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On a group extension involving the Suzuki group $Sz(8)$

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Abstract
The Suzuki simple group $Sz(8)$ has an automorphism group 3. Using the electronic Atlas [22], the group $Sz(8){:}3$ has an absolutely irreducible module of dimension 12 over $\mathbb{F}_2$. Therefore a split extension group of the form $2^{12}{:}(Sz(8){:}3) := \overline{G}$ exists. In this paper we study this group, where we determine its conjugacy classes and character table using the coset analysis technique together with Clifford-Fischer Theory. We determined the inertia factor groups of $\overline{G}$ by analysing the maximal subgroups of $Sz(8){:}3$ and maximal of the maximal subgroups of $Sz(8){:}3$ together with various other information. It turns out that the character table of $\overline{G}$ is a 43×43 complex valued matrix, while the Fischer matrices are all integer valued matrices with sizes ranging from 1 to 7.

Keywords Group extensions · Suzuki simple group · Inertia groups · Fischer matrices · Character table

Mathematics Subject Classification 20C15 · 20C40

1 Introduction

Let n be a non-negative integer and $q = 2^{1+2n}$ and $r = 2^n$. In 1960, M. Suzuki constructed a new type of groups as subgroups of $SL(4, q)$ generated by certain explicit matrices. These infinite family of groups nowadays are known as Suzuki groups and denoted by $Sz(q)$. For any $q = 2^{1+2n}$, $n \in \mathbb{N}$, the Suzuki group $Sz(q)$ is simple and has order $q^2(q^2+1)(q-1)$. The order of $Sz(q)$ is always divisible by 5, but not 3. A remarkable point here is that the Suzuki groups are the only non-abelian simple groups whose order is not divisible by 3. The Schur multiplier is trivial for all $n > 1$, and for $n = 1$, i.e., for the group $Sz(8)$, the Schur multiplier is isomorphic to the Klein 4-group $\mathbb{V}_4 \cong 2^2$. The outer automorphism group of $Sz(q)$ is cyclic of order $2n + 1$. In the special case, the group $Sz(8)$ has order $29120 = 2^6 \times 5 \times 7 \times 13$ and has outer automorphism group isomorphic to the cyclic group $\mathbb{Z}_3$. Therefore the group $Sz(8){:}3$ exists and using the electronic Atlas [22], we can see that both $Sz(8)$ and $Sz(8){:}3$ can be represented in terms of matrices with small dimensions

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over finite fields, or in terms of permutations on 65 points. More precisely one can see that the group $Sz(8){:}3$ has a 12-dimensional absolutely irreducible module over $\mathbb{F}_2$. Therefore a split extension group of the form $2^{12}{:}(Sz(8){:}3) := \overline{G}$ exists. In this article we focus on the group $\overline{G}$, where we will determine its conjugacy classes, the inertia factors of this extension with the fusions of their conjugacy classes into the classes of $Sz(8){:}3$, the character tables of these inertia factors and finally the full character table of the full extension $\overline{G}$. We used the coset analysis method together with the Clifford-Fischer theory (see [1, 19]). The most interesting part is the determination of the inertia factor groups, where there are three inertia factor groups, namely $H_1 = Sz(8){:}3$, H_2 and H_3. The main method used to determine the structures of H_2 and H_3, is by analysing the maximal subgroups of $Sz(8){:}3$ and maximal of these maximal subgroups. Sometimes we consider the third level of maximal subgroups of $Sz(8){:}3$. We ended up with finding that H_2 and H_3 are groups of the forms $2^{3+3}{:}3$ and $2 \times A_4$ respectively. The Fischer matrices of $\overline{G}$ have all been determined in this paper and their sizes range between 1 and 7. The character table of $\overline{G}$ is a 43×43 complex valued matrix and it is partitioned into 51 parts corresponding to the 3 inertia factor groups and the 17 conjugacy classes of $G = Sz(8){:}3$. The character table of any finite group extension $\overline{G} = N{\cdot}G$ (here N is the kernel of the extension and G is isomorphic to $\overline{G}/N$) produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP [17] or Magma [13]. Also there is an interesting interplay between the coset analysis and Clifford-Fischer Theory. Indeed the size of each Fischer matrix is $c(g_i)$, the number of $\overline{G}$-classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. That is computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

By the electronic Atlas [22], we can see that $Sz(8){:}3$ has an absolutely irreducible module of dimension 12 over $\mathbb{F}_2$. The following two elements g_1 and g_2 are 12×12 matrices over $\mathbb{F}_2$ that generate $Sz(8){:}3$.

$$g_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 0&0&0&0&1&0&0&1&1&1&00&0\\ 0&1&0&1&0&0&1&0&1&0&0&1\\ 0&1&1&0&0&1&1&0&1&0&0&1\\ 0&0&0&1&1&0&0&0&1&1&0&0\\ 0&0&1&0&0&0&1&1&1&0&0&0\\ 0&1&1&0&0&0&1&0&1&0&1&0\\ 1&0&0&0&1&1&0&1&1&1&1&1\\ 0&0&0&1&1&1&1&0&1&0&1&0\\ 0&0&0&1&0&0&0&1&1&1&0&1\\ 1&0&0&0&1&0&1&0&1&1&1&1\\ 1&1&1&1&0&1&0&1&1&0&1&0\\ 0&1&0&1&0&1&1&1&0&1&1&0 \end{pmatrix},$$

where $o(g_1) = 2$ and $o(g_2) = 3$.

Using the above two generators of $Sz(8){:}3$ together with few GAP commands we were able to construct our split extension group $\overline{G} = 2^{12}{:}(Sz(8){:}3)$ in terms of 13×13 matrices over $\mathbb{F}_2$. The following three elements $\overline{g}_1$, $\overline{g}_2$ and $\overline{g}_3$ generate $\overline{G}$.

$$\overline{g}_1 = \begin{pmatrix} 0&0&0&0&1&0&0&1&1&1&0&0&0\\ 0&1&0&1&0&0&1&0&1&0&0&1&0\\ 0&1&1&0&0&1&1&0&1&0&0&1&0\\ 0&0&0&1&1&0&0&0&1&1&0&0&0\\ 0&0&1&0&0&0&1&1&1&0&0&0&0\\ 0&1&1&0&0&0&1&0&1&0&1&0&0\\ 1&0&0&0&1&1&0&1&1&1&1&1&0\\ 0&0&0&1&1&1&1&0&1&0&1&0&0\\ 0&0&0&1&0&0&0&1&1&1&0&1&0\\ 1&0&0&0&1&0&1&0&1&1&1&1&0\\ 1&1&1&1&0&1&0&1&1&0&1&0&0\\ 0&1&0&1&0&1&1&1&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$\overline{g}_2 = \begin{pmatrix} 0&1&1&0&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0\\ 1&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&1&1&0&1&0&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&1&0&0&1&1&0\\ 1&1&1&0&1&1&0&0&1&0&1&0&0\\ 0&1&0&1&0&0&1&0&1&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

$$\overline{g}_3 = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix},$$

where $o(\overline{g}_1) = 3$, $o(\overline{g}_2) = 2$ and $o(\overline{g}_3) = 2$.

For the sake of computations, we used few GAP commands to convert the representation of our group $\overline{G}$ from matrix into permutation representation, where we were able to represent $\overline{G}$ in terms of the set $\{1, 2, \ldots, 4096\}$.

Having $\overline{G}$ being constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\overline{G}$ possesses two proper normal subgroups of orders 4096 and 119275520. The normal subgroup of order 4096 is an elementary abelian group isomorphic to N. In GAP one can check for the complements of N in $\overline{G}$, where in our case we obtained two complements both isomorphic to $Sz(8){:}3$ and any of these two complements together with N gives the split extension in consideration.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1–12, 14, 16].

