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Existence and controllability results for stochastic impulsive integro-differential equations with infinite delay

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Abstract
In this work, the existence and the controllability of impulsive stochastic integro-differential equations with infinite delay are investigated. Unlike previous papers, the result of this one relies upon some weaker assumptions using a recently defined measure of noncompactness, resolvent operator solution in sense of Grimmer and Mönch fixed point theorem. The semigroup is only required to be strongly continuous. At the end, examples are presented to illustrate the obtained result.

Keywords Hausdorff measure of noncompactness · Impulsive stochastic integro-differential equations · Controllability · Infinite delay · Resolvent operator · Fixed point theory

Mathematics Subject Classification 60H10 · 34A37 · 47G20 · 47H10

1 Introduction

Generally, ordinary differential equations are most commonly used to describe real-world phenomena. Because of the uncertainty, stochastic differential equations are widely spread in almost all applied sciences, as they are more convenient and natural than deterministic models (for the theory of stochastic differential equations in infinite dimensional space see [5, 23]). Simultaneously, nature can encounter some unexpected disturbances that might affect the system for a short time such as earthquakes and disasters, this can be described as

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impulsive differential equations (see the monographs in [4, 20, 25]). As a consequence, impulsive stochastic differential equations have received extensive attention. In [14] Guendouzi and Khadem discussed the existence of mild solutions for impulsive fractional stochastic equations with infinite delay by Sadovskii's fixed point theorem. In [6] the existence and exponential stability results have been established for impulsive neutral stochastic functional differential equations driven by fractional Brownian motion with infinite delay using an analytic semigroup and Mönch fixed point theorem. In [11] the existence and mean-square exponential stability of mild solutions for impulsive stochastic partial differential equations with finite delay have been discussed by employing Hausdorff measures of noncompactness and Mönch fixed point.

Additionally, the authors also focused on the controllability of impulsive stochastic differential equations. Whatever the exact or approximate controllability, the purpose is to determine whether or not it is possible to find a control function that steers the system towards a prescribed final state. In the infinite-dimensional systems, it is noteworthy that the exact controllability using a compact semigroup is invalid, instead, the approximate controllability can be achieved (see [1, 2, 21, 27]).

In recent years, there is a great interest in integral differential equations related to the resolvent operators, the study developed by Grimmer [12] frequently used in heat conduction in material with memory and reaction-diffusion problems. The resolvent operator does not satisfy the algebraic property of the semigroup operator due to a memory term. In [22] Lizama and Pozo established the existence results of mild solutions for a semilinear integro-differential equation when the resolvent operator is norm continuous. Later on, Ezzinbi et al. [9] proved the existence of mild solutions when the norm continuity of the resolvent operator is equivalent to the norm continuity of the semigroup operator. Recently, Ezzinbi et al. [10] ensured the existence results of mild solutions with only the fact that A generates a strongly continuous semigroup. The proof is based on a new measure of noncompactness and some estimations related to integral operators. First we focus on the existence of mild solutions for stochastic impulsive differential equations with infinite delay of the form:

$$\begin{cases} du(t) = \big[Au(t) + \displaystyle\int_0^t \Upsilon(t-s)u(s)ds\big]dt + f(t, u_t)\, d\mathbb{W}(t), \ \ t \neq t_i, t \in [0, a], \\ u(t_i^+) = u(t_i^-) + \mathcal{J}_i\big(u(t_i)\big), \qquad i = 1, 2, \ldots, n, \\ u(t) = \xi(t) \in \mathfrak{B}. \end{cases} \tag{1}$$

Here, the state $u(\cdot)$ takes values in a real separable Hilbert space $\big(\mathbb{H}, (\cdot,\cdot)_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}}\big)$. $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$. Υ is a closed linear operator on $\mathbb{H}$ with domain $D(A) \subset D(\Upsilon)$. $\{\mathbb{W}(t) : t \geq 0\}$ is a given $\mathbb{K}$-valued Brownian motion, and $f, \mathcal{J}_i$ are appropriate functions to be specified latter. $0 = t_0 < t_1 < \ldots < t_i < t_{i+1} = a$ are impulse points, and $u(t_i^+), u(t_i^-)$ represent the right and left limits of $u(t)$ at time $t = t_i$, respectively, the history function $u_t : (-\infty, 0] \to \mathbb{H}$ defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$, belong to some abstract phase space $\mathfrak{B}$ defined axiomatically, and the initial data $\{\xi(t), t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$ measurable, $\mathfrak{B}$-valued random variable independent of $\mathbb{W}$.

