corresponding to the first row of (11), and next rearrange this equation by using (5), then we get finitely many solutions, for the matrix equation (7), based on the parameter $t \in \mathbb{R}^*$ such that $(c_1, c_2) = \left(t, \frac{b_3}{G^{(m-k+2)}_{(a_2,b_2)}} - \frac{tb_1}{G^{(m-n+2)}_{(a_2,b_2)}}\right)$.

Now, consider the case $G^{(m)}_{(a_2,b_2)} = 0$. In this case, the augmented matrix (10) turns into the matrix

$$\left[\begin{array}{cc|c} 0 & 0 & a_3F_{k-2} + b_3F_{k-1} \\ \frac{b_1 G^{(m-1)}_{(a_2,b_2)}}{G^{(m-n+2)}_{(a_2,b_2)}} & a_2F_{m-3} + b_2F_{m-2} & a_3F_{k-3} + b_3F_{k-2} \end{array}\right]. \tag{12}$$

Consistency of the system corresponding to the augmented matrix (12) is possible if $a_3F_{k-2} + b_3F_{k-1} = 0$. It is clear that $k \neq 2$, since $k = 2$ leads to $b_3 = 0$, which is a contradiction. If we first write $a_3 = -\frac{b_3F_{k-1}}{F_{k-2}}$ in the equation corresponding to the second row of the matrix in (12), and next rearrange this row by using (2) and (3), then we obtain the general solution of the matrix equation (7) as

$$(c_1, c_2) = \left(t, \left(\frac{(-b_3(-1)^{k-2})}{F_{k-2}} - \frac{tb_1}{F_{2-n}}\right)\frac{F_{m-2}}{(-b_2)(-1)^{m-2}}\right),\ t \in \mathbb{R}^*.$$

Thus, we have proved the following theorem.

Theorem 1 *For the solutions of the matrix equation $c_1Q^{(n)}_{g(a_1,b_1)} + c_2Q^{(m)}_{g(a_2,b_2)} = Q^{(k)}_{g(a_3,b_3)}$, where c_1, c_2 are unknowns, $m, n, k \in \mathbb{Z}$, and $a_i, b_i \in \mathbb{R}^*$, $i = 1, 2, 3$, the followings are true.*

(1) In case $-a_1G^{(m-n+2)}_{(a_2,b_2)} + b_1G^{(m-n+1)}_{(a_2,b_2)} \neq 0$, there is unique solution such that

$$(c_1, c_2) = \left(\frac{(-1)^k(-a_3G^{(m-k+2)}_{(a_2,b_2)} + b_3G^{(m-k+1)}_{(a_2,b_2)})}{(-1)^n(-a_1G^{(m-n+2)}_{(a_2,b_2)} + b_1G^{(m-n+1)}_{(a_2,b_2)})}, \frac{(-a_1G^{(k-n+2)}_{(a_3,b_3)} + b_1G^{(k-n+1)}_{(a_3,b_3)})}{(-a_1G^{(m-n+2)}_{(a_2,b_2)} + b_1G^{(m-n+1)}_{(a_2,b_2)})}\right).$$

(2) In case $-a_1G^{(m-n+2)}_{(a_2,b_2)} + b_1G^{(m-n+1)}_{(a_2,b_2)} = 0$, if $G^{(m-n+2)}_{(a_2,b_2)} \neq 0$, then

(i) There is no solution if $a_3G^{(m-k+2)}_{(a_2,b_2)} - b_3G^{(m-k+1)}_{(a_2,b_2)} \neq 0$, $G^{(m)}_{(a_2,b_2)} \neq 0$.
(ii) In case $a_3G^{(m-k+2)}_{(a_2,b_2)} - b_3G^{(m-k+1)}_{(a_2,b_2)} = 0$,
(a) There are finitely many solutions such that

$$(c_1, c_2) = \left(t, \frac{b_3}{G^{(m-k+2)}_{(a_2,b_2)}} - \frac{tb_1}{G^{(m-n+2)}_{(a_2,b_2)}}\right),\ t \in \mathbb{R}^*,$$

under the conditions $G^{(m-k+2)}_{(a_2,b_2)} \neq 0$ and $G^{(m)}_{(a_2,b_2)} \neq 0$,
(b) There are finitely many solutions such that

$$(c_1, c_2) = \left(t, \left(\frac{(-b_3(-1)^{k-2})}{F_{k-2}} - \frac{tb_1}{F_{2-n}}\right)\frac{F_{m-2}}{(-b_2)(-1)^{m-2}}\right)$$

under the condition $G^{(m)}_{(a_2,b_2)} = 0$.

□

3 Numerical examples

Now, we give some examples illustrating the theorem.