2 Conjugacy classes of $\overline{G} = 2^{12}{:}(Sz(8){:}3)$

In this section we compute the conjugacy classes of the group $\overline{G}$ using the coset analysis technique (see Basheer [1], Basheer and Moori [2, 3, 5] or Moori [19] and [20] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of $Sz(8){:}3$. Firstly note that $Sz(8){:}3$ has 17 conjugacy classes (see the Atlas [22] or Table 4 of this paper). Corresponding to these 17 classes of $Sz(8){:}3$, we obtained 43 classes in $\overline{G}$.

In Table 1, we list the conjugacy classes of $\overline{G}$, where in this table:

- k_i is the number of orbits $Q_{i1}, Q_{i2}, \ldots, Q_{ik_i}$ for the action of N on the coset $N\overline{g}_i = Ng_i$, where g_i is a representative of a class of the complement $Sz(8){:}3$ of N in $\overline{G}$. In particular, the action of N on the identity coset N produces 4096 orbits each consists of singleton. Thus for $\overline{G}$, we have $k_1 = 4096$.
- f_{ij} is the number of orbits fused together under the action of $C_G(g_i)$ on $Q_1, Q_2, \ldots, Q_k$. In particular, the action of $C_G(1_G) = G = Sz(8){:}3$ on the orbits $Q_1, Q_2, \ldots, Q_k$ affords three orbits of lengths 1, 455 and 3640 (with corresponding point stabilizers $Sz(8){:}3$, $2^{3+3}{:}3$ and $2 \times A_4$. Thus $f_{11} = 1$, $f_{12} = 455$ and $f_{13} = 3640$.
- m_{ij}'s are weights (attached to each class of $\overline{G}$) that will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}, \tag{1}$$

Table 1 The conjugacy classes of $\overline{G}$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 4096$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	357826560
		$f_{12} = 1575$	$m_{12} = 455$	g_{12}	2	455	786432
		$f_{13} = 2520$	$m_{13} = 2520$	g_{13}	2	3640	98304
$g_2 = 2A$	$k_2 = 64$	$f_{21} = 1$	$m_{21} = 64$	g_{21}	2	29120	12288
		$f_{22} = 1$	$m_{22} = 64$	g_{22}	4	29120	12288
		$f_{23} = 3$	$m_{23} = 192$	g_{23}	4	87360	4096
		$f_{24} = 3$	$m_{24} = 192$	g_{24}	4	87360	4096
		$f_{25} = 8$	$m_{25} = 512$	g_{25}	4	232960	1536
		$f_{26} = 24$	$m_{26} = 1536$	g_{26}	4	698880	512
		$f_{27} = 24$	$m_{27} = 1536$	g_{27}	4	698880	512
$g_3 = 3A$	$k_3 = 16$	$f_{31} = 1$	$m_{31} = 256$	g_{31}	3	372736	960
		$f_{32} = 5$	$m_{32} = 1280$	g_{32}	6	1863680	192
		$f_{33} = 10$	$m_{33} = 2560$	g_{33}	6	3727360	96
$g_4 = 3B$	$k_4 = 16$	$f_{41} = 1$	$m_{41} = 256$	g_{41}	3	372736	960
		$f_{42} = 5$	$m_{42} = 1280$	g_{42}	6	1863680	192
		$f_{43} = 10$	$m_{43} = 2560$	g_{43}	6	3727360	96
$g_5 = 4A$	$k_5 = 8$	$f_{51} = 1$	$m_{51} = 512$	g_{51}	4	931840	384
		$f_{52} = 2$	$m_{52} = 512$	g_{52}	8	931840	384
		$f_{53} = 3$	$m_{53} = 1536$	g_{53}	8	2795520	128
		$f_{54} = 3$	$m_{54} = 1536$	g_{54}	8	2795520	128
$g_6 = 4B$	$k_6 = 8$	$f_{61} = 1$	$m_{61} = 512$	g_{61}	4	931840	384
		$f_{62} = 2$	$m_{62} = 512$	g_{62}	8	931840	384
		$f_{63} = 3$	$m_{63} = 1536$	g_{63}	8	2795520	128
		$f_{64} = 3$	$m_{64} = 1536$	g_{64}	8	2795520	128
$g_7 = 5A$	$k_7 = 1$	$f_{71} = 1$	$m_{71} = 4096$	g_{71}	5	23855104	15
$g_8 = 6A$	$k_8 = 4$	$f_{81} = 1$	$m_{81} = 1024$	g_{81}	6	7454720	48
		$f_{82} = 1$	$m_{82} = 1024$	g_{82}	12	7454720	48
		$f_{83} = 2$	$m_{83} = 2048$	g_{83}	12	14909440	24
$g_9 = 6B$	$k_9 = 4$	$f_{91} = 1$	$m_{91} = 1024$	g_{91}	6	7454720	48
		$f_{92} = 1$	$m_{92} = 1024$	g_{92}	12	7454720	48
		$f_{93} = 2$	$m_{93} = 2048$	g_{93}	12	14909440	24
$g_{10} = 7A$	$k_{10} = 1$	$f_{10,1} = 1$	$m_{10,1} = 4096$	$g_{10,1}$	7	51118080	7
$g_{11} = 12A$	$k_{11} = 2$	$f_{11,1} = 1$	$m_{11,1} = 2048$	$g_{11,1}$	12	14909440	24
		$f_{11,2} = 1$	$m_{11,2} = 2048$	$g_{11,2}$	24	14909440	24
$g_{12} = 12B$	$k_{12} = 2$	$f_{12,1} = 1$	$m_{12,1} = 2048$	$g_{12,1}$	12	14909440	24
		$f_{12,2} = 1$	$m_{12,2} = 2048$	$g_{12,2}$	24	14909440	24
$g_{13} = 12C$	$k_{13} = 2$	$f_{13,1} = 1$	$m_{13,1} = 2048$	$g_{13,1}$	12	14909440	24
		$f_{13,2} = 1$	$m_{13,2} = 2048$	$g_{13,2}$	24	14909440	24
$g_{14} = 12D$	$k_{14} = 2$	$f_{14,1} = 1$	$m_{14,1} = 2048$	$g_{14,1}$	12	14909440	24

Table 1 continued

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\vert[g_{ij}]_{\overline{G}}\vert$	$\vert C_{\overline{G}}(g_{ij})\vert$
		$f_{14,2}=1$	$m_{14,2}=2048$	$g_{14,2}$	24	14909440	24
$g_{15}=13A$	$k_{15}=1$	$f_{15,1}=1$	$m_{15,1}=4096$	$g_{15,1}$	13	27525120	13
$g_{16}=15A$	$k_{16}=1$	$f_{16,1}=1$	$m_{16,1}=4096$	$g_{16,1}$	15	23855104	15
$g_{17}=15B$	$k_{17}=1$	$f_{17,1}=1$	$m_{17,1}=4096$	$g_{17,1}$	15	23855104	15

Table 2 The maximal subgroups of $G = Sz(8){:}3$

M_i	$\vert M_i\vert$	$[(Sz(8){:}3) : M_i]$
$Sz(8)$	29120	3
$2^{3+3}{:}(7{:}3)$	1344	65
13:12	156	560
$3\times(5{:}4)$	60	1456
7:6	42	2080

where N is the kernel of an extension $\overline{G}$ that is in consideration.

3 Inertia factor groups of $\overline{G} = 2^{12}{:}(Sz(8){:}3)$

We have seen in Sect. 2 that the action of $\overline{G}$ on N produced three orbits of lengths 1, 455 and 3640. By a theorem of Brauer (for example see Theorem 5.1.1 of Basheer [1]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce three orbits of lengths 1, r and s, where $1+r+s = |\mathrm{Irr}(N)| = 4096$; that is

$$r + s = 4095. \tag{2}$$

The values of r and s will be determined through deep investigation on the maximal subgroups of $Sz(8){:}3$ or maximal of the maximal subgroups of $Sz(8){:}3$ together with various information including the sizes of the Fischer matrices, fusions of the the conjugacy classes of some subgroups into the group $Sz(8){:}3$ and other information. In Table 2 we supply the maximal subgroups of $Sz(8){:}3$, where we need these subgroups in the process of the determination of H_2 and H_3.