Our goal is to ensure the existence and the exact controllability using a new measure of noncompactness without imposing any restrictive assumptions. The resolvent operator is neither compact nor equicontinuous.

This paper is organized as follows. In Sect. 2, we give some necessary definitions concerning Brownian motion, resolvent operator and some estimations to be used in the rest of this paper. In Sect. 3, we prove the existence of mild solutions for our system (1). In Sect.

4, we discuss the exact controllability. At the end, examples are provided to illustrate the obtained results.

2 Preliminaries

Let $(\mathbb{H}, (\cdot,\cdot)_{\mathbb{H}}, \|\cdot\|_{\mathbb{H}})$, and $(\mathbb{K}, (\cdot,\cdot)_{\mathbb{K}}, \|\cdot\|_{\mathbb{K}})$ be two real separable Hilbert spaces. $\mathrm{L}(\mathbb{K}, \mathbb{H})$ denotes the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space equipped with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e it is right continuous increasing family and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Suppose that $\{\mathbb{W}(t) : t \geq 0\}$ is a Q-Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. We assume that there exists a complete orthonormal system $\{e_k\}_{k=1}^{\infty}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers $\{\lambda_k\}_{k=1}^{\infty}$ such that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$, and a sequence β_k of independent Brownian motions such that

$$\mathbb{W}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k e_k, \quad t \in [0, a],$$

and $\mathcal{F}_t = \mathcal{F}_t^{\mathbb{W}}$, where $\mathcal{F}_t^{\mathbb{W}}$ is the σ-algebra generated by $\{\mathbb{W}(t) : 0 \leq s \leq t\}$. We define $\mathrm{L}_2^0 = \mathrm{L}_2(Q^{1/2}\mathbb{K}, \mathbb{H})$ the space of all Hilbert-Schmidt operators from $Q^{1/2}\mathbb{K}$ to $\mathbb{H}$, with the norm $\|\psi\|^2_{\mathrm{L}_2^0} = Tr\big((\psi Q^{1/2})(\psi Q^{1/2})^*\big)$, for any $\psi \in \mathrm{L}_2^0$. Clearly, for any bounded operator $\psi \in \mathrm{L}(\mathbb{K}, \mathbb{H})$, we have

$$\|\psi\|^2_{\mathrm{L}_2^0} = Tr(\psi Q \psi^*) = \sum_{k=1}^{\infty} \|\sqrt{\lambda_k}\psi e_k\|^2.$$

The collection of all strongly measurable $p-$integrable, $\mathbb{H}$ valued random variables denoted by $\mathrm{L}^p(\Omega, \mathbb{H})$ is a Banach space equipped with the norm

$$\|u(\cdot)\|_{\mathrm{L}^p} = \big(\mathrm{E}\|u(\cdot, \mathbb{W})\|^p\big)^{1/p},$$

where $\mathrm{E}(u) = \int_\Omega u(\mathbb{W})d\mathcal{P}(\mathbb{W})$. And the subspace $\mathrm{L}_0^p(\Omega, \mathbb{H})$ is given by

$$\mathrm{L}_0^p(\Omega, \mathbb{H}) = \Big\{f \in \mathrm{L}^p(\Omega, \mathbb{H})\ f\ is\ \mathcal{F}_0\text{-measurable}\Big\}.$$

We denote by $\mathcal{C}\big([0, a], \mathrm{L}^p(\Omega, \mathbb{H})\big)$ the Banach space of all continuous, $\mathcal{F}_t$-adapted measurable process from $[0, a]$ into $\mathrm{L}^p(\Omega, \mathbb{H})$, satisfying $\sup_{t\in[0,a]} E\|u(t)\|^p < \infty$.

For the infinite delay case, we employ the axiomatic definition of the phase space $\mathfrak{B}$ introduced by Hale and Kato [18], with an appropriate change to treat the impulsive differential equations (see [15]).

