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Existence of solutions for some quasilinear elliptic system with weight and measure-valued right hand side

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Abstract

Let Ω be an open bounded domain in $I\!R^n$, we prove the existence of a solution u for the nonlinear elliptic system

$$(\text{QES}) \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where μ is Radon measure on Ω with finite mass. In particular, we show that if the coercivity rate of σ lies in the range $]\frac{s+1}{s}, (\frac{s+1}{s})(2-\frac{1}{n})]$ with $s \in \left(\frac{n}{p}\ \infty\right) \cap \left(\frac{1}{p-1}\ \infty\right)$, then u is approximately differentiable and the equation holds with Du replaced by apDu. The proof relies on an approximation of μ by smooth functions f_k and a compactness result for the corresponding solutions u_k. This follows from a detailed analysis of the Young measure $\{\delta_u(x) \otimes \vartheta(x)\}$ generated by the sequence (u_k, Du_k), and the div-curl type inequality $\langle\vartheta(x), \sigma(x, u, \cdot)\rangle \leq \overline{\sigma}(x)\langle\vartheta(x), \cdot\rangle$ for the weak limit $\overline{\sigma}$ of the sequence.

Keywords Nonlinear elliptic system · Mesure-valued · Young measure · The div-curl type inequality

Mathematics Subject Classification 35J46 · 35J62

1 Introduction

We consider the existence and compactness questions for elliptic systems of the form

$$(\text{QES}) \begin{cases} -div\sigma\left(x, u\left(x\right), Du\left(x\right)\right) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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with measure-valued right hand side $\mu \in M(\Omega, I\!R^m)$ on an open, bounded domain Ω in $I\!R^n$, we denote by $M(\Omega, I\!R^m)$, with $m \in I\!N^*$, the Banach space of vectors μ of bounded Radon measures

$$\mu = (\mu_1, \dots \mu_m) \quad with \quad \mu_i \in M(\Omega) \quad for \ i \ = 1, \dots, m.$$

With $M(\Omega)$ be a vector space of bounded Radon measures.
$\omega = \{\omega_{ij} \mid 0 \le i \le n,\ 1 \le j \le m\}$ is a family of weight functions defined on Ω with $\omega_{ij}(x) > 0$ for almost every $x \in \Omega$ and $\omega^* = \{\omega^*_{ij} = \omega_{ij}^{1-p'}, 0 \le i \le n,\ 1 \le j \le m\}$, $(\frac{1}{p} + \frac{1}{P'} = 1)$. In this paper we are interested in the solution u in the Sobolev space $W_0^{1,p}(\Omega, \omega, I\!R^m)$, and estimations in the weak Lebesgues spaces. We assume that σ satisfies the following hypotheses (H_0)–(H_3) explained below. We denote by $I\!M^{m\times n}$ the real vector space of $m \times n$ matrices equipped with the inner product $M : N = \sum_{ij} M_{ij} N_{ij}$. The Jacobian matrix of a function $u : \Omega \longrightarrow I\!R^m$ is denoted by

$$Du(x) = (D_1 u(x), D_2 u(x), \dots, D_n u(x)) \ \text{ with } D_i = \partial/\partial x_i.$$

Let $\omega = \{\omega_{ij} \mid 0 \le i \le n,\ 1 \le j \le m\}$, and $\overline{\omega_0} = (\omega_{0j})$ for all $1 \le j \le m$ the weight functions system defined in Ω satisfying the following integrability conditions

$$\omega_{ij} \in L^1_{loc}(\Omega), \quad \omega_{ij}^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega) \tag{1.2}$$

$$\omega_{ij}^{-s} \in L^1(\Omega) \tag{1.3}$$

for some $s \in \left(\frac{n}{p}\ \infty\right) \cap \left(\frac{1}{p-1}\ \infty\right)$. The space $W^{1,p}(\Omega, \omega, I\!R^m)$ is the set of functions

$$\left\{u = u(x) \mid u \in L^p(\Omega, \overline{\omega_0}, I\!R^m),\ D_{ij} u = \frac{\partial u^i}{\partial x_j} \in L^p(\Omega, \omega_{ij}, I\!R^m),\ 1 \le i \le n,\ 1 \le j \le m\}\right\}$$

with

$$L^p(\Omega, \omega_{ij}, I\!R^m) = \left\{u = u(x) \mid | u \mid \omega_{ij}{}^{\frac{1}{p}} \in L^p(\Omega, I\!R^m)\right\}.$$

The weighted space $W^{1,p}(\Omega, \omega, I\!R^m)$ can be equipped by the norm

$$\|u\|_{1,p,\omega} = \left(\sum_{j=1}^m \int_\Omega |u_j|^p w_{0j} dx + \sum_{1\le i\le n, 1\le j\le m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx\right)^{\frac{1}{p}}.$$

The norm $\|\cdot\|_{1,\omega,p}$ is equivalent to the norm $|||\ \cdot\ |||$ on $W_0^{1,p}(\Omega, \omega, I\!R^m)$, defined by $|||\ u\ |||= (\sum_{1\le i\le n, 1\le j\le m} \int_\Omega |D_{ij} u|^p \omega_{ij} dx)^{\frac{1}{p}}$. The condition (1.2) implies $(W^{1,p}(\Omega, \omega, I\!R^m), \|\cdot\|_{1,p,\omega})$ is a Banach space and $C_0^\infty(\Omega, I\!R^m)$ subspace of $(W^{1,p}(\Omega, \omega, I\!R^m)$. The space $(W_0^{1,p}(\Omega, \omega, I\!R^m)$ is the closure of $C_0^\infty(\Omega, I\!R^m)$ in $W^{1,p}(\Omega, \omega, I\!R^m)$ for the norm $\|\cdot\|^p_{1,p,\omega}$. The condition (1.3), implies

$$W^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W^{1,p_s}(\Omega, I\!R^m) \hookrightarrow L^r(\Omega, I\!R^m), \tag{1.4}$$

for all $1 \le r \le p_s^*$ if $p \times s \le n(s+1)$, and $\forall r \ge 1$ if $p \times s > n(s+1)$ with $p_s = \frac{p\times s}{s+1}$ and $p_s^* = \frac{n\times p\times s}{n(s+1) - p\times s}$,for proof see [1].

Our article deals with the existence of a weak solution of system declared at the top in each of the four cases located in the part of the hypotheses in (H_2) and in a Sobolev space with weights, but the article in [2] treats in a weightless Sobolev space.

2 Hypothesis

(H_0) (Continuity) $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \longrightarrow I\!M^{m\times n}$ is a Carathéodory function, i.e: $x \longmapsto \sigma(x,u,p)$ is measurable for every (u,p) and $(u,p) \longmapsto \sigma(x,u,p)$ is continuous for almost every $x \in \Omega$.

(H_1) (Coercivity and growth): There exist constants $c_1,\ c_2,\ \beta > 0$ and $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega)$, $0<\alpha<p$, $1<q<\frac{p^2}{\alpha}$, $0<\theta<\frac{n(p_s-1)}{n-1}$, such that, for all $1\le r\le n$, and $1\le s\le m$

$$|\sigma_{rs}(x,u,F)| \le \beta\omega_{rs}^{\frac{1}{p}}\left[\lambda_1 + c_1\sum_{j=1}^{m}\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + c_1\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{\theta}\right]$$

$$\sigma(x,u,F):F \ge -\lambda_2 - \sum_{j=1}^{m}\lambda_3\gamma_j^{\frac{\alpha}{p}}|u_j|^{\frac{q\alpha}{p}} + c_2\sum_{i,j}\omega_{ij}|F_{ij}|^p.$$

(H_2) (Monotonicity) σ satisfies one of the following conditions:

a) For all $x\in\Omega,\ u\in I\!R^m$ the function $F\longmapsto\sigma(x,u,F)$ is a C^1 and monotone function, which means $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)\ge 0$, for all $x\in\Omega$, $u\in I\!R^m$, and $F,G\in I\!M^{m\times n}$.

b) There exist a function $W:\Omega\times I\!R^m\times I\!M^{m\times n}\to R$ such that $\sigma(x,u,F)=\frac{\partial W}{\partial F}(x,u,F)$, and the function $F\longmapsto W(x,u,F)$ is a convex C^1 function.

c) σ is strictly monotone, i.e. σ is monotone, i.e., $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)\ge 0$ and $(\sigma(x,u,F)-\sigma(x,u,G)):(F-G)=0$. implies $F=G$.

d) The function $F\longmapsto\sigma(x,u,F)$ is strictly p-quasi-monotone, i.e., $\int_{I\!M^{m\times n}}(\sigma(x,u,\lambda)-\sigma(x,u,\bar\lambda)):(\lambda-\bar\lambda)d\nu(\lambda)>0$, for all homogeneous $W^{1,p}$-gradient Young measures ν with center of mass $\bar\lambda=\langle\nu;\mathrm{Id}\rangle=\displaystyle\int_{I\!M^{m\times n}}\lambda d\nu(\lambda)$ which are not a single Dirac mass.

