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A thermoviscoelastic contact problem with friction, damage and wear diffusion

Soumia Latreche[1] · **Lynda Selmani**[2]

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Abstract
In this paper we present a model for quasistatic frictional contact between a thermoviscoelastic body and a moving foundation that involves wear of contacting surface and diffusion of wear debris. The damage effect is taken into account in the thermoviscoelastic constitutive law, its evolution is described by a parabolic inclusion with the homogeneous Neumann boundary condition. Contact is modeled with a normal compliance condition and is associated to a dry friction. The wear takes place on a part of the contact surface, when the wear debris surface density diffuse on the whole of the contact surface and is accompanied by frictional heat exchange. We derive a variational formulation of the problem and state that, under a smallness assumption on the problem data, there exists a unique weak solution for the model. The proof is based on elliptic variational inequalities, parabolic variational inequalities, first order evolution equations and fixed point arguments.

Keywords Thermoviscoelastic materials · Friction · Normal compliance · Damage · Wear diffusion · Frictional heat generation

Mathematics Subject Classification 74F05 · 74M10 · 74M15

1 Introduction

This work studies a quasistatic model for the process of frictional contact between a thermoviscoelastic body and a moving foundation when wear debris is generated and diffuses on the contact surface. The damage effect is included in the thermoviscoelastic constitutive law. The contact is described with a normal compliance condition and the associated a version of Coulomb law of dry friction in which the coefficient of friction is assumed to depend on the density of the wear particles and on the slip rate. The motion is accompanied by wear diffusion and frictional heat generation. The wear takes place on a part of the contact surface

✉ Soumia Latreche
latreche.soumiaa@gmail.com

1 Department of Sciences, Teacher Education College of Setif, El-Eulma, Algeria

2 Laboratory of Applied Mathematics, Faculty of Sciences, University Ferhat Abbas of Setif 1, Setif, Algeria

and its rate is described by the Archard differential condition. So, our interest is to describe a physical process in which thermal effect, damage effect, friction, wear diffusion and frictional heat generation are involved, and to show that the resulting model leads to well-posed mathematical problem. Then we present the result on the existence and uniqueness of a weak solution to the system. The model is set as a system of an evolutionary variational inequality for the displacements, a parabolic variational equation for the density of the wear particles, a parabolic variational inequality for the damage and an evolution equation for the temperature.

Frictional contact arise in structural and mechanical systems, a considerable progress has been achieved in modeling and mathematical analysis. Models of frictional contact problems are investigated in [13, 15, 18, 20, 21]. Frictional contact problems with wear, both in the dynamic and the quasistatic case, can be found in [16, 17, 19, 20]. Mathematical models for frictional contact with wear under thermodynamic considerations have been considered in [1, 5]. General dynamic thermoelastic models, which were derived from thermodynamical principles, can be found in [12, 23]. Quasistatic or dynamic thermoviscoelastic frictional contact problems can be found in [4, 5, 14]. A quasistatic thermoviscoelastic problem for a beam can be found in [10, 11], where the wear of the contacting surface is included. Quasistatic thermoviscoelastic problem with normal compliance, multivalued friction and wear diffusion can be found in [9].

Following [6, 7], the evolution of the microscopic cracks responsible for the damage is determined by a parabolic inclusion with a constitutive function describing the source of damage in the system which results from tension or compression. Using the subdifferential of indicator function of the interval [0, 1] guarantees that the damage function β which measures the decrease in the load-bearing capacity of the material, varies between 0 and 1. When $\beta = 1$ there is no damage in the material, when $\beta = 0$ the material is completely damaged, when $0 < \beta < 1$ there is partial damage and the system has a reduced load carrying capacity. Contact problems with damage have been investigated in [8, 17, 20] and the monograph [22].

The rest of the manuscript is structured as follows. In Sect. 2 we present the notation we shall use as well as some preliminary material. In Sect. 3 we present the physical setting, describe the mechanical problem, list the assumptions on the data and give the variational formulation of the problem. In Sect. 4 we state our main existence and uniqueness result based on arguments of elliptic variational inequalities, parabolic variational inequalities, first order evolution equations and fixed point.

2 Notations and preliminaries

In this section we present some notations and preliminary material we shall use later in this paper. For further details, we refer the reader to [3]. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary Γ and $\boldsymbol{\nu}$ denote the unit outward normal on Ω. We denote by S_3 the space of second order symmetric tensors on $\mathbb{R}^3$ while "." and | . | will represent the inner product and the Euclidean norm on the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Everywhere in the sequel the index i and j run from 1 to 3. The summation convention over repeated indices is adopted and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. We introduce the following spaces

$$H = \left\{ \boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega) \right\} = L^2(\Omega)^3,$$
$$\mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\} = L^2(\Omega)_s^{3\times 3},$$

$$H_1 = \{\boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\} = H^1(\Omega)^3,$$
$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H\}.$$

Here $\boldsymbol{\varepsilon}$ and Div are the deformation and divergence operators, respectively, defined by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$, $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$, $Div\boldsymbol{\sigma} = (\sigma_{ij,\,j})$. The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, dx,$$
$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\,\boldsymbol{\sigma}, Div\,\boldsymbol{\tau})_H.$$

The associated norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $|\cdot|_H$, $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$ and $|\cdot|_{\mathcal{H}_1}$, respectively. For an element $\boldsymbol{v} \in H_1$ we denote by $\boldsymbol{v}$ its trace on Γ and by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$, $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$ its normal and tangential components on the boundary. For an element $\boldsymbol{\sigma} \in \mathcal{H}_1$, by $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$ we denote the normal and the tangential traces of $\boldsymbol{\sigma}$. The following two Green formulas hold

$$(div\ \boldsymbol{v}, u)_{L^2(\Omega)} + (\boldsymbol{v}, \nabla u)_H = \int_\Gamma u(\boldsymbol{v} \cdot \boldsymbol{\nu})\, da \quad \text{for all } u \in H^1(\Omega) \text{ and } \boldsymbol{v} \in H_1, \tag{1}$$

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\,\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\boldsymbol{v}\, da \qquad \forall\, \boldsymbol{v} \in H_1 \text{ and } \boldsymbol{\sigma} \in \mathcal{H}_1. \tag{2}$$

Let $T > 0$. For every real Banach space X we denote by $C(0, T; X)$ and $C^1(0, T; X)$ the spaces of continuous and continuously differentiable functions from $[0, T]$ to X, with norms

$$|\,\boldsymbol{f}\,|_{C(0,T;X)} = \max_{t\in[0,T]} |\,\boldsymbol{f}(t)\,|_X, \qquad |\,\boldsymbol{f}\,|_{C^1(0,T;X)} = \max_{t\in[0,T]} |\,\boldsymbol{f}(t)\,|_X + \max_{t\in[0,T]} |\,\dot{\boldsymbol{f}}(t)\,|_X.$$

For $k \in \mathbb{N}$ and $p \in [1, \infty]$, we use the standard notation for the Lebesgue spaces $L^p(0, T; X)$ and for the Sobolev spaces $W^{k,p}(0, T; X)$. Moreover, if X_1 and X_2 are two real Hilbert spaces, then $X_1 \times X_2$ denotes the product space endowed with the canonical inner product $(\cdot, \cdot)_{X_1 \times X_2}$ and norm $|.|_{X_1 \times X_2}$.

3 Problem statement and variational formulation

A thermoviscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$ and $meas(\Gamma_3) > 0$. Let $[0, T]$ be the time interval of interest, for $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$ and a body force of density $\boldsymbol{f}_0$ is applied in $\Omega \times (0, T)$. An initial gap g exists between the potential contact surface Γ_3 and the foundation, and it is measured along the outward normal $\boldsymbol{\nu}$. We assume that the coordinate system is such that Γ_3 occupies a regular domain in the Ox_1x_2 plane and the foundation is moving with velocity $\boldsymbol{v}^*$ in the Ox_1x_2 plane. Furthermore, Γ_3 is divided into two subdomains D_d and D_ω by a smooth curve γ^*. The wear takes place only on D_ω, while the wear particles diffuse on the whole of the contact surface Γ_3. The boundary $\partial\Gamma_3$ of Γ_3 is assumed to be Lipschitz and is composed of two parts γ_d and γ_ω. Then $\partial D_\omega = \gamma_\omega \cup \gamma^*$ and $\partial D_d = \gamma_d \cup \gamma^*$.