Definition 1 [18] Let $\mathfrak{B}$ be a linear space of $\mathcal{F}_0$-measurable functions mapping from $(-\infty, 0]$ into $\mathbb{H}$, endowed with a seminorm $\|\cdot\|_{\mathfrak{B}}$, satisfying the following axioms:

(A) if $u : (\infty, \sigma + a] \to \mathbb{H}$, $a > 0$, such that $u\mid_{[\sigma,\sigma+a]} \in \mathcal{PC}\big([\sigma, \sigma + a], \mathbb{H}\big)$ and $u_0 \in \mathfrak{B}$, then for every $t \in [\sigma, \sigma + a]$, the following conditions hold:

1. $u_t \in \mathfrak{B}$.
2. $\|u(t)\|_{\mathbb{H}} \leq H\|u_t\|_{\mathfrak{B}}$.

3. $\|u_t\|_{\mathfrak{B}} \leq K(t-\sigma)\sup\{\|u(s)\|, \sigma \leq s \leq t\} + M(t-\sigma)\|u_\sigma\|_{\mathfrak{B}}$, where $H > 0$ is a constant. $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $u(\cdot)$.

(B) The space $\mathfrak{B}$ is complete.

Lemma 1 [28] *Let $u : (-\infty, a] \to \mathbb{H}$ be an $\mathcal{F}_t$-adapted measurable process, such that the $\mathcal{F}_0$-adapted process $u(t) = \xi(t) \in \mathrm{L}^p(\Omega, \mathfrak{B})$, and $u\mid_{[0,a]} \in \mathcal{PC}([0, a], \mathbb{H})$, then*

$$\| u_s \|_{\mathfrak{B}} \leq M_a\, E\| \xi \|_{\mathfrak{B}} + K_a \sup_{0\leq s\leq a} E\| u(s) \|_{\mathbb{H}},$$

where $M_a = \sup\{M(t), 0 \leq t \leq a\}$, $K_a = \sup\{K(t), 0 \leq t \leq a\}$.

Now, we define the space of all piecewise continuous functions $\mathcal{PC}([0, a], \mathbb{H})$ formed by all $\mathcal{F}_t$-adapted measurable, $\mathbb{H}$ valued stochastic process $\{u(t) : [0, a] \to \mathbb{H}$, $u(t)$ is continuous everywhere except a finite number of points $t = t_i$, at which $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exist, for $i = 1, 2, \ldots, n\}$ endowed with the norm

$$\| u \|_{\mathcal{PC}} = \left(\sup_{0\leq t\leq a} \mathrm{E}\| u(t) \|_{\mathbb{H}}^p\right)^{\frac{1}{p}}.$$

Now, let us recall the general definition of measure of noncompactness.

Definition 2 [3, 19] A function ψ defined on the set of all bounded subsets of a Banach space X with values in $\mathbb{R}^+$ is called a measure of noncompactness (MNC) on X, if for any bounded subset $D \subset X$, we have $\psi(\overline{co}D) = \psi(D)$, where $\overline{co}D$ stands for the closed convex hull of D.

Definition 3 [19] A MNC is called

- monotone: if for any bounded subset $D, C \subset X$, $D \subset C \Rightarrow \psi(D) \leq \psi(C)$,
- nonsingular: if $\psi(D \cup a) = \psi(D)$, for any $a \in X$ and $D \subset X$,
- regular: $\psi(D) = 0$ if and only if D is relatively compact in X.

One of the most known measures of noncompactness that fulfills all the above properties is the Hausdorff measure of noncompactness defined by

$$\chi(D) = inf \Big\{\varepsilon > 0, D \text{ has a finite cover by balls of radius } \varepsilon \Big\}.$$

Moreover, it enjoys the following additional properties:

Lemma 2 [3]

- $\chi(\lambda D) = \mid \lambda \mid \chi(D)$, *for any* $\lambda \in \mathbb{R}$.
- $\chi(D + C) \leq \chi(D) + \chi(C)$.
- *if $(V_n)_{n=1}^\infty$ is a nondecreasing sequence of bounded closed nonempty subsets of X and* $\lim_{n\to+\infty} \chi(V_n) = 0$, *then $\bigcap_{n=1}^\infty V_n$ is nonempty and compact in X.*
- *if $Q : X \to X$ is Lipshitz continuous map with constant κ, then $\chi(Q(D)) \leq \kappa\chi(D)$, for any bounded subset D of X.*

Now, let us introduce other examples of measures of noncompactness that was investigated in [19].