(H_3) (structure conditions) i) (Angle condition) for all $x\in\Omega$, $u\in I\!R^m$ and $F\in I\!M^{m\times n}$ there holds
$\sigma(x,u,F):MF\ge 0$, for all matrices $M\in I\!M^{m\times m}$ of the form $M=\mathrm{Id}-a\otimes a$ with $|a|\le 1$. ii) (The sign condition) for all $x\in\Omega$, $u\in I\!R^m$ and $F\in I\!M^{m\times n}$, we have $\sigma_j(x,u,F):F_j\ge 0$, for all $1\le j\le m$ where F_j and σ_j are the columns j of the matrix F and σ, respectively.

(H_4) (The Hardy-Type Inequality) There exist $c>0$, a weight function $\gamma=(\gamma_j)_{1\le j\le m}$, and a parameter $1<q<\frac{p^2}{\alpha}$ (H_1), such that:

$$\left(\sum_{j=1}^{m}\int_\Omega\gamma_j\mid u_j\mid^q dx\right)^{\frac{1}{q}} \le c\left(\sum_{i,j}\int_\Omega\omega_{ij}\mid D_{ij}u\mid^p dx\right)^{\frac{1}{p}}.$$

Remark 2.1 1. Assumption (H_0) ensures that $\sigma(x,u(x),U(x))$ is measurable on Ω for measurable function $u:\Omega\longrightarrow I\!R^m$ and $U:\Omega\longrightarrow I\!M^{m\times n}$. A typical example for a function σ satisfying (H_0) is $\sigma(x, u, p) = \xi(x, u, p)p$ with a real valued non-negative function ξ.

2. A serious technical obstacle is that for $p_s \in (1, 2 - 1/n]$ solutions of the system (1.1) in general do not belong to the Sobolev space $W^{1,1}(\Omega, \omega, I\!R^m)$ [2].

This fact has led to the use of normalized solutions in [2] and generalized entropy solutions in [3] for elliptic equations of the above type. We will use a notion of solution where the weak derivative Du is replaced by the approximate derivative apDu. Recall that a measurable function u is said to be approximately differentiable at $x \in \Omega$ if there exists a matrix $F_x \in I\!M^{m\times n}$ such that for all $\epsilon > 0$, $\lim_{r\to 0} \frac{1}{r^n}\text{meas}\,\{y \in B(x, r) : |u(y) - u(x) - F_x(y - x)| > \epsilon r\} = 0$. We write $\text{apDu}(x) = F_x$.

Definition 2.1 A measurable function $u : \Omega \longrightarrow I\!R^m$ is called a solution of the system (1.1) if:

- **(i)** u is almost everywhere approximately differentiable.
- **(ii)** $\eta \circ u \in W^{1,1}(\Omega, \omega, I\!R^m)$, for all, $\eta \in C_0^1(I\!R^m, I\!R^m)$.
- **(iii)** $\sigma(x, u, \text{apDu}) \in L^1(\Omega, I\!M^{m\times n})$;
- **(iv)** The equation $-div\sigma(x, u(x), Du(x)) = \mu$ holds in the sense of distributions. Moreover we say that u satisfies the boundary condition (1.2) if $\eta \circ u \in W_0^{1,1}(\Omega, \omega, I\!R^m)$, for all, $\eta \in C^1(I\!R^m, I\!R^m) \cap L^\infty(I\!R^m, I\!R^m)$ with $\eta = \text{Id}$ on $B(0, r)$, for some $r > 0$, and $|D\eta(y)| \le c(1 + |y|)^{-1}$, with $c < \infty$.

Remark 2.2 1. The conditions in Definition (2.1) (except (ii)) are the weakest possible in order to define the system (1.1) in the sense of distributions. Note that if u is approximately differentiable, then apDu is measurable, so $\sigma(\cdot, u, \text{apDu})$ is measurable.

2. The assumption $\eta \circ u \in W^{1,1}(\Omega, I\!R^m)$ ensures minimal regularity of u. For example, if $\mu = 0$, and $\sigma(x, u, p) = \sigma(p)$ with $\sigma(0) = 0$, then piecewise constant functions u satisfy $\text{apDu} = 0$ a.e. but are not admissible solutions. The following theorem is the main result in this paper.

Theorem 2.1 *Let Ω be a bounded, open set. We suppose that the hypotheses (H_0)–(H_2)–(H_3) and the coercivity condition in (H_1) are satisfied. Let μ denote a $I\!R^m$-valued Radon measure on Ω with finite mass. Then the system (1.1) has a solution u in the sense of Definition 2.1, which satisfies the weak Lebesgue space estimate*

$$\|u\|^*_{L^{t^*_{p_s},\infty}(\Omega, I\!R^m)} + \|apDu\|^*_{L^{t_{p_s},\infty}(\Omega, I\!M^{m\times n})} \le C, \tag{2.1}$$

with the constant C depending on $|\Omega|$, c, c_2, and $\| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}(\Omega)}$, with $t_{p_s} = \frac{n(p_s-1)}{n-1}$ and $t^*_{p_s} = \frac{n(p_s-1)}{n-p_s}$ is the Sobolev exponent of t_{p_s}. If $c_2 = 0$ the right hand side of (1.2) reduces to $C(c_1)\left\|\mu^{\frac{1}{p-1}}\right\|_M$.

Remark 2.3 1. If $p_s > 2 - \frac{1}{n}$, then $t_{p_s} > 1$ and $Du \in L^1(\Omega, M^{m\times n})$.

2. If $p > n$ one can replace the $L^{s,\infty}$-norm of u in (1.3) by the $C^{0,\beta}$-norm with $\beta = 1 - \frac{n}{p}$. For $p = q = n$ it is an open question whether $Du \in L^{n,\infty}$. See Section 7 [4] for the (weaker) inclusion $u \in BMO_{loc}$.
3. The exponent in (1.2) are optimal as can be seen from the nonlinear Green's function $G_p(x) = c\,|x|^{\frac{-n}{s^*}}$ for the p-Laplace equation: $-div(|Du|^{p-2}Du) = \delta_0$ in $I\!R^m$, $n \ge 3$. In particular, $L^{s,\infty}$ cannot be replaced by L^s where $L^{s,\infty}$, is a Laurent space.

4. The pointwise monotonicity condition can be replaced by a weaker integrated version, called quasi-monotonicity.
   The key point in the proof of the theorem, is the div-curl inequality for the Young measure $\{\vartheta_x\}_{x\epsilon\Omega}$ generated by a sequence Du_k of gradients of approximate solutions. Together with the identity. (1.4)
   apDu$(x) =< \vartheta_x, Id >$. The div-curl inequality implies easily that $\sigma(\cdot, u_k, Du_k)$ converges weakly in L^1 to $\sigma(\cdot, u, \text{apDu})$. (1.4) is a consequence of general properties of young measures if $p > 2 - \frac{1}{n}$ since in this case Du_k is bounded in L^s for some $s > 1$. If $1 < p \leq 2 - \frac{1}{n}$ one only has the weaker bounds.

3 Some preliminary lemmas

In this section, we will also use the Young measures, and Inequality div-curl for assume the convergence of subsequence $u_k \longrightarrow u$ in measure and for almost every subsequence, with u is approximately differentiable, and apDu $=< \nu_x, \text{Id} >$, ν_x is the Young measures generated by a sequence Du_k.

Lemma 3.1 *Let $u_{k:} : \Omega \longrightarrow I\!R^m$ a sequence of measurable functions such that:*

$$\sup_{k\in I\!N} \int_\Omega |u_k|^s dx < +\infty \text{ for some } s > 0. \tag{3.1}$$

We suppose that for each $\alpha > 0$ the sequence of truncated functions $\{T_\alpha(u_k)\}_{k\in I\!N}$ is precompact in $L^1(\Omega, I\!R^m)$. Then there exists a measurable function u on Ω such that for a subsequence $u_k \longrightarrow u$ in measure.

