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Characterization of *k*-regularities in semigroups

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Abstract
In this paper, we highlight an analogous study to characterize the k-regular, k-intra-regular, k-completely regular semigroups in terms of interval-valued (in short, (i-v)) fuzzy ideals with interval-valued fuzzy k-closure.

Keywords k-regular semigroup · k-intra-regular semigroup · k-completely regular semigroup

Mathematics Subject Classification 08A72

1 Introduction

In semigroup theory, the concept of regularity, analogous to J. Von Neumann's regularity [10] for rings, was initiated by J. A. Green [3]. Later on, the corresponding semigroups are examined and characterized by various authors in terms of different aspects, particularly the ideals of semigroups. In this paper, we evaluate the analog results, but in the k-regularity [4] of Harinath and k-intra-regularity, k-completely regularity [2] of Bogdanović et al. by the help of (i-v) fuzzy ideals with (i-v) fuzzy k-closure.

2 Preliminaries

All preliminaries are considered from the prior works [1, 5–9, 11–13] of interval-valued fuzzy set theory.

Definition 2.1 ([14]) An interval number on [0, 1], denoted by $\widetilde{a}$, is defined as a closed subinterval $[a^-, a^+]$ of [0, 1] satisfying $0 \leq a^- \leq a^+ \leq 1$.

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In this paper, $[0, 0] := \widetilde{0}$, $[1, 1] := \widetilde{1}$ and $D[0, 1]$ represents the set of interval numbers on $[0, 1]$.

Definition 2.2 ([1, 14]) For two interval numbers $\widetilde{a} = [a^-, a^+]$, $\widetilde{b} = [b^-, b^+] \in D[0, 1]$,

1. $\widetilde{a} \leq \widetilde{b}$ if $a^- \leq b^-$ and $a^+ \leq b^+$.
2. $\widetilde{a} = \widetilde{b}$ if $a^- = b^-$ and $a^+ = b^+$.
3. $\widetilde{a} < \widetilde{b}$ if $\widetilde{a} \neq \widetilde{b}$ and $\widetilde{a} \leq \widetilde{b}$.
4. $\widetilde{a} \geq \widetilde{b}$ means $\widetilde{b} \leq \widetilde{a}$ and $\widetilde{a} > \widetilde{b}$ means $\widetilde{b} < \widetilde{a}$.
5. $Min^i(\widetilde{a}, \widetilde{b}) = [min(a^-, b^-), min(a^+, b^+)]$.
6. $Max^i(\widetilde{a}, \widetilde{b}) = [max(a^-, b^-), max(a^+, b^+)]$.

In this paper, two interval numbers $\widetilde{a}, \widetilde{b} \in D[0, 1]$ are either $\widetilde{a} \leq \widetilde{b}$ or $\widetilde{a} > \widetilde{b}$.

Definition 2.3 ([5]) Let X be a non-empty set and $A \subseteq X$. The $(i\text{-}v)$ characteristic function $\widetilde{\chi}_A$ of A is an $(i\text{-}v)$ fuzzy subset of X defined as

$$\widetilde{\chi}_A(x) = \begin{cases} \widetilde{1} \text{ if } x \in A, \\ \widetilde{0} \text{ if } x \in X \setminus A, \end{cases}$$

where $x \in X$.

Definition 2.4 ([5]) Let X be a non-empty set, $x \in X$ and $\widetilde{a} \in D[0, 1] \setminus \{\widetilde{0}\}$. An $(i\text{-}v)$ fuzzy point $x_{\widetilde{a}}$ of X is an $(i\text{-}v)$ fuzzy subset of X defined by

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a} \text{ if } y = x, \\ \widetilde{0} \text{ otherwise}, \end{cases}$$

where $y \in X$.

Definition 2.5 ([5]) If $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are two $(i\text{-}v)$ fuzzy subsets of a non-empty set X,

1. $\widetilde{\mu}_1 \subseteq \widetilde{\mu}_2$ if $\widetilde{\mu}_1(y) \leq \widetilde{\mu}_2(y)$ for all $y \in X$.
2. $(\widetilde{\mu}_1 \cup \widetilde{\mu}_2)(x) = Max^i\left(\widetilde{\mu}_1(x), \widetilde{\mu}_2(x)\right)$ and $(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x) = Min^i\left(\widetilde{\mu}_1(x), \widetilde{\mu}_2(x)\right)$ for $x \in X$.

Definition 2.6 ([5]) Let $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ be two $(i\text{-}v)$ fuzzy subsets of a semigroup S. The product $\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2$ of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ is an $(i\text{-}v)$ fuzzy subset of S defined as

$$(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x) = \begin{cases} \sup\limits_{x=pq} \left\{Min^i\left(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q)\right)\right\} & \text{when } x = pq \text{ for some } p, q \in S; \\ \widetilde{0} & \text{otherwise}, \end{cases}$$

where $x \in S$.

Definition 2.7 ([9, 13]) A non-empty $(i\text{-}v)$ fuzzy subset $\widetilde{\mu}$ of a semigroup S is called

1. an $(i\text{-}v)$ fuzzy subsemigroup of S if $\widetilde{\mu}(xy) \geq Min^i\left(\widetilde{\mu}(x), \widetilde{\mu}(y)\right)$.
2. an $(i\text{-}v)$ fuzzy left ideal of S if $\widetilde{\mu}(xy) \geq \widetilde{\mu}(y)$.
3. an $(i\text{-}v)$ fuzzy right ideal of S if $\widetilde{\mu}(xy) \geq \widetilde{\mu}(x)$.
4. an $(i\text{-}v)$ fuzzy ideal of S if $\widetilde{\mu}(xy) \geq \widetilde{\mu}(y)$ and $\widetilde{\mu}(xy) \geq \widetilde{\mu}(x)$.
5. an $(i\text{-}v)$ fuzzy quasi-ideal of S if $(\widetilde{\mu} \, o \, \widetilde{\chi}_S) \cap (\widetilde{\chi}_S \, o \, \widetilde{\mu}) \subseteq \widetilde{\mu}$.
6. an $(i\text{-}v)$ fuzzy bi-ideal of S if $\widetilde{\mu}(xyz) \geq Min^i(\widetilde{\mu}(x), \widetilde{\mu}(z))$ and $\widetilde{\mu}(xy) \geq Min^i\left(\widetilde{\mu}(x), \widetilde{\mu}(y)\right)$

for any $x, y, z \in S$.

Lemma 2.8 ([13]) *A non-empty* $(i\text{-}v)$ *fuzzy subset* $\widetilde{\mu}$ *of a semigroup* S *is an* $(i\text{-}v)$ *fuzzy*

1. *subsemigroup of* S *if and only if* $\widetilde{\mu}\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$.
2. *left ideal of* S *if and only if* $\widetilde{\chi}_S\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$.
3. *right ideal of* S *if and only if* $\widetilde{\mu}\, o\, \widetilde{\chi}_S \subseteq \widetilde{\mu}$.
4. *ideal of* S *if and only if* $\widetilde{\mu}\, o\, \widetilde{\chi}_S \subseteq \widetilde{\mu}$ *and* $\widetilde{\chi}_S\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$.
5. *bi-ideal of* S *if and only if* $\widetilde{\mu}\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$ *and* $\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$.