For every bounded subset $D \subset \mathcal{C}([0, a], X)$, we define

$$\mathrm{mod}_{\mathcal{C}}(D) = \sup\{\mathrm{mod}_{\mathcal{C}}(D(t))\ \ t \in [0, a]\},$$

where

$$\mathrm{mod}_{\mathcal{C}}\big(D(t)\big) = \lim_{\delta\to 0}\sup_{x\in D}\big\{\sup\big\{|\,x(t_2)-x(t_1)\,|:\ t_1,t_2\in(t-\delta,t+\delta)\big\}\big\},$$

and

$$\chi_\infty(D) = \sup\big\{\ \chi\big(D(t)\big)\ :\ t\in[0,a]\big\},$$

where χ denotes the Hausdorff measure of noncompactness in X. and since we miss the fullness in such MNCs, we define the function ψ on the family of bounded subsets in $\mathcal{C}([0,a],X)$ by taking

$$\psi(D) = \chi_\infty(D) + \mathrm{mod}_{\mathcal{C}}(D),$$

Then, ψ is a full monotone and nonsingular MNC on the space $\mathcal{C}([0,a],X)$. for more details we refer to [17, 19].

Theorem 1 [10] *Let $\mathcal{F}$ be a function from $\mathbb{R}^+$ into $\mathrm{L}(\mathbb{H})$. Suppose that $\mathcal{F}$ is continuous for the strong operator topology. Let D be a bounded subset of $\mathbb{H}$, and $\big\{\mathcal{F}(\cdot)u\ :\ u\in D\big\}\subset\mathcal{C}(\mathbb{R}^+,\mathbb{H})$. Then, for any $t\geq 0$, we have*

$$mod_{\mathcal{C}}\big(\mathcal{F}(t)D\big)\leq\omega\big(\mathcal{F}(t)\big)\chi(D).$$

In particular, for any $t\in[0,a]$, we have

$$mod_{\mathcal{C}}\big(\mathcal{F}(t)D\big)\leq 2M\chi(D),$$

where

$$\omega\big(\mathcal{F}(t)\big) = \lim_{\delta\to 0}\sup_{\|x\|\leq 1}\big\{\ \|\mathcal{F}(t_2)x-\mathcal{F}(t_1)x\|_{\mathbb{H}}\ :\ t_1,t_2\in(t-\delta,t+\delta)\big\},$$

and

$$M := \sup_{t\in[0,a]}\|\mathcal{F}(t)\|_{\mathrm{L}(\mathbb{H})}<\infty. \tag{2}$$

Lemma 3 [5] *For any $p\geq 1$ and for arbitrary $\mathrm{L}_2^0(\mathbb{K},\mathbb{H})$-valued predictable process $\Phi(\cdot)$, we have*

$$\sup_{s\in[0,t]}E\left\|\int_0^s\Phi(u)d\mathbb{W}(u)\right\|_{\mathbb{H}}^{2p}\leq\big(p(2p-1)\big)^p\left(\int_0^t\left(E\|\Phi(s)\|_{\mathrm{L}_2^0}^{2p}\right)^{\frac{1}{p}}ds\right)^p,\ t\geq 0,$$

in the rest of our paper, we denote by $C_p=\big(p(p-1)/2\big)^{\frac{p}{2}}$.

Lemma 4 [26] *If $\Phi(\cdot)\subset\mathrm{L}^p([0,a],\mathrm{L}_2^0(\mathbb{K},\mathbb{H}))$, $\mathbb{W}(t)$ is a Q-Wiener process. For any $p\geq 2$, the Hausdorff measure of noncompactness χ satisfies*

$$\chi\Big(\int_0^t\Phi(s)\,d\mathbb{W}(s)\Big)\leq\sqrt{a\frac{p}{2}(p-1)}\,\chi\big(\Phi(t)\big),$$

where

$$\int_0^t\Phi(s)\,d\mathbb{W}(s)=\left\{\int_0^t x(s)\,d\mathbb{W}(s)\ :\ \textit{for all } x\in\Phi,\ t\in[0,a]\right\}.$$

Theorem 2 [24] *Let $D \subset \mathbb{H}$ be a closed, convex subset of a Banach space $\mathbb{H}$, with $0 \in \mathbb{H}$. Suppose that there is a continuous map $T : D \to D$, satisfies Mönch's condition, that is*

$$D_0 \subseteq D \text{ countable}, \; D_0 \subseteq \left(\overline{co}\{0\} \cup T(D_0)\right) \Longrightarrow D_0 \text{ is relatively compact}.$$

Then T has at least one fixed point in D.