Proof Choose a subsequence of $\{u_k\}$ (not relabeled) which generates a Young measure $\{\vartheta_x\}_{x\in\Omega}$. By 3.1 and Theorem (Young, Tartar, Ball) the measure ν_x are probability measure for almost every a $x \in \Omega$ and $T_\alpha(u_k) \longrightarrow v_\alpha = \langle \nu_x; T_\alpha \rangle$, weakly in $L^1(\Omega, I\!R^m)$ and in fact strongly since $T_\alpha(u_k)$ is precompact in L^1. Consequently there exists a subsequence such that $T_\alpha(u_{k_l}) \longrightarrow v_\alpha$ almost uniformly, i.e.

$$T_\alpha(u_{k_l}) \longrightarrow v_\alpha \text{ uniformly up to a set of arbitrary small measure.} \tag{3.2}$$

Let $M_\alpha = \{x \in \Omega : |v_\alpha(x)| < \alpha\}$. Then for each $\epsilon > 0$ and $\delta > 0$ there exists a set E_ϵ of measure meas$(E_\epsilon) < \epsilon$ and an index $l_0(\epsilon; \delta)$ such that: $|T_\alpha(u_{k_l})| < |v_\alpha(x)| + \delta$ for all $x \in M_\alpha \backslash E_\epsilon$ and all $l > l_0$. It follows that $u_{k_l}(x) \longrightarrow v_\alpha(x)$ for almost every$x \in M_\alpha \backslash E_\epsilon$ consider first $x \in M_\beta$; $\beta < \alpha$ and then the union over $\beta < \alpha$). Since $\varepsilon > 0$ was arbitrary it follows that $\nu_x = \delta_{v_\alpha}(x)$ for almost every $x \in M_\alpha$ In view of the Ball's theorem it suffices to show that $\cup M_\alpha$ has full measure. Now clearly $M_\alpha \subset M_\beta$ for $\alpha < \beta$ since $T_\beta(u_{k_l}) \longrightarrow T_\beta(v_\alpha) = v_\alpha$ almost everywhere in M_α and therefore$v_\alpha = v_\beta$ on M_α. By (3.2) there exists for each $\epsilon > 0$ a set E_ϵ, and an index $l_0(\epsilon, \alpha)$ such that meas $(E_\epsilon) < \epsilon$ and $| u_{k_l} | \geq | T_\alpha(u_{k_l}) | \geq \frac{\alpha}{2}$ on $(\Omega\backslash E_\epsilon)\backslash M_\alpha$ for all $l \geq l_0$. In view of (3.2) this implies meas $((\Omega\backslash E_\epsilon)\backslash M_\alpha) \leq \frac{c}{\alpha^s}\epsilon \longrightarrow 0$ we deduce meas$(\Omega\backslash \cup M_\alpha) = \lim_{\alpha\longrightarrow\infty} \text{meas}(\Omega\backslash M_\alpha) = 0$ □

Lemma 3.2 *Let Ω be a domain in $I\!R^n$ with $|\Omega| < \infty$ and $u_k \in W^{1,1}(\Omega, I\!R^m)$. Suppose that there exist $p > 1$ and $s > 0$ such that:*

$$\sup_k \sum_{i,j} \int_{|u_k|\leq\alpha} \omega_{ij} |D_{ij} u_k|^p dx \leq c(\alpha) < \infty, \ \forall\alpha > 0, \tag{3.3}$$

and $\sup_{k\in I\!N}\int_\Omega |u_k|^s dx \le c < \infty$. *Then there exist a subsequence* u_{k_j} *and a measurable function* $u:\Omega\longrightarrow I\!R^m$ *such that* $u_{k_j}\longrightarrow u$ *in measure. Moreover u is for almost every* $x\in\Omega$ *approximately differentiable, for all* $\eta\in C_0^1(\Omega, I\!R^m)$ *there holds* $\eta\circ u\in W^{1,p}(\Omega,\omega, I\!R^m)$. *if* $u_k\in W_0^{1,1}(\Omega, I\!R^m)$ *then* $\eta\circ u\in W_0^{1,1}(\Omega, I\!R^m)\cap W^{1,p}(\Omega,\omega, I\!R^m)$ *provided that* $\eta = Id$ *on* $B(0,r)$ *for some* $r>0$.

Proof Choose

$$(u_k)_\alpha = \begin{cases} u_k & \text{if} \quad |u_k|\le\alpha, \\ 0 & \text{if} \quad |u_k|>\alpha. \end{cases}$$

For the hypotheses:

$$\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}(u_k)_\alpha|^p dx = \sum_{i,j}\int_{|u_k|\le\alpha}\omega_{ij}|D_{ij}u_k|^p dx \le c(\alpha) < \infty.$$

Then, $(u_k)_\alpha\in W_0^{1,1}(\Omega, w, I\!R^m)$ and for (1.4), (H_4) and $|D|u||\le|Du|$ we have

$$\begin{aligned}\int_\Omega |DT_\alpha(|u_k|)|^{p_s}dx &= \int_{|u_k|\le\alpha}|D|u_k|)|^{p_s}dx \\ &\le \sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}(u_k)_\alpha|^p dx \\ &\le c(\alpha) < +\infty\end{aligned}$$

Hence by the compact Sobolev embedding $W_s^{1,p_s}(\Omega)\hookrightarrow\hookrightarrow L^{p_s}(\Omega)$, we have $\{T_\alpha(|u_k|)\}$ is precompact in $L^1(\Omega)$. And, if $\eta\in C_0^\infty(B(0,3\alpha), I\!R^m)$ a symmetric radial such that $\eta = \text{Id}$ on $B(0,2\alpha)$, then by (1.2) and (3.3) $\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}(\eta(u_k))|^p dx = \sum_{ij}\int_{|u_k|\le\alpha}\omega_{ij}|D_{ij}(u_k)|^p dx + \sum_{i,j}\int_{\alpha<|u_k|\le 2\alpha}\omega_{ij}|D_{ij}(id)|^p dx + \sum_{i,j}\int_{2\alpha<|u_k|\le 3\alpha}\omega_{ij}|D_{ij}(\eta(u_k))|^p dx \le c(\alpha) + c\sum_{i,j}\|\omega_{ij}\|_{L^1_{loc}(\Omega)} + c < \infty$. Then, by (1.4), $\eta(u_k)$ is precompact in $L^{p_s}(\Omega, I\!R^m)$, and as in Lemma 8 [2], there exist a measurable function $u:\Omega\longrightarrow I\!R^m$ such that $u_k\longrightarrow u$ in measure, with $u(x) =<\vartheta_x, \text{Id}>$ for almost every $x\in\Omega$ and u is approximately differentiable because $\eta(u_k)\rightharpoonup\eta(u)$ in $W^{1,P}(\Omega,\omega, I\!R^m)$ and $apDu = ap(\eta\circ u)$. □

Lemma 3.3 *Let* u_k *be as in Lemma (3.2) with* $p>1$. *Then the Young measure* ϑ_x *generated by (a subsequence of)* Du_k *has the following properties:*

(a) ϑ_x *is a probability measure for almost every* $x\in\Omega$.
(b) ϑ_x *has finite* p_s*-th- moment for almost every* $x\in\Omega$, *i.e.,* $\int_{M^{m\times n}}|\lambda|^{p_s}d\vartheta_x(\lambda)$ *is finite for almost every* $x\in\Omega$.
(c) ϑ_x *satisfies* $<\vartheta_x, Id>= apDu(x)$ *almost everywhere in* Ω.
(d) ϑ_x *is a homogeneous* W^{1,P_s}*-gradient young measure for almost every* $x\in\Omega$.

Proof Let $\widetilde{\vartheta_x}$ denote the Young measure generated by (a subsequence of) the sequence $\{u_k, Du_k\}$. By Lemma 3.2 we have:

$$\widetilde{\vartheta_x} = \delta_{u(x)}\otimes\vartheta_x.$$

Let $\eta \in C_0^\infty(B(0,2\alpha), I\!R^m)$, $\eta = Id$ on $B(0,\alpha)$, and let ϑ^η be the Young measure generated by

$$D(\eta \circ u_k) = (D\eta)(u_k)Du(x),$$

then ϑ^η is a probability measure, has finite p-th moment and

$$< \vartheta^\eta, Id > = (D(\eta \circ u))(x) = D\eta(u(x))\text{apDu}(x).$$

It follows for $\varphi \in C_0^\infty(M^{m\times n})$, that:

$$\varphi(D(\eta \circ u_k)) \rightharpoonup \langle \vartheta^\eta, \varphi\rangle = \int_{M^{m\times n}} \varphi(\lambda) d\vartheta_x^\eta(\lambda).$$

Based on the proof (3.2), we have $\sum_{i,j}\int_\Omega |\omega_{ij} D_{ij}(\eta \circ u_k)|^p dx < \infty$, and by (1.4)

$\sup_{k\in I\!N}\int_\Omega |D(\eta \circ u_k)|^{p_s} dx < \infty$, and the (Ball's Theorem, proof lemma 9 [2]) we conclude (a)-(b)-(c)- and (d). □