Firstly since 1, r and s are the lengths of the orbits on the action of $\overline{G}$ on N (which can be reduced to the action of G on N), it follows that $[G : H_1] = 1$, $[G : H_2] = r$ and $[G : H_3] = s$, where H_1, H_2 and H_3 are the inertia factor groups in $G = Sz(8){:}3$. It follows that $H_1 = G = Sz(8){:}3$ and $r, s \mid |G|$; that is $r, s|87360$. Now 87360 has 112 positive divisors, where 96 divisors are less than 4095. Out of these 96 divisors, only two pairs (r, s) satisfy Eq. (2). These are the pairs:

$$(r, s) \in \{(455, 3640), (1365, 2730)\}. \tag{3}$$

Here we do not distinguish between the pair (r, s) and (s, r) and therefore we excluded the other two pairs (3640, 455) and (2730, 1365) from our consideration and we restrict ourselves only to those in Eq. (3). Another point that we put in mind is that since the extension $\overline{G}$ splits over N and N is an elementary abelian group, it follows that all the character tables of H_1, H_2 and H_3 that we will use to construct the character table of $\overline{G}$ are the ordinary ones. From

Tables 1 and 4 we have $|\mathrm{Irr}(\overline{G})| = 43$ and $|\mathrm{Irr}(H_1)| = |\mathrm{Irr}(G)| = |\mathrm{Irr}(Sz(8){:}3)| = 17$. Since $\sum_{i=1}^{3} |\mathrm{Irr}(H_i)| = |\mathrm{Irr}(\overline{G})| = 43$, we have $|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = |\mathrm{Irr}(\overline{G})| = 43$, that is

$$|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 26. \tag{4}$$

Our next task is to show that $(r, s) = (455, 3640)$ and that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will be dual to the action of $\overline{G}$ on the classes of N. This will be achieved by excluding the other possible pair by getting a contradiction to some fact, which we show in the next proposition.

Proposition 1 $(r, s) \neq (1365, 2730)$.

Proof For the purpose of contradiction assume $(r, s) = (1365, 2730)$; that is $r = 1365$ and $s = 2730$ (or $[Sz(8){:}3 : H_2] = 1365$ and $[Sz(8){:}3 : H_3] = 2730$) and consequently $|H_2| = 64$ and $|H_3| = 32$. By looking at the maximal subgroups of $Sz(8){:}3$, given in Table 2, it follows that H_2 is either an index 455 subgroup of $Sz(8)$ or an index 21 subgroup of $2^{3+3}{:}(7{:}3)$. If H_2 is an index 455 subgroup of $Sz(8)$, then by looking at the maximal subgroups $Sz(8)$, available in the Atlas, it follows that H_2 must be an index 7 subgroup of $2^{3+3}{:}7$. In this case we can see clearly that H_2 will be isomorphic to the group 2^{3+3}. On the other hand, if H_2 is an index 21 subgroup of $2^{3+3}{:}(7{:}3)$, then it will also be isomorphic to the group 2^{3+3}. In Table 3 we list the character table of the group 2^{3+3} together with the fusions of the conjugacy classes of this group into the classes of $Sz(8){:}3$. The interplay between the coset analysis and Clifford-Fischer Theory was mentioned in some details in [6]. In particular, the size of the Fischer matrix correspond to a conjugacy class $[g]_G$ is equal to $c(g)$, where $c(g)$ is the number of conjugacy classes of the full extension $\overline{G}$ that correspond to the conjugacy class $[g]_G$ obtained using the coset analysis technique. Now from Table 1 we can see that $\overline{G} = 2^{12}{:}(Sz(8){:}3)$ has four conjugacy classes correspond to the class $[g_5]_{Sz(8):3} = [4A]_{Sz(8):3}$. Therefore the Fischer matrix $\mathcal{F}_5$ will be a 4×4 matrix. We also know that the rows of any Fischer matrix $\mathcal{F}_i$ (corresponds to the class $[g_i]_G$) are partitioned into submatrices correspond to the inertia factors, where there is possible fusions from the conjugacy classes of these inertia factors into the class $[g_i]_G$. For the Fischer matrix $\mathcal{F}_5$, which we found to be of size 4, we have one row corresponds to the first inertia factor $H_1 = Sz(8){:}3$. From Table 3, we can see that there are 7 conjugacy classes of H_2 that fuse to the class $g_5 = 4A$. Thus the contribution of H_1 and H_2 together to $\mathcal{F}_5$ will be 8 rows, which is a contradiction to the fact that $\mathcal{F}_5$ is a 4×4 matrix. Therefore H_2 can not be the group 2^{3+3} and we deduce that $(r, s) = (1365, 2730)$ is not the required pair. □

Corollary 2 *The action of* $Sz(8){:}3$ *on* $\mathrm{Irr}(2^{12})$ *is dual to the action of* $Sz(8){:}3$ *on the conjugacy classes of* $N = 2^{12}$.

Proof The application of Eq. (3) and Proposition 1 shows that $(r, s) = (455, 3640)$ and it follows that the action of $Sz(8){:}3$ on $\mathrm{Irr}(2^{12})$ is dual to the action of $Sz(8){:}3$ on the conjugacy classes of $N = 2^{12}$ as claimed. □

Proposition 3 *The inertia factor groups have the forms* $2^{3+3}{:}3$ *and* $2 \times A_4$.

Proof We found that the orbit lengths on the action of $Sz(8){:}3$ on $\mathrm{Irr}(2^{12})$ are 1, 455 and 3640. It follows that $[G : H_1] = 1$, $[G : H_2] = 455$ and $[G : H_3] = 3640$ and consequently $H_1 = G = Sz(8){:}3$, $|H_2| = 192$ and $|H_3| = 24$. By Eq. (4) we also have $|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 26$. Now we investigate the maximal subgroups of $Sz(8){:}3$ to locate H_2 and H_3. Since $|H_2| = 192$ and by looking at the maximal subgroups of $Sz(8){:}3$, given in Table 2, it follows that H_2

Table 3 The character table of 2^{3+3}

	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$2f$	$2g$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$4g$	$4h$	$4i$	$4j$	$4k$	$4l$	$4m$	$4n$
$\hookrightarrow Sz(8){:}3$	$1A$	$2A$	$2A$	$2A$	$2A$	$2A$	$2A$	$2A$	$4A$	$4A$	$4B$	$4A$	$4B$	$4B$	$4B$	$4A$	$4B$	$4B$	$4A$	$4B$	$4A$	$4A$
$\lvert C_{2^{3+3}}(g)\rvert$	64	64	64	64	64	64	64	64	16	16	16	16	16	16	16	16	16	16	16	16	16	16
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	−1	−1	−1	1	1	1	1	1	−1	−1	1	−1	−1	−1
χ_3	1	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	1	−1	1	−1	−1	1	1	−1
χ_4	1	1	1	1	1	1	1	1	−1	1	1	1	−1	−1	1	−1	−1	1	−1	−1	−1	1
χ_5	1	1	1	1	1	1	1	1	1	1	−1	−1	−1	1	−1	1	−1	1	−1	1	−1	−1
χ_6	1	1	1	1	1	1	1	1	−1	−1	1	−1	−1	1	−1	1	1	−1	−1	−1	1	1
χ_7	1	1	1	1	1	1	1	1	1	−1	1	−1	1	−1	−1	−1	−1	−1	1	1	−1	1
χ_8	1	1	1	1	1	1	1	1	−1	1	−1	−1	1	−1	−1	−1	1	1	1	−1	1	−1
χ_9	2	2	−2	−2	2	−2	−2	2	0	0	0	$-2i$	0	0	$2i$	0	0	0	0	0	0	0
χ_{10}	2	2	−2	−2	2	−2	−2	2	0	0	0	$2i$	0	0	$-2i$	0	0	0	0	0	0	0
χ_{11}	2	−2	2	−2	2	−2	2	−2	0	0	0	0	0	$2i$	0	$-2i$	0	0	0	0	0	0
χ_{12}	2	−2	2	−2	2	−2	2	−2	0	0	0	0	0	$-2i$	0	$2i$	0	0	0	0	0	0
χ_{13}	2	−2	−2	2	2	2	−2	−2	0	0	0	0	$2i$	0	0	0	0	0	$-2i$	0	0	0
χ_{14}	2	−2	−2	2	2	2	−2	−2	0	0	0	0	$-2i$	0	0	0	0	0	$2i$	0	0	0
χ_{15}	2	2	2	−2	−2	2	−2	−2	$-2i$	0	0	0	0	0	0	0	0	0	0	$2i$	0	0
χ_{16}	2	2	2	−2	−2	2	−2	−2	$2i$	0	0	0	0	0	0	0	0	0	0	$-2i$	0	0
χ_{17}	2	2	−2	2	−2	−2	2	−2	0	0	0	0	0	0	0	0	$2i$	0	0	0	$-2i$	0
χ_{18}	2	2	−2	2	−2	−2	2	−2	0	0	0	0	0	0	0	0	$-2i$	0	0	0	$2i$	0
χ_{19}	2	−2	2	2	−2	−2	−2	2	0	$2i$	0	0	0	0	0	0	0	$-2i$	0	0	0	0
χ_{20}	2	−2	2	2	−2	−2	−2	2	0	$-2i$	0	0	0	0	0	0	0	$2i$	0	0	0	0
χ_{21}	2	−2	−2	−2	−2	2	2	2	0	0	$2i$	0	0	0	0	0	0	0	0	0	0	$-2i$
χ_{22}	2	−2	−2	−2	−2	2	2	2	0	0	$-2i$	0	0	0	0	0	0	0	0	0	0	$2i$