In order to obtain our results, we need to recall some details about resolvent operators for integro-differential equations.

In what follows, $\mathbb{H}$ is a Banach space, A and $\Upsilon(t)$ are closed linear operators on H. Let $\mathbb{K}$ be the Banach space $D(A)$ equipped with the graph norm defined by

$$\|y\|_{\mathbb{K}} = \|Ay\| + \|y\|, \; y \in \mathbb{K}.$$

The notation $\mathcal{C}(\mathbb{R}^+, \mathbb{K})$ stands for the space of all continuous functions from $\mathbb{R}^+$ into $\mathbb{K}$. Let us consider the following Cauchy problem

$$\begin{aligned} u^{'}(t) &= Au(t) + \int_0^t \Upsilon(t-s)u(s)ds, \quad t \geq 0, \\ u(0) &= u_0 \in \mathbb{H}. \end{aligned} \tag{3}$$

Definition 4 ([12]): A resolvent operator for the problem (3) is a bounded linear operator $R(t) \in \mathrm{L}(\mathbb{H})$, for $t \geq 0$, having the following properties:

- (i) $R(0) = I$ (The Identity operator of $\mathbb{H}$) and $\|R(t)\| \leq Me^{\beta t}$ for some constants $M > 0$ and $\beta \in \mathbb{R}$.
- (ii) For each $u \in \mathbb{H}$, $R(t)u$ is strongly continuous for $t \geq 0$.
- (iii) For $u \in \mathbb{K}$, $R(\cdot)u \in \mathcal{C}^1(\mathbb{R}^+, \mathbb{H}) \cap \mathcal{C}(\mathbb{R}^+, \mathbb{K})$ and

$$\begin{aligned} R^{'}(t)u &= AR(t)u + \int_0^t \Upsilon(t-s)R(s)u \, ds \\ &= R(t)Au + \int_0^t R(t-s)\Upsilon(s)u \, ds, \text{ for } t \geq 0. \end{aligned}$$

Next, we make the following hypotheses:

- $H(1)$ The operator A is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $\mathbb{H}$.
- $H(2)$ For all $t \geq 0$, $\Upsilon(t)$ is a closed linear operator from $D(A)$ to $\mathbb{H}$ and $\Upsilon(t) \in \mathrm{L}(\mathbb{K}, \mathbb{H})$. For any $u \in \mathbb{K}$, the map $t \to \Upsilon(t)u$ is bounded, differentiable and the derivative $t \to \Upsilon^{'}(t)u$ is bounded and uniformly continuous on $\mathbb{R}^+$.

Theorem 3 ([12]): *Assume that $H(1) - H(2)$ hold. Then there exists a unique resolvent operator to the Cauchy problem* (3).

For more details concerning resolvent operators of integral equations we refer to [7, 12, 13]. Now we can define a mild solution of our problem 1.

Definition 5 A $\mathcal{F}_t$-adapted stochastic process $\{u(t) \;\; t \in (-\infty, a]\}$ is called a mild solution of (1), if $u_0 = \xi \in \mathfrak{B}$, $u \mid_{[0,a]} \in \mathcal{PC}$ and

$$u(t) = R(t)\xi(0) + \int_0^t R(t-s)f(s, u_s) \, d\mathbb{W}s + \sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(u(t_i^-)\big),$$

3 Existence of mild solutions results

The following assumptions will be required throughout this paper

H(3) The nonlinear function $f : [0, a] \times \mathfrak{B} \to \mathrm{L}_2^0(\mathbb{K}, \mathbb{H})$ satisfying the following conditions:

1. The function $t \to f(t, u)$ is strongly measurable for any $u \in \mathfrak{B}$.
2. The function $u \to f(t, u)$ is continuous for any $t \in [0, a]$.
3. There exists a positive constant $r > 0$, a function $m \in \mathrm{L}^1\big([0, a], \mathbb{R}^+\big)$, and a nondecreasing continuous function $\varphi_f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$E\big(\|f(t, u)\|^p_{\mathrm{L}_2^0} \big) \leq m(t)\varphi_f(\|x\|^p_{\mathfrak{B}}) \qquad \lim_{r\to+\infty} \inf \frac{\varphi_f(r)}{r} = 0 < \infty,$$

4. there exists a function $\zeta_f \in \mathrm{L}^1\big([0, a], \mathbb{R}^+\big)$ such that for any bounded set $D \subset \mathfrak{B}$

$$\chi(f(t, D)) \leq \zeta_f(t) \sup_{-\infty\leq\theta\leq 0} \chi\big(D(\theta)\big).$$

H(4) The impulsive function $\mathcal{J}_i : \mathbb{H} \to \mathbb{H}$ is continuous and for all $u \in \mathbb{H}$

1. there exist $N_i > 0$, and for some positive number $r > 0$, we have

$$E\|\mathcal{J}_i(u)\|^p \leq N_i(\|u\|^p_{\mathfrak{B}}) \qquad \lim_{r\to+\infty} \inf \frac{N_i(r)}{r} = 0 < \infty,$$

2. there exist constants $\sigma_i > 0$, such that for any bounded set $D \subset \mathfrak{B}$

$$\chi(\mathcal{J}_i(D)) \leq \sigma_i \sup_{-\infty\leq\theta\leq 0} \chi\big(D(\theta)\big).$$

Theorem 4 *Assume that the hypotheses H(1) - H(4) are satisfied. Then the impulsive stochastic integro-differential problem* (1) *has at least one mild solution, provided that*

$$l_0 = M\Big(2\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\big([0,a],\mathbb{R}^+\big)} + 3\sum_{i=1}^{n}\sigma_i\Big) < 1. \tag{4}$$

Proof Let us consider the space Ξ_a of all functions $u : (-\infty, a] \to \mathbb{H}$ such that $u_0 \in \mathfrak{B}$, and the restriction $u\mid_{[0,a]} \in \mathcal{PC}$, with the seminorm $\|\cdot\|_a$, defined by

$$\| u \|_a = \| u_0\|_{\mathfrak{B}} + \Big(\sup_{0\leq t\leq a} \mathrm{E}\| u(t) \|^p_{\mathbb{H}}\Big)^{\frac{1}{p}}, u \in \Xi_a.$$

Now, consider the operator $S : \Xi_a \to \Xi_a$ defined by

$$(Su)(t) = \begin{cases} \xi(t), & t \in (-\infty, 0], \\ R(t)\xi(0) + \displaystyle\int_0^t R(t-s)f(s, u_s)\,d\mathbb{W}s & \\ \quad + \displaystyle\sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(u(t_i^-)\big), & t \in [0, a]. \end{cases} \tag{5}$$

For $\xi \in \mathfrak{B}$, we define $\tilde{\xi}$ by

$$\tilde{\xi}(t) = \begin{cases} \xi(t), & t \in (-\infty, 0], \\ R(t)\xi(0), & t \in [0, a]. \end{cases}$$

Then, $\tilde{\xi} \in \Xi_a$, we can decompose $u(t) = z(t) + \tilde{\xi}(t)$, $t \in (-\infty, a]$, if and only if $z_0 = 0$, and

$$z(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \displaystyle\int_0^t R(t-s) f\big(s, z_s + \tilde{\xi}_s\big)\, d\mathbb{W}s & \\ \displaystyle + \sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(z(t_i^-) + \tilde{\xi}(t_i^-)\big), & t \in [0, a]. \end{cases}$$

Let $\Xi_a^0 = \big\{z \in \Xi_a \;\; z_0 = 0 \in \mathfrak{B}\big\}$, for any $z \in \Xi_a$,we have

$$\| z \|_a = \| z_0 \|_{\mathfrak{B}} + \Big(\sup_{0\le t\le a} \mathrm{E}\| z(t) \|_{\mathbb{H}}^p\Big)^{\frac{1}{p}} = \Big(\sup_{0\le t\le a} \mathrm{E}\| z(t) \|_{\mathbb{H}}^p\Big)^{\frac{1}{p}}.$$

Thus, $\big(\Xi_a^0, \|\cdot\|_{\Xi_a^0}\big)$ is a Banach space.