4 Approximate solutions and a priori bounds

We introduce the following approximating problems

$$-div\sigma(x, u_k, Du_k) = f_k \ in\, \Omega. \tag{4.1}$$

$$u_k = 0 \ \ on\, \partial\Omega. \tag{4.2}$$

With $f_k \in W^{-1,p'}(\Omega, \omega^*, I\!R^m) \cap L^1(\Omega, I\!R^m)$ and $f_k \rightharpoonup^* \mu \ \ in \ \ M(\Omega, I\!R^m)$ such that $\| f_k \|_{L^1(\Omega,I\!R^m)} \leq \| \mu \|_{M(\Omega,I\!R^m)}$. By [5] and using the assumptions (H_0), (H_1),(H_2) and (H_4), the problem (4.1) and (4.2) has a solution u_k $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ u_k is the subsequence approximates solutions of (1.1). The results of Theorem (2.1) is the consequence of the following proposition

Proposition 4.1 *Let, $f \in L^1(\Omega, I\!R^m)$ and σ satisfies (H_0), the coercivity of (H_1) and (H_3). If $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ is a solution of*

$$-\, div\sigma(x, u, Du) = f \quad in\, \Omega. \tag{4.3}$$

in the sense of distributions. Then

$$u \in L^{t^*_{p_s},\infty}(\Omega, I\!R^m), \quad Du \in L^{t_{p_s},\infty}(\Omega, I\!R^m)$$

and

$$\begin{aligned}
&\| u \|^*_{L^{t^*_{p_s},\infty}(\Omega,I\!R^m)} + \| Du \|^*_{L^{t_{p_s},\infty}(\Omega,M^{m\times n})} \\
&\leq C\Big(|\Omega|, \| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}(\Omega)}, \| f \|_{L^1(\Omega,I\!R^m)}\Big) \sum_{i,j}\int_{|u|\leq\alpha} \omega_{ij}|D_{ij}u|^p dx \\
&\leq M\alpha + L, \quad \forall \alpha > 0,
\end{aligned}$$

M and L are the constants depending on:

$$\| \lambda_1 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}(\Omega)}, \| f \|_{L^1(\Omega,I\!R^m)}, c_2.$$

Proof We suppose the condition of l'angle in (H_3). Let $\alpha > 0$. Testing $T_\alpha(u)$ in (4.3) and we use the coercivity condition in (H_1), and Hölder inequality, we have

$$c_2.\sum_{i,j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx \leq \alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \parallel \lambda_2 \parallel_{L^1(\Omega)} + c \parallel \lambda_3 \parallel_{L(\frac{p}{\alpha})'} \left(\sum_{j=1}^{m}\int_{|u|\leq\alpha}\gamma_j \mid u_j \mid^q dx\right)^{\frac{\alpha}{p}} \tag{4.4}$$

Choose:

$$(u)_\alpha = \begin{cases} u & \text{if} \quad |u| \leq \alpha, \\ 0 & \text{if} \quad |u| > \alpha. \end{cases}$$

Then $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ because $u \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and by Hardy-Type inequality

$$\sum_{j}^{m}\int_{|u|\leq\alpha}\gamma_j|u_j|^q dx = \sum_{j}^{m}\int_{|u|\leq\alpha}\gamma_j|(u_\alpha)_j|^q dx \leq c\left(\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{q}{p}} \leq c\left(\sum_{i;j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{q}{p}}$$

By (4.4)

$$c_2\left(\sum_{ij}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right) \leq \alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \parallel \lambda_2 \parallel_{L^1(\Omega)} + c \parallel \lambda_3 \parallel_{L(\frac{p}{\alpha})'} \cdot \left(\sum_{i,j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{\alpha q}{p^2}}$$

and $\frac{\alpha q}{p^2} < 1$. Then

$$\left(\sum_{ij}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right) \leq c\left(\alpha \parallel f \parallel_{L^1(\Omega,I\!R^m)} + \parallel \lambda_2 \parallel_{L^1(\Omega)}\right) \leq M\alpha + L, \tag{4.5}$$

with $L = L\left(c, \parallel \lambda_2 \parallel_{L^1(\Omega)}, c \parallel \lambda_3 \parallel_{L(\frac{p}{\alpha})'}\right)$ and $M = M\left(c_1, c_2 \parallel \lambda_3 \parallel, \parallel f \parallel_{L^1(\Omega,I\!R^m)}\right)$, we choose $u^\alpha = \min(|u|, \alpha)$, then by $|D|u|| \leq |Du|$

$$\int_\Omega |Du^\alpha|^{p_s} dx = \int_{|u|\leq\alpha}|D|u||^{p_s} dx + 0 \leq \int_{|u|\leq\alpha}|Du|^{p_s} dx = \int_\Omega |Du_\alpha|^{p_s} dx \leq \left(\sum_{i,j}\int_\Omega \omega_{ij}|D_{ij}u_\alpha|^p dx\right)^{\frac{p_s}{p}} = \left(\sum_{i,j}\int_{|u|\leq\alpha}\omega_{ij}|D_{ij}u|^p dx\right)^{\frac{p_s}{p}}$$

And by (4.5), and $p_s \le p$, we have:

$$\int_\Omega |Du^\alpha|^{p_s} dx \le c\left(\alpha \parallel f \parallel_{|L^1(\Omega,\mathbb{R}^m)} + \parallel \lambda_1 \parallel_{L^1(\Omega)}\right). \tag{4.6}$$

By (1.4) and (4.6), we have

$$\int_\Omega |u^\alpha|^{p_s^*} dx \le c\left(\int_\Omega |Du^\alpha|^{p_s} dx\right)^{\frac{p_s^*}{p}} \le c\left(\alpha \parallel f \parallel_{L^1(\Omega;\mathbb{R}^m)} + \parallel \lambda_1 \parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}} \tag{4.7}$$

Then

$$\begin{aligned}\lambda_{|u|}(\alpha) &= \alpha^{-p_s^*}\int_{|u|>\alpha} \alpha^{p_s^*} dx \le \alpha^{-p_s^*}\int_{|u|>\alpha} |u^\alpha|^{p_s^*} dx\\ &\le c\alpha^{-p_s^*}\left(\alpha \parallel f \parallel_{L^1(\Omega,\mathbb{R}^m)} + \parallel \lambda_2 \parallel_{L^1(\Omega)}\right)^{\frac{p_s^*}{p}}\end{aligned} \tag{4.8}$$

and we continue in the same way as in a case that is non-degenerated [2] by replacing p by p_s as well as

$$\begin{aligned}\|u\|^*_{L^{t^*_{p_s},\infty}}(\Omega,\mathbb{R}^m) &= \sup_{\alpha>0} \alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{p_s}}}\\ &\le |\Omega| + \sup_{\alpha>1} \alpha|\lambda_{|u|}(\alpha)|^{\frac{1}{t^*_{p_s}}}\\ &\le |\Omega| + c(\parallel f \parallel^{\frac{1}{p_s-1}}_{L^1(\Omega,\mathbb{R}^m)}, \parallel \lambda_2 \parallel^{\frac{1}{p_s-1}}_{L^1(\Omega)})\end{aligned}$$

i.e.

$$\|u\|^*_{L^{t^*_{p_s},\infty}}(\Omega,\mathbb{R}^m) \le c\left(|\Omega|, \parallel \lambda_2 \parallel_{L^1(\Omega)}, \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega)}; c_2, \parallel f \parallel_{L^1(\Omega,\mathbb{R}^m)}\right), \tag{4.9}$$

on the other hen, by using ($p_s \le p$) and thinks to (1.4), we obtain

$$\begin{aligned}\lambda_{|Du|}(s) &\le s^{-p_s}\int_{|u|\le\alpha} |Du|^{p_s} dx + \lambda_{|u|}(\alpha)\\ &= s^{-p_s}\int_{|u|\le\alpha} |Du_\alpha|^{p_s} dx + \lambda_{|u|}(\alpha)\\ &\le s^{-p_s}\left(\sum_{i,j}\int_{|u|\le\alpha} \omega_{ij}|D_{ij}u_\alpha|^p dx\right) + \lambda_{|u|}(\alpha)\\ &\le s^{-p_s}\left(\sum_{i,j}\int_{|u|\le\alpha} \omega_{ij}|D_{ij}u|^p dx\right) + \lambda_{|u|}(\alpha)\end{aligned}$$