The wear function $\omega = \omega(\boldsymbol{x}, t)$ is defined on D_ω and the wear particle surface density function $\zeta = \zeta(\boldsymbol{x}, t)$ is defined on Γ_3. The function ζ measures the surface density of the diffusing wear particles and the wear function ω measures the depth of the wear i.e., the amount of material per unit surface that has been removed, then $\omega = \lambda\zeta$ in D_ω, where λ is a conversion factor from wear debris surface density to wear depth, which we assume to be a positive constant. For the sake of convenience we extend ω by zero to the whole of Γ_3, and below when confusion is unlikely we use the same symbol for the function and its extension. Thus,

$$\omega = \lambda\zeta\,\chi_{[D_w]} \text{ on } \Gamma_3 \times (0, T), \tag{3}$$

where $\chi_{[D_w]}$ is the characteristic function of the set D_ω (i.e., $\chi_{[D_w]}(\mathbf{x}) = 1$ when $\boldsymbol{x} \in D_\omega$ and $\chi_{[D_w]}(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin D_\omega$). The wear diffusion coefficient k is given by

$$k = k(\boldsymbol{x}) = \begin{cases} k_w \text{ in } D_\omega, \\ k_d \text{ in } D_d. \end{cases}$$

Here, wear diffusion is described by the following nonlinear diffusion equation

$$\dot{\zeta} - div(k\nabla\zeta) = \kappa\, |\boldsymbol{\sigma}_\tau|\, R^*(\left|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*\right|)\chi_{[D_\omega]} \qquad \text{in } \Gamma_3 \times (0, T), \tag{4}$$

where $R^* : \mathbb{R}_+ \to \mathbb{R}_+$ is the truncation operator

$$R^*(r) = \begin{cases} r & \text{if } r \le R, \\ R & \text{if } r > R, \end{cases} \tag{5}$$

R is a fixed positive constant and κ is the wear rate coefficient. We need this operator in order to avoid some mathematical difficulties, however, from the physical point of view the use of R^* is not restrictive since, in practice, the slip velocity is bounded and no smallness assumption will be made on R.

Then, the classical model for the above process is as follows:

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^3$, *a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to S_3$, *a temperature field* $\theta : \Omega \times [0, T] \to \mathbb{R}$, *a damage field* $\beta : \Omega \times [0, T] \to \mathbb{R}$ *and a surface particle density field* $\zeta : \Gamma_3 \times [0, T] \to \mathbb{R}$ *such that*

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}), \beta) - C(\theta, \beta) \qquad \text{in } \Omega \times (0, T), \tag{6}$$

$$\dot{\beta} - k_1 \triangle \beta + \partial\varphi_Y(\beta) \ni \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}), \theta, \beta) \qquad \text{in } \Omega \times (0, T), \tag{7}$$

$$Div\boldsymbol{\sigma} + \boldsymbol{f}_0 = 0 \qquad \text{in } \Omega \times (0, T), \tag{8}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1 \times (0, T), \tag{9}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2 \times (0, T), \tag{10}$$

$$\begin{cases} -\sigma_\nu = p_\nu, \quad |\, \boldsymbol{\sigma}_\tau \,| \le \mu p_\nu, \\ \boldsymbol{\sigma}_\tau = -\mu p_\nu \frac{\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*}{|\dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^*|} \quad \text{if } \dot{\boldsymbol{u}}_\tau \neq \boldsymbol{v}^* \end{cases} \qquad \text{on } \Gamma_3 \times (0, T), \tag{11}$$

$$\dot{\zeta} - div(k\nabla\zeta) = \kappa\mu p_\nu R^*(|\, \dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^* \,|)\chi_{[D_\omega]} \qquad \text{on } \Gamma_3 \times (0, T), \tag{12}$$

$$\zeta = 0 \qquad \text{on } \partial\Gamma_3 \times (0, T), \tag{13}$$

$$\dot{\theta} - div(K_c\nabla\theta) = \psi(\dot{\boldsymbol{u}}, \theta, \beta) + q \qquad \text{in } \Omega \times (0, T), \tag{14}$$

$$-k_{ij}\theta_{,i}\eta_j = k_e(\theta - \theta_R) \qquad \text{on } \Gamma_3 \times (0, T), \tag{15}$$

$$\theta = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \times (0, T), \tag{16}$$

$$\frac{\partial\beta}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0, T), \tag{17}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \beta(0) = \beta_0 \qquad \text{in } \Omega, \tag{18}$$

$$\zeta(0) = \zeta_0 \qquad \text{in } \Gamma_3. \tag{19}$$

Here $p_\nu = p_\nu(u_\nu - \lambda\zeta\chi_{[D_\omega]} - g)$ and $\mu = \mu(\zeta, |\, \dot{\boldsymbol{u}}_\tau - \boldsymbol{v}^* \,|)$ is the coefficient of friction which depends on the density of the wear particles and on the slip rate. Equation (6) represents the thermoviscoelastic constitutive law, where $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{u}$ represents the displacement field, $\dot{\boldsymbol{u}}$ the velocity, θ is the temperature field and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the small strain tensor. Here $\mathcal{A}$ and $\mathcal{G}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively and C represents the thermal expansion tensor. Equation (7) represents the inclusion used for the evolution of the damage field, where the set of admissible damage functions defined by

$$Y = \{\xi \in H^1(\Omega) \;/\; 0 \le \xi \le 1 \quad \text{a.e. } \Omega\},$$

k_1 is a positive coefficient, $\partial\varphi_Y$ is the subdifferential of the indicator function φ_Y and ϕ is a given constitutive function which describes the sources of the damage in the system. Equation (8) represents the equilibrium equation, since the process is assumed to be quasistatic. Equations (9)–(10) are the displacement-traction conditions. Equation (12) represents the nonlinear diffusion equation, Eq. (13) is the absorbing boundary condition. In (18) $\boldsymbol{u}_0$ is the given initial displacement field, θ_0 is the initial temperature and β_0 is the given initial damage field. In (19), ζ_0 is the given initial surface particle density field. To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $\boldsymbol{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$.

The evolution of the temperature field θ is governed by the heat equation (see [1, 13]), obtained from the conservation of energy, and defined by the differential equation for the temperature given in (14), where $K_c = (k_{ij})$ represents the thermal conductivity tensor, $div(K_c\nabla\theta) = (k_{ij}\theta_{,i})_{,i}$ and $q(t)$ the density of volume heat sources. The associated temperature boundary condition is given by (15), where θ_R is the temperature of the foundation and k_e is the heat exchange coefficient between the body and the obstacle. Condition (11) represents the normal compliance condition with wear and the associated general law of dry friction on the contact surface Γ_3. In (16) the temperature vanishes on $\Gamma_1 \cup \Gamma_2$. Equation (17) represents the Neumann boundary condition. To obtain a variational formulation of the problem (6)–(19) we need additional notation. Let V be the closed subspace of H_1 defined by

$$V = \{\boldsymbol{v} \in H_1 \;/\; \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1\},$$

and let E be the closed subspace of $H^1(\Omega)$ given by

$$E = \left\{y \in H^1(\Omega) \;/\; y = 0 \text{ on } \Gamma_1 \cup \Gamma_2\right\}.$$

Since Γ is Lipschitz continuous and $meas(\Gamma_1) > 0$, Korn's and Poincare's inequalities hold true

$$|\, \boldsymbol{\varepsilon}(\boldsymbol{v}) \,|_{\mathcal{H}} \ge C \,|\, \boldsymbol{v} \,|_{H_1} \;\forall\, \boldsymbol{v} \in V, \tag{20}$$

$$|\, \nabla y \,|_H \ge C \,|\, y \,|_{H^1(\Omega)} \qquad \forall\, y \in E, \tag{21}$$

where here and below C is a positive constant depending on the problem data but is independent of the solutions, its value may change from line to line. We define the inner products on V and on E by

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \;\forall \boldsymbol{u}, \boldsymbol{v} \in V, \tag{22}$$

$$(y, z)_E = (\nabla y, \nabla z)_H \;\forall y, z \in E, \tag{23}$$

respectively. It follows from (20) and (22) that $|\cdot|_{H_1}$ and $|\cdot|_V$ are equivalent norms on V and from (21) and (23), it follows that $|\cdot|_{H^1(\Omega)}$ and $|\cdot|_E$ are equivalent norms on E. Therefore $(V, |\cdot|_V)$ and $(E, |\cdot|_E)$ are real Hilbert spaces. By the Sobolev's trace theorem, there exists a constant $C_\Gamma > 0$ which depends only on Ω, Γ_1 and Γ_3 such that

$$|\, \boldsymbol{v} \,|_{L^2(\Gamma_3)^3} \leq C_\Gamma \,|\, \boldsymbol{v} \,|_V \qquad \forall\, \boldsymbol{v} \in V. \tag{24}$$

There exists $\hat{C}_\Gamma > 0$ depending on Ω, Γ_1, Γ_2 and Γ_3 such that

$$|\, \theta \,|_{L^2(\Gamma_3)} \leq \hat{C}_\Gamma \,|\, \theta \,|_E \qquad \forall \theta \in E. \tag{25}$$

E' is the dual of the space E. Identifying $L^2(\Omega)$ with its own dual we can write $E \subset L^2(\Omega) \subset E'$. Below $\langle \cdot, \cdot \rangle$ represents the duality pairing between E' and E, and $|\cdot|_{E'}$ denotes the norm on E'. Also, $\langle \theta, \eta \rangle = (\theta, \eta)_{L^2(\Omega)}$ for $\theta \in L^2(\Omega)$ and $\eta \in E$.