Example 1 Suppose that $m = 5, n = 4, k = 6, a_1 = 2, a_2 = 5, a_3 = 6, b_1 = 3, b_2 = 7, b_3 = 4$. Under these assumptions, since $-2G^{(3)}_{(5,7)} + 3G^{(2)}_{(5,7)} = -3 \neq 0$, we get $c_1 = \frac{22}{3}$ and $c_2 = -\frac{2}{3}$ by Theorem 1(1).

Actually, it is easily seen that c_1 and c_2 obtained above hold the equality $c_1 Q^{(4)}_{g(2,3)} + c_2 Q^{(5)}_{g(5,7)} = Q^{(6)}_{g(6,4)}$, that is the matrix equality

$$c_1 \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} + c_2 \begin{bmatrix} 50 & 31 \\ 31 & 19 \end{bmatrix} = \begin{bmatrix} 62 & 38 \\ 38 & 24 \end{bmatrix}$$

which has the unique solution such that $c_1 = \frac{22}{3}$ and $c_2 = -\frac{2}{3}$.

Example 2 Assume that $n = 4, m = 5, k = 6, a_1 = 14, a_2 = 5, a_3 = 6, b_1 = 24, b_2 = 7, b_3 = 4$. So, we have $-14G^{(3)}_{(5,7)} + 24G^{(2)}_{(5,7)} = 0$, $G^{(3)}_{(5,7)} = 12 \neq 0$, $6G^{(1)}_{(5,7)} - 4G^{(0)}_{(5,7)} = 22 \neq 0$ (by (1)), and $G^{(5)}_{(5,7)} = 31 \neq 0$. Hence, there is no solution of the matrix equation $c_1 Q^{(4)}_{g(14,24)} + c_2 Q^{(5)}_{g(5,7)} = Q^{(6)}_{g(6,4)}$ by Theorem 1(2)-(i).

Actually, there is no pair (c_1, c_2) satisfying the matrix equation $c_1 Q^{(4)}_{g(14,24)} + c_2 Q^{(5)}_{g(5,7)} = Q^{(6)}_{g(6,4)}$, that is the matrix equality

$$c_1 \begin{bmatrix} 100 & 62 \\ 62 & 38 \end{bmatrix} + c_2 \begin{bmatrix} 50 & 31 \\ 31 & 19 \end{bmatrix} = \begin{bmatrix} 62 & 38 \\ 38 & 24 \end{bmatrix}$$

which has no solution.

Example 3 Take $n = 4, m = 5, k = 6, a_1 = 14, a_2 = 5, a_3 = 6, b_1 = 24, b_2 = 7, b_3 = 15$. So, it is clear that $-14G^{(3)}_{(5,7)} + 24G^{(2)}_{(5,7)} = 0$, $G^{(3)}_{(5,7)} = 12 \neq 0$, and $6G^{(1)}_{(5,7)} - 15G^{(0)}_{(5,7)} = 0$. Thus, by Theorem 1-(ii)-(a), there are finitely many solutions such that $(c_1, c_2) = \left(t, \frac{15}{G^{(1)}_{(5,7)}} - \frac{24t}{G^{(3)}_{(5,7)}}\right) = (t, 3 - 2t)$ with the conditions $G^{(1)}_{(5,7)} = -2 \neq 0$ (by (1)) and $G^{(5)}_{(5,7)} = 31 \neq 0$.

Actually, if we try to find the values c_1 and c_2 satisfying the equation $c_1 Q^{(4)}_{g(14,24)} + c_2 Q^{(5)}_{g(5,7)} = Q^{(6)}_{g(6,15)}$, or equivalently, the matrix equality

$$c_1 \begin{bmatrix} 100 & 62 \\ 62 & 38 \end{bmatrix} + c_2 \begin{bmatrix} 50 & 31 \\ 31 & 19 \end{bmatrix} = \begin{bmatrix} 150 & 93 \\ 93 & 57 \end{bmatrix},$$

then we get $2c_1 + c_2 = 3$, and therefore $(c_1, c_2) = (t, 3 - 2t)$ with $t \in \mathbb{R}^*$.

4 Applications

In this section, we give some new identities related to generalized Fibonacci sequences by using Theorem 1.

Firstly, consider the equations (8). Adding these equations side by side leads to the equation

$$c_1(a_1 F_{n-1} + b_1 F_n) + c_2(a_2 F_{m-1} + b_2 F_m) = a_3 F_{k-1} + b_3 F_k, \tag{13}$$

or equivalently,

$$c_1 G^{(n+1)}_{(a_1,b_1)} + c_2 G^{(m+1)}_{(a_2,b_2)} = G^{(k+1)}_{(a_3,b_3)}. \tag{14}$$

From now on, suppose that $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$. We consider Theorem 1(1). The condition $-a_1 G_{(a_2,b_2)}^{(m-n+2)} + b_1 G_{(a_2,b_2)}^{(m-n+1)} \neq 0$ in the theorem turns into the conditions $m \neq n$ and $\mu \neq 0$.