By (4.5) and (4.8):

$$\lambda_{|Du|}(s) \le c\left(\max\left(\frac{\alpha}{s^{p_s}}, \frac{1}{s^{p_s}}\right) + \max\left(\alpha^{-p_s^*}, \alpha^{\frac{p_s^*}{p_s} - p_s^*}\right)\right)$$

or $-t^*_{p_s} = \frac{p_s^*}{p_s} - p_s^*$, so as in [6]

$$\parallel Du \parallel^*_{L^{t_{p_s},\infty}(\Omega,M^{m\times n})} \le c\left(|\Omega|, \parallel \lambda_2 \parallel_{L^1(\Omega)}, \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega)}, c_2, \parallel f \parallel_{L^1(\Omega;\mathbb{R}^m)}\right). \tag{4.10}$$

From (4.5), (4.9) and (4.10), we obtain the result of the proposition (4.1) in case i). ii)-Suppose the angle condition in (H_3), let $S_\alpha(y) = (T_\alpha(y_1); T_\alpha(y_2); \dots T_\alpha(y_m))$, $y \in I\!R^m$, the cubic truncation, we have $Ds_\alpha(y) = \mathrm{Id}$ if $|y|_{max} = \max_{1\le i\le m}|y_i| \le \alpha$, in the same way as in i)-by testing $S_\alpha(u)$ in (4.3). Then $\int_\Omega \sigma(x,u,Du) : D(S_\alpha(u))dx = \int_\Omega f\cdot S_\alpha(u)dx$ or

$$\int_\Omega \sigma(x,u,Du) : D(S_\alpha(u))dx = \sum_{i=1}^m {}_{|u_i|\le\alpha}\sigma_i(x,u,Du) : Du_i dx$$

$$\ge \int_{|u|=\max(|u_1|;\dots;|u_m|)} \sum_{i=1}^m \sigma_i(x,u,Du) : Du_i dx$$

and like $\sum_{i=1}^m \sigma_i(x,u,Du) : Du_i dx = \sigma(x,u,Du) : Du$. By the coercivity condition in (H_1) and the Hölder Inequality we obtain:

$$c_2 \sum_{i,j} \int_{|u|\le\alpha} \omega_{ij}|D_{ij}u|^p dx \le \sqrt{m}.\alpha \parallel f \parallel_{L^1(\Omega;I\!R^m)} + \parallel \lambda_2 \parallel_{L^1(\Omega)}$$

$$+ c \parallel \lambda_3 \parallel_{L^{(\frac{p}{\alpha})'}(\Omega)} \left(\sum_{j=1}^m \int_{|u|\le\alpha} \gamma_j|u_j|^q dx\right)^{\frac{q}{p}}$$

and we continue in the same way as in i), this completes the proof of the Proposition (4.1) □

5 A div-curl inequality

The result of this section is the key ingredient for the proof that one can pass to the limit in the Eq. (4.1) for the solution $\{u_k\}_{k\in I\!N}$ of approximating problems. Since it is independent of the differential equation we state it a more general form using only the hypotheses (5.1)–(5.8) below:

$$\sigma;\tau : \Omega \times I\!R^m \times M^{m\times n} \longrightarrow M^{m\times n}, \tag{5.1}$$

is a Carathéodory function.

$$\sigma \text{ and } \tau \text{ satisfing one of the fellowing conditions:} \tag{5.2}$$

- **(i)** $\sigma(x,u,F) : MF \ge 0$, $\tau(x,u,F) : MF \ge 0$; $M = \mathrm{Id} - b\otimes b \in M^{m\times n}$, with $|b| \le 1$.
- **(ii)** $\sigma_j(x;u;F) : F_j \ge 0$, and $\tau_j(x,u,F) : F_j \ge 0$; $1 \le j \le m$, σ_j, τ_j and F_j is the j^{eme} columns of σ, τ, F.

$$u_k \in W^{1;1}(\Omega, I\!R^m) \text{ and there exists an } s \ge 0 \text{ such that } \int_\Omega |\, Du_k\,|^s\, dx \le c \text{ uniformly in } k \tag{5.3}$$

The sequence $\sigma_k(x) = \sigma(x,u_k,Du_k)$ is equiintegrale. (5.4)

The sequence u_k converges in measure to some function u,
and u is almost everywhere approximately differentiable. (5.5)

The sequence $f_k = -div(\sigma_k + \tau_k) - \mu$ is bounded in $L^1(\Omega, I\!R^m)$. (5.6)

$D_{ij}u_k \in L^r_{loc}(\Omega, \omega_{ij}, M^{m\times n})$ and $(\sigma_k + \tau_k) \in L^{r'}_{loc}(\Omega, w^*, M^{m\times n})$, for some $1 \leq r \prec \infty$ and $(1 \leq i \leq n,\ 1 \leq j \leq m.)$ (5.7)

The sequence $\tau_k(x) = \tau[x](x, u, Du_k)$ converges to weakly to 0 in $L^1(\Omega, M^{m\times n})$. (5.8)

Lemma 5.1 *Suppose (5.1)–(5.8). Then (after passage to a subsequence) the sequence σ_k converges weakly in $L^1(\Omega, M^{m\times n})$ and the weak limit $\overline{\sigma}$ is given by $\overline{\sigma}(x) = \langle \nu_x; \sigma(x, u(x), .)\rangle$. Moreover the following inequality holds:*

$$\int_{M^{m\times n}}, \sigma(x, u(x), \lambda) : \lambda d\nu_x(\lambda) \leq \overline{\sigma}(x) : apDu(x) \textit{ for a.e. } x \in \Omega. \tag{5.9}$$

Proof See [6] □

6 Passage to the limit

Proposition 6.1 *Suppose that the sequence $(u_k)_{k\in I\!N}$ satisfies the hypotheses (5.1)–(5.7), (H_2) and that the Young measure ν generated by the sequence $(Du_k)_{k\in I\!N}$ satisfies: a)-c) and d)- in Lemma (3.3). Then the sequence (σ_k) is weakly converge in $L^1(\Omega, M^{m\times n})$, with $\overline{\sigma}$ is the limit and $\overline{\sigma(x)} = \langle \nu_x, u(x), apDu(x)\rangle$. If in (H_2) b)- c)-or d)-holds, $\sigma(x, u_k, Du_k) \to \sigma(x, u, apDu)$ strongly in $L^1(\Omega, M^{m\times n})$.*

In the cases (c) and (d) it follows addition that $Du_k \to apDu$ in measure.

Proof See [6].

Proof of the Theorem 2.1

Case: $\theta = p - 1$ For using the results of Proposition (6.1): we assume that (5.1)–(5.7) and the Young measure ν_x generated by the sequence Du_k satisfies:(i), (ii) and (iii) in Lemma(3.3), for the approximate systems (4.1) and (4.2). By the proposition 6.1, with $u_k \in W_0^{1,p}(\Omega, w, I\!R^m)$, we have: $\| \mu \|_{L^{t^*_{p_s},\infty}(\Omega, I\!R^m)} \leq c\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}_{(\Omega)}}, c_2, \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}\right)$ and

$$\sum_{i,j} \int_{|u_k|\leq\alpha} \omega_{ij}|D_{ij}u_k|^p dx \leq M\alpha + L < \infty. \tag{6.1}$$

By $L^{t^*_{p_s},\iota\infty}(\Omega, I\!R^m) \hookrightarrow L^p(\Omega, I\!R^m)$ for all $1 < p < t^*_{p_s}$, then

$$\| u_k \|_{L^p(\Omega, I\!R^m)} \leq c < \infty. \tag{6.2}$$

Now

- (5.1) is (H_0)
- (5.2) is (H_3)
- (5.3): $u_k \in W_0^{1,p}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega, I\!R^m)$ with $p_s > 1$, then $u_k \in W^{1,1}(\Omega, I\!R^m)$. Moreover, by the proposition

$$\| Du_k \|_{L^{t_{p_s},\infty}(\Omega, I\!R^m)} \leq c\left(|\Omega|, \| \lambda_2 \|_{L^1(\Omega)}, \| \lambda_3 \|_{L^{(\frac{p}{\alpha})'}_{(\Omega)}}, c_2, \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}\right)$$

hence

$$\| Du_k \|_{L^s(\Omega,M^{m\times n})} \leq c > \infty,\ \forall\, 1 < s < t_{p_s}$$

with $\sup_{k\in I\!N} \int_\Omega |Du_k|^s dx < \infty$.

