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Jensen-Mercer inequality for uniformly convex functions with some applications

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Abstract
In this study, the Jensen-Mercer inequality for a uniformly convex function is established. There are also certain application-related inequalities that are presented.

Keywords Jensen's inequality · Mercer's inequality · Uniformly convex function

Mathematics Subject Classification 37B40 · 26B25 · 94A17 · 26D15 · 26D20

1 Introduction and basic notions

The relationship between inequalities and the concept of convexity is strong. Many researchers have been studied inequalities such as Jensen inequality, Jensen-Mercer inequality, Hermite-Hadamard inequality (see [6–8, 14, 34]) and etc. for some functions with concept of convextiy such as convex functions, m-convex functions and etc. In reality, several areas of science, especially information theory, have benefited greatly from the study of convex functions (also known as functions with convexity) [2, 8, 10, 11, 13, 16–20, 26, 27, 29–32]. In this article, we develop basic results concerning uniformly convex functions, Jensen's inequality, and Mercer's inequality. Analytical applications are also studied. We require the following notations on all of the paper.

Definition 1 ([9, 12]) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Then f is uniformly convex with modulus $\phi : \mathbb{R}_{\geq 0} \longrightarrow [0, +\infty)$ if ϕ is increasing, vanishes only at 0, and

$$f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(|x-y|) \leq \alpha f(x) + (1-\alpha) f(y)$$

for every $\alpha \in [0, 1]$ and $x, y \in [a, b]$.

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Theorem 1 [25] (Jensen's inequality) *If f is a convex function on an interval I, $x_i \in I$, $1 \le i \le n$ and $\sum_{i=1}^{n} p_i = 1$, $p_i \ge 0$, then*

$$0 \le \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right).$$

Theorem 2 [24] (Mercer's inequality) *If f is a convex function on $I := [a, b]$, $x_i \in I$, $1 \le i \le n$ and $\sum_{i=1}^{n} p_i = 1$, $p_i \ge 0$, then*

$$f\left(a + b - \sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(x_i) \le f(a) + f(b). \tag{1}$$

Theorem 3 [28] *Let $f : I \longrightarrow \mathbb{R}$ be a uniformly convex function with modulus $\phi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ on I, $\{x_k\}_{k=1}^{n} \subseteq [a, b]$ be a sequence and let π be a permutation on $\{1, ..., n\}$ such that $x_{\pi(1)} \le x_{\pi(2)} \le ... \le x_{\pi(n)}$. Then the inequality*

$$f\left(\sum_{k=1}^{n} p_k x_k\right) \le \sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n-1} p_{\pi(k)} p_{\pi(k+1)} \phi(x_{\pi(k+1)} - x_{\pi(k)}) \tag{2}$$

holds for every convex combination $\sum_{k=1}^{n} p_k x_k$ of points $x_k \in I$.

Let $\phi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ be a function and $\{x_i\}_{i=1}^{n} \subseteq [a, b]$ be an increasing sequence. Define

$$\begin{aligned} J_1^{\phi,\bar{x}}(x_\mu, x_\nu) &:= \frac{1}{\sum_{i \ne \mu,\nu} p_i} \sum_{i \notin A_{\mu,\nu}} p_i p_{i+1} \phi(x_{i+1} - x_i) + \frac{p_{\mu-1} p_{\mu+1} \phi(x_{\mu+1} - x_{\mu-1})}{\sum_{i \ne \mu,s} p_i} \\ &+ \frac{p_{\nu-1} p_{\nu+1} \phi(x_{\nu+1} - x_{\nu-1})}{\sum_{i \ne \mu,s} p_i} + (p_\mu + p_\nu)\left(\sum_{i \ne \mu,\nu} p_i\right) \phi \\ &\times \left(\left|\frac{\sum_{i \ne \mu,\nu} p_i x_i}{\sum_{i \ne \mu,\nu} p_i} - \frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}\right|\right) \end{aligned}$$

Theorem 4 [28] *If f is uniformly convex with modulus $\phi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ on I and $x_1 \le x_2 \le ... \le x_n$. Then the inequality*

$$\begin{aligned} \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \ge \max_{1 \le \mu < \nu \le n} \Bigg\{ & p_\mu f(x_\mu) + p_\nu f(x_\nu) \\ & - \left((p_\mu + p_\nu) f\left(\frac{p_\mu x_\mu + p_\nu x_\nu}{p_\mu + p_\nu}\right) + J_1^{\phi,\bar{x}}(x_\mu, x_\nu)\right)\Bigg\} \ge 0 \end{aligned}$$

holds for every convex combination $\sum_{i=1}^{n} p_i x_i$ of points $x_i \in I$.

2 Main results

In this section, we give an improvement of Mercer's inequality via uniformly convex functions.