Now, let's handle the cases $a = \mp b$ and $a^2 \neq b^2$, separately.

First, suppose that $a = b$. In this case, from (13), we get

$$c_1 F_{n+1} + c_2 F_{m+1} = F_{k+1} \tag{15}$$

taking into account $a \neq 0$. On the other hand, in this case, the solutions c_1 and c_2 in Theorem 1(1) becomes as follows:

$$c_1 = (-1)^{k-n} \frac{F_{m-k}}{F_{m-n}} \quad \text{and} \quad c_2 = \frac{F_{k-n}}{F_{m-n}}.$$

If these solutions are written in (15), the identity

$$(-1)^{k-n} F_{m-k} F_{n+1} + F_{k-n} F_{m+1} = F_{m-n} F_{k+1} \tag{16}$$

is obtained. Notice that the identity (16) holds for all $m, n, k \in \mathbb{Z}$ even if $m \neq n$ in the hypothesis. Also, since $a = b$ and $a, b \in \mathbb{R}^*$, in this particular case the condition $\mu \neq 0$ is already satisfied. So, writing $n - 1 + c$, $m - 1 + c$, $k - 1 + c$ instead of n, m, and k in (16) leads to the following result.

Corollary 1 *The identity*

$$(-1)^{k-n} F_{m-k} F_{n+c} = F_{m-n} F_{k+c} - F_{k-n} F_{m+c}$$

holds for all $c, m, n, k \in \mathbb{Z}$. □

Note that when proceeding with the condition $a = -b$, the same result as above is obtained.

Now, suppose that $a^2 \neq b^2$. Writing c_1 and c_2 obtained in Theorem 1(1) in the equation (14) yields

$$\begin{aligned}(-1)^{k-n} & (-a_3 G_{(a_2,b_2)}^{(m-k+2)} + b_3 G_{(a_2,b_2)}^{(m-k+1)}) G_{(a_1,b_1)}^{(n+1)} \\ &= (-a_1 G_{(a_2,b_2)}^{(m-n+2)} + b_1 G_{(a_2,b_2)}^{(m-n+1)}) G_{(a_3,b_3)}^{(k+1)} \\ &\quad - (-a_1 G_{(a_3,b_3)}^{(k-n+2)} + b_1 G_{(a_2,b_2)}^{(k-n+1)}) G_{(a_3,b_3)}^{(k+1)}.\end{aligned} \tag{17}$$

Since $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$, from the equality (17), we get

$$\begin{aligned}(-1)^{k-n} & (-aG_{m-k+2} + bG_{m-k+1}) G_{n+1} \\ &= (-aG_{m-n+2} + bG_{m-n+1}) G_{k+1} - (-aG_{k-n+2} + bG_{k-n+1}) G_{m+1}.\end{aligned} \tag{18}$$

From this, we obtain

$$(-1)^{k-n} (b^2 - a^2) F_{m-k} G_{n+1} = (b^2 - a^2) F_{m-n} G_{k+1} - (b^2 - a^2) F_{k-n} G_{m+1}. \tag{19}$$

Since $a^2 \neq b^2$, from the equality (19), it is obtained that

$$(-1)^{k-n} F_{m-k} G_{n+1} = F_{m-n} G_{k+1} - F_{k-n} G_{m+1}. \tag{20}$$

Notice that in case of $m = n$, the identity is also (20) provided. Also, if it is written $n - 1 + c$, $m - 1 + c$, $k - 1 + c$ instead of n, m, and k in (20), the following result is obtained.

Corollary 2 *For all integers c, m, n, k, the identity*

$$(-1)^{k-n} F_{m-k} G_{n+c} = F_{m-n} G_{k+c} - F_{k-n} G_{m+c}$$

holds with $a^2 \neq b^2$ and $\mu \neq 0$. □

Now, consider Theorem 1(2-ii)–(a). Suppose that $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$. So, the conditions in the first case of Theorem 1(2-ii)–(a) turn into the conditions

$$\begin{aligned}&(b^2 - a^2 - ab)F_{m-n} = 0,\\&aF_{m-n} + bF_{m-n+1} \neq 0,\\&(a^2 - b^2 + ab)F_{m-k} = 0,\\&aF_{m-k} + bF_{m-k+1} \neq 0,\\&aF_{m-2} + bF_{m-1} \neq 0.\end{aligned}$$

Also, we have $c_1 = t$ and $c_2 = b\left(\frac{1}{G_{m-k+2}} - \frac{t}{G_{m-n+2}}\right)$ with $t \in \mathbb{R}^*$. If we write these solutions in (14) by taking $t = 1$, then we get the following result.