Let $T : \Xi_a^0 \to \Xi_a^0$ be the operator defined as follows

$$(Tz)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \displaystyle\int_0^t R(t-s) f\big(s, z_s + \tilde{\xi}_s\big)\, d\mathbb{W}s & \\ \displaystyle + \sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(z(t_i^-) + \tilde{\xi}(t_i^-)\big), & t \in [0, a]. \end{cases} \tag{6}$$

Clearly, It turns out that the operator S has a fixed point is equivalent to T has one. So we only need to prove that T has one. This will be achieved in several Lemmas.

For $r > 0$, let $\Omega_r^0 = \big\{z \in \Xi_a^0 \;\; E\|z\|_{\mathbb{H}}^p \le r\,\big\}$.

Clearly. Ω_r^0 is a bounded closed and convex set, and for any $z \in \Omega_r^0$, and from Lemma 1, we have

$$\begin{aligned} \| z_s + \tilde{\xi}_s \|_{\mathfrak{B}}^p &\le 2^{p-1}\Big(\| z_s \|_{\mathfrak{B}}^p + \| \tilde{\xi}_s \|_{\mathfrak{B}}^p\Big) \\ &\le 4^{p-1}\Big[M_a^p\, E\| z_0 \|_{\mathfrak{B}}^p + K_a^p\Big(\sup_{0\le t\le a} \mathrm{E}\| z(t) \|_{\mathbb{H}}^p\Big) \\ &\quad + M_a^p\, E\| \xi_0 \|_{\mathfrak{B}}^p + K_a^p\Big(\sup_{0\le t\le a} \mathrm{E}\| R(t)\xi(0) \|_{\mathbb{H}}^p\Big)\Big] \\ &\le 4^{p-1}\Big[K_a^p\Big(\sup_{0\le t\le a} \mathrm{E}\| z(t) \|_{\mathbb{H}}^p\Big) + \Big(M_a^2 + K_a^p M^p H^p\Big)E\|\xi\|_{\mathfrak{B}}^p\Big] \\ &\le 4^{p-1}\Big[K_a^p r + \Big(M_a^p + K_a^p M^p H^p\Big)E\| \xi \|_{\mathfrak{B}}^p\Big] = r^*. \end{aligned} \tag{7}$$

Step 01: We claim that there exists $r > 0$, such that $T(\Omega_r^0) \subseteq \Omega_r^0$. If it is not true, then for each positive number r, there exists a function $z^r \in \Omega_r^0$, such that $E\|T(z^r)(t)\|^p > r$, then for any $t \in [0, a]$, by (2), (7), and Lemma 3, we have

$$\begin{aligned} r < E\Big\| T(z^r)(t) \Big\|^p &\le 2^{p-1} M^p C_p \Big[\int_0^t \Big(E\Big\| f\big(s, z_s^r + \tilde{\xi}_s\big) \Big\|_{\mathrm{L}_2^0}^p\Big)^{\frac{2}{p}} ds\Big]^{\frac{p}{2}} \\ &\quad + 2^{p-1} M^p \sum_{0<t_i<t} E\Big\|\mathcal{J}_i\Big(z^r(t_i^-) + \tilde{\xi}(t_i^-)\Big)\Big\|^p \\ &\le 2^{p-1} M^p C_p a^{p/2-1}\int_0^t m(s)\varphi_f(\| z_s^r + \tilde{\xi}_s \|_{\mathfrak{B}}^p)\, ds \end{aligned}$$

$$
\begin{aligned}
&+2^{p-1}M^p\sum_{i=1}^{n}N_i(\| z_s^r+\tilde{\xi}_s \|_{\mathfrak{B}}^p)\\
&\leq 2^{p-1}M^pC_pa^{p/2-1}\| m \|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}\varphi_f(r^*)\\
&+2^{p-1}M^p\sum_{i=1}^{n}N_i(r^*).
\end{aligned}
$$

Dividing both sides by r, and taking the lower limit as $r\to\infty$, we get that $1\leq 0$, witch is a contradiction. Hence, $T(\Omega_r^0)\subseteq\Omega_r^0$.