Proposition 2.9 ([5, 6]) *For an* $(i\text{-}v)$ *fuzzy point* $x_{\widetilde{a}}$ *of a semigroup* S, *the* $(i\text{-}v)$ *fuzzy*

1. *left ideal generated by* $x_{\widetilde{a}}$, $\langle x_{\widetilde{a}}\rangle_l = x_{\widetilde{a}} \cup (\widetilde{\chi}_S\, o\, x_{\widetilde{a}})$.
2. *right ideal generated by* $x_{\widetilde{a}}$, $\langle x_{\widetilde{a}}\rangle_r = x_{\widetilde{a}} \cup (x_{\widetilde{a}}\, o\, \widetilde{\chi}_S)$.
3. *ideal generated by* $x_{\widetilde{a}}$, $\langle x_{\widetilde{a}}\rangle = x_{\widetilde{a}} \cup (\widetilde{\chi}_S\, o\, x_{\widetilde{a}}) \cup (x_{\widetilde{a}}\, o\, \widetilde{\chi}_S) \cup (\widetilde{\chi}_S\, o\, x_{\widetilde{a}}\, o\, \widetilde{\chi}_S)$.
4. *quasi-ideal generated by* $x_{\widetilde{a}}$, $\langle x_{\widetilde{a}}\rangle_q = x_{\widetilde{a}} \cup ((\widetilde{\chi}_S\, o\, x_{\widetilde{a}}) \cap (x_{\widetilde{a}}\, o\, \widetilde{\chi}_S))$.

Proposition 2.10 ([5, 6]) *The following results hold in a semigroup* S:

1. $\langle x_{\widetilde{a}}\rangle(z) = \begin{cases} \widetilde{a} \text{ when } z \in < x >, \\ \widetilde{0} \text{ otherwise,} \end{cases}$
2. $\big(\langle x_{\widetilde{a}}\rangle\, o\, \langle y_{\widetilde{b}}\rangle\big)(z) = \begin{cases} Min^i(\widetilde{a}, \widetilde{b}) & \text{when } z \in < x >< y >, \\ \widetilde{0} & \text{otherwise,} \end{cases}$

   where $z \in S$.

3 *k*-regularities in semigroups

In this section, first, we establish a new abstract idea, entitle it by $(i\text{-}v)$ fuzzy k-closure in semigroups, and then study its basic properties to fulfill our required goal.

Definition 3.1 Let S be a semigroup and k be a fixed positive integer. For a non-empty subset A of S, we define a subset $(A)_k$, call it as "k-closure" of A, by $(A)_k = \{x \in S : x^k \in A\}$.

Definition 3.2 Let k be a fixed positive integer. An $(i\text{-}v)$ fuzzy k-closure, denoted by $cl_k(\widetilde{\mu})$, of an $(i\text{-}v)$ fuzzy subset $\widetilde{\mu}$ of a semigroup S is an $(i\text{-}v)$ fuzzy subset of S defined by $cl_k(\widetilde{\mu})(x) = \widetilde{\mu}(x^k)$ for every $x \in S$.

The following results are immediate from the proposed definitions.

Lemma 3.3 *If* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *are two non-empty* $(i\text{-}v)$ *fuzzy subsets of a semigroup* S, *then*

(i) $cl_k(\widetilde{\mu}_1) \subseteq cl_k(\widetilde{\mu}_2)$, *if* $\widetilde{\mu}_1 \subseteq \widetilde{\mu}_2$.
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1) \cap cl_k(\widetilde{\mu}_2)$.
(iii) $cl_k(\widetilde{\mu}_1 \cup \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1) \cup cl_k(\widetilde{\mu}_2)$.

Since the proofs are simple, we omit the proofs.

Lemma 3.4 *If* $x_{\widetilde{a}}$ *is an* $(i\text{-}v)$ *fuzzy point of a semigroup* S, *then for any* $y \in S$,

(i) $cl_k((x^k)_{\widetilde{a}})(y) = \begin{cases} \widetilde{a} & \text{if } y^k = x^k, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

(ii) $cl_k(\langle (x^k)_{\widetilde{a}}\rangle_r)(y) = \begin{cases} \widetilde{a} & \text{if } y \in (< x^k >_r)_k, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

(iii) $cl_k(\langle (x^k)_{\widetilde{a}}\rangle_l)(y) = \begin{cases} \widetilde{a} & \text{if } y \in (< x^k >_l)_k, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

(iv) $cl_k(\langle(x^k)_{\widetilde{a}}\rangle)(y) = \begin{cases} \widetilde{a} & \textit{if } y \in (< x^k >)_k, \\ \widetilde{0} & \textit{otherwise.} \end{cases}$

Proof (i) For any $y \in S$, we have $cl_k((x^k)_{\widetilde{a}})(y) = (x^k)_{\widetilde{a}}(y^k) = \begin{cases} \widetilde{a} & \text{if } y^k = x^k, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

(ii) For any $y \in S$, we have $cl_k(\langle(x^k)_{\widetilde{a}}\rangle_r)(y) = \langle(x^k)_{\widetilde{a}}\rangle_r(y^k) = \begin{cases} \widetilde{a} & \text{if } y^k \in < x^k >_r, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

$= \begin{cases} \widetilde{a} & \text{if } y \in (< x^k >_r)_k, \\ \widetilde{0} & \text{otherwise.} \end{cases}$

Proofs of (iii) and (iv) are similar. □

According to Harinath and Bogdanović et al., k-regular, k-intra-regular, and k-completely regular semigroups are as follows:

Definition 3.5 ([4]) A semigroup S is said to be k-regular if for every $x \in S$, there exists $y \in S$ such that $x^k = x^k y x^k$.

Definition 3.6 ([2]) A semigroup S is said to be k-intra-regular if for every $x \in S$, there exist $u, v \in S$ such that $x^k = u x^k x^k v$.

Definition 3.7 ([2]) A semigroup S is said to be k-completely regular if for every $x \in S$, there exists $y \in S$ such that $x^k = x^k y x^k$ and $x^k y = y x^k$.

Remark 3.8 Note that every regular (intra-regular) semigroup is k-regular (resp. k-intra regular) semigroup.