Recall that Γ_3 is assumed to be a regular domain in the Ox_1x_2 plane with Lipschitz boundary $\partial\Gamma_3$. Keeping in mind the boundary condition (13), for the surface particle density function, we shall use the space

$$H_0^1(\Gamma_3) = \left\{ \xi \in H^1(\Gamma_3) \,/\, \xi = 0 \text{ on } \partial\Gamma_3 \right\}.$$

This is a real Hilbert space endowed with the inner product

$$(\zeta, \xi)_{H_0^1(\Gamma_3)} = (\nabla\zeta, \nabla\xi)_{L^2(\Gamma_3)^2},$$

where $\nabla : H_0^1(\Gamma_3) \to L^2(\Gamma_3)^2$ denotes the gradient operator, that is $\nabla\xi = \left(\xi_{,x_1}, \xi_{,x_2}\right)$. By the Friedrichs–Poincaré inequality there exists a constant $\widetilde{C}_\Gamma > 0$, which depends on Γ_3, such that

$$|\, \zeta \,|_{L^2(\Gamma_3)} \leq \widetilde{C}_\Gamma \,|\, \zeta \,|_{H_0^1(\Gamma_3)} \quad \forall \zeta \in H_0^1(\Gamma_3). \tag{26}$$

We use the notation $H^{-1}(\Gamma_3)$ for the dual of the space $H_0^1(\Gamma_3)$. Identifying $L^2(\Gamma_3)$ with its own dual we can write $H_0^1(\Gamma_3) \subset L^2(\Gamma_3) \subset H^{-1}(\Gamma_3)$. Below $\langle \cdot, \cdot \rangle$ represents the duality pairing between $H^{-1}(\Gamma_3)$ and $H_0^1(\Gamma_3)$, and $|\cdot|_{H^{-1}(\Gamma_3)}$denotes the norm on $H^{-1}(\Gamma_3)$. Also, $\langle \zeta, \xi \rangle = (\zeta, \xi)_{L^2(\Gamma_3)}$ for $\zeta \in L^2(\Gamma_3)$ and $\xi \in H_0^1(\Gamma_3)$.

For our existence and uniqueness result we will need the following hypotheses.

The viscosity operator $\mathcal{A}$: $\Omega \times S_3 \to S_3$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\, \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2) \,| \leq L_{\mathcal{A}} \,|\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \,| \;\; \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_3, \;\text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}} \,|\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \,|^2 \;\; \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_3, \;\text{ a.e. } \boldsymbol{x} \in \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in S_3. \\ (d) \text{ The mapping } \boldsymbol{x} \to \mathcal{A}(\boldsymbol{x}, \boldsymbol{0}) \;\in \mathcal{H}. \end{cases} \tag{27}$$

The elasticity operator $\mathcal{G} : \Omega \times S_3 \times \mathbb{R} \to S_3$ satisfies

$$\begin{cases} (a) \text{ There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ |\, \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \beta_1) - \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \beta_2) \,| \leq L_{\mathcal{G}}(|\, \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \,| + |\, \beta_1 - \beta_2 \,|) \\ \quad \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in S_3, \; \forall \beta_1, \beta_2 \in \mathbb{R}, \;\text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x}, \boldsymbol{\varepsilon}, \beta) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in S_3 \text{ and } \beta \in \mathbb{R}. \\ (c) \text{ The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x}, \boldsymbol{0}, 0) \in \mathcal{H}. \end{cases} \tag{28}$$

The damage source function $\phi : \Omega \times S_3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists } L_\phi > 0 \text{ such that} \\ \mid \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \theta_1, \beta_1) - \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \theta_2, \beta_2) \mid \le L_\phi(\mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2 \mid + \mid \theta_1 - \theta_2 \mid + \mid \beta_1 - \beta_2 \mid) \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \in S_3 \text{ and } \theta_1, \theta_2, \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \to \phi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \theta, \beta) \text{ is Lebesgue measurable on } \Omega \text{ for any } \boldsymbol{\varepsilon} \in S_3 \text{ and } \theta, \beta \in \mathbb{R}. \\ (c) \text{ The mapping } \boldsymbol{x} \to \phi(\boldsymbol{x}, \boldsymbol{0}, 0, 0) \in L^2(\Omega). \end{cases} \tag{29}$$

The thermal expansion operator C: $\Omega \times \mathbb{R} \times \mathbb{R} \to S_3$ satisfies

$$\begin{cases} (a) \text{ There exists } L_C > 0 \text{ such that} \\ \mid C(\boldsymbol{x}, \theta_1, \beta_1) - C(\boldsymbol{x}, \theta_2, \beta_2) \mid \le L_C(\mid \theta_1 - \theta_2 \mid + \mid \beta_1 - \beta_2 \mid) \, \forall \theta_1, \theta_2, \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \to C(\boldsymbol{x}, \theta, \beta) \text{ is Lebesgue measurable on } \Omega \text{ for any } \theta, \beta \in \mathbb{R}. \\ (c) \text{ The mapping } \boldsymbol{x} \to C(\boldsymbol{x}, 0, 0) \in \mathcal{H}. \end{cases} \tag{30}$$

The normal compliance functions $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfy

$$\begin{cases} (a) \text{ There exists } L_\nu > 0 \text{ such that} \\ \mid p_\nu(\boldsymbol{x}, u_1) - p_\nu(\boldsymbol{x}, u_2) \mid \le L_\nu \mid u_1 - u_2 \mid \ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) \text{ the mapping } \boldsymbol{x} \to p_\nu(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3 \text{ for any } u \in \mathbb{R}. \\ (c) \ p_\nu(\boldsymbol{x}, u) = 0 \text{ for all } u \le 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ (d) \text{ There exists } p_\nu^* > 0 \text{ such that } p_\nu(\boldsymbol{x}, u) \le p_\nu^* \, \forall u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{31}$$

The coefficient of friction $\mu : \Gamma_3 \times \mathbb{R}^2 \to \mathbb{R}_+$ satisfies

$$\begin{cases} (a) \text{ There exists } L_\mu > 0 \text{ such that} \\ \mid \mu(\boldsymbol{x}, a_1, b_1) - \mu(\boldsymbol{x}, a_1, b_1) \mid \le L_\mu(\mid a_1 - a_2 \mid + \mid b_1 - b_2 \mid) \\ \forall a_1, a_2, b_1, b_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ (b) \text{ The mapping } \boldsymbol{x} \to \mu(\boldsymbol{x}, a, b, c) \text{ is Lebesgue measurable on } \Gamma_3, \forall a, b, c \in \mathbb{R}. \\ (c) \text{ There exists } \mu^* > 0 \text{ such that } \mu(\boldsymbol{x}, a, b) \le \mu^* \, \forall a, b \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{cases} \tag{32}$$

The operator in the heat equation $\psi : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists } L_\psi > 0 \text{ such that} \\ \mid \psi(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \theta_1, \beta_1) - \psi(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \theta_2, \beta_2) \mid \le L_\psi(\mid \boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_1 \mid + \mid \theta_1 - \theta_2 \mid + \mid \beta_1 - \beta_2 \mid) \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{R}^3 \ \ \forall \theta_1, \theta_2, \beta_1, \beta_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ (b) \text{ The mapping } \boldsymbol{x} \to \psi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \theta, \beta) \text{ is Lebesgue measurable on } \Omega, \ \forall \boldsymbol{\varepsilon} \in \mathbb{R}^3 \, \forall \theta, \beta \in \mathbb{R}. \\ (c) \ \psi(\boldsymbol{x}, \boldsymbol{\varepsilon}, \theta, \beta) \in L^2(\mathcal{H}). \end{cases} \tag{33}$$

For some $c_k > 0$, for all $(\xi_i) \in \mathbb{R}^3$

$$K_c = (k_{ij}), \ \ k_{ij} = k_{ji} \in L^\infty(\Omega), \ \ k_{ij}\xi_j\xi_i \ge c_k\xi_i\xi_i. \tag{34}$$

For the initial gap function, wear diffusion coefficient, wear rate coefficient, velocity of the foundation, heat source density, body forces and surface traction we make the following assumptions

$$g \in L^2(\Gamma_3), \quad g \ge 0 \text{ a.e. on } \Gamma_3. \tag{35}$$

$$k \in L^\infty(\Gamma_3), \quad k \ge k^* > 0 \text{ a.e. on } \Gamma_3. \tag{36}$$

$$\kappa \in L^\infty(\Gamma_3), \quad \kappa > 0 \text{ a.e. on } \Gamma_3. \tag{37}$$

$$\boldsymbol{v}^* : \Gamma_3 \times [0, T] \to \mathbb{R}^3 \text{ is a continuous function.} \tag{38}$$

$$q \in L^2(0, T; L^2(\Omega)). \tag{39}$$

$$\boldsymbol{f}_0 \in C(0, T; H), \ \ \boldsymbol{f}_2 \in C(0, T; L^2(\Gamma_2)^3). \tag{40}$$