must be an index 7 subgroup of $2^{3+3}{:}(7{:}3)$ and since the index is 7, then H_2 must be maximal in $2^{3+3}{:}(7{:}3)$. Now the group $2^{3+3}{:}(7{:}3)$ has 3 maximal subgroups of orders 448, 192 and 168. The group of order 192 has the structure $2^{3+3}{:}3$ and therefore H_2 is isomorphic to group $2^{3+3}{:}3$, which has 18 irreducible characters. Using this together with Eq. (4) we deduce that the third inertia factor group that we are looking for must have 8 ordinary irreducible characters.

Next we consider H_3. Since $|H_3| = 24$ and by looking at the maximal subgroups of $Sz(8){:}3$, given in Table 2, it follows that H_3 is an index 56 subgroup of $2^{3+3}{:}(7{:}3)$. From the above we know that $2^{3+3}{:}(7{:}3)$ has 3 maximal subgroups of orders 448, 192 and 168 with respective structures $2^{3+3}{:}7$, $2^{3+3}{:}3$ and $2^3{:}(7{:}3)$. Therefore H_3 is either:

- an index 8 subgroup of $2^{3+3}{:}3$ or
- an index 7 subgroup of $2^3{:}(7{:}3)$.

Now the group $2^{3+3}{:}3$ has 3 conjugacy classes of maximal subgroups represented by $2^{3+2}{:}3$, 2^{3+3} and $4 \times A_4$ with respective orders 96, 64 and 48. Since $24 \nmid 64$, then it is clear that either $H_3 \leq 2^{3+2}{:}3$ with index 4 or $H_3 \leq 4 \times A_4$ with index 2. The group $2^{3+2}{:}3$ has two conjugacy classes of maximal subgroups represented by $(2 \times 4){:}4$ and $2 \times A_4$ with orders 32 and 24. Therefore if $H_3 \leq 2^{3+2}{:}3$ with index 4, then it must be isomorphic to the group $2 \times A_4$, which has 8 irreducible characters. On the other hand in the case that $H_3 \leq 4 \times A_4$ with index 2, we notice that the group $4 \times A_4$ has 3 conjugacy classes of maximal subgroups represented by $2 \times A_4$, $4 \times 2 \times 2$ and $\mathbb{Z}_{12}$ with respective orders 24, 16 and 12. Therefore if $H_3 \leq 4 \times A_4$ with index 2, then it must be isomorphic to the group $2 \times A_4$, which has 8 irreducible characters. Next we turn to the case that H_3 is an index 7 subgroup of $2^3{:}(7{:}3)$. In this case it is clear that H_3 is maximal in $2^3{:}(7{:}3)$ since the index is prime. Now the group $2^3{:}(7{:}3)$ has 3 conjugacy classes of maximal subgroups represented by $2^3{:}7$, $2 \times A_4$ and $7{:}3$ with respective orders 56, 24 and 21. Therefore if $H_3 \leq 2^3{:}(7{:}3)$ with index 7, then it must be isomorphic to the group $2 \times A_4$, which has 8 irreducible characters. In either case we can see that the group H_3 has the structure $2 \times A_4$. This completes the investigation on the two inertia factor groups H_2 and H_3. □

As subgroups of the full extension $\overline{G} = 2^{12}{:}(Sz(8){:}3)$ that is generated by $\overline{g}_1$, $\overline{g}_2$ and $\overline{g}_3$ given in Section 1, the two inertia factor groups $H_2 = 2^{3+3}{:}3$ and $H_3 = 2 \times A_4$ are generated as follows: $H_2 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and $H_3 = \langle \beta_1, \beta_2 \rangle$, where

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta_1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\beta_2 = \begin{pmatrix} 0&0&0&0&0&1&0&1&1&0&0&1&0\\ 0&1&0&1&1&0&1&1&0&1&1&1&0\\ 0&1&0&0&1&0&1&1&1&1&0&0&0\\ 0&0&1&1&1&0&0&1&0&1&1&0&0\\ 0&0&0&1&0&0&1&1&1&1&1&1&0\\ 1&0&0&0&0&1&0&0&1&1&0&1&0\\ 1&0&0&1&1&0&0&1&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&1&1&1&0\\ 1&0&0&0&0&0&0&1&1&1&0&1&0\\ 1&1&0&0&0&1&1&1&0&1&1&1&0\\ 0&0&0&0&0&1&0&0&0&0&0&1&0\\ 1&1&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$

The character table of the simple Suzuki group $Sz(8)$ is available in the Atlas and thus the extension $Sz(8)$:3 can easily be constructed using Clifford-Fischer Theory. However we used the two generators g_1 and g_2 given in Sect. 1 together with GAP to construct the character table of $Sz(8)$:3, which we list in Table 4. Also at the end of this paper we list the character tables of H_2 and H_3 that we can easily obtain through GAP or Magma. Recall that H_2 and H_3 are not maximal subgroups of $Sz(8)$:3, but they are maximal of some maximal subgroups of $Sz(8)$:3. We determined the fusions of the conjugacy classes of H_2 and H_3 into the classes $Sz(8)$:3 using the permutation characters of $Sz(8)$:3 on 2^{3+3}:(7:3) and 2^{3+3}:3; the permutation characters of 2^{3+3}:(7:3) and 2^{3+3}:3 on H_2 and H_3 respectively; together with the size of centralizers. We found the following proposition to be very useful in the process of determining the fusions.

Proposition 4 *Let $K_1 \leq K_2 \leq K_3$ and let ψ be a class function on K_1. Then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} = \psi\uparrow_{K_1}^{K_3}$. More generally if $K_1 \leq K_2 \leq \cdots \leq K_n$ is a nested sequence of subgroups of K_n and ψ is a class function on K_1, then $(\psi\uparrow_{K_1}^{K_2})\uparrow_{K_2}^{K_3} \cdots \uparrow_{K_{n-1}}^{K_n} = \psi\uparrow_{K_1}^{K_n}$.*

Proof See Proposition 3.5.6 of Basheer [1]. □

In Tables 5 and 6 we supply the full character tables of the inertia factor groups H_2 and H_3 together with the fusions of their conjugacy classes into the classes of $Sz(8)$:3.