- (5.4): Let A a measurable in Ω, by (H_1) and Hölder we have

$$\int_A |\sigma(x,u_k,Du_k)|dx \leq c\left(\sum_{r,s}\int_\Omega \omega_{rs}dx\right)^{\frac{1}{p}} \cdot \Bigg[\| \lambda_1 \|_{L^{p'}(\Omega)} + \left(\sum_{j=1}^m \int_\Omega \gamma_j |(u_k)_j|^q dx\right)^{\frac{1}{p'}} + \left(\sum_{i;j}\int_\Omega \omega_{ij}|D_{ij}u_k|^p dx\right)^{\frac{1}{p'}}\Bigg],$$

and with (1.4) and (1.2):

$$\int_A |\sigma(x,u_k,Du_k)|dx \leq c\left(\sum_{r,s} \| \omega_{rs} \|_{L^1_{loc}(\Omega)}^{\frac{1}{p}}\right) \times\left[\| \lambda_1 \|_{L^{p'}(\Omega)} + \| u_k \|_{1,p,\omega}^{\frac{q}{pp'}} + \| u_k \|_{1,p,\omega}^{\frac{p}{p'}}\right] < \infty.$$

- (5.5): By (6.1) and (6.2) and Lemma (3.2).
- (5.6): $\| f_k \|_{L^1(\Omega,I\!R^m)} \leq \| \mu \|_{M(\Omega,\omega^*,I\!R^m)}$.
- (5.7): $\forall\varepsilon > 0$ and $x_0 \in \Omega$ $\int_{B(x_0,\varepsilon)} |D_{ij}u_k|^p \omega_{ij} dx \leq \| u_k \|_{1,p,\omega}^p < \infty$ and by (H_3) we implies

$$\begin{aligned}\int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'}\omega_{rs}^* dx &= \int_{B(x_0,\varepsilon)} |\sigma_{rs}(x,u_k,Du_k)|^{p'}\omega_{rs}^{1-p'} dx \\ &\leq c\int_{B(x_0,\varepsilon)} w^{1-p'+\frac{p'}{p}}\Bigg[|\lambda_1|^{p'} + \sum_{j=1}^m \gamma_j|(u_k)_j|^q \\ &\quad + \sum_{i,j}\omega_{ij}|D_{ij}u_k|^p\Bigg]dx \\ &\leq c\left(\| \lambda_1 \|_{L^{p'}(\Omega)}^{p'} + \| u_k \|_{1,p,w}^{\frac{q}{p}} + \| u_k \|_{1,p,\omega}^{p}\right) < \infty.\end{aligned}$$

Then, by the Proposition (6.1) $\sigma(x,u_k,Du_k) \longrightarrow \sigma(x,u,apDu)$ in $L^1(\Omega, I\!M^{m\times n})$ and $\forall\varphi \in C_0^\infty(\Omega, I\!R^m)$; $D\varphi \in L^\infty(\Omega, I\!M^{m\times n})$ hence

$$\int_\Omega \sigma(x,u_k,Du_k) : D\varphi dx \longrightarrow \int_\Omega \sigma(x,u,apDu) : D\varphi dx$$

i.e.

$$-div\sigma(x,u_k,apDu_k) \longrightarrow -div\sigma(x,u,\text{apDu})$$

In the sense of distributions. On the other hand $f_k \overset{*}{\longrightarrow} \mu$ in $L^1(\Omega, I\!R^m)$. Then $\int_\Omega f_k.\varphi dx \longrightarrow \int_\Omega \mu.\varphi dx \quad \forall\varphi \in C_0^\infty(\Omega, I\!R^m)$ so μ is the solution in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ of the system:

$$\begin{aligned} -div\sigma(x, u, \text{apDu}) &= \mu \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

to show the estimation (2.1), we take the function η in $C_0^1(B(0, 2\alpha), I\!R^m)$; $\eta = \text{Id}$ in $B(0, \alpha)$ and $|D\eta| \leq c$, then:

$$\begin{aligned} \sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}\eta(u_k)|^p dx &= \sum_{i,j} \int_\Omega \omega_{ij} |(D_{ij}\eta)(u_k)|^p |Du_k\|^p dx \\ &\leq c^p. \sum_{ij} \int_{|u_k|\leq\alpha} \omega_{ij} |D_{ij}u_k)| dx \\ &\quad + c \sum_{i,j} \int_{|u_k|\leq 2\alpha} \omega_{ij} D_{ij}u_k|^p dx \\ &\leq c.c(\alpha) + c.c(2\alpha) < \infty, \end{aligned}$$

thanks to (6.1).
Now, we have $\eta(u_k) \longrightarrow \eta(u)$, for every $x \in \Omega$ because η is C^∞. Then $\eta(u_k) \rightharpoonup \eta(u)$, in $W_0^{1,p}(\Omega, \omega, I\!R^m)$ and $\text{apDu} = \text{apD}(\eta \circ u)$ on $\{|u| \prec \alpha\}$. Hence,

$$\begin{aligned} \sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}(\eta \circ u)|^p dx &\leq \liminf_{k\longrightarrow\infty} \int_\Omega w|D(\eta \circ u_k)|^p dx \\ &\leq \liminf_{k\longrightarrow\infty} \sum_{ij} \int_{|u_k|\leq 2\alpha} |D_{ij}\eta(u_k)|^p |D_{ij}u_k| \omega_{ij} dx \\ &\leq \leq c \liminf_{k\longrightarrow\infty} \int_{|u_k|\leq 2\alpha} \omega_{ij} |D_{ij}u_k)|^p dx \\ &\leq c.c(2\alpha) < \infty. \end{aligned}$$

Then:

$$\sum_{i,j} \int_{|u|\leq\alpha} \omega_{ij} |apDu|^p dx = \sum_{i,j} \int_{|u_k|\leq 2\alpha} \omega_{ij} |D(\eta \circ u)|^p dx < \infty,$$

in the same as in the proof of the Proposition (4.1) by replacing u_k by u and f_k by μ, we obtain the estimation (2.1) and this completes the proof of the Theorem 2.1

□

Case: $0 < \theta < \frac{n(p_s-1)}{n-1}$ (the general case) The idea is to consider the regularized problems:

$$-div\phi_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = \mu \quad \text{in } \Omega, \tag{6.3}$$

$$u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{6.4}$$

with

$$\phi_{\varepsilon,r,s}(x, u, F) = \sigma_{rs}(x, u, F) + \varepsilon\beta \left(\sum_{ij} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-2} \right) \omega_{rs}^{\frac{1}{p}} F_{rs},$$

$\forall 1 \leq r \leq n$, $\forall 1 \leq s \leq m$ with $s > n+1$, and $\varepsilon < \frac{1}{2}$, we have $p < s$, then $s' < p'$, and $(\frac{s}{\alpha})' < (\frac{p}{\alpha})'$. Moreover $\exists c > 0$ which doesn't depend on p, s, such that $\omega_{rs}^{\frac{1}{p}} \leq c\omega_{rs}^{\frac{1}{s}}$ $\forall\, 1 \leq r \leq n$ and $1 \leq s \leq m$.

By (H_1) for σ, we obtain

$$
\begin{aligned}
|\phi_{\varepsilon,r,s}(x,u,F)| &\leq \beta'.|\omega_{rs}|^{\frac{1}{p}}\left[\lambda_1 + \sum_{j=1}^{m}\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + \sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{\theta}\right] \\
&\quad + \varepsilon\beta\omega_{rs}^{\frac{1}{p}}\left(\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-1}\right)\left(\theta < \frac{n(p_s-1)}{n-1} < n(s-1)\right) \\
&\leq \leq \beta'\omega_{rs}^{\frac{1}{p}}\left[\lambda_1 + \sum_{j=1}^{m}\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}} + \sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-1}\right].
\end{aligned}
$$

And $p < s$, then $\frac{1}{p'} < \frac{1}{s'}$ and like $\omega_{rs}^{\frac{1}{p}} \leq c\omega_{rs}^{\frac{1}{s}}$, then: $|\phi_{\varepsilon,r,s}(x,u,F)| \leq \beta'.|\omega_{rs}|^{\frac{1}{s}}\left[\lambda_1 + \sum_{j=1}^{m}\gamma_j^{\frac{1}{s'}}|u_j|^{\frac{q}{s'}} + \sum_{ij}\omega_{ij}^{\frac{1}{s'}}|F_{ij}|^{s-1}\right]$, and by (H_3), we conclude that

$$
\phi_\varepsilon(x,u,F) : F = \sigma(x,u,F) : F + \varepsilon \sum_{i,j,r,s}\omega_{ij}^{\frac{1}{p'}}\omega_{rs}^{\frac{1}{p}}|F_{ij}|^{s-2}F_{ij}.F_{rs}
$$

$\geq -\lambda_2 - \sum_{j=1}^{m}\lambda_3\gamma_j^{\frac{q}{s}}.|u_j|^{\frac{q\alpha}{s}} + \varepsilon\sum_{ij}\omega_{ij}|F_{ij}|^s$.