Corollary 3 *For all integers m, n, k, the identity*

$$(G_{n+1} - G_{k+1})G_{m-k+2}G_{m-n+2} = bG_{m+1}(G_{m-k+2} - G_{m-n+2})$$

holds, where $-\mu F_{m-n} = \mu F_{m-k} = 0$ and $G_{m-n+2}, G_{m-k+2}, G_m \neq 0$. □

Finally, we handle the second case of Theorem 1(2-ii)-(b). Assuming that $a_1 = a_2 = a_3 = a$ and $b_1 = b_2 = b_3 = b$, we see that the conditions in the second case of Theorem 1(2-ii)-(b) turn into the conditions

$$\begin{aligned}&(b^2 - a^2 - ab)F_{m-n} = 0,\\&aF_{m-n} + bF_{m-n+1} \neq 0,\\&(a^2 - b^2 + ab)F_{m-k} = 0,\\&aF_{m-2} + bF_{m-1} = 0.\end{aligned}$$

If the solutions c_1 and c_2 in this part of the theorem are written in (14), and it is rearranged the equality obtained by using (1), then the following result is obtained.

Corollary 4 *For all integers m, n, k, the identity*

$$G_{k+1} - G_{n+1} = F_{2-m}G_{m+1}\left(\frac{1}{F_{2-k}} + \frac{1}{F_{2-n}}\right)$$

holds, where $-\mu F_{m-n} = \mu F_{m-k} = G_m = 0$ and $G_{m-n+2} \neq 0$. □

NOTE: In case $a_2 = b_2$, the matrix equation (7) turns into the matrix equation

$$d_1 Q^{(n)}_{g(a_1,b_1)} + d_2 Q^m = Q^{(k)}_{g(a_3,b_3)},\ d_i \in \mathbb{R}^*,\ i = 1, 2$$

which was handled in [11]. If $a_2 = b_2$ and $a_3 = b_3$, then the matrix equation (7) turns into the matrix equation

$$h_1 Q^{(n)}_{g(a_1,b_1)} + h_2 Q^m = Q^k,\ h_i \in \mathbb{R}^*,\ i = 1, 2$$

which was considered a special case the main result in [11]. Finally, in case $a_1 = b_1, a_2 = b_2$, and $a_3 = b_3$, the matrix equation (7) turns into the equation

$$e_1 Q^n + e_2 Q^m = Q^k,\ e_i \in \mathbb{R}^*,\ i = 1, 2$$

which was discussed in the studies [9] and [10].

References

1. Euler, R., Sadek, J.: Elementary problems and solutions. Fibonacci Q. **42**(1), 86–91 (2004)
2. Horadam, A.: A generalized Fibonacci sequence. Am. Math. Month. **68**(5), 455–459 (1961)
3. Gupta, V., Panwar ve, Y. K., Sikhwal, O.: "Generalized Fibonacci sequences." Theor. Math. Appl. **2**(2) (2012)
4. Koshy, T.: "Fibonacci and Lucas numbers with applications", John Wiley and Sons (2001)
5. Koshy, T.: "Fibonacci and Lucas identities, Fibonacci and lucas numbers with applications", John Wiley & Sons (2018)
6. King, C.H.: Some properties of the Fibonacci numbers, Master's Thesis, San Jose State College (1960)
7. Keskin, R., Demirtürk, B.: Some new Fibonacci and Lucas identities by matrix methods. Internat. J. Math. Ed. Sci. Tech. **41**(3), 379–387 (2009)
8. Hoggatt, V. E.: Fibonacci and Lucas numbers, The Fibonacci association University of Santa Clara (1979)
9. Karakaya, S.: On the linear combinations of matrices associated with Fibonacci and Lucas numbers, Master's Thesis (2018)
10. Özdemir, H., Karakaya, S., Petik, T.: On characterization of some linear combinations involving the matrices Q and R. Honam Math. J. **42**(2), 235–249 (2020)
11. Öndül, A., Özdemir, H., Petik, T.: On characterization of being a matrix $Q^{(k)}_{g(a_2,b_2)}$ of linear combinations of a matrix, $Q^{(n)}_{g(a_1,b_1)}$ and a matrix Q^m. Konuralp J. Math. **8**(2), 361–364 (2020)
12. Sinha, S.: The Fibonacci numbers and its amazing applications. Int. J. Eng. Sci. Invent. **6**(9), 07–14 (2017)
13. Uçar, S., Taş, N., Özgür, N. Yılmaz: "A new application to coding theory via Fibonacci and Lucas numbers". Math. Sci. Appl. E-Notes **7**(1), 62–70 (2019)
14. Vajda, S.: "Fibonacci and Lucas numbers and the golden section: theory and applications", Courier Corporation (1989)