Theorem 5 *If f is a uniformly convex function with modulus ϕ on $[a, b]$ and $a < x < b$, then*

$$f(a+b-x)+f(x)+\frac{2(b-x)(x-a)}{(b-a)^2}\phi(b-a)\leq f(a)+f(b). \tag{3}$$

Proof Let $x \in [a, b]$ be arbitrary. So, there exists a $\lambda \in [0, 1]$ such that $x = \lambda a + (1-\lambda)b$. Then

$$\begin{aligned} f(a+b-x) &= f((1-\lambda)a+\lambda b) \leq (1-\lambda)f(a)+\lambda f(b)-\lambda(1-\lambda)\phi(b-a) \\ &= f(a)+f(b)-[\lambda f(a)+(1-\lambda)f(b)]-\lambda(1-\lambda)\phi(b-a) \\ &\leq f(a)+f(b)-f(\lambda a+(1-\lambda)b)-2\lambda(1-\lambda)\phi(b-a) \\ &= f(a)+f(b)-f(x)-\frac{2(b-x)(x-a)}{(b-a)^2}\phi(b-a). \end{aligned}$$

So, the proof is complete. □

Theorem 6 *Let f be a uniformly convex function with modulus ϕ on I, $\{x_i\} \subseteq I$ be a non-increasing sequence, $1 \leq i \leq n$ and $\sum_{i=1}^{n} p_i = 1$, then*

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)+\frac{1}{2}\phi\left(\frac{|b+a-2\sum_{i=1}^{n}p_i x_i|}{2}\right)+\sum_{i=1}^{n-1}p_i p_{i+1}\phi\left(x_i-x_{i+1}\right) \\ &\leq f\left(a+b-\sum_{i=1}^{n}p_i x_i\right)+\sum_{i=1}^{n}p_i f(x_i)\leq f(a)+f(b) \\ &-\frac{2\phi(b-a)}{(b-a)^2}\sum_{i=1}^{n}p_i(b-x_i)(x_i-a)-\sum_{i=1}^{n-1}p_i p_{i+1}\phi(x_i-x_{i+1}). \end{aligned} \tag{4}$$

Proof Since $\{x_i\}_i \subseteq [a, b]$, there is a sequence $\{\lambda_i\}_i (0 \leq \lambda_i \leq 1)$, such that $x_i = \lambda_i a + (1-\lambda_i)b$. Hence,

$$\begin{aligned} I &:= f\left(a+b-\sum_{i=1}^{n}p_i x_i\right)+\sum_{i=1}^{n}p_i f(x_i) \\ &= f\left(a+b-\sum_{i=1}^{n}p_i(\lambda_i a+(1-\lambda_i)b)\right)+\sum_{i=1}^{n}p_i f(\lambda_i a+(1-\lambda_i)b) \\ &\geq f\left(a+b-a\sum_{i=1}^{n}p_i\lambda_i-b\sum_{i=1}^{n}p_i(1-\lambda_i)\right)+f\left(a\sum_{i=1}^{n}p_i\lambda_i+b\sum_{i=1}^{n}p_i(1-\lambda_i)\right) \\ &+\sum_{i=1}^{n-1}p_i p_{i+1}\phi(x_i-x_{i+1}). \end{aligned}$$

Set $p := \sum_{i=1}^{n} p_i\lambda_i$ and $q := 1 - \sum_{i=1}^{n} p_i\lambda_i$. Consequently,

$$\begin{aligned}
I &\geq f(a+b-pa-qb) + f(pa+qb) + \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) \\
&= f(qa+qb) + f(pa+qb) + \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) \\
&\geq 2f\left(\frac{pa+qb}{2} + \frac{qa+pb}{2}\right) + \frac{1}{2}\phi\left(\frac{(b-a)|p-q|}{2}\right) + \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) \\
&= 2f\left(\frac{a+b}{2}\right) + \frac{1}{2}\phi\left(\frac{(b-a)|p-q|}{2}\right) + \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}). \qquad (5)
\end{aligned}$$

Since

$$p - q = 2\sum_{i=1}^{n} p_i\lambda_i - 1 = 2\sum_{i=1}^{n} p_i\left(\frac{b-x_i}{b-a}\right) - 1 = \frac{a+b-2\sum_{i=1}^{n} p_i x_i}{b-a},$$

the first inequality holds. On the other hand, by the Theorem 3, we have

$$\begin{aligned}
&f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(x_i) = f\left(\sum_{i=1}^{n} p_i(a+b-x_i)\right) + \sum_{i=1}^{n} p_i f(x_i) \\
&\leq \sum_{i=1}^{n} p_i f(a+b-x_i) - \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) + \sum_{i=1}^{n} p_i f(x_i).
\end{aligned}$$