On the other hand, $0 < \alpha < p-1 < s-1$, $1 < q < \frac{p^2}{\alpha} < \frac{s^2}{\alpha'}$, $\lambda_1 \in L^{p'}(\Omega) \hookrightarrow L^{s'}(\Omega)$, and $\lambda_3 \in L^{(\frac{p}{\alpha})'}(\Omega) \hookrightarrow L^{(\frac{s}{\alpha})'}(\Omega)$ and as σ_ε verifies the conditions of the structures (of l'angle and sign), the strict monotony, the s-quasi monotonous with regard to F is a C^1 monotony in relation with F or accepting a convex potential because: $F \longrightarrow \varepsilon\beta\left(\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|F_{ij}|^{s-2}\right)\omega_{rs}^{\frac{1}{p}}F_{rs}$ verify them as well, hence σ_ε verifies the hypotheses (H_0)–(H_5), for the regularized Problems (6.3) and (6.4), thus for the previous case, $\theta = s-1$ of Theorem 2.1, there exists a solution, $u_\varepsilon \in W_0^{1,s}(\Omega, \omega, I\!R^m)$ of the system (6.3) and (6.4). Now showing that the conditions: i), ii) and iii), of lemma (3.3), and the hypotheses (5.1)–(5.8) of the div-curl inequality are verified for u_ε with order s in the place of p.

We suppose the condition of l'angle verifying that ϕ_ε by testing, $T_\alpha(u_\varepsilon)$ $\alpha > 0$ in (5.3) and (5.4), we get: $\int_\Omega \phi_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : DT_\alpha(u_\varepsilon)dx = \int_\Omega f.T_\alpha(u_\varepsilon)dx$, so

$$
\begin{aligned}
&\int_{|u_\varepsilon|\leq\alpha}\sigma(x,u_\varepsilon,Du_\varepsilon) : Du_\varepsilon dx + \int_{|u_\varepsilon|>\alpha}\frac{\alpha}{|u_\varepsilon|}\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon) : \left(\mathrm{Id} - \frac{u_\varepsilon}{|u_\varepsilon|}\otimes\frac{u_\varepsilon}{|u_\varepsilon|}\right)Du_\varepsilon dx \\
&\quad + \varepsilon\beta\int_{|u_\varepsilon|\leq\alpha}\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\sum_{r,s}\omega_{rs}^{\frac{1}{p}}|D_{rs}u_\varepsilon|^2 dx \\
&\quad + \varepsilon\beta\int_{|u_\varepsilon|>\alpha}\sum_{ij}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\sum_{r,s}\omega_{rs}D_{rs}u_\varepsilon\left(Id - \frac{u_\varepsilon}{|u_\varepsilon|}\otimes\frac{u_\varepsilon}{|u_\varepsilon|}\right) \\
&\leq \alpha.\parallel f \parallel_{L^1(\Omega,I\!R^m)}.
\end{aligned}
$$

since

$$\sum_{rs} |D_{rs}u_\varepsilon|^{s-2} D_{rs}u_\varepsilon \left(\mathrm{Id} - \frac{\alpha}{|u_\varepsilon|} \left(\frac{u_\varepsilon}{|u_\varepsilon|} \otimes \frac{u_\varepsilon}{|u_\varepsilon|} \right) \right) \geq 0$$

so

$$\int_{|u_\varepsilon| \leq \alpha} \sigma(x, u_\varepsilon, Du_\varepsilon) : Du_\varepsilon dx \leq \alpha \parallel f \parallel_{L^1(\Omega, \mathbb{R}^m)} .$$

And by the coercivity condition of σ in (H_1) and Hölder inequality, we get as in the proof of the Proposition 4.1

$$\sum_{ij} \int_{|u_\varepsilon| \leq \alpha} \omega_{ij} |D_{ij}u_\varepsilon|^p dx \leq M'\alpha + L', \tag{6.5}$$

And the following a priori estimation:

$$\parallel u_\varepsilon \parallel^*_{L^{t^*_{ps};\infty}(\Omega, \mathbb{R}^m)} + \parallel Du_\varepsilon \parallel^*_{L^{t_{ps},\infty}(\Omega, \mathbb{R}^{m\times n})} < c < \infty, \tag{6.6}$$

and by the injection $L^{\beta',\infty} \hookrightarrow L^{\alpha'}, \forall 0 < \alpha' < \beta'$, then $\forall, \ \ 0 < r < t^*_{p_s}, \forall 0 < p < t_{p_s}$

$$\parallel u_\varepsilon \parallel_{L^r(\Omega, \mathbb{R}^m)} + \parallel Du_\varepsilon \parallel_{L^p(\Omega, \mathbb{M}^{m\times n})} + \parallel Du_\varepsilon \parallel^*_{L^{t_{ps},\infty}(\Omega, \mathbb{M}^{m\times n})} < \infty. \tag{6.7}$$

We suppose that the condition of the sign is verify.

As in the same way in the proof of the Proposition (4.1), we test $S_\alpha(u_\varepsilon)$ in (6.3) and (6.4), we obtain (6.5) and (6.7).

Starting with verifying that i), ii) et iii) of lemma (3.3) and the hypotheses (5.1) and(5.7) for σ_ε. By (6.5)and(6.7), the points i), ii) et iii) are a direct consequence of Lemmas (3.2) and (3.3). On the other hand:

- (5.1): for σ is (H_0) and $\tau_{rs}(x, u, F) = \varepsilon\beta \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-2} \right) .\omega_{rs}^{\frac{1}{p}} F_{rs}$ is a Carathéodory function, because $x \longmapsto \omega_{ij}(x)$, is measurable, so σ_ε is a Cathéodory function.
- (5.2)

(i) $\phi_\varepsilon(x, u, F) : MF = \sigma(x, u, F) : MF + \left(\sum_{rs} (\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-2}) \omega_{rs}^{\frac{1}{p}} F_{rs} \right)$ $(MF)_{rs} \geq 0$, with $M = \mathrm{Id} - a \otimes a$ and $|a| \leq 1$.

(ii)

$$\begin{aligned} \phi_{rs}(x, u, F) \cdot F_j &= \sigma_j(x, u, F) : F_j + \tau_j(x, u, F) \cdot F_j \\ &= \sigma_j(x, u, F) : F_j + \sum_{l=1}^{m} \varepsilon\beta \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |F_{ij}|^{s-1} \right) \cdot w_{lj}^{\frac{1}{p}} |F_{lj}|^2 \geq 0, \end{aligned}$$

$\forall 1 \leq j \leq m.$

- $(5,3)$: $u_\varepsilon \in W_0^{1,s}(\Omega, \omega, \mathbb{R}^m) \hookrightarrow W_0^{1,s_s}(\Omega, \mathbb{R}^m)$, $s_s > 1$, so $u_\varepsilon \in W^{1,1}(\Omega, \mathbb{R}^m)$, and by (6.7) $\sup_{\varepsilon>0} \int_\Omega |Du_\varepsilon|^p dx < \infty, \forall, 0 < p < t_{p_s}$.

  (4.5): $\sigma(x, u_\varepsilon, Du_\varepsilon)$ is equi-integrable as previously $\forall\ \Omega' \subset \Omega$, measurable, we have:

$$\int_{\Omega'} \left| \sum_{i,j} (\omega_{ij}^{\frac{1}{p'}} \right| D_{ij}u_\varepsilon|^{s-2}) \omega_{rs}^{\frac{1}{p}} D_{rs}u_\varepsilon |dx$$

$$\leq \left(\sum_{i,j} \int_{\Omega'} \omega_{ij} |D_{ij} u_\varepsilon|^{s-1} dx \right)$$
$$\leq c \sum_{ij} \int_{\Omega'} \omega_{ij} |D_{ij} u_\varepsilon|^s dx \leq c \parallel u_\varepsilon \parallel_{1,s,w}^s .$$

- (5.5): by (6.7) and the Lemma (3.2).
- (5.6): by (6.3), $-div(\sigma_l + \tau_k) - \mu = 0$, with $\mu \in M(\Omega, I\!R^m)$ is bounded in $L^1(\Omega, I\!R^m)$.
- (5.7): $\forall\ \varepsilon > 0$ and $x_0 \in \Omega$, by the growth condition of σ_ε and previously with s in the place of p, $\int_{B(x,\varepsilon)} |\sigma_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|^s \omega_{rs}^* dx < \infty$
 and
- (5.8): $\int_{B(x,\varepsilon)} |D_{ij} u_\varepsilon|^s \omega_{rs} dx < \parallel u_\varepsilon \parallel_{1,s,w}^\varepsilon < \infty$.