But the converse may not be true always. This follows from the following example:

Example 3.9 Consider the semigroup $(\mathbb{Z}_4, \bullet_4)$ whose composition table is as follows:

$\bullet_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Here $\mathbb{Z}_4$ is not a regular as well as intra-regular semigroup because $2 \neq 2.x.2$ and $2 = x.2^2.y$ for any $x, y \in \mathbb{Z}_4$. On the other hand, it observe that $1^3 = 1^3.1.1^3$, $2^3 = 2^3.3.2^3$, $3^3 = 3^3.3.3^3$ and $1^3 = 3.(1^3)^2.3$, $2^3 = 2.(2^3)^2.3$, $3^3 = 3.(3^3)^2.1$. This shows that $\mathbb{Z}_4$ is 3-regular and 3-intra regular semigroup.

Now in the following, we try to extend a few important results from regular semigroups to k-regular semigroups to characterize them using (*i*-*v*) fuzzy ideals of semigroups.

Theorem 3.10 [13] *The following statements are equivalent in a semigroup S:*

(i) *S is regular.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 = \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2$, *where $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are resp. (i-v) fuzzy right ideal and (i-v) fuzzy left ideal of S.*

Theorem 3.11 *For a semigroup S, the following statements are equivalent:*

(i) *S is k-regular.*
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$, *where $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are resp. (i-v) fuzzy right ideal and (i-v) fuzzy left ideal of S.*

Proof $(i) \Longrightarrow (ii)$. Let S be k-regular semigroup. Then for $x \in S$, $x^k = x^k y x^k$ for some $y \in S$. Consequently, for an (*i-v*) fuzzy right ideal $\widetilde{\mu}_1$ and an (*i-v*) fuzzy left ideal $\widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x^k) = \sup\limits_{\substack{p,q \in S \\ x^k = pq}} \left\{Min^i(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q))\right\} \geq Min^i(\widetilde{\mu}_1(x^k y), \widetilde{\mu}_2(x^k)) \geq Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k))$ (by the property of $\widetilde{\mu}_1) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Thus $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. Also, $\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1 \cap \widetilde{\mu}_2$ implies $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$, by Lemma 3.3. So we find that $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$.

$(ii) \Longrightarrow (i)$. Suppose (ii) holds and $x \in S$. Then for an (*i-v*) fuzzy point $x_{\widetilde{a}}$ of S, $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_r \cap \langle (x^k)_{\widetilde{a}} \rangle_l)(x) = cl_k(\langle (x^k)_{\widetilde{a}} \rangle_r \, o \, \langle (x^k)_{\widetilde{a}} \rangle_l)(x)$ (by given condition (ii)) $= (\langle (x^k)_{\widetilde{a}} \rangle_r \, o \, \langle (x^k)_{\widetilde{a}} \rangle_l)(x^k) = (((x^k)_{\widetilde{a}} \cup ((x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S)) \, o \, ((x^k)_{\widetilde{a}} \cup (\widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}})))(x^k) \leq (((x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}}) \cup ((x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}}))(x^k)$. But $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_r \cap \langle (x^k)_{\widetilde{a}} \rangle_l)(x) = \big(\langle (x^k)_{\widetilde{a}} \rangle_r \cap \langle (x^k)_{\widetilde{a}} \rangle_l\big)(x^k) = \widetilde{a}$. Therefore, $(((x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}}) \cup ((x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}}))(x^k) \geq \widetilde{a}$, that means, $(x^k)_{\widetilde{a}} \in (((x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}}) \cup ((x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}}))$. Consequently, it imply that either $(x^k)_{\widetilde{a}} = (x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}}$ or $(x^k)_{\widetilde{a}} = (x^k)_{\widetilde{a}} \, o \, y_{\widetilde{b}} \, o \, (x^k)_{\widetilde{a}}$ for some (*i-v*) fuzzy point $y_{\widetilde{b}}$ of S. It implies that either $x^k = x^k x^k$ or $x^k = x^k y x^k$, that means, for both cases, $x^k = x^k y x^k$ for some $y \in S$. Thus it indicates that S is k-regular. □

Theorem 3.12 [13] *The following conditions are equivalent in a semigroup S:*

(i) S *is regular.*
(ii) $\widetilde{\mu} = \widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}$ *for an* (*i-v*) *fuzzy bi-ideal* $\widetilde{\mu}$ *of* S.
(iii) $\widetilde{\mu} = \widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}$ *for an* (*i-v*) *fuzzy quasi-ideal* $\widetilde{\mu}$ *of* S.

Theorem 3.13 *The following statements are equivalent in a semigroup S:*

(i) S *is* k*-regular.*
(ii) $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})$ *for an* (*i-v*) *fuzzy bi-ideal* $\widetilde{\mu}$ *of* S.
(iii) $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})$ *for an* (*i-v*) *fuzzy quasi-ideal* $\widetilde{\mu}$ *of* S.

Proof $(i) \Longrightarrow (ii)$. Let S be a k-regular semigroup. For an (*i-v*) fuzzy bi-ideal $\widetilde{\mu}$ of S, $\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu} \subseteq \widetilde{\mu}$ implies $cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}) \subseteq cl_k(\widetilde{\mu})$. Suppose $x \in S$. Then $cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})(x) = (\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})(x^k) = \sup\limits_{\substack{p,q,r \in S \\ x^k = pqr}} \left\{Min^i\big(\widetilde{\mu}(p), Min^i(\widetilde{\chi}_S(q), \widetilde{\mu}(r))\big)\right\} \geq Min^i\big(\widetilde{\mu}(x^k), Min^i(\widetilde{\chi}_S(y), \widetilde{\mu}(x^k))\big)$ (as $x^k = x^k y x^k$ for some $y \in S$, by assumption) $= Min^i(\widetilde{\mu}(x^k), \widetilde{\mu}(x^k)) = \widetilde{\mu}(x^k) = cl_k(\widetilde{\mu})(x)$. Therefore, $cl_k(\widetilde{\mu}) \subseteq cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})$. Hence it concludes that $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})$ holds.

$(ii) \Longrightarrow (iii)$. The implication follows easily from Lemma 3.18 [13].

$(iii) \Longrightarrow (i)$. Suppose $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$ are resp. the (*i-v*) fuzzy right ideal and (*i-v*) fuzzy left ideal of S. Then $\widetilde{\mu}_1 \cap \widetilde{\mu}_2$ is an (*i-v*) fuzzy quasi ideal of S, by Lemma 3.16 [13]. Thus by (iii), it follows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k((\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \, o \, \widetilde{\chi}_S \, o \, (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$, that is, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. Again $\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1 \cap \widetilde{\mu}_2$ implies $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. Thus it shows that $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. On that account, it concludes from Theorem 3.11 that S is k-regular. □

Theorem 3.14 [13] *The following statements are equivalent in a semigroup S:*

(i) S *is regular.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 = \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1$ *for an* (*i-v*) *fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an* (*i-v*) *fuzzy ideal* $\widetilde{\mu}_2$ *of* S.
(iii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 = \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1$ *for an* (*i-v*) *fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an* (*i-v*) *fuzzy ideal* $\widetilde{\mu}_2$ *of* S.