Then from (3), we have

$$\begin{aligned}
&f\left(a+b-\sum_{i=1}^{n} p_i x_i\right) + \sum_{i=1}^{n} p_i f(x_i) \\
&\leq \sum_{i=1}^{n} p_i f(a+b-x_i) - \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) + \sum_{i=1}^{n} p_i f(x_i) \\
&\leq \sum_{i=1}^{n} p_i\left[f(a) + f(b) - f(x_i) - \frac{2(b-x_i)(x_i-a)}{(b-a)^2}\phi(b-a)\right] \\
&\quad - \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}) + \sum_{i=1}^{n} p_i f(x_i) \\
&= f(a) + f(b) - \frac{2\phi(b-a)}{(b-a)^2}\sum_{i=1}^{n} p_i(b-x_i)(x_i-a) - \sum_{i=1}^{n-1} p_i p_{i+1}\phi(x_i - x_{i+1}),
\end{aligned}$$

which completes the proof. □

3 Applications

Finding upper and lower bounds is one of the most important applications of the Jensen-mercer inequality. In this section, we give some applications in A–G inequality(see [1–5])

Lemma 1 [28] *If $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ defined by $f(x) = \log(\frac{1}{x})$, then f is uniformly convex with modulus $\phi(r) = \frac{r^2}{2b^2}$.*

Let $\mathbf{x} = \{x_i\}_{i=1}^n$ be a positive real sequence and

$$A := \sum_{i=1}^{n} p_i x_i \text{ and } G := \prod_{i=1}^{n} x_i{}^{p_i}$$

denote the usual arithmetic and geometric means of $\{x_i\}$, respectively. Denote $\mu := \min\{x_i\}$, $\nu := \max\{x_i\}$, $\tilde{A} := \mu + \nu - A$ and $\tilde{G} := \frac{\mu\nu}{G}$. From (4) we conclude the following result.

Proposition 1 *Let $\mathbf{x} = \{x_i\}_{i=1}^n$ be a sequence and $x_i > 0$ for all $i = 1, ..., n$, $\mu = \min\{x_i\}$ and $\nu = \max\{x_i\}$.*

1. *If $\mathbb{E}(\mathbf{x}^2) := \sum_{i=1}^n p_i x_i^2$ is the 2-th moment of the function x, then*

$$\begin{aligned} \tilde{G} &\leq \tilde{G} \exp\left(\frac{1}{\nu^2}\left[(\mu+\nu)A - \mathbb{E}(\mathbf{x}^2) - \mu\nu\right]\right) \\ &\leq \tilde{A} \leq \frac{(\mu+\nu)^2}{4G} \exp\left(-\frac{1}{16\nu^2}(\tilde{A} - A)^2\right) \leq \frac{(\mu+\nu)^2}{4G}. \end{aligned} \tag{6}$$

2. *Under the above notation, we have*

$$\begin{aligned} \mu\nu &\leq \mu\nu \exp\left(\frac{1}{\nu^2}\left[(\mu+\nu)A - \mathbb{E}(\mathbf{x}^2) - \mu\nu\right]\right) \\ &\leq G\tilde{A} \leq \frac{(\mu+\nu)^2}{4} \exp\left(-\frac{1}{16\nu^2}(\tilde{A} - A)^2\right) \leq \frac{(\mu+\nu)^2}{4}. \end{aligned} \tag{7}$$

Proof 1. Applying Theorem 6 and Lemma 1 with $f(x) = -\log x$, have

$$\begin{aligned} &\log\left(\frac{4}{(\mu+\nu)^2}\right) + \frac{1}{16\nu^2}(\tilde{A} - A)^2 + \frac{1}{2\nu^2}\sum_{i=1}^{n-1} p_i p_{i+1}(x_i - x_{i+1})^2 \\ &\leq -\log \tilde{A} - \log G \leq -\log(\mu\nu) \\ &\quad - \frac{1}{\nu^2}\left[(\mu+\nu)A - \mathbb{E}(\mathbf{x}^2) - \mu\nu\right] - \frac{1}{2\nu^2}\sum_{i=1}^{n-1} p_i p_{i+1}(x_i - x_{i+1})^2, \end{aligned}$$

after some calculations we have

$$\begin{aligned} \tilde{G} &\leq \tilde{G} \exp\left(\frac{1}{\nu^2}\left[(\mu+\nu)A - \mathbb{E}(\mathbf{x}^2) - \mu\nu\right]\right) \\ &\leq \tilde{G} \exp\left\{\frac{1}{\nu^2}[(\mu+\nu)A - \mathbb{E}(\mathbf{x}^2) - \mu\nu] + \frac{1}{2\nu^2}\sum_{i=1}^{n-1} p_i p_{i+1}(x_i - x_{i+1})^2\right\} \\ &\leq \tilde{A} \leq \frac{(\mu+\nu)^2}{4G} \exp\{-\frac{(\tilde{A} - A)^2}{16\nu^2} - \frac{1}{2\nu^2}\sum_{i=1}^{n-1} p_i p_{i+1}(x_i - x_{i+1})^2\} \\ &\leq \frac{(\mu+\nu)^2}{4G} \exp\left(-\frac{1}{16\nu^2}(\tilde{A} - A)^2\right) \leq \frac{(\mu+\nu)^2}{4G}. \end{aligned}$$

Thus, the desired assertion follows.

2. Multiplying both sides of Inequality (6) by G, (7) follows.

□

Conclusion 1 In this paper we establish Jensen-Mercer inequality for uniformly convex function. Some related inequalities with applications are also presented.

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