The boundary and initial data satisfy

$$\boldsymbol{u}_0 \in V, \ \ \zeta_0 \in L^2(\Gamma_3), \ \ \beta_0 \in Y, \theta_0 \in L^2(\Omega), \ \ \theta_R \in L^2(0, T; L^2(\Gamma_3)), \ \ k_e \in L^\infty(\Omega, \mathbb{R}_+). \tag{41}$$

We define the vector valued function $\boldsymbol{f} : [0, T] \to V$ and the bilinear forms $a : H_0^1(\Gamma_3) \times H_0^1(\Gamma_3) \to \mathbb{R}$ and $b : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v} \, da \quad \forall \boldsymbol{v} \in V. \tag{42}$$

$$a(\zeta, \xi) = \int_{\Gamma_3} k \nabla \zeta \cdot \nabla \xi \, da. \tag{43}$$

$$b(\xi, \boldsymbol{\varphi}) = k_1 \int_\Omega \nabla \xi \cdot \nabla \boldsymbol{\varphi} \, dx. \tag{44}$$

Finally, the functional $j : L^2(\Gamma_3) \times V^3 \to \mathbb{R}$ and the operator $F : H_0^1(\Gamma_3) \times V^3 \to H^{-1}(\Gamma_3)$ are given by

$$\begin{aligned} j(\zeta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = & \int_{\Gamma_3} p_\nu(u_\nu - \lambda \zeta \chi_{[D_\omega]} - g) w_\nu da \\ & + \int_{\Gamma_3} \mu(\zeta, | \boldsymbol{v}_\tau - \boldsymbol{v}^* |) p_\nu(u_\nu - \lambda \zeta \chi_{[D_\omega]} - g) \, | \boldsymbol{w}_\tau \\ & - \boldsymbol{v}^* | \, da \quad \forall \zeta \in L^2(\Gamma_3), \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \end{aligned} \tag{45}$$

$$\begin{aligned} (F(\zeta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}), \xi)_{H^{-1}(\Gamma_3) \times H_0^1(\Gamma_3)} = & \int_{\Gamma_3} \kappa \mu(\zeta, | \boldsymbol{v}_\tau - \boldsymbol{v}^* |) p_\nu(u_\nu - \lambda \zeta \chi_{[D_\omega]} - g) \\ & R^*(| \boldsymbol{w}_\tau - \boldsymbol{v}^* |) \xi da \quad \forall \zeta, \xi \in H_0^1(\Gamma_3), \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V. \end{aligned} \tag{46}$$

Using standard arguments based on Green formulas given on (1) and (2), we obtain the following formulation of the mechanical problem (6)–(19).

Problem PV. *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$, *a stress field* $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$, *a temperature field* $\theta : [0, T] \to E$, *a damage field* $\beta : [0, T] \to H^1(\Omega)$ *and a surface particle density field* $\zeta : [0, T] \to H_0^1(\Gamma_3)$ *such that*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \beta(t)) - C(\theta(t), \beta(t)) \tag{47}$$

$$\begin{aligned} & (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \boldsymbol{v}) - j(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)) \\ & \quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \end{aligned} \tag{48}$$

$$\begin{aligned} & \beta(t) \in Y, (\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + b(\beta(t), \xi - \beta(t)) \\ & \quad \geq (\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \theta(t), \beta(t)), \xi - \beta(t))_{L^2(\Omega)} \quad \forall \xi \in Y \quad \text{a.e. } t \in (0, T), \end{aligned} \tag{49}$$

$$\dot{\theta}(t) + K\theta(t) = S(\dot{\boldsymbol{u}}(t), \theta(t), \beta(t)) + Q(t) \quad \text{in } E' \quad \text{a.e. } t \in (0, T), \tag{50}$$

$$(\dot{\zeta}(t), \xi)_{H^{-1}(\Gamma_3)\times H_0^1(\Gamma_3)} + a(\zeta(t), \xi) = (F(\zeta(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)), \xi)_{H^{-1}(\Gamma_3)\times H_0^1(\Gamma_3)}$$
$$\forall \xi \in H_0^1(\Gamma_3) \quad \text{a.e. } t \in (0, T), \tag{51}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \beta(0) = \beta_0, \quad \zeta(0) = \zeta_0, \quad \theta(0) = \theta_0, \tag{52}$$

where $Q : [0, T] \to E'$, $K : E \to E'$ and $S : V \times E \times L^2(\Omega) \to E'$ are given by

$$(Q(t), \eta)_{E'\times E} = \int_{\Gamma_3} k_e \theta_R(t)\eta da + \int_{\Omega} q(t)\eta dx, \tag{53}$$

$$(K\tau, \eta)_{E'\times E} = \sum_{i,j=1}^{3} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \cdot \eta da, \tag{54}$$

$$(S(\boldsymbol{u}, \theta, \beta), \eta)_{E'\times E} = \int_{\Omega} \psi(\boldsymbol{u}, \theta, \beta)\eta dx, \tag{55}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\theta, \eta, \tau \in E$, $\beta \in L^2(\Omega)$ and $\zeta \in L^2(\Gamma_3)$. Below in this section $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$ represent elements of V, θ_1, θ_2 are elements of E and ζ_1, ζ_2 are in $L^2(\Gamma_3)$. Finally, we use (45), the assumption (31) on p_ν, the assumption (32) on μ, (24) and (25) to see that

$$j(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{v}_2) - j(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{v}_1) + j(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{v}_1) - j(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{v}_2)$$
$$\leq C_\Gamma^2(L_\nu + \mu^* L_\nu)|\boldsymbol{u}_1 - \boldsymbol{u}_2|_V |\boldsymbol{v}_1 - \boldsymbol{v}_2|_V + C_\Gamma(L_\nu\lambda + \mu^* L_\nu\lambda + p_\nu^* L_\mu)|\zeta_1$$
$$- \zeta_2|_{L^2(\Gamma_3)} |\boldsymbol{v}_1 - \boldsymbol{v}_2|_V + p_\nu^* L_\mu C_\Gamma^2 |\boldsymbol{v}_1 - \boldsymbol{v}_2|_V^2 \tag{56}$$

This inequality will be used in the following section.

4 An existence and uniqueness result

Our main existence and uniqueness result is the following.

Theorem 1 *Let the assumptions (27)–(41) hold. There exists a constant $C^* > 0$, depending on C_Γ, $m_{\mathcal{A}}$ and L_μ such that if $p_\nu^* < C^*$ then problem PV has a unique solution $\{\boldsymbol{u}, \boldsymbol{\sigma}, \beta, \theta, \zeta\}$ satisfying*

$$\boldsymbol{u} \in C^1(0, T; V), \quad \boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \tag{57}$$
$$\beta \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \tag{58}$$
$$\theta \in C(0, T; L^2(\Omega)) \cap L^2(0, T; E) \cap W^{1,2}(0, T; E'), \tag{59}$$
$$\zeta \in L^2(0, T; H_0^1(\Gamma_3)) \cap C(0, T; L^2(\Gamma_3)), \quad \dot{\zeta} \in L^2(0, T; H^{-1}(\Gamma_3)). \tag{60}$$

We conclude that, under assumptions (27)–(41), the mechanical problem (6)–(19) has a unique weak solution with the regularities (57)–(60).

The proof of this theorem will be carried out in several steps. We assume in what follows that assumptions (27)–(41) are satisfied and moreover

$$C_\Gamma^2 p_\nu^* L_\mu < m_{\mathcal{A}}. \tag{61}$$

First let $\alpha \in L^2(0, T; H^{-1}(\Gamma_3))$, we solve the following parabolic equation.