Theorem 3.15 *The following statements are equivalent in a semigroup S:*

(i) *S is k-regular.*
(ii) *$cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ for an (i-v) fuzzy quasi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy ideal $\widetilde{\mu}_2$ of S.*
(iii) *$cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ for an (i-v) fuzzy bi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy ideal $\widetilde{\mu}_2$ of S.*

Proof $(i) \Longrightarrow (iii)$. Let us suppose S be k-regular semigroup. Then for every $x \in S$, $x^k = x^k y x^k$ for some $y \in S$ and hence, $x^k = x^k (y x^k y) x^k$. Moreover, for an (*i-v*) fuzzy bi-ideal $\widetilde{\mu}_1$ and an (*i-v*) fuzzy ideal $\widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)(x^k) = \sup\limits_{\substack{p,q,r \in S \\ x^k = pqr}} \left\{ Min^i\left(\widetilde{\mu}_1(p), Min^i(\widetilde{\mu}_2(q), \widetilde{\mu}_1(r))\right)\right\} \geq Min^i\left(\widetilde{\mu}_1(x^k), Min^i(\widetilde{\mu}_2(y x^k y), \widetilde{\mu}_1(x^k))\right)$ $\geq Min^i\left(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k)\right) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Thus it follows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$. Also $\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}_1$ and $\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1 \subseteq \widetilde{\chi}_S \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\chi}_S \subseteq \widetilde{\mu}_2$. Therefore, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}_1) = cl_k(\widetilde{\mu}_1)$ (by Theorem 3.13). In addition from Lemma 3.3, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1) \subseteq cl_k(\widetilde{\mu}_2)$. Thus $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1) \subseteq cl_k(\widetilde{\mu}_1) \cap cl_k(\widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$, by Lemma 3.3. Consequently, it follows that $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_1) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$.

$(iii) \Longrightarrow (ii)$. The implication follows easily.

$(ii) \Longrightarrow (i)$. Let $\widetilde{\mu}$ be an (*i-v*) fuzzy quasi-ideal of S. Then $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu} \cap \widetilde{\chi}_S) = cl_k(\widetilde{\mu} \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu})$ (by (ii)). As a consequence, it follows from Theorem 3.13 that S is k-regular. □

Theorem 3.16 [13] *The following statements are equivalent in a semigroup S:*

(i) *S is regular.*
(ii) *$\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2$ for an (i-v) fuzzy quasi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy left ideal $\widetilde{\mu}_2$ of S.*
(iii) *$\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2$ for an (i-v) fuzzy bi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy left ideal $\widetilde{\mu}_2$ of S.*

Theorem 3.17 *The following statements are equivalent in a semigroup S:*

(i) *S is k-regular.*
(ii) *$cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$ for an (i-v) fuzzy quasi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy left ideal $\widetilde{\mu}_2$ of S.*
(iii) *$cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$ for an (i-v) fuzzy bi-ideal $\widetilde{\mu}_1$ and an (i-v) fuzzy left ideal $\widetilde{\mu}_2$ of S.*

Proof $(i) \Longrightarrow (iii)$. Suppose S is k-regular semigroup. Then for an (*i-v*) fuzzy bi-ideal $\widetilde{\mu}_1$ and an (*i-v*) fuzzy left ideal $\widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x^k) = \sup\limits_{\substack{p,q \in S \\ x^k = pq}} \left\{ Min^i\left(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q)\right)\right\} \geq Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(y x^k))$ (since $x^k = x^k y x^k$ for some $y \in S$) $\geq Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k)) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Therefore, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$ holds.

$(iii) \Longrightarrow (ii)$. The implication follows clearly.

$(ii) \Longrightarrow (i)$. Let (ii) hold and suppose $\widetilde{\mu}_1, \widetilde{\mu}_2$ are respectively the (*i-v*) fuzzy right ideal and (*i-v*) fuzzy left ideal of S. Then $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$ and $\widetilde{\mu}_1$ is an (*i-v*) fuzzy quasi-ideal of S, by Lemma 3.14 [13]. Therefore, by given condition (ii), $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. Hence it follows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. This concludes that S is k-regular. □

Theorem 3.18 [13] *The following statements are equivalent in a semigroup S:*

(i) *S is regular.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$*, an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_3$ *of S.*
(iii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$*, an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_3$ *of S.*

Theorem 3.19 *The following statements are equivalent in a semigroup S:*

(i) *S is k-regular.*
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3)$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$*, an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_3$ *of S.*
(iii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3)$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$*, an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_3$ *of S.*

Proof $(i) \Longrightarrow (iii)$. Let S be k-regular semigroup. Then for $x \in S$, $x^k = x^k y x^k$ for some $y \in S$ and so $x^k = (x^k y) x^k (y x^k)$. Therefore, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3)(x^k) = \sup\limits_{\substack{p,q,r \in S \\ x^k = pqr}} \left\{Min^i(\widetilde{\mu}_1(p), Min^i(\widetilde{\mu}_2(q), \widetilde{\mu}_3(r)))\right\} \geq Min^i(\widetilde{\mu}_1(x^k y), Min^i(\widetilde{\mu}_2(x^k), \widetilde{\mu}_3(y x^k)))$
$\geq Min^i\left(\widetilde{\mu}_1(x^k), Min^i(\widetilde{\mu}_2(x^k), \widetilde{\mu}_3(x^k))\right)$ (by the properties of $\widetilde{\mu}_1$ and $\widetilde{\mu}_3$) $= (\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3)(x)$. As a result, it follows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \cap \widetilde{\mu}_3) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2 \, o \, \widetilde{\mu}_3)$.

$(iii) \Longrightarrow (ii)$. It clearly holds.

$(ii) \Longrightarrow (i)$. Suppose (ii) holds and $\widetilde{\mu}_1, \widetilde{\mu}_2$ are resp. (*i-v*) fuzzy right ideal and (*i-v*) fuzzy left ideal of S. Then $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. Again $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\chi}_S \cap \widetilde{\mu}_2)$ (from condition (ii)) $\subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\chi}_S \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. Hence it shows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$ and accordingly, S is k-regular. □

Theorem 3.20 [12] *In a semigroup S, the following statements are equivalent:*

(i) *S is intra-regular.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1 \, o \, \widetilde{\mu}_2$ *for an (i-v) fuzzy left ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*

Theorem 3.21 *The following statements are equivalent in a semigroup S:*

(i) *S is k-intra-regular.*
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$ *for an (i-v) fuzzy left ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*

Proof $(i) \Longrightarrow (ii)$. Suppose S is a k-intra-regular semigroup and $\widetilde{\mu}_1, \widetilde{\mu}_2$ are resp. (*i-v*) fuzzy left ideal and (*i-v*) fuzzy right ideal of S. Then for every $x \in S$, $x^k = u(x^k x^k)v$ for some $u, v \in S$. Now $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x^k) = \sup\limits_{\substack{p,q \in S \\ x^k = pq}} \left\{Min^i(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q))\right\} \geq Min^i(\widetilde{\mu}_1(u x^k), \widetilde{\mu}_2(x^k v)) \geq Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k))$ (by the properties of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$) $= (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Consequently, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$.