4 Fischer matrices of $\overline{G} = 2^{12}$:($Sz(8)$:3)

In this section we calculate the Fischer matrices of $\overline{G} = 2^{12}$:($Sz(8)$:3). From Sect. 3 of Basheer and Moori [2] we recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of g_{ij}, $1 \leq j \leq c(g_i)$, in $\overline{G}$ and m_{ij} respectively. Also the rows of $\mathcal{F}_i$ are partitioned into parts $\mathcal{F}_{ik}$, $1 \leq k \leq t$, corresponding to the inertia factors $H_1, H_2, \ldots, H_t$, where each $\mathcal{F}_{ik}$ consists of $c(g_{ik})$ rows correspond to the α_k^{-1}−regular classes (those are the H_k−classes that fuse to class $[g_i]_G$). Thus every row of $\mathcal{F}_i$ is labeled by the pair (k, m), where $1 \leq k \leq t$ and $1 \leq m \leq c(g_{ik})$. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \leq i \leq 17$, $1 \leq j \leq c(g_i)$. Also the fusions of classes of H_2 and H_3 into classes of G are given in Tables 5 and 6 respectively. Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G} = 2^{12}$:($Sz(8)$:3) range between 1 and 7 for every $i \in \{1, 2, \ldots, 17\}$.

We have used the arithmetical properties of the Fischer matrices, given in Proposition 3.6 of [2], to calculate some of the entries of these matrices and to build a system of algebraic equations. With the help of the symbolic mathematical package Maxima [18], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, which we list below.

$\mathcal{F}_1$				
g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	2	2
$\|C_{\overline{G}}(g_{1j})\|$		357826560	786432	98304
(k, m)	$\|C_{H_k}(g_{1km})\|$			
(1, 1)	87360	1	1	1
(2, 1)	192	455	−57	7
(3, 1)	24	3640	56	−8
m_{1j}		1	455	3640

$\mathcal{F}_2$								
g_2		g_{21}	g_{22}	g_{23}	g_{24}	g_{25}	g_{26}	g_{27}
$o(g_{2j})$		2	4	4	4	4	4	4
$\|C_{\overline{G}}(g_{2j})\|$		12288	12288	4096	4096	1536	512	512
(k, m)	$\|C_{H_k}(g_{2km})\|$							
(1, 1)	192	1	1	1	1	1	1	1
(2, 1)	192	1	1	1	1	−1	1	−1
(2, 2)	64	3	3	3	3	−3	−1	1
(2, 3)	64	3	3	3	3	3	−1	−1
(3, 1)	24	8	−8	8	−8	0	0	0
(3, 2)	8	24	24	−8	−8	0	0	0
(3, 3)	8	24	−24	−8	8	0	0	0
m_{2j}		64	64	192	192	512	1536	1536

$\mathcal{F}_3$				
g_3		g_{31}	g_{32}	g_{33}
$o(g_{3j})$		3	6	6
$\|C_{\overline{G}}(g_{3j})\|$		960	192	96
(k, m)	$\|C_{H_k}(g_{3km})\|$			
(1, 1)	60	1	1	1
(2, 1)	12	5	−3	1
(3, 1)	6	10	2	−2
m_{3j}		256	1280	2560

$\mathcal{F}_4$				
g_4		g_{41}	g_{42}	g_{43}
$o(g_{4j})$		3	6	6
$\lvert C_{\overline{G}}(g_{4j})\rvert$		960	192	96
(k, m)	$\lvert C_{H_k}(g_{4km})\rvert$			
(1, 1)	60	1	1	1
(2, 1)	12	5	−3	1
(3, 1)	6	10	2	−2
m_{4j}		256	1280	2560

$\mathcal{F}_5$					
g_5		g_{51}	g_{52}	g_{53}	g_{54}
$o(g_{5j})$		4	8	8	8
$\lvert C_{\overline{G}}(g_{5j})\rvert$		384	384	128	128
(k, m)	$\lvert C_{H_k}(g_{5km})\rvert$				
(1, 1)	48	1	1	1	1
(2, 1)	48	1	−1	1	−1
(2, 2)	16	3	−3	−1	1
(2, 3)	16	3	3	−1	−1
m_{5j}		512	512	1536	1536

$\mathcal{F}_6$					
g_6		g_{61}	g_{62}	g_{63}	g_{64}
$o(g_{6j})$		4	8	8	8
$\lvert C_{\overline{G}}(g_{6j})\rvert$		384	384	128	128
(k, m)	$\lvert C_{H_k}(g_{6km})\rvert$				
(1, 1)	48	1	1	1	1
(2, 1)	48	1	−1	1	−1
(2, 2)	16	3	−3	−1	1
(2, 3)	16	3	3	−1	−1
m_{6j}		512	512	1536	1536

$\mathcal{F}_7$		
g_7		g_{71}
$o(g_{7j})$		5
$\lvert C_{\overline{G}}(g_{7j})\rvert$		15
(k, m)	$\lvert C_{H_k}(g_{7km})\rvert$	
(1, 1)	15	1
m_{7j}		4096

$\mathcal{F}_8$				
g_8		g_{81}	g_{82}	g_{83}
$o(g_{8j})$		6	12	12
$\lvert C_{\overline{G}}(g_{8j})\rvert$		48	48	24
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$			
(1, 1)	12	1	1	1
(2, 1)	12	1	1	−1
(3, 1)	6	2	−2	0
m_{8j}		1024	1024	2048

$\mathcal{F}_9$				
g_9		g_{91}	g_{92}	g_{93}
$o(g_{9j})$		6	12	12
$\lvert C_{\overline{G}}(g_{9j})\rvert$		48	48	24
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$			
(1, 1)	12	1	1	1
(2, 1)	12	1	1	−1
(3, 1)	6	2	−2	0
m_{9j}		1024	1024	2048

$\mathcal{F}_{10}$		
g_{10}		$g_{10,1}$
$o(g_{10j})$		7
$\lvert C_{\overline{G}}(g_{10j})\rvert$		7
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$	
(1, 1)	7	1
m_{10j}		4096

$\mathcal{F}_{11}$			
g_{11}		$g_{11,1}$	$g_{11,2}$
$o(g_{11j})$		12	24
$\lvert C_{\overline{G}}(g_{11j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$		
(1, 1)	24	1	1
(2, 1)	24	1	−1
m_{11j}		2048	2048

$\mathcal{F}_{12}$			
g_{12}		$g_{12,1}$	$g_{12,2}$
$o(g_{12j})$		12	24
$\lvert C_{\overline{G}}(g_{12j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$		
(1, 1)	24	1	1
(2, 1)	24	1	−1
m_{12j}		2048	2048

$\mathcal{F}_{13}$			
g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		12	24
$\lvert C_{\overline{G}}(g_{13j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{13j}		2048	2048

$\mathcal{F}_{14}$			
g_{14}		$g_{14,1}$	$g_{14,2}$
$o(g_{14j})$		12	24
$\lvert C_{\overline{G}}(g_{14j})\rvert$		24	24
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$		
$(1,1)$	24	1	1
$(2,1)$	24	1	-1
m_{14j}		2048	2048

$\mathcal{F}_{15}$		
g_{15}		$g_{15,1}$
$o(g_{15j})$		13
$\lvert C_{\overline{G}}(g_{15j})\rvert$		13
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$	
$(1,1)$	13	1
m_{15j}		4096

$\mathcal{F}_{16}$		
g_{16}		$g_{16,1}$
$o(g_{16j})$		15
$\lvert C_{\overline{G}}(g_{16j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$	
$(1,1)$	15	1
m_{16j}		4096

$\mathcal{F}_{17}$		
g_{17}		$g_{17,1}$
$o(g_{17j})$		15
$\lvert C_{\overline{G}}(g_{17j})\rvert$		15
(k,m)	$\lvert C_{H_k}(g_{17km})\rvert$	
$(1,1)$	15	1
m_{17j}		4096

5 Character table of $\overline{G} = 2^{12}{:}(Sz(8){:}3)$

Through Sects. 2, 3 and 4, we have determined

- the conjugacy classes of $\overline{G} = 2^{12}{:}(Sz(8){:}3)$ (Table 1),

- the inertia factors H_1, H_2 and H_3.
- the character tables of all the inertia factor groups of G (Tables 4, 5 and 6). In Tables 5 and 6 we also supplied the fusions of the classes of the groups H_2 and H_3 into classes of G.
- the Fischer matrices of $\overline{G}$ (see Sect. 4).