Problem PV_α. *Find* $\zeta_\alpha : [0, T] \to H_0^1(\Gamma_3)$ *such that*

$$(\dot{\zeta}_\alpha(t), \xi)_{H^{-1}(\Gamma_3)\times H_0^1(\Gamma_3)} + a(\zeta_\alpha(t), \xi) = (\alpha(t), \xi)_{H^{-1}(\Gamma_3)\times H_0^1(\Gamma_3)}$$

$$\forall \xi \in H_0^1(\Gamma_3),\ \text{a.e. } t \in (0,T)\,, \tag{62}$$

$$\zeta_\alpha(0) = \zeta_0. \tag{63}$$

Lemma 1 *There exists a unique solution of problem* (62)–(63) *satisfying*

$$\zeta_\alpha \in L^2(0,T; H_0^1(\Gamma_3)) \cap C(0,T; L^2(\Gamma_3)),\ \dot{\zeta}_\alpha \in L^2(0,T; H^{-1}(\Gamma_3)). \tag{64}$$

Moreover, if ζ_i *is the solution to Problem* PV_α *corresponding to* $\alpha = \alpha_i \in L^2(0,T; H^{-1}(\Gamma_3))$, *for* $i = 1, 2$, *then*

$$|\zeta_1(t) - \zeta_2(t)|^2_{L^2(\Gamma_3)} + \int_0^t |\nabla\zeta_1(s) - \nabla\zeta_2(s)|^2_{L^2(\Gamma_3)^2} ds$$
$$\leq C \int_0^t |\alpha_1(s) - \alpha_2(s)|^2_{H^{-1}(\Gamma_3)} ds \qquad \forall t \in [0,T]. \tag{65}$$

Proof The proof follows from an evolution equation result with linear continuous operators, see for example [24].

Now let $\rho \in L^2(0,T; E')$, we solve the following evolution equation.

Problem PV_ρ. *Find* $\theta_\rho : [0,T] \to E$ *such that*

$$\dot{\theta}_\rho(t) + K\theta_\rho(t) = \rho(t) \quad \text{in } E',\ \text{a.e. } t \in (0,T)\,, \tag{66}$$

$$\theta_\rho(0) = \theta_0. \tag{67}$$

We have the following result.

Lemma 2 *Problem* PV_ρ *has a unique solution satisfying the regularity* (59). *Moreover,* $\exists C > 0$ *such that* $\forall \rho_i \in L^2(0,T; E')$, *denote* $\theta_{\rho_i} = \theta_i$, $i = 1, 2$,

$$|\,\theta_1(t) - \theta_2(t)\,|^2_{L^2(\Omega)} + \int_0^t |\,\theta_1(s) - \theta_2(s)\,|^2_E\, ds \leq C \int_0^t |\,\rho_1(s) - \rho_2(s)\,|^2_{E'}\, ds \quad \forall t \in [0,T]. \tag{68}$$

Proof The proof follows from classical first order evolution equation given in [2, 22]. Here the Gelfand triple is given by $E \subset L^2(\Omega) \subset E'$. The operator K is linear and coercive. By Korn's inequality, we have $(K\tau, \tau)_{E' \times E} \geq C \mid \tau \mid^2_E$.

Next, for $\forall \rho_1, \rho_2 \in L^2(0,T; E')$ we have for a.e. $s \in (0,T)$

$$(\dot{\theta}_1(s) - \dot{\theta}_2(s), \theta_1(s) - \theta_2(s))_{L^2(\Omega)} + (K\theta_1(s) - K\theta_2(s), \theta_1(s) - \theta_2(s))_{E' \times E}$$
$$= (\rho_1(s) - \rho_2(s), \theta_1(s) - \theta_2(s))_{E' \times E},$$

then by integrating over $(0,t)$ for $t \in [0,T]$, (68) follows by using (34) and (54).

In the third step, we let $\gamma \in L^2(0,T; L^2(\Omega))$ be given and consider the following variational problem for the damage field.

Problem PV_γ. *Find a damage field* $\beta_\gamma : [0,T] \to H^1(\Omega)$ *such that*

$$\beta_\gamma(t) \in Y,\ (\dot{\beta}_\gamma(t), \xi - \beta_\gamma(t))_{L^2(\Omega)} + b(\beta_\gamma(t), \xi - \beta_\gamma(t))$$
$$\geq (\gamma(t), \xi - \beta_\gamma(t))_{L^2(\Omega)} \quad \forall \xi \in Y \text{ a.e. } t \in (0,T)\,, \tag{69}$$

$$\beta_\gamma(0) = \beta_0. \tag{70}$$

Lemma 3 *Problem PV_γ has a unique solution β_γ such that*

$$\beta_\gamma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{71}$$

Moreover, if β_i is the solution to Problem PV_γ corresponding to $\gamma = \gamma_i \in L^2(0,T;L^2(\Omega))$, for $i = 1,2$, then

$$|\beta_1(t)-\beta_2(t)|^2_{L^2(\Omega)} + \int_0^t |\beta_1(s)-\beta_2(s)|^2_{L^2(\Omega)} ds \le C \int_0^t |\gamma_1(s)-\gamma_2(s)|^2_{L^2(\Omega)} ds \qquad \forall t \in [0,T]. \tag{72}$$

Proof We use (44), β_0 in (41) and a classical existence and uniqueness result on parabolic inequalities (see for example [2, 22]).

Now we substitute (47) in (48) and we consider the obtained variational inequality with $\zeta = \zeta_\alpha, \theta = \theta_\rho$ and $\beta = \beta_\gamma$. Let $(\boldsymbol{z}, \boldsymbol{h}) \in C(0,T;V)^2$ be given, we consider the following variational problem.

Problem $PV_{\alpha\rho\gamma zh}$. *Find a displacement field* $\boldsymbol{v}_{\alpha\rho\gamma zh} : [0,T] \to V$ *such that*

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\alpha\rho\gamma zh}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma zh}(t)))_{\mathcal{H}} + j(\zeta_\alpha(t), \boldsymbol{z}(t), \boldsymbol{h}(t), \boldsymbol{v}) - j(\zeta_\alpha(t), \boldsymbol{z}(t), \boldsymbol{h}(t), \boldsymbol{v}_{\alpha\rho\gamma zh}(t)) \\
&\ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma zh}(t))_V - (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}(t)), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma zh}(t)))_{\mathcal{H}} \\
&\quad + (C(\theta_\rho(t), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma zh}(t)))_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in V, t \in (0,T).
\end{aligned} \tag{73}$$

Lemma 4 *There exists a unique solution of problem (73) satisfying $\boldsymbol{v}_{\alpha\rho\gamma zh} \in C(0,T;V)$.*

Proof By using variational inequalities results (see for example [3]), we conclude that there exists a unique solution $\boldsymbol{v}_{\alpha\rho\gamma zh}(t)$ of problem (73) for $t \in (0,T)$. Now we show that $\boldsymbol{v}_{\alpha\rho\gamma zh} : [0,T] \to V$ is continuous. Let $t_1, t_2 \in (0,T)$, we denote $\boldsymbol{v}_{\alpha\rho\gamma zh}(t_i) = \boldsymbol{v}_i$, $\zeta_\alpha(t_i) = \zeta_i$, $\theta_\rho(t_i) = \theta_i$, $\beta_\gamma(t_i) = \beta_i$, $\boldsymbol{z}(t_i) = \boldsymbol{z}_i$, $\boldsymbol{h}(t_i) = \boldsymbol{h}_i$ and $\boldsymbol{f}(t_i) = \boldsymbol{f}_i$ for $i = 1,2$. We use (73) to find

$$\begin{aligned}
(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \le\ & j(\zeta_1, \boldsymbol{z}_1, \boldsymbol{h}_1, \boldsymbol{v}_2) - j(\zeta_1, \boldsymbol{z}_1, \boldsymbol{h}_1, \boldsymbol{v}_1) \\
&+ j(\zeta_2, \boldsymbol{z}_2, \boldsymbol{h}_2, \boldsymbol{v}_1) - j(\zeta_2, \boldsymbol{z}_2, \boldsymbol{h}_2, \boldsymbol{v}_2) \\
&+ (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}_1), \beta_1) - \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}_2), \beta_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \\
&+ (C(\theta_1, \beta_1) - C(\theta_2, \beta_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \\
&+ (\boldsymbol{f}_1 - \boldsymbol{f}_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_V.
\end{aligned}$$

Condition (22), the estimate (56) and assumptions (27), (28) and (30) give us

$$\begin{aligned}
m_{\mathcal{A}} \mid \boldsymbol{v}_1 - \boldsymbol{v}_2 \mid_V \le\ & (L_{\mathcal{G}} + C_\Gamma^2 (L_\nu + \mu^* L_\nu)) \mid \boldsymbol{z}_1 - \boldsymbol{z}_2 \mid_V \\
&+ C_\Gamma^2 p_\nu^* L_\mu \mid \boldsymbol{h}_1 - \boldsymbol{h}_2 \mid_V + (L_{\mathcal{G}} + L_C) \mid \beta_1 - \beta_2 \mid_{L^2(\Omega)} \\
&+ C_\Gamma (\lambda L_\nu + \mu^* L_\nu \lambda + p_\nu^* L_\mu) \mid \zeta_1 - \zeta_2 \mid_{L^2(\Gamma_3)} \\
&+ \mid \boldsymbol{f}_1 - \boldsymbol{f}_2 \mid_V + L_C \mid \theta_1 - \theta_2 \mid_E,
\end{aligned} \tag{74}$$

which implies that $\boldsymbol{v}_{\alpha\rho\gamma zh} : [0,T] \to V$ is a continuous function.