$(ii) \Longrightarrow (i)$. Let (ii) hold and $x \in S$. Then for an (*i-v*) fuzzy point $x_{\widetilde{a}}$ of S, $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r) \subseteq cl_k(\langle (x^k)_{\widetilde{a}} \rangle_l \, o \, \langle (x^k)_{\widetilde{a}} \rangle_r)$ (by (ii)) $= cl_k\Big(\big((x^k)_{\widetilde{a}} \cup (\widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}})\big) \, o \, \big((x^k)_{\widetilde{a}} \cup ((x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S)\big)\Big) \subseteq cl_k(\widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S)$. But $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r)(x) = (\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r)(x^k) = \widetilde{a}$. As a consequence, $cl_k(\widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S)(x) \geq \widetilde{a}$. This implies that $(\widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S)(x^k) \geq \widetilde{a}$, implies $(x^k)_{\widetilde{a}} \in \widetilde{\chi}_S \, o \, (x^k)_{\widetilde{a}} \, o \, (x^k)_{\widetilde{a}} \, o \, \widetilde{\chi}_S$. This implies that $x^k \in S(x^k x^k)S$. Thus it signify that S is k-intra-regular. □

Theorem 3.22 [12] *In a semigroup S, the following statements are equivalent:*

(i) S *is regular and intra-regular semigroup.*
(ii) $\widetilde{\mu} = \widetilde{\mu}\, o\, \widetilde{\mu}$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}$ *of S.*
(iii) $\widetilde{\mu} = \widetilde{\mu}\, o\, \widetilde{\mu}$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}$ *of S.*
(iv) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(v) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vi) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(vii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*

Theorem 3.23 *In a semigroup S, the following statements are equivalent:*

(i) S *is k-regular and k-intra-regular semigroup.*
(ii) $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu}\, o\, \widetilde{\mu})$ *for every (i-v) fuzzy quasi-ideal* $\widetilde{\mu}$ *of S.*
(iii) $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu}\, o\, \widetilde{\mu})$ *for every (i-v) fuzzy bi-ideal* $\widetilde{\mu}$ *of S.*
(iv) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(v) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vi) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(vii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*

Proof $(i) \Longrightarrow (iii)$. Let us suppose S be k-regular and k-intra-regular semigroup. Then for every $x \in S$, $x^k = x^k y x^k$ and $x^k = u(x^k x^k)v$ for some $u, v, y \in S$. Therefore, $x^k = x^k y x^k = x^k y x^k y x^k = (x^k y u x^k)(x^k v y x^k)$. If $\widetilde{\mu}$ is an (i-v) fuzzy bi-ideal of S then $\widetilde{\mu}\, o\, \widetilde{\mu} \subseteq \widetilde{\mu}$ implies $cl_k(\widetilde{\mu}\, o\, \widetilde{\mu}) \subseteq cl_k(\widetilde{\mu})$. On the other hand, $cl_k(\widetilde{\mu}\, o\, \widetilde{\mu})(x) = (\widetilde{\mu}\, o\, \widetilde{\mu})(x^k) = \sup\limits_{\substack{x^k = pq \\ p, q \in S}} \{Min^i(\widetilde{\mu}(p), \widetilde{\mu}(q))\} \geq Min^i(\widetilde{\mu}(x^k y u x^k), \widetilde{\mu}(x^k v y x^k)) \geq Min^i(Min^i(\widetilde{\mu}(x^k), \widetilde{\mu}(x^k)), Min^i(\widetilde{\mu}(x^k), \widetilde{\mu}(x^k)))$ (by the property of $\widetilde{\mu}$) $= \widetilde{\mu}(x^k) = cl_k(\widetilde{\mu})(x)$. This implies that $cl_k(\widetilde{\mu}) \subseteq cl_k(\widetilde{\mu}\, o\, \widetilde{\mu})$. Hence it follows that $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu}\, o\, \widetilde{\mu})$.

$(iii) \Longrightarrow (ii)$. Proof follows easily.

$(ii) \Longrightarrow (i)$. Suppose (ii) holds and let $\widetilde{\mu}_1, \widetilde{\mu}_2$ be resp. (i-v) fuzzy right ideal and (i-v) fuzzy left ideal of S. Then $cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. Again, by (ii), it follows that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) = cl_k((\widetilde{\mu}_1 \cap \widetilde{\mu}_2)\, o\, (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$, that means, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$. Thus it follows that $cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. Hence from Theorem 3.11, we find that S is k-regular.

Again consider $x_{\widetilde{a}}$ is an (i-v) fuzzy point of S. If $\langle (x^k)_{\widetilde{a}} \rangle_q$ is an (i-v) fuzzy quasi-ideal of S, $\widetilde{a} = \langle (x^k)_{\widetilde{a}} \rangle_q (x^k) = cl_k(\langle (x^k)_{\widetilde{a}} \rangle_q)(x) = cl_k\big(\langle (x^k)_{\widetilde{a}} \rangle_q\, o\, \langle (x^k)_{\widetilde{a}} \rangle_q\big)(x)$ (by (ii)) $= \big(\langle (x^k)_{\widetilde{a}} \rangle_q\, o\, \langle (x^k)_{\widetilde{a}} \rangle_q\big)(x^k)$. This implies that $x^k \in < x^k >_q < x^k >_q$, where $< x^k >_q$ is a quasi-ideal of S generated by x^k, i.e. $x^k \in S x^k x^k S$. Thus S is k-intra-regular.

$(i) \Longrightarrow (v)$. Let (i) hold and $\widetilde{\mu}_1, \widetilde{\mu}_2$ be (i-v) fuzzy bi-ideals of S. Since S is k-regular and k-intra-regular semigroup, for every $x \in S$; $x^k = x^k y x^k$ and $x^k = u(x^k x^k)v$ for some $u, v, y \in S$. Now $cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)(x) = (\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)(x^k) = \sup\limits_{\substack{x^k = pq \\ p, q \in S}} \{Min^i(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q))\} \geq Min^i(\widetilde{\mu}_1(x^k y u x^k), \widetilde{\mu}_2(x^k v y x^k)) \geq Min^i(Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_1(x^k)), Min^i(\widetilde{\mu}_2(x^k), \widetilde{\mu}_2(x^k)))$ (by the properties of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$) $= Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k)) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Consequently, it argues that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2)$.