It follows by [1, 2] that the full character table of $\overline{G}$ can be constructed easily in the format of Clifford-Fischer theory. This table will be partitioned into 51 parts corresponding to the 17 cosets and the three inertia factor groups. The full character table of $\overline{G}$ is 43×43 $\mathbb{C}$-valued matrix. In Table 7, we supply the character table of $\overline{G}$ in the format of Clifford-Fischer Theory. Finally we would like to remark that the accuracy of this character table has been tested using GAP.

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Declarations

Conflict of interest The author declares that he has no competing interests.

Appendix

In Table 4, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = -2 - 2i\sqrt{3}$ and $C = -E(12)^7$.
In Table 5, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $B = E(12)^7$.
In Table 6, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
In Table 7, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = -2 + 2i\sqrt{3}$, $C = -\frac{5}{2} + 5i\frac{\sqrt{3}}{2}$, $D = \frac{3}{2} - 3i\frac{\sqrt{3}}{2}$, $E = -5 + 5i\sqrt{3}$, $F = -1 + i\sqrt{3}$ and $G = -E(12)^7$.

Table 4 The character table of $G = Sz(8){:}3$

$[g]_G$	$1A$	$2A$	$3A$	$3B$	$4A$	$4B$	$5A$	$6A$	$6B$	$7A$	$12A$	$12B$	$12C$	$12D$	$13A$	$15A$	$15B$
$\|C_G(g)\|$	87360	192	60	60	48	48	15	12	12	7	12	12	12	12	13	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	A	$\overline{A}$	1	1	1	$\overline{A}$	A	1	A	$\overline{A}$	A	$\overline{A}$	1	A	$\overline{A}$
χ_3	1	1	$\overline{A}$	A	1	1	1	A	$\overline{A}$	1	$\overline{A}$	A	$\overline{A}$	A	1	$\overline{A}$	A
χ_4	14	-2	-1	-1	$-2i$	$2i$	-1	1	1	0	$-i$	$-i$	i	i	1	-1	-1
χ_5	14	-2	-1	-1	$2i$	$-2i$	-1	1	1	0	i	i	$-i$	$-i$	1	-1	-1
χ_6	14	-2	$-\overline{A}$	$-A$	$-2i$	$2i$	-1	A	$\overline{A}$	0	C	$-\overline{C}$	$-C$	$\overline{C}$	1	$-\overline{A}$	$-A$
χ_7	14	-2	$-\overline{A}$	$-A$	$2i$	$-2i$	-1	A	$\overline{A}$	0	$-C$	$\overline{C}$	C	$-\overline{C}$	1	$-\overline{A}$	$-A$
χ_8	14	-2	$-A$	$-\overline{A}$	$-2i$	$2i$	-1	$\overline{A}$	A	0	$-\overline{C}$	C	$\overline{C}$	$-C$	1	$-A$	$-\overline{A}$
χ_9	14	-2	$-A$	$-\overline{A}$	$2i$	$-2i$	-1	$\overline{A}$	A	0	$\overline{C}$	$-C$	$-\overline{C}$	C	1	$-A$	$-\overline{A}$
χ_{10}	64	0	4	4	0	0	-1	0	0	1	0	0	0	0	-1	-1	-1
χ_{11}	64	0	B	$\overline{B}$	0	0	-1	0	0	1	0	0	0	0	-1	$-A$	$-\overline{A}$
χ_{12}	64	0	$\overline{B}$	B	0	0	-1	0	0	1	0	0	0	0	-1	$-\overline{A}$	$-A$
χ_{13}	91	-5	1	1	-1	-1	1	1	1	0	-1	-1	-1	-1	0	1	1
χ_{14}	91	-5	$\overline{A}$	A	-1	-1	1	A	$\overline{A}$	0	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$	0	$\overline{A}$	A
χ_{15}	91	-5	A	$\overline{A}$	-1	-1	1	$\overline{A}$	A	0	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$	0	A	$\overline{A}$
χ_{16}	105	9	0	0	-3	-3	0	0	0	0	0	0	0	0	1	0	0
χ_{17}	195	3	0	0	3	3	0	0	0	-1	0	0	0	0	0	0	0

Table 5 The character table of $H_2 = 2^{3+3}{:}3$

$[g]_{H_2}$	$1a$	$2a$	$2b$	$2c$	$3a$	$3b$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$6a$	$6b$	$12a$	$12b$	$12c$	$12d$
$\lvert C_{H_2}(g)\rvert$	192	192	64	64	12	12	48	48	16	16	16	16	12	12	12	12	12	12
$\hookrightarrow Sz(8){:}3$	$1A$	$2A$	$2A$	$2A$	$3A$	$3B$	$4A$	$4B$	$4A$	$4B$	$4B$	$4A$	$6A$	$6B$	$12C$	$12D$	$12B$	$12A$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
χ_3	1	1	1	1	$\overline{A}$	A	-1	-1	-1	-1	1	1	A	$\overline{A}$	$-\overline{A}$	$-A$	$-A$	$-\overline{A}$
χ_4	1	1	1	1	A	$\overline{A}$	-1	-1	-1	-1	1	1	$\overline{A}$	A	$-A$	$-\overline{A}$	$-\overline{A}$	$-A$
χ_5	1	1	1	1	$\overline{A}$	A	1	1	1	1	1	1	A	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$
χ_6	1	1	1	1	A	$\overline{A}$	1	1	1	1	1	1	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A
χ_7	2	-2	2	-2	$-\overline{A}$	$\overline{A}$	$-2i$	$2i$	0	0	0	0	A	$\overline{A}$	B	$-\overline{B}$	$\overline{B}$	$-B$
χ_8	2	-2	2	-2	$\overline{A}$	$-\overline{A}$	$-2i$	$2i$	0	0	0	0	$\overline{A}$	A	$-\overline{B}$	B	$-B$	$\overline{B}$
χ_9	2	-2	2	-2	$-\overline{A}$	$\overline{A}$	$2i$	$-2i$	0	0	0	0	A	$\overline{A}$	$-B$	$\overline{B}$	$-\overline{B}$	B
χ_{10}	2	-2	2	-2	$-A$	$-\overline{A}$	$2i$	$-2i$	0	0	0	0	$\overline{A}$	A	$\overline{B}$	$-B$	B	$-\overline{B}$
χ_{11}	2	-2	2	-2	-1	-1	$-2i$	$2i$	0	0	0	0	1	1	i	i	$-i$	$-i$
χ_{12}	2	-2	2	-2	-1	-1	$2i$	$-2i$	0	0	0	0	1	1	$-i$	$-i$	i	i
χ_{13}	3	3	3	3	0	0	3	3	-1	-1	-1	-1	0	0	0	0	0	0
χ_{14}	3	3	3	3	0	0	-3	-3	1	1	-1	-1	0	0	0	0	0	0
χ_{15}	6	-6	-2	2	0	0	0	0	$-2i$	$2i$	0	0	0	0	0	0	0	0
χ_{16}	6	-6	-2	2	0	0	0	0	$2i$	$-2i$	0	0	0	0	0	0	0	0
χ_{17}	6	6	-2	-2	0	0	0	0	0	0	$-2i$	$2i$	0	0	0	0	0	0
χ_{18}	6	6	-2	-2	0	0	0	0	0	0	$2i$	$-2i$	0	0	0	0	0	0