We now consider the operator $\Lambda_{\alpha\rho\gamma z} : C(0, T; V) \to C(0, T; V)$ defined by

$$\Lambda_{\alpha\rho\gamma z}\boldsymbol{h} = \boldsymbol{v}_{\alpha\rho\gamma z h}. \tag{75}$$

We have the following result.

Lemma 5 *The operator $\Lambda_{\alpha\rho\gamma z}$ has a unique fixed point $\boldsymbol{h}_{\alpha\rho\gamma z} \in C(0, T; V)$.*

Proof Let $\boldsymbol{h}_1, \boldsymbol{h}_2 \in C(0, T; V)$ and let $\boldsymbol{v}_i$ denote the solution of (73) for $\boldsymbol{h} = \boldsymbol{h}_i$, i.e. $\boldsymbol{v}_i = \boldsymbol{v}_{\alpha\rho\gamma z h_i}$, $i = 1, 2$. Using the definition of the operator $\Lambda_{\alpha\rho\gamma z}$ given in (75) we find that

$$| \Lambda_{\alpha\rho\gamma z}\boldsymbol{h}_1(t) - \Lambda_{\alpha\rho\gamma z}\boldsymbol{h}_2(t) |_V = | \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t) |_V \quad \forall t \in [0, T].$$

We use arguments like those used for the estimate (74) to obtain

$$m_{\mathcal{A}} \mid \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t) \mid_V \le C_\Gamma^2 p_\nu^* L_\mu \mid \boldsymbol{h}_1(t) - \boldsymbol{h}_2(t) \mid_V \ \forall t \in [0, T].$$

The two previous inequalities and the assumption (61) imply that the operator $\Lambda_{\alpha\rho\gamma z}$ is a contraction on the Banach space $C(0, T; V)$.

In the sequel let $\boldsymbol{h}_{\alpha\rho\gamma z}$ be the fixed point obtained in Lemma 5 and let $\boldsymbol{v}_{\alpha\rho\gamma z} \in C(0, T; V)$ be the function defined by

$$\boldsymbol{v}_{\alpha\rho\gamma z} = \boldsymbol{v}_{\alpha\rho\gamma z h_{\alpha\rho\gamma z}}. \tag{76}$$

We have $\Lambda_{\alpha\rho\gamma z}\boldsymbol{h}_{\alpha\rho\gamma z} = \boldsymbol{h}_{\alpha\rho\gamma z}$ and

$$\boldsymbol{v}_{\alpha\rho\gamma z} = \boldsymbol{h}_{\alpha\rho\gamma z}. \tag{77}$$

We take $\boldsymbol{h} = \boldsymbol{h}_{\alpha\rho\gamma z}$ in (73) and we use (76) and (77), to see that $\boldsymbol{v}_{\alpha\rho\gamma z}$ satisfies

$$\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\alpha\rho\gamma z}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma z}(t)))_{\mathcal{H}} + j(\zeta_\alpha(t), \boldsymbol{z}(t), \boldsymbol{v}_{\alpha\rho\gamma z}(t), \boldsymbol{v}) \\
&\quad - j(\zeta_\alpha(t), \boldsymbol{z}(t), \boldsymbol{v}_{\alpha\rho\gamma z}(t), \boldsymbol{v}_{\alpha\rho\gamma z}(t)) \\
&\quad \ge (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma z}(t))_V - (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{z}(t)), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma z h}(t)))_{\mathcal{H}} \\
&\quad + (C(\theta_\rho(t), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}_{\alpha\rho\gamma z h}(t)))_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in V, \quad t \in (0, T).
\end{aligned} \tag{78}$$

Let now $\boldsymbol{u}_{\alpha\rho\gamma z} \in C^1(0, T; V)$ be the function defined by

$$\boldsymbol{u}_{\alpha\rho\gamma z}(t) = \int_0^t \boldsymbol{v}_{\alpha\rho\gamma z}(s) ds + \boldsymbol{u}_0 \quad \forall t \in [0, T]. \tag{79}$$

We define the operator $\Lambda_{\alpha\rho\gamma} : C(0, T; V) \to C(0, T; V)$ by

$$\Lambda_{\alpha\rho\gamma}\boldsymbol{z} = \boldsymbol{u}_{\alpha\rho\gamma z}. \tag{80}$$

We have the following result.

Lemma 6 *The operator $\Lambda_{\alpha\rho\gamma}$ has a unique fixed point $\boldsymbol{z}_{\alpha\rho\gamma} \in C(0, T; V)$.*

Proof Let $\boldsymbol{z}_1, \boldsymbol{z}_2 \in C(0, T; V)$ and denote $\boldsymbol{v}_i = \boldsymbol{v}_{\alpha\rho\gamma z_i}$, $\boldsymbol{u}_i = \boldsymbol{u}_{\alpha\rho\gamma z_i}$ for $i = 1, 2$. We use (78) and arguments like those used for the estimate (74) in the proof of Lemma 4 to have

$$(m_{\mathcal{A}} - C_\Gamma^2 p_\nu^* L_\mu) |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|_V \le (L_{\mathcal{G}} + C_\Gamma^2 (L_\nu + \mu^* L_\nu)) |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)|_V, \tag{81}$$

for all $s \in [0, T]$. Using now (80)–(81) we obtain

$$\left|\Lambda_{\alpha\rho\gamma}\boldsymbol{z}_1(t) - \Lambda_{\alpha\rho\gamma}\boldsymbol{z}_2(t)\right|_V \le C \int_0^t |\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)|_V \, ds,$$

for all $t \in [0, T]$ and C is a positive constant. By reiterating this inequality we obtain that a power of $\Lambda_{\alpha\rho\gamma}$ is a contraction mapping on $C(0, T; V)$, which concludes the proof.

We are now ready to prove the unique solvability of the variational problem.

Problem $PV_{\alpha\rho\gamma}$. *Find a displacement field* $\boldsymbol{u}_{\alpha\rho\gamma} : [0, T] \to V$ *such that*

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} + j(\zeta_\alpha(t), \boldsymbol{u}_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t), \boldsymbol{v}) \\
&- j(\zeta_\alpha(t), \boldsymbol{u}_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t))_V \\
&- (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_{\alpha\rho\gamma}(t)), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} + (C(\theta_\rho(t), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} \\
&\forall \boldsymbol{v} \in V, \quad t \in (0, T), \qquad (82) \\
&\boldsymbol{u}_{\alpha\rho\gamma}(0) = \boldsymbol{u}_0. \qquad (83)
\end{aligned}
$$

Lemma 7 *There exists a unique function* $\boldsymbol{u}_{\alpha\rho\gamma} \in C^1(0, T; V)$ *satisfying* (82)–(83).

Proof Let $z_{\alpha\rho\gamma} \in C(0, T; V)$ be the fixed point guaranteed by Lemma 6 and let $\boldsymbol{u}_{\alpha\rho\gamma} \in C^1(0, T; V)$ be defined by (79), for $\boldsymbol{z} = z_{\alpha\rho\gamma}$. We have $\dot{\boldsymbol{u}}_{\alpha\rho\gamma} = \boldsymbol{v}_{\alpha\rho\gamma z_{\alpha\rho\gamma}}$ and, writing (78) for $\boldsymbol{z} = z_{\alpha\rho\gamma}$, we find

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} + j(\zeta_\alpha(t), z_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t), \boldsymbol{v}) \\
&\quad - j(\zeta_\alpha(t), z_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t), \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)) \\
&\quad \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t))_V - (\mathcal{G}(\boldsymbol{\varepsilon}(z_{\alpha\rho\gamma}(t)), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} \\
&\quad + (C(\theta_\rho(t), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}_{\alpha\rho\gamma}(t)))_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in V, \quad t \in (0, T). \qquad (84)
\end{aligned}
$$