The implications $(v) \Longrightarrow (iv) \Longrightarrow (ii)$, $(v) \Longrightarrow (vi) \Longrightarrow (iv)$ and $(v) \Longrightarrow (vii) \Longrightarrow (iv)$ follow easily. □

Theorem 3.24 [12] *In a semigroup S, the following statements are equivalent:*

(i) S *is regular and intra-regular semigroup.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(iii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*
(iv) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(v) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vi) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*

Theorem 3.25 *In a semigroup S, the following statements are equivalent:*

(i) S *is k-regular and k-intra-regular semigroup.*
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(iii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*
(iv) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for an (i-v) fuzzy right ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(v) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vi) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(vii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*

Proof $(i) \Longrightarrow (vi)$. Let S be k-regular and k-intra-regular semigroup. Then for $x \in S$, $x^k = x^k y x^k$ and $x^k = u(x^k x^k)v$ for some $u, v, y \in S$, that means, $x^k = x^k y u(x^k x^k) v y x^k$. So for $(i\text{-}v)$ fuzzy bi-ideals $\widetilde{\mu}_1, \widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x) = (\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)(x^k) = \sup\limits_{\substack{x^k = pq \\ p,q \in S}} \{Min^i(\widetilde{\mu}_1(p), \widetilde{\mu}_2(q))\} \geq Min^i(\widetilde{\mu}_1(x^k y u x^k), \widetilde{\mu}_2(x^k v y x^k)) \geq Min^i(Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_1(x^k)), Min^i(\widetilde{\mu}_2(x^k), \widetilde{\mu}_2(x^k)))$ (by the properties of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2$) $= Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k)) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Hence $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. By Similar arguments, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)$. Consequently, it indicates that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k\big((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1)\big)$.

$(vi) \Longrightarrow (v) \Longrightarrow (ii)$. The implications follow easily.

$(ii) \Longrightarrow (i)$. Suppose (ii) holds. Then for $(i\text{-}v)$ fuzzy right ideal $\widetilde{\mu}_1$ and $(i\text{-}v)$ fuzzy left ideal $\widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. In addition, $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k((\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) \cap (\widetilde{\mu}_2 \, o \, \widetilde{\mu}_1))$ (by assumption (ii)) $\subseteq cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2)$. Thus it implies that $cl_k(\widetilde{\mu}_1 \, o \, \widetilde{\mu}_2) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)$. As a consequence, S is k-regular.

Again $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r)(x) \leq cl_k\big(\big(\langle (x^k)_{\widetilde{a}} \rangle_l \, o \, \langle (x^k)_{\widetilde{a}} \rangle_r\big) \cap \big(\langle (x^k)_{\widetilde{a}} \rangle_r \, o \, \langle (x^k)_{\widetilde{a}} \rangle_l\big)\big)(x)$ (by (ii)) $= (\langle (x^k)_{\widetilde{a}} \rangle_l \, o \, \langle (x^k)_{\widetilde{a}} \rangle_r) \cap \big(\langle (x^k)_{\widetilde{a}} \rangle_r \, o \, \langle (x^k)_{\widetilde{a}} \rangle_l\big)(x^k) = Min^i\big((\langle (x^k)_{\widetilde{a}} \rangle_l \, o \, \langle (x^k)_{\widetilde{a}} \rangle_r)(x^k), \big(\langle (x^k)_{\widetilde{a}} \rangle_r \, o \, \langle (x^k)_{\widetilde{a}} \rangle_l\big)(x^k)\big)$. But $cl_k(\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r)(x) = (\langle (x^k)_{\widetilde{a}} \rangle_l \cap \langle (x^k)_{\widetilde{a}} \rangle_r)(x^k) = \widetilde{a}$. Thus it follows that $x^k \in (< x^k >_l < x^k >_r) \cap (< x^k >_r < x^k >_l)$, that is, $x^k \in S x^k x^k S$. This shows that S is k-intra-regular semigroup.

The implications $(vi) \Longrightarrow (vii) \Longrightarrow (ii)$ and $(vi) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (ii)$ follow easily. □

Theorem 3.26 [12] *In a semigroup S, the following statements are equivalent:*

(i) *S is regular and intra-regular semigroup.*
(ii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(iii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*
(iv) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(v) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*
(vi) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(vii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*
(viii) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(ix) $\widetilde{\mu}_1 \cap \widetilde{\mu}_2 \subseteq \widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*

Theorem 3.27 *In a semigroup S, the following statements are equivalent:*

(i) *S is k-regular and k-intra regular semigroup.*
(ii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(iii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*
(iv) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy left ideal* $\widetilde{\mu}_2$ *of S.*
(v) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy right ideal* $\widetilde{\mu}_2$ *of S.*
(vi) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_2$ *of S.*
(vii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for an (i-v) fuzzy bi-ideal* $\widetilde{\mu}_1$ *and an (i-v) fuzzy quasi-ideal* $\widetilde{\mu}_2$ *of S.*
(viii) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for (i-v) fuzzy quasi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*
(ix) $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ *for (i-v) fuzzy bi-ideals* $\widetilde{\mu}_1$ *and* $\widetilde{\mu}_2$ *of S.*

Proof $(i) \Longrightarrow (ix)$. Let S be k-regular and k-intra-regular semigroup. Then for $x\in S$, $x^k = x^k y x^k$ and $x^k = u(x^k x^k)v$ for some $u, v, y \in S$, that means, $x^k = (x^k yuux^k)(x^k vux^k)(x^k vvyx^k)$. Therefore, for (*i-v*) fuzzy bi-ideals $\widetilde{\mu}_1, \widetilde{\mu}_2$ of S, $cl_k(\widetilde{\mu}_1\, o \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)(x) = (\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)(x^k) = \sup\limits_{\substack{x^k=p\,q\,r\\ p,q,r\in S}} \left\{Min^i\left(\widetilde{\mu}_1(p), Min^i(\widetilde{\mu}_2(q), \widetilde{\mu}_1(r))\right)\right\}$
$\geq Min^i(\widetilde{\mu}_1(x^k yuux^k), Min^i(\widetilde{\mu}_2(x^k vux^k), \widetilde{\mu}_1(x^k vvyx^k)))$ (by the properties of $\widetilde{\mu}_1$ and $\widetilde{\mu}_2) = Min^i(\widetilde{\mu}_1(x^k), \widetilde{\mu}_2(x^k)) = (\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x^k) = cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2)(x)$. Thus it concludes that $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$.

The implications $(ix) \Longrightarrow (viii) \Longrightarrow (ii)$ hold clearly.

$(ii) \Longrightarrow (i)$. Let us suppose (ii) hold and $\widetilde{\mu}_1,\ \widetilde{\mu}_2$ be (*i-v*) fuzzy right ideal and (*i-v*) fuzzy left ideal of S. Then by (ii), $cl_k(\widetilde{\mu}_1 \cap \widetilde{\mu}_2) \subseteq cl_k(\widetilde{\mu}_1\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ (since $\widetilde{\mu}_1$ is (*i-v*) fuzzy quasi-ideal of S) $\subseteq cl_k(\widetilde{\chi}_S\, o\, \widetilde{\mu}_2\, o\, \widetilde{\mu}_1) \subseteq cl_k(\widetilde{\mu}_2\, o\, \widetilde{\mu}_1)$ (by the property of $\widetilde{\mu}_2$). Thus from Theorem 3.21, it indicates that S is k-intra-regular.