Table 6 The character table of $H_3 = 2 \times A_4$

$[g]_{H_3}$	$1a$	$2a$	$2b$	$2c$	$3a$	$3b$	$6a$	$6b$
$\|C_{H_3}(g)\|$	24	24	8	8	6	6	6	6
$\hookrightarrow Sz(8){:}3$	$1A$	$2A$	$2A$	$2A$	$3A$	$3B$	$6A$	$6B$
χ_1	1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1	-1
χ_3	1	-1	1	-1	A	$\overline{A}$	$-\overline{A}$	$-A$
χ_4	1	-1	1	-1	$\overline{A}$	A	$-A$	$-\overline{A}$
χ_5	1	1	1	1	A	$\overline{A}$	$\overline{A}$	A
χ_6	1	1	1	1	$\overline{A}$	A	A	$\overline{A}$
χ_7	3	-3	-1	1	0	0	0	0
χ_8	3	3	-1	-1	0	0	0	0

Table 7 The character table of $\overline{G} = 2^{12}{:}(Sz(8){:}3)$

$[g_i]_G$	$1A$			$2A$							$3A$			$3B$			$4A$			
$[g_{ij}]_{\overline{G}}$	$1a$	$2a$	$2b$	$2c$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$3a$	$6a$	$6b$	$3b$	$6c$	$6d$	$4g$	$8a$	$8b$	$8c$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	357826560	786432	98304	12288	12288	4096	4096	1536	512	512	960	192	96	960	192	96	384	384	128	128
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	1	1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	1	1	1	1
χ_3	1	1	1	1	1	1	1	1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	1	1	1	1
χ_4	14	14	14	−2	−2	−2	−2	−2	−2	−2	−1	−1	−1	−1	−1	−1	$-2i$	$-2i$	$-2i$	$-2i$
χ_5	14	14	14	−2	−2	−2	−2	−2	−2	−2	−1	−1	−1	−1	−1	−1	$2i$	$2i$	$2i$	$2i$
χ_6	14	14	14	−2	−2	−2	−2	−2	−2	−2	$-\overline{A}$	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	$-A$	$-2i$	$-2i$	$-2i$	$-2i$
χ_7	14	14	14	−2	−2	−2	−2	−2	−2	−2	$-\overline{A}$	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	$-A$	$2i$	$2i$	$2i$	$2i$
χ_8	14	14	14	−2	−2	−2	−2	−2	−2	−2	$-A$	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	$-\overline{A}$	$-2i$	$-2i$	$-2i$	$-2i$
χ_9	14	14	14	−2	−2	−2	−2	−2	−2	−2	$-A$	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	$-\overline{A}$	$2i$	$2i$	$2i$	$2i$
χ_{10}	64	64	64	0	0	0	0	0	0	0	4	4	4	4	4	4	0	0	0	0
χ_{11}	64	64	64	0	0	0	0	0	0	0	B	B	B	$\overline{B}$	$\overline{B}$	$\overline{B}$	0	0	0	0
χ_{12}	64	64	64	0	0	0	0	0	0	0	$\overline{B}$	$\overline{B}$	$\overline{B}$	B	B	B	0	0	0	0
χ_{13}	91	91	91	−5	−5	−5	−5	−5	−5	−5	1	1	1	1	1	1	−1	−1	−1	−1
χ_{14}	91	91	91	−5	−5	−5	−5	−5	−5	−5	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	−1	−1	−1	−1
χ_{15}	91	91	91	−5	−5	−5	−5	−5	−5	−5	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	−1	−1	−1	−1
χ_{16}	105	105	105	9	9	9	9	9	9	9	0	0	0	0	0	0	−3	−3	−3	−3
χ_{17}	195	195	195	3	3	3	3	3	3	3	0	0	0	0	0	0	3	3	3	3
χ_{18}	455	−57	7	7	7	7	7	−1	−1	−1	5	−3	1	5	−3	1	7	−1	−1	−1
χ_{19}	455	−57	7	7	7	7	7	−1	−1	−1	5	−3	1	5	−3	1	−1	7	−1	−1
χ_{20}	455	−57	7	7	7	7	7	−1	−1	−1	C	D	$\overline{A}$	$\overline{C}$	$\overline{D}$	A	7	−1	−1	−1
χ_{21}	455	−57	7	7	7	7	7	−1	−1	−1	$\overline{C}$	$\overline{D}$	A	C	D	$\overline{A}$	7	−1	−1	−1
χ_{22}	455	−57	7	7	7	7	7	−1	−1	−1	C	D	$\overline{A}$	$\overline{C}$	$\overline{D}$	A	−1	7	−1	−1

Table 7 continued

$[g_i]_G$	$1A$			$2A$							$3A$			$3B$			$4A$			
$[g_{ij}]_{\overline{G}}$	$1a$	$2a$	$2b$	$2c$	$4a$	$4b$	$4c$	$4d$	$4e$	$4f$	$3a$	$6a$	$6b$	$3b$	$6c$	$6d$	$4g$	$8a$	$8b$	$8c$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	357826560	786432	98304	12288	12288	4096	4096	1536	512	512	960	192	96	960	192	96	384	384	128	128
χ_{23}	455	−57	7	7	7	7	7	−1	−1	−1	$\overline{C}$	$\overline{D}$	A	C	D	$\overline{A}$	−1	7	−1	−1
χ_{24}	910	−114	14	−2	−2	−2	−2	−10	6	−2	−5	3	−1	−5	3	−1	$2i$	$-2i$	$-2i$	$2i$
χ_{25}	910	−114	14	−2	−2	−2	−2	−10	6	−2	−5	3	−1	−5	3	−1	$-2i$	$2i$	$2i$	$-2i$
χ_{26}	910	−114	14	−2	−2	−2	−2	−10	6	−2	$-\overline{C}$	$-\overline{D}$	$-A$	$-C$	$-D$	$-\overline{A}$	$2i$	$-2i$	$-2i$	$2i$
χ_{27}	910	−114	14	−2	−2	−2	−2	−10	6	−2	$-\overline{C}$	$-\overline{D}$	$-A$	$-C$	$-D$	$-\overline{A}$	$-2i$	$2i$	$2i$	$-2i$
χ_{28}	910	−114	14	−2	−2	−2	−2	−10	6	−2	$-C$	$-D$	$-\overline{A}$	$-\overline{C}$	$-\overline{D}$	$-A$	$2i$	$-2i$	$-2i$	$2i$
χ_{29}	910	−114	14	−2	−2	−2	−2	−10	6	−2	$-C$	$-D$	$-\overline{A}$	$-\overline{C}$	$-\overline{D}$	$-A$	$-2i$	$2i$	$2i$	$-2i$
χ_{30}	1365	−171	21	21	21	21	21	−3	−3	−3	0	0	0	0	0	0	−3	−3	−3	5
χ_{31}	1365	−171	21	21	21	21	21	−3	−3	−3	0	0	0	0	0	0	−3	−3	5	−3
χ_{32}	2730	−342	42	−6	−6	−6	−6	−6	−6	10	0	0	0	0	0	0	$6i$	$6i$	$-2i$	$-2i$
χ_{33}	2730	−342	42	−6	−6	−6	−6	−6	−6	10	0	0	0	0	0	0	$-6i$	$-6i$	$2i$	$2i$
χ_{34}	2730	−342	42	−6	−6	−6	−6	18	2	−6	0	0	0	0	0	0	$-6i$	$6i$	$-2i$	$2i$
χ_{35}	2730	−342	42	−6	−6	−6	−6	18	2	−6	0	0	0	0	0	0	$6i$	$-6i$	$2i$	$-2i$
χ_{36}	3640	56	−8	56	−8	−8	−8	0	0	0	10	2	−2	10	2	−2	0	0	0	0
χ_{37}	3640	56	−8	−8	56	−8	−8	0	0	0	10	2	−2	10	2	−2	0	0	0	0
χ_{38}	3640	56	−8	−8	56	−8	−8	0	0	0	E	F	$-F$	$\overline{E}$	$\overline{F}$	$-\overline{F}$	0	0	0	0
χ_{39}	3640	56	−8	−8	56	−8	−8	0	0	0	$\overline{E}$	$\overline{F}$	$-\overline{F}$	E	F	$-F$	0	0	0	0
χ_{40}	3640	56	−8	56	−8	−8	−8	0	0	0	E	F	$-F$	$\overline{E}$	$\overline{F}$	$-\overline{F}$	0	0	0	0
χ_{41}	3640	56	−8	56	−8	−8	−8	0	0	0	$\overline{E}$	$\overline{F}$	$-\overline{F}$	E	F	$-F$	0	0	0	0
χ_{42}	10920	168	−24	−24	−24	−24	40	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{43}	10920	168	−24	−24	−24	40	−24	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 7 continued