Testing that u_ε in (6.3) and (6.4)

$$\varepsilon\beta \int_\Omega \left(\sum_{ij} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \right) \left(\sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 \right) dx$$
$$\leq \parallel u_\varepsilon \parallel_{L^\infty(\Omega, I\!R^m)} \parallel \mu \parallel_{M(\Omega, \omega^*, I\!R^m)} \tag{6.8}$$

We have $W_0^{1,s}(\Omega, w, I\!R^m) \hookrightarrow W_0^{1,s_s}(\Omega, I\!R^m) \hookrightarrow L^\infty(\Omega, I\!R^m)$. Then

$$\parallel u_\varepsilon \parallel_{L^\infty(\Omega, I\!R^m)} \leq c \left(\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij} u_\varepsilon|^s dx \right)^{\frac{1}{s}}$$
$$\leq c \left(\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \omega_{ij}^{\frac{1}{p}} |D_{ij} u_\varepsilon|^2 dx \right)^{\frac{1}{s}} \tag{6.9}$$
$$\leq c \left(\int_\Omega \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \right) \cdot \left(\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 dx \right) \right)^{\frac{1}{s}} .$$

Thanks to (6.8) and (6.9), we have

$$\int_\Omega \sum_{ij} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 dx$$
$$\leq \frac{c \|\mu\|_{M(\Omega, \omega^*, I\!R^m)}}{\varepsilon} \left(\int_\Omega \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \right) \cdot \left(\sum_{rs} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 dx \right) \right) \text{ So:}$$

$$\left(\int_\Omega \left(\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2} \right) \cdot \left(\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 dx \right) \right)^{\frac{s-1}{s}} \leq \frac{c \parallel \mu \parallel_M}{\varepsilon},$$

which mean that

$$\left(\int_\Omega (\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij} u_\varepsilon|^{s-2}) . (\sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs} u_\varepsilon|^2 dx \right)^{\frac{1}{s}} \leq \frac{c \parallel \mu \parallel_M}{\varepsilon}, \tag{6.10}$$

and

$$\| u_\varepsilon \|_{L^\infty(\Omega, I\!R^m)} \leq c \left(c \frac{\| \mu \|_M}{\varepsilon} \right)^{\frac{1}{s-1}}. \tag{6.11}$$

On the other hand and $\forall\ 1 < p < \frac{s}{s-1}$, can write

$$\begin{aligned}
&\| \epsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-2} \omega_{rs}^{\frac{1}{p}} |F_{rs}| \|_{L^{\frac{s}{s-1}}(\Omega, I\!M^{m\times n})} \\
&\leq \varepsilon^{\frac{s}{s-1}} \left(\int_\Omega |\sum_{i,j} \omega_{ij}^{\frac{1}{p'}} D_{ij}u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} |^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \\
&\leq c\varepsilon^{\frac{s}{s-1}} \left(|\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{rs}^{\frac{1}{p}} |^{\frac{s}{s-1}} dx \right)^{\frac{s-1}{s}} \\
&\leq c\varepsilon^{\frac{s}{s-1}} \left(\sum_{i,j} \int_\Omega \omega_{ij} |D_{ij}u_\varepsilon|^{s-2} \sum_{r,s} \omega_{rs}^{\frac{s}{(s-1)p}} |D_{rs}u_\varepsilon|^2 dx \right) < \infty.
\end{aligned}$$

thanks to (6.10). Now, since $u_\varepsilon \in W_0^{1,s}(\Omega, \omega, I\!R^m) \hookrightarrow W_0^{1,s_s}(\Omega, I\!R^m) \hookrightarrow W_0^{1,p_s}(\Omega, I\!R^m)$, so by testing $T_\alpha(u_\varepsilon)$ in (6.3) and (6.4), we obtain as in the proof of the proposition (4.1)

$$\| Du_\varepsilon \|^*_{L^{\frac{n(p_s-1)}{n-1},\infty}(\Omega, I\!M^{m\times n})} \leq c. \tag{6.12}$$

By the Hölder inequality for the exponent a with a and ξ are the solutions of systems:

$$\begin{cases} a'\xi = \tau > \frac{n(p_s-1)}{n-1} \\ a\left((s-1)\rho - \xi\right) = s \end{cases}$$

a given system accepting the solution when $\rho < \frac{s}{s-1}$. So

$$\begin{aligned}
&\int_\Omega |\varepsilon \sum_{i,j} \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-1} \omega_{ij}^{\frac{1}{p}} |^\rho dx \\
&\leq c \int_\Omega \varepsilon^\rho \left(\sum_{i,j} \omega_{ij}^{\frac{\rho}{p'}} |D_{ij}u_\varepsilon|^{(s-1)\rho-\xi} \omega_{ij}^{\frac{\rho}{p}} |D_{ij}u_\varepsilon|^\xi \right)^\rho dx \\
&\leq c\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{a\rho} |D_{ij}u_\varepsilon|^{a((s-1)\rho-\xi)} dx \right)^{\frac{1}{a}} . \left(\int_\Omega |Du_\varepsilon|^{a'\xi} dx \right)^{\frac{1}{a'}} \\
&\leq c\varepsilon^\rho \left(\sum_{i,j} \int_\Omega \omega_{ij}^{\frac{1}{p'}} |D_{ij}u_\varepsilon|^{s-2} \sum_{r,s} \omega_{rs}^{\frac{1}{p}} |D_{rs}u_\varepsilon|^2 \right)^{\frac{1}{a}} . \| Du_\varepsilon \|^{\frac{\tau}{a}}_{L^\tau(\Omega, I\!M^{m\times n})} .
\end{aligned}$$

And by the injection: $L^{\frac{n(p_s-1)}{n-1}} \hookrightarrow L^\tau \quad \forall \tau > \frac{n(p_s-1)}{n-1}$ and thanks to (6.10)–(6.12), we get:

$$\begin{aligned}\int_{\Omega}|\varepsilon\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-1}\omega_{ij}^{\frac{1}{p}}|^{\rho}dx &\leq c.\varepsilon^{\rho}\left(\frac{c\parallel\mu\parallel_M}{\varepsilon}\right)^{\frac{s}{(s-1)a}}.c^{\frac{\tau}{a}}\\ &\leq c.c^{\frac{\tau}{a}}\varepsilon^{\frac{a((s-1)\rho-s))}{a(s-1)}}\\ &\leq c.c^{\frac{\tau}{a}}\varepsilon^{\frac{a\xi}{a(s-1)}}\\ &\leq c.c^{\frac{\tau}{a}}.\varepsilon^{\frac{\xi}{s-1}}\end{aligned}$$

with $\frac{\xi}{s-1}>0$. Hence

$$\lim_{\varepsilon\longrightarrow}\parallel\varepsilon\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-1}\omega_{rs}^{\frac{1}{p}}D_{rs}u_\varepsilon\parallel_{L^{\rho}(\Omega,\mathbb{M}^{m\times n})}=0,\quad\forall\rho<\frac{s}{s-1}.$$

In particular for $\rho=1$

$$\lim_{\varepsilon\longrightarrow}\int_{\Omega}|\varepsilon\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-1}\omega_{rs}^{\frac{1}{p}}D_{rs}u_\varepsilon|dx=0,$$

which mean that

$$\tau[\varepsilon](x,u_\varepsilon,Du_\varepsilon)=\varepsilon\sum_{i,j}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\omega_{rs}^{\frac{1}{p}}D_{rs}u_\varepsilon\rightharpoonup 0$$

in $L^1(\Omega,\mathbb{M}^{m\times n})$.

As well as by the Proposition 6.1, $div\sigma(x,u_\varepsilon,Du_\varepsilon)$ converges to $div\sigma(x,u,\mathrm{apD}u)$, in the sense of the distributions, and as

$$\tau[\varepsilon](x,u_\varepsilon,Du_\varepsilon)=\varepsilon\sum_{i,j}\omega_{ij}^{\frac{1}{p'}}|D_{ij}u_\varepsilon|^{s-2}\omega_{rs}^{\frac{1}{p}}D_{rs}u_\varepsilon\rightharpoonup 0,$$

in $L^1(\Omega,\mathbb{M}^{m\times n})$. Then $div\sigma_\varepsilon(x,u_\varepsilon,Du_\varepsilon)$ converge to $div\sigma(x,u,\mathrm{apD}u)$ in the sense of distributions, i-e: u is the solution of the system

$$\begin{cases}-div\sigma(x,u,\mathrm{apD}u)=\mu & \mathrm{in}\Omega\\ u=0 & \mathrm{on}\partial\Omega.\end{cases}$$

In the same way as in the case of $\theta=p-1$, we have

$$\int_{|u|\leq\alpha}|apDu|^{s}dx<c(\alpha)<\infty\text{ and }p<s.$$

So we conclude as in the proof of the Proposition 6.1, in order to get the estimation of Theorem (2.1). This completes the proof of the theorem. □

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