Again consider $\widetilde{\mu}$ is an (*i-v*) fuzzy quasi-ideal of S. Then $cl_k(\widetilde{\mu}) = cl_k(\widetilde{\mu} \cap \widetilde{\chi}_S) \subseteq cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu})$, since $\widetilde{\chi}_S$ is (*i-v*) fuzzy left ideal of S. But $cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu}) \subseteq cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\chi}_S) \subseteq cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S)$ and $cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu}) \subseteq cl_k(\widetilde{\chi}_S\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu}) \subseteq cl_k(\widetilde{\chi}_S\, o\, \widetilde{\mu})$.

As a result, $cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu}) \subseteq cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S) \cap cl_k(\widetilde{\chi}_S\, o\, \widetilde{\mu}) = cl_k\big((\widetilde{\mu}\, o\, \widetilde{\chi}_S) \cap (\widetilde{\chi}_S\, o\, \widetilde{\mu})\big)$ (by Lemma 3.3) $\subseteq cl_k(\widetilde{\mu})$ (by the property of $\widetilde{\mu}$). This implies that $cl_k(\widetilde{\mu}\, o\, \widetilde{\chi}_S\, o\, \widetilde{\mu}) = cl_k(\widetilde{\mu})$. Consequently, by Theorem 3.13, S is k-regular.

The implications $(ix) \Longrightarrow (vii) \Longrightarrow (v) \Longrightarrow (iii) \Longrightarrow (i)$, $(vii) \Longrightarrow (iv) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (ix) \Longrightarrow (vi) \Longrightarrow (viii)$ follow easily. □

Finally, in the end, we consider a few definitions defined in [7] to illustrate our final results.

Definition 3.28 [7] Let S be a semigroup and $x, z \in S$. Then for an (i-v) fuzzy subset $\widetilde{\mu}$ of S,

$$(x\widetilde{\mu}z)(u) = \begin{cases} \sup\limits_{u=xyz} \widetilde{\mu}(y) & \text{if } u = xyz \text{ for some } y \in S, \\ \widetilde{0} & \text{otherwise,} \end{cases}$$

$$(x\,\widetilde{\mu})(u) = \begin{cases} \sup\limits_{u=xy} \widetilde{\mu}(y) & \text{if } u = xy \text{ for some } y \in S, \\ \widetilde{0} & \text{otherwise;} \end{cases}$$

and

$$(\widetilde{\mu}\,x)(u) = \begin{cases} \sup\limits_{u=yx} \widetilde{\mu}(y) & \text{if } u = yx \text{ for some } y \in S, \\ \widetilde{0} & \text{otherwise,} \end{cases}$$

where $u \in S$.

Definition 3.29 [7] An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is said to be an (i-v) fuzzy weakly completely regular subsemigroup of S if for every $x \in S$, $R_x \cap C_x \neq \emptyset$ and $\sup\limits_{x' \in R_x \cap C_x} \widetilde{\mu}(x') \geq \widetilde{\mu}(x)$, when $\widetilde{\mu}(x) \neq \widetilde{0}$.

Definition 3.30 An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is said to be an (i-v) fuzzy weakly k-completely regular subsemigroup of S if for every $x \in S$, $R_{x^k} \cap C_{x^k} \neq \emptyset$ and $\sup\limits_{x' \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(x') \geq cl(\widetilde{\mu})(x)$, when $cl(\widetilde{\mu})(x) \neq \widetilde{0}$.

Theorem 3.31 [7] *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy weakly completely regular subsemigroup of S if and only if $\big(x^2\,\widetilde{\mu} \cap \widetilde{\mu}\,x^2\big)(x) \geq \widetilde{\mu}(x)$ for all $x \in S$ with $\widetilde{\mu}(x) \neq \widetilde{0}$.*

Theorem 3.32 *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy weakly k-completely regular subsemigroup of S if and only if $cl(x^{2k}\,\widetilde{\mu} \cap \widetilde{\mu}\,x^{2k}\big)(x) \geq cl(\widetilde{\mu})(x)$ for all $x \in S$ with $cl(\widetilde{\mu})(x) \neq \widetilde{0}$.*

Proof Let $\widetilde{\mu}$ be an (i-v) fuzzy weakly k-completely regular subsemigroup of S and suppose $x \in S$ such that $cl(\widetilde{\mu})(x) \neq \widetilde{0}$. Then by our hypothesis, there exists $y^* \in S$ such that $x^k = x^k y^* x^k$ and $x^k y^* = y^* x^k$. As a result, $x^k = x^{2k} y^* = y^* x^{2k}$, that means, $x^k \in x^{2k} S \cap S x^{2k}$. Therefore,

$$\begin{aligned} cl(x^{2k}\,\widetilde{\mu} \cap \widetilde{\mu}\,x^{2k})(x) &= (x^{2k}\,\widetilde{\mu} \cap \widetilde{\mu}\,x^{2k})(x^k) \\ &= Min^i\Big(\big(x^{2k}\,\widetilde{\mu}\big)(x^k), \big(\widetilde{\mu}\,x^{2k}\big)(x^k)\Big) \\ &= Min^i\big(\sup_{\substack{u \in S \\ x^k = x^{2k}u}} \widetilde{\mu}(u), \sup_{\substack{v \in S \\ x^k = vx^{2k}}} \widetilde{\mu}(v)\big) \\ &\geq Min^i\big(\sup_{u \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(u), \sup_{u \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(u)\big) \\ &\geq cl(\widetilde{\mu})(x). \end{aligned}$$

Conversely, suppose the condition hold in S. Let $x \in S$ and $\widetilde{\mu}$ be an (*i*-*v*) fuzzy subsemigroup of S. If $cl(\widetilde{\mu})(x) \neq \widetilde{0}$, then $cl(x^{2k}\,\widetilde{\mu} \cap \widetilde{\mu}\,x^{2k})(x) \geq cl(\widetilde{\mu})(x)$ (by our hypothesis). This implies $Min^i\Big(\big(x^{2k}\,\widetilde{\mu}\big)(x^k), \big(\widetilde{\mu}\,x^{2k}\big)(x^k)\Big) \geq cl(\widetilde{\mu})(x)$, implies $\big(x^{2k}\,\widetilde{\mu}\big)(x^k) \neq \widetilde{0}$ and $\big(\widetilde{\mu}\,x^{2k}\big)(x^k) \neq \widetilde{0}$. That means $x^k = x^{2k}u$ and $x^k = vx^{2k}$ for some $u, v \in S$. As a consequence, $x^k ux^k = (vx^{2k})ux^k = v(x^{2k}u)x^k = vx^{2k} = x^k$ and $x^k vx^k = x^k v(x^{2k}u) = x^k(vx^{2k})u = x^{2k}u = x^k$. But $x^k(vx^k u)x^k = (x^k vx^k)ux^k = x^k ux^k = x^k$, $x^k(vx^k u) = (x^k vx^k)u = x^k u = vx^{2k}u = vx^k = v(x^k ux^k) = (vx^k u)x^k$. This shows that $vx^k u \in R_{x^k} \cap C_{x^k}$, that means, $R_{x^k} \cap C_{x^k} \neq \emptyset$. Finally,