$[g_i]_G$	$4B$				$5A$	$6A$			$6B$			$7A$	$12A$		$12B$		$12C$		$12D$		$13A$	$15A$	$15B$
$[g_{ij}]_{\overline{G}}$	$4h$	$8d$	$8e$	$8f$	$5a$	$6e$	$12a$	$12b$	$6f$	$12c$	$12d$	$7a$	$12e$	$24a$	$12f$	$24b$	$12g$	$24c$	$12h$	$24d$	$13a$	$15a$	$15b$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	384	384	128	128	15	48	48	24	48	48	24	7	24	24	24	24	24	24	24	24	13	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	1	A	A	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	1	A	$\overline{A}$
χ_3	1	1	1	1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	1	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A	A	1	$\overline{A}$	A
χ_4	$2i$	$2i$	$2i$	$2i$	-1	1	1	1	1	1	1	0	$-i$	$-i$	$-i$	$-i$	i	i	i	i	1	-1	-1
χ_5	$-2i$	$-2i$	$-2i$	$-2i$	-1	1	1	1	1	1	1	0	i	i	i	i	$-i$	$-i$	$-i$	$-i$	1	-1	-1
χ_6	$2i$	$2i$	$2i$	$2i$	-1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	0	G	G	$-\overline{G}$	$-\overline{G}$	$-G$	$-G$	$\overline{G}$	$\overline{G}$	1	$-\overline{A}$	$-A$
χ_7	$-2i$	$-2i$	$-2i$	$-2i$	-1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	0	$-G$	$-G$	$\overline{G}$	$\overline{G}$	G	G	$-\overline{G}$	$-\overline{G}$	1	$-\overline{A}$	$-A$
χ_8	$2i$	$2i$	$2i$	$2i$	-1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	0	$-\overline{G}$	$-\overline{G}$	G	G	$\overline{G}$	$\overline{G}$	$-G$	$-G$	1	$-A$	$-\overline{A}$
χ_9	$-2i$	$-2i$	$-2i$	$-2i$	-1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	0	$\overline{G}$	$\overline{G}$	$-G$	$-G$	$-\overline{G}$	$-\overline{G}$	G	G	1	$-A$	$-\overline{A}$
χ_{10}	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1	-1	-1
χ_{11}	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1	$-\overline{A}$	$-A$
χ_{12}	0	0	0	0	-1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-1	$-A$	$-\overline{A}$
χ_{13}	-1	-1	-1	-1	1	1	1	1	1	1	1	0	-1	-1	-1	-1	-1	-1	-1	-1	0	1	1
χ_{14}	-1	-1	-1	-1	1	$\overline{A}$	$\overline{A}$	$\overline{A}$	A	A	A	0	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	0	$\overline{A}$	A
χ_{15}	-1	-1	-1	-1	1	A	A	A	$\overline{A}$	$\overline{A}$	$\overline{A}$	0	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	0	A	$\overline{A}$
χ_{16}	-3	-3	-3	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
χ_{17}	3	3	3	3	0	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0	0
χ_{18}	7	-1	-1	-1	0	1	1	-1	1	1	-1	0	1	-1	1	-1	1	-1	1	-1	0	0	0
χ_{19}	-1	7	-1	-1	0	1	1	-1	1	1	-1	0	-1	1	-1	1	-1	1	-1	1	0	0	0
χ_{20}	7	-1	-1	-1	0	$\overline{A}$	$\overline{A}$	$-\overline{A}$	A	A	$-A$	0	$\overline{A}$	$-\overline{A}$	A	$-A$	$\overline{A}$	$-\overline{A}$	A	$-A$	0	0	0

Table 7 continued

$[g_i]_G$	$4B$				$5A$	$6A$			$6B$			$7A$	$12A$		$12B$		$12C$		$12D$		$13A$	$15A$	$15B$
$[g_{ij}]_{\overline{G}}$	$4h$	$8d$	$8e$	$8f$	$5a$	$6e$	$12a$	$12b$	$6f$	$12c$	$12d$	$7a$	$12e$	$24a$	$12f$	$24b$	$12g$	$24c$	$12h$	$24d$	$13a$	$15a$	$15b$
$\vert C_{\overline{G}}(g_{ij})\vert$	384	384	128	128	15	48	48	24	48	48	24	7	24	24	24	24	24	24	24	24	13	15	15
χ_{21}	7	-1	-1	-1	0	A	A	$-A$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	0	A	$-A$	$\overline{A}$	$-\overline{A}$	A	$-A$	$\overline{A}$	$-\overline{A}$	0	0	0
χ_{22}	-1	7	-1	-1	0	$\overline{A}$	$\overline{A}$	$-\overline{A}$	A	A	$-A$	0	$-\overline{A}$	$\overline{A}$	$-A$	A	$-\overline{A}$	$\overline{A}$	$-A$	A	0	0	0
χ_{23}	-1	7	-1	-1	0	A	A	$-A$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	0	$-A$	A	$-\overline{A}$	$\overline{A}$	$-A$	A	$-\overline{A}$	$\overline{A}$	0	0	0
χ_{24}	$-2i$	$2i$	$-2i$	$2i$	0	1	1	-1	1	1	-1	0	i	$-i$	i	$-i$	$-i$	i	$-i$	i	0	0	0
χ_{25}	$2i$	$-2i$	$2i$	$-2i$	0	1	1	-1	1	1	-1	0	$-i$	i	$-i$	i	i	$-i$	i	$-i$	0	0	0
χ_{26}	$-2i$	$2i$	$-2i$	$2i$	0	A	A	$-A$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	0	$\overline{G}$	$-\overline{G}$	$-G$	G	$-\overline{G}$	$\overline{G}$	G	$-G$	0	0	0
χ_{27}	$2i$	$-2i$	$2i$	-2i	0	A	A	$-A$	$\overline{A}$	$\overline{A}$	$-\overline{A}$	0	$-\overline{G}$	$\overline{G}$	G	$-G$	$\overline{G}$	$-\overline{G}$	$-G$	G	0	0	0
χ_{28}	$-2i$	$2i$	$-2i$	$2i$	0	$\overline{A}$	$\overline{A}$	$-\overline{A}$	A	A	$-A$	0	$-G$	G	$\overline{G}$	$-\overline{G}$	G	$-G$	$-\overline{G}$	$\overline{G}$	0	0	0
χ_{29}	$2i$	$-2i$	$2i$	$-2i$	0	$\overline{A}$	$\overline{A}$	$-\overline{A}$	A	A	$-A$	0	G	$-G$	$-\overline{G}$	$\overline{G}$	$-G$	G	$\overline{G}$	$-\overline{G}$	0	0	0
χ_{30}	-3	-3	5	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	-3	-3	-3	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	$-6i$	$-6i$	$2i$	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	$6i$	$6i$	$-2i$	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	$6i$	$-6i$	$-2i$	$2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	$-6i$	$6i$	$2i$	$-2i$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	0	0	0	2	-2	0	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	0	0	0	-2	2	0	-2	2	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{38}	0	0	0	0	0	$-F$	F	0	$-\overline{F}$	$\overline{F}$	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{39}	0	0	0	0	0	$-\overline{F}$	$\overline{F}$	0	$-F$	F	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{40}	0	0	0	0	0	F	$-F$	0	$\overline{F}$	$-\overline{F}$	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{41}	0	0	0	0	0	$\overline{F}$	$-\overline{F}$	0	F	$-F$	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{42}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{43}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

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