Inequality (82) follows now from (84) and (75) since $\boldsymbol{u}_{\alpha\rho\gamma} = z_{\alpha\rho\gamma}$. Moreover, (83) results from (79). We conclude that $\boldsymbol{u}_{\alpha\rho\gamma}$ is a solution of (82)–(83). For the uniqueness, let $\boldsymbol{u}_{\alpha\rho\gamma}$ be the solution of (82)–(83) and let $\boldsymbol{u}^*_{\alpha\rho\gamma}$ be any other solution such that $\boldsymbol{u}^*_{\alpha\rho\gamma} \in C^1(0, T; V)$. Let $\boldsymbol{v}^*_{\alpha\rho\gamma} = \dot{\boldsymbol{u}}^*_{\alpha\rho\gamma}$. Using (82) we obtain that $\boldsymbol{v}^*_{\alpha\rho\gamma}$ satisfies

$$
\begin{aligned}
&(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^*_{\alpha\rho\gamma}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}^*_{\alpha\rho\gamma}(t)))_{\mathcal{H}} + j(\zeta_\alpha(t), \boldsymbol{u}^*_{\alpha\rho\gamma}(t), \boldsymbol{v}^*_{\alpha\rho\gamma}(t), \boldsymbol{v}) \\
&\quad - j(\zeta_\alpha(t), \boldsymbol{u}^*_{\alpha\rho\gamma}(t), \boldsymbol{v}^*_{\alpha\rho\gamma}(t), \boldsymbol{v}^*_{\alpha\rho\gamma}(t)) \\
&\geq (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{v}^*_{\alpha\rho\gamma}(t))_V - (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^*_{\alpha\rho\gamma}(t)), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}^*_{\alpha\rho\gamma}(t)))_{\mathcal{H}} \\
&\quad + (C(\theta_\rho(t), \beta_\gamma(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}^*_{\alpha\rho\gamma}(t)))_{\mathcal{H}} \qquad \forall \boldsymbol{v} \in V, \quad t \in (0, T). \qquad (85)
\end{aligned}
$$

This inequality has a form of (78) with $\boldsymbol{z} = \boldsymbol{u}^*_{\alpha\rho\gamma}$ and, therefore, it follows from (73) that it has a unique solution, already denoted by $\boldsymbol{v}_{\alpha\rho\gamma u^*_{\alpha\rho\gamma}}$. We conclude that $\boldsymbol{v}^*_{\alpha\rho\gamma} = \boldsymbol{v}_{\alpha\rho\gamma u^*_{\alpha\rho\gamma}}$. Since $\boldsymbol{v}^*_{\alpha\rho\gamma} = \dot{\boldsymbol{u}}^*_{\alpha\rho\gamma}$ it follows from (79) that

$$
\boldsymbol{u}^*_{\alpha\rho\gamma}(t) = \int_0^t \boldsymbol{v}_{\alpha\rho\gamma u^*_{\alpha\rho\gamma}}(s)ds + \boldsymbol{u}_0 \quad \forall t \in [0, T]. \tag{86}
$$

From (79) and (86) we obtain $\boldsymbol{u}^*_{\alpha\rho\gamma} = \boldsymbol{u}_{\alpha\rho\gamma u^*_{\alpha\rho\gamma}}$, which shows that $\boldsymbol{u}^*_{\alpha\rho\gamma}$ is a fixed point of the operator $\Lambda_{\alpha\rho\gamma}$ defined by (80). Using now Lemma 5 we deduce that

$$
\boldsymbol{u}^*_{\alpha\rho\gamma} = z^*_{\alpha\rho\gamma}. \tag{87}
$$

The uniqueness of the problem (82)–(83) is now a consequence of the fact that $\boldsymbol{u}_{\alpha\rho\gamma} = z_{\alpha\rho\gamma}$ and equality (87).

Next, we need to investigate the properties of the operator $F : H^1_0(\Gamma_3) \times E \times V^3 \to H^{-1}(\Gamma_3)$ given by (46).

Lemma 8 *The following inequality holds*

$$
\begin{aligned}
&|F(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{w}_1) - F(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{w}_2)|_{H^{-1}(\Gamma_3)} \\
&\quad \leq L_F(|\ \zeta_1 - \zeta_2\ |_{H_0^1(\Gamma_3)} + |\ \boldsymbol{u}_1 - \boldsymbol{u}_2\ |_V + |\ \boldsymbol{v}_1 - \boldsymbol{v}_2\ |_V + |\ \boldsymbol{w}_1 - \boldsymbol{w}_2\ |_V) \\
&\quad \forall \zeta_1, \zeta_2 \in H_0^1(\Gamma_3),\ \forall \boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}_1, \boldsymbol{w}_2 \in V,
\end{aligned} \tag{88}
$$

where

$$
L_F = |\ \kappa\ |_{L^\infty(D_\omega)}\ C_\Gamma \max\left\{\mu^* p_\nu^* C_\Gamma,\ \mu^* L_\nu R C_\Gamma,\ (L_\mu p_\nu^* + \lambda\mu^* L_\nu) R\tilde{C}_\Gamma,\ p_\nu^* L_\mu R C_\Gamma\right\}.
$$

Proof The estimate (88) follows from the definition (46) on F, the assumptions (31) on p_ν, (32) on μ, (24), (25), (26) and the definition of R^*.

Now we define the operator $\Lambda : L^2(0, T; H^{-1}(\Gamma_3) \times E' \times L^2(\Omega)) \to L^2(0, T; H^{-1}(\Gamma_3) \times E' \times L^2(\Omega))$ by

$$
\Lambda(\alpha, \rho, \gamma) = (\Lambda_1(\alpha, \rho, \gamma), \Lambda_2(\alpha, \rho, \gamma), \Lambda_3(\alpha, \rho, \gamma)), \tag{89}
$$

such that

$$
\Lambda_1(\alpha(t), \rho(t), \gamma(t)) = F(\zeta_\alpha(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t), \dot{\boldsymbol{u}}(t)), \tag{90}
$$

$$
\Lambda_2(\alpha(t), \rho(t), \gamma(t)) = S(\dot{\boldsymbol{u}}(t), \theta_\rho(t), \beta_\gamma) + Q(t), \tag{91}
$$

$$
\Lambda_3(\alpha(t), \rho(t), \gamma(t)) = \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \theta_\rho(t), \beta_\gamma(t)). \tag{92}
$$

Lemma 9 *The operator Λ has a unique fixed point $(\alpha, \rho, \gamma) \in L^2(0, T; H^{-1}(\Gamma_3) \times E' \times L^2(\Omega))$ such that $\Lambda(\alpha, \rho, \gamma) = (\alpha, \rho, \gamma)$.*

Proof Let $(\alpha_i, \rho_i, \gamma_i) \in L^2(0, T; H^{-1}(\Gamma_3) \times E' \times L^2(\Omega))$ $i = 1, 2$. Denote $\boldsymbol{u}_{\alpha_i \rho_i \gamma_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\alpha_i \rho_i \gamma_i} = \boldsymbol{v}_i$, $\zeta_{\alpha_i} = \zeta_i$, $\theta_{\rho_i} = \theta_i$ and $\beta_{\gamma_i} = \beta_i$ for $i = 1, 2$. Using (90)–(92), (29), (30), (31), (32), (33), the definition of R^* and Lemma 8 we deduce that

$$
\begin{aligned}
&|\ (\Lambda_1(\alpha_1, \rho_1, \gamma_1) - \Lambda_1(\alpha_2, \rho_2, \gamma_2), \xi)\ |^2_{H^{-1}(\Gamma_3) \times H_0^1(\Gamma_3)} \\
&\quad \leq \int_{\Gamma_3} |F(\zeta_1, \boldsymbol{u}_1, \dot{\boldsymbol{u}}_1, \dot{\boldsymbol{u}}_1) - F(\zeta_2, \boldsymbol{u}_2, \dot{\boldsymbol{u}}_2, \dot{\boldsymbol{u}}_2)|^2_{H^{-1}(\Gamma_3)}\ |\ \xi\ |^2_{H_0^1(\Gamma_3)}\ da \\
&\quad \leq C(|\ \zeta_1 - \zeta_2\ |^2_{L^2(\Gamma_3)} + |\ \boldsymbol{u}_1 - \boldsymbol{u}_2\ |^2_V + |\ \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\ |^2_V)\ |\ \xi\ |^2_{L^2(\Gamma_3)}. \\
&|\ (\Lambda_2(\alpha_1, \rho_1, \gamma_1) - \Lambda_2(\alpha_2, \rho_2, \gamma_2), \eta)\ |^2_{E' \times E} \\
&\quad \leq \int_\Omega |\big(\psi(\dot{\boldsymbol{u}}_1, \theta_1, \beta_1) - \psi(\dot{\boldsymbol{u}}_2, \theta_2, \beta_2)\big)\eta|^2 dx \\
&\quad \leq C(|\ \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2\ |^2_V + |\ \theta_1 - \theta_2\ |^2_{L^2(\Omega)} + |\beta_1 - \beta_2|^2_{L^2(\Omega)})\ |\ \eta\ |^2_{L^2(\Omega)}. \\
&|\ \Lambda_3(\alpha_1, \rho_1, \gamma_1) - \Lambda_3(\alpha_2, \rho_2, \gamma_2)\ |^2_{L^2(\Omega)} = |\ \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \theta_1, \beta_1) - \phi(\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \theta_2, \beta_2)\ |^2_{L^2(\Omega)} \\
&\quad \leq C(|\boldsymbol{u}_1 - \boldsymbol{u}_2|^2_V + |\theta_1 - \theta_2|^2_{L^2(\Omega)} + |\beta_1 - \beta_2|^2_{L^2(\Omega)}).
\end{aligned}
$$