$$\begin{aligned}
\sup_{y \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(y) &\geq \sup_{\substack{u,\, v \in S \\ vx^k u \in R_{x^k} \cap C_{x^k}}} \widetilde{\mu}(vx^k u) \\
&\geq \sup_{\substack{u,\, v \in S \\ vx^k u \in R_{x^k} \cap C_{x^k}}} \Big\{ Min^i\Big(Min^i\big(\widetilde{\mu}(v), \widetilde{\mu}(x^k)\big), \widetilde{\mu}(u)\Big)\Big\} \\
&= Min^i\Big(\sup_{\substack{u,\, v \in S \\ x^k = vx^{2k} \\ x^k = x^{2k}u}} \Big\{ Min^i\big(\widetilde{\mu}(u), \widetilde{\mu}(v)\big)\Big\}, \widetilde{\mu}(x^k)\Big) \\
&\geq Min^i\Big(Min^i\Big(\sup_{\substack{u \in S \\ x^k = x^{2k}u}} \big\{ \widetilde{\mu}(u)v\big\}, \sup_{\substack{v \in S \\ x^k = vx^{2k}}} \big\{ \widetilde{\mu}(v) \big\}\Big), \widetilde{\mu}(x^k)\Big) \\
&= Min^i\Big(cl(x^{2k}\,\widetilde{\mu} \cap \widetilde{\mu}\,x^{2k})(x), cl(\widetilde{\mu})(x)\Big) \\
&\geq cl(\widetilde{\mu})(x).
\end{aligned}$$

This indicates that $\widetilde{\mu}$ is an (*i*-*v*) fuzzy weakly k-completely regular subsemigroup of S. □

Theorem 3.33 *An (i-v) fuzzy subsemigroup $\widetilde{\mu}$ of a semigroup S is an (i-v) fuzzy weakly k-completely regular subsemigroup of S if and only if $cl(x^{2k}\,\widetilde{\mu}\,x^{2k})(x) \geq cl(\widetilde{\mu})(x)$ for $x \in S$ with $cl(\widetilde{\mu})(x) \neq \widetilde{0}$.*

Proof Let us assume that $\widetilde{\mu}$ is an (*i*-*v*) fuzzy weakly k-completely regular subsemigroup of S. Let $x \in S$. If $cl(\widetilde{\mu})(x) \neq \widetilde{0}$, by our hypothesis, $R_{x^k} \cap C_{x^k} \neq \emptyset$ and hence, $x^k = x^k x^* x^k = x^k x^*(x^k x^* x^k)x^* x^k = x^{2k}(x^*)^3 x^{2k} \in x^{2k}Sx^{2k}$, where $x^* \in R_{x^k} \cap C_{x^k}$. Therefore, $cl\big(x^{2k}\,\widetilde{\mu}\,x^{2k}\big)(x) = (x^{2k}\,\widetilde{\mu}\,x^{2k})(x^k) = \sup_{\substack{z \in S \\ x^k = x^{2k}zx^{2k}}} \widetilde{\mu}(z) \geq \sup_{\substack{x^* \in R_{x^k} \cap C_{x^k} \\ x = x^{2k}(x^* x^* x^*)x^{2k}}} \widetilde{\mu}(x^* x^* x^*) \geq \sup_{x^* \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(x^*) \geq cl(\widetilde{\mu})(x)$ (by our hypothesis) and hence the given condition holds.

Conversely, let the condition hold and $\widetilde{\mu}$ be an (*i*-*v*) fuzzy subsemigroup of S. Let $x \in S$ such that $cl(\widetilde{\mu})(x) \neq \widetilde{0}$. Then by our hypothesis $cl\big(x^2\,\widetilde{\mu}\,x^2\big)(x) \neq \widetilde{0}$. This suggests that there exists $x^* \in S$ such that $x^k = x^{2k}x^* x^{2k}$. Also $x^k = x^{2k}x^* x^{2k} = x^k(x^k x^* x^k)x^k$ and $x^k(x^k x^* x^k) = x^{2k}x^*(x^{2k}x^* x^{2k}) = (x^{2k}x^* x^{2k})x^* x^{2k} = x^k x^* x^{2k} = (x^k x^* x^k)x^k$. Thus $x^k x^* x^k \in R_{x^k} \cap C_{x^k}$ and hence $R_{x^k} \cap C_{x^k} \neq \emptyset$. Now

$$\begin{aligned}
\sup_{x* \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(x^*) &\geq \sup_{x^k x^* x^k \in R_{x^k} \cap C_{x^k}} \widetilde{\mu}(x^k x^* x^k) \\
&\geq \sup_{\substack{x^* \in S \\ x^k x^* x^k \in R_{x^k} \cap C_{x^k}}} \Big\{ Min^i\Big(\widetilde{\mu}(x^k), \widetilde{\mu}(x^*)\Big)\Big\} \text{ (by the property of } \widetilde{\mu}) \\
&= Min^i\Big(\widetilde{\mu}(x^k), \sup_{\substack{x^* \in S \\ x^k x^* x^k \in R_{x^k} \cap C_{x^k}}} \Big\{\widetilde{\mu}(x^*)\Big\}\Big)
\end{aligned}$$

$$\begin{aligned}
&\geq Min^i\Big(\widetilde{\mu}(x^k), \sup_{\substack{x^* \in S \\ x^k = x^{2k}x^*x^{2k}}} \Big\{\widetilde{\mu}(x^*)\Big\}\Big) \\
&= Min^i\Big(\widetilde{\mu}(x^k), (x^{2k}\widetilde{\mu}x^{2k})(x^k)\Big) \\
&= Min^i\Big(cl(\widetilde{\mu})(x), cl(x^{2k}\widetilde{\mu}x^{2k})(x)\Big) \\
&\geq Min^i\Big(cl(\widetilde{\mu})(x), cl(\widetilde{\mu})(x)\Big) \text{ (by given condition)} \\
&= cl(\widetilde{\mu})(x).
\end{aligned}$$

This concludes that $\widetilde{\mu}$ is an (i-v) fuzzy weakly k-completely regular subsemigroup of S. □

Conclusion

k-closure plays a significant role in case of semiring theory. For this reason, we try to define (i-v) fuzzy k-closure in semigroup theory and study different kind of k-regular semigroups in terms of (i-v) fuzzy k-closure. As a result, this work disclose a new direction to develop the theory of k-regular semigroups in terms of (i-v) fuzzy k-closure.

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