Hence

$$| \Lambda(\alpha_1, \rho_1, \gamma_1) - \Lambda(\alpha_2, \rho_2, \gamma_2) |^2_{H^{-1}(\Gamma_3)\times E'\times L^2(\Omega)}$$
$$\leq C(|\zeta_1 - \zeta_2|^2_{L^2(\Gamma_3)} + |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2_V + | \dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2 |^2_V + | \theta_1 - \theta_2 |^2_{L^2(\Omega)} + |\beta_1 - \beta_2|^2_{L^2(\Omega)}). \tag{93}$$

Since $\boldsymbol{u}_1(0) = \boldsymbol{u}_2(0) = \boldsymbol{u}_0$ we have

$$| \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t) |^2_V \leq \int_0^t | \boldsymbol{v}_1(s) - \boldsymbol{v}_2(s) |^2_V \, ds. \tag{94}$$

Moreover, from (82) we obtain that

$$\begin{aligned}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_1) - \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \leq\ & j(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{v}_2) - j(\zeta_1, \boldsymbol{u}_1, \boldsymbol{v}_1, \boldsymbol{v}_1)\\ & + j(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{v}_1) - j(\zeta_2, \boldsymbol{u}_2, \boldsymbol{v}_2, \boldsymbol{v}_2)\\ & + (C(\theta_1, \beta_1) - C(\theta_2, \beta_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}}\\ & + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \beta_1) - \mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}_2), \beta_2), \boldsymbol{\varepsilon}(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}}.\end{aligned} \tag{95}$$

The assumption (27) on $\mathcal{A}$, the assumption (61) and the estimate (56) on j give us

$$| \boldsymbol{v}_1 - \boldsymbol{v}_2 |^2_V \leq C\left(| \boldsymbol{u}_1 - \boldsymbol{u}_2 |^2_V + | \zeta_1 - \zeta_2 |^2_{L^2(\Gamma_3)} + | \theta_1 - \theta_2 |^2_E + |\beta_1 - \beta_2|^2_{L^2(\Omega)}\right). \tag{96}$$

From (94), (96) and by using Gronwall inequality we see that

$$| \boldsymbol{v}_1 - \boldsymbol{v}_2 |^2_V \leq C\left(| \zeta_1 - \zeta_2 |^2_{L^2(\Gamma_3)} + | \theta_1 - \theta_2 |^2_E + |\beta_1 - \beta_2|^2_{L^2(\Omega)}\right). \tag{97}$$

From (93), (94) and (97) we find that

$$\begin{aligned}& | \Lambda(\alpha_1(t), \rho_1(t), \gamma_1(t)) - \Lambda(\alpha_2(t), \rho_2(t), \gamma_2(t)) |^2_{H^{-1}(\Gamma_3)\times E'\times L^2(\Omega)}\\ & \leq C\Big(|\zeta_1(t) - \zeta_2(t)|^2_{L^2(\Gamma_3)} + | \theta_1(t) - \theta_2(t) |^2_E + | \theta_1(t) - \theta_2(t) |^2_{L^2(\Omega)} + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)}\Big)\\ & \quad + C\Big(\int_0^t |\nabla\zeta_1(s) - \nabla\zeta_2(s)|^2_{L^2(\Gamma_3)^2} ds + \int_0^t |\theta_1(s) - \theta_2(s)|^2_E ds + \int_0^t |\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)} ds\Big)\\ & \leq C\Big(|\zeta_1(t) - \zeta_2(t)|^2_{L^2(\Gamma_3)} + \int_0^t |\nabla\zeta_1(s) - \nabla\zeta_2(s)|^2_{L^2(\Gamma_3)^2} ds + |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega)}\\ & \quad + \int_0^t |\theta_1(s) - \theta_2(s)|^2_E ds + |\beta_1(t) - \beta_2(t)|^2_{L^2(\Omega)} + \int_0^t |\beta_1(s) - \beta_2(s)|^2_{L^2(\Omega)} ds\Big).\end{aligned}$$

Using estimates (64), (68) and (72), we obtain

$$\begin{aligned}& | \Lambda(\alpha_1(t), \rho_1(t), \gamma_1(t)) - \Lambda(\alpha_2(t), \rho_2(t), \gamma_2(t)) |^2_{H^1_0(\Gamma_3)\times E'\times L^2(\Omega)}\\ & \leq C\int_0^t \left(| \alpha_1(s) - \alpha_2(s) |^2_{H^{-1}(\Gamma_3)} + | \rho_1(s) - \rho_2(s) |^2_{E'} + | \gamma_1(s) - \gamma_2(s) |^2_{L^2(\Omega)} \right) ds.\\ & \leq C\int_0^t | (\alpha_1(s), \rho_1(s), \gamma_1(s)) - (\alpha_2(s), \rho_2(s), \gamma_2(s)) |^2_{H^{-1}(\Gamma_3)\times E'\times L^2(\Omega)} \, ds.\end{aligned}$$

Reiterating this inequality n times leads to

$$\begin{aligned}&\mid \Lambda^n(\alpha_1,\rho_1,\gamma_1)-\Lambda^n(\alpha_2,\rho_2,\gamma_2)\mid^2_{L^2(0,T;H^{-1}(\Gamma_3)\times E'\times L^2(\Omega))}\\&\quad\le \frac{(CT)^n}{n!}\mid(\alpha_1,\rho_1,\gamma_1)-(\alpha_2,\rho_2,\gamma_2)\mid^2_{L^2(0,T;H^{-1}(\Gamma_3)\times E'\times L^2(\Omega))}.\end{aligned}$$

Thus, for n sufficiently large, Λ^n is a contraction on the Banach space $L^2(0,T;H^{-1}(\Gamma_3)\times E'\times L^2(\Omega))$, and so Λ has a unique fixed point.

Next, we show the existence of a solution to problem PV.

Let $(\alpha^*,\rho^*,\gamma^*)$ be the fixed point of the operator Λ defined by (89)–(92) and obtained in lemma 9.

Let ζ_{α^*} be the solution to problem PV_α for $\alpha=\alpha^*$ (see Lemma 1), let θ_{ρ^*} be the solution to problem PV_ρ for $\rho=\rho^*$ (see Lemma 2) and let β_{γ^*} be the solution to problem PV_γ for $\gamma=\gamma^*$ (see Lemma 3). Denote $\boldsymbol{u}^*=\boldsymbol{u}_{\alpha^*\rho^*\gamma^*}$, $\dot{\boldsymbol{u}}^*=\dot{\boldsymbol{u}}_{\alpha^*\rho^*\gamma^*}$, $\zeta^*=\zeta_{\alpha^*}$, $\theta^*=\theta_{\rho^*}$ and $\beta^*=\beta_{\gamma^*}$.

$$\begin{aligned}\Lambda_1(\alpha^*,\rho^*,\gamma^*)&=\alpha^*=F(\zeta^*,\boldsymbol{u}^*,\dot{\boldsymbol{u}}^*,\dot{\boldsymbol{u}}^*),\\\Lambda_2(\alpha^*,\rho^*,\gamma^*)&=\rho^*=S(\dot{\boldsymbol{u}}^*,\theta^*,\beta^*)+Q,\\\Lambda_3(\alpha^*,\rho^*,\gamma^*)&=\gamma^*=\phi(\boldsymbol{\varepsilon}(\boldsymbol{u}^*),\theta^*,\beta^*).\end{aligned}$$

$\boldsymbol{u}^*=\boldsymbol{u}_{\alpha^*\rho^*\gamma^*}$ is a solution to the problem $PV_{\alpha\rho\gamma}$ for $\alpha=\alpha^*$, $\rho=\rho^*$ and $\gamma=\gamma^*$

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}^*(t)))_{\mathcal{H}}+j(\zeta^*(t),\boldsymbol{u}^*(t),\dot{\boldsymbol{u}}^*(t),\boldsymbol{v})\\&\quad-j(\zeta^*(t),\boldsymbol{u}^*(t),\dot{\boldsymbol{u}}^*(t),\dot{\boldsymbol{u}}^*(t))+(\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^*(t)),\beta^*(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}^*(t)))_{\mathcal{H}}\\&\quad-(C(\theta^*(t),\beta^*(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}-\dot{\boldsymbol{u}}^*(t)))_{\mathcal{H}}\ge(\boldsymbol{f}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}^*(t))_V\quad\forall\boldsymbol{v}\in V,\quad t\in(0,T),\end{aligned}\tag{98}$$

$$\boldsymbol{u}^*(0)=\boldsymbol{u}_0,\tag{99}$$

and $\boldsymbol{\sigma}^*=\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^*)+\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}^*),\beta^*)-C(\theta^*,\beta^*)$. The uniqueness of the solution is a consequence of the uniqueness of the solution of problems $PV_{\alpha\rho\gamma}$, PV_α, PV_ρ, PV_γ and the uniqueness of the operator Λ.

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