Step 02: We show that the operator $T:\Omega_r^0\to\Omega_r^0$ is continuous. For that purpose, let us consider a sequence $\{z^n\}_{n=1}^{\infty}\subseteq\Omega_r^0$, such that $\lim_{n\to\infty}z^n=z\in\Omega_r^0$. By H(3)- H(4), we have

$$
\lim_{n\to\infty} f\big(s,z_s^n+\tilde{\xi}_s\big)=f\big(s,z_s+\tilde{\xi}_s\big), \tag{8}
$$

$$
\lim_{n\to\infty} \mathcal{J}_i\big(z^n(t_i)+\tilde{\xi}(t_i)\big)=\mathcal{J}_i\big(z(t_i)+\tilde{\xi}(t_i)\big). \tag{9}
$$

And

$$
E\Big\|f\big(s,z_s^n+\tilde{\xi}_s\big)-f\big(s,z_s+\tilde{\xi}_s\big)\Big\|_{\mathrm{L}_2^0}^p\leq 2^{p-1}m(s)\varphi_f(r^*). \tag{10}
$$

Then, from the Lebesgue dominated convergence theorem, we obtain that

$$
\begin{aligned}
&E\Big\|(Tz^n)(t)-(Tz)(t)\Big\|^p\\
&\leq 2^{p-1}E\Big\|\int_0^t R(t-s)f\Big(s,z_s^n+\tilde{\xi}_s\Big)-f\Big(s,z_s+\tilde{\xi}_s\Big)d\mathbb{W}s\Big\|^p\\
&\quad+2^{p-1}E\Big\|\sum_{0<t_i<t}R(t-t_i)\mathcal{J}_i\Big(z^n(t_i)+\tilde{\xi}(t_i)\Big)-\mathcal{J}_i\Big(z(t_i)+\tilde{\xi}(t_i)\Big)\Big\|^p\\
&\leq 2^{p-1}M^pC_p\Big[\int_0^t\Big(E\Big\| f\Big(s,z_s^n+\tilde{\xi}_s\Big)-f\Big(s,z_s+\tilde{\xi}_s\Big)\Big\|_{\mathrm{L}_2^0}^p\Big)^{\frac{2}{p}}ds\Big]^{\frac{p}{2}}\\
&\quad+2^{p-1}M^p\sum_{0<t_i<t}E\Big\|\mathcal{J}_i\Big(z^n(t_i)+\tilde{\xi}(t_i)\Big)-\mathcal{J}_i\Big(z(t_i)+\tilde{\xi}(t_i)\Big)\Big\|^p\\
&\longrightarrow 0 \ \text{ as } \ n\longrightarrow\infty.
\end{aligned}
$$

Then, we conclude that

$$
\|(Tz^n)-(Tz)\|_a^2\longrightarrow 0,\ \text{as}\, n\longrightarrow\infty.
$$

Therefore T is continuous.

Step 03: We show that the Mönch condition hold. Let $D=\{z^n\}_{n=1}^{\infty}$ be a countable subset of Ω_r^0, such that

$$
D\subseteq\big(\overline{co}\{z_0\}\cup T(D)\big). \tag{11}
$$

We show that the set D is relatively compact.

From (H3)-(H4), using (2) and Lemma 4, we have

$$
\begin{aligned}
\chi\big(TD(t)\big) \leq\ & \chi\Big(\Big\{\int_0^t R(t-s)f\big(s, z_s^n+\tilde{\xi}_s\big)d\mathbb{W}s\Big\}_{n=1}^{\infty}\Big)\\
& +\chi\Big(\Big\{\sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(z^n(t_i)+\tilde{\xi}(t_i)\big)\Big\}_{n=1}^{\infty}\Big)\\
\leq\ & M\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}\sup_{-\infty\leq\theta\leq 0}\chi\left(\left\{z^n(s+\theta)+\tilde{\xi}(s+\theta)\right\}_{n=1}^{\infty}\right)\\
& +M\sum_{i=1}^{n}\sigma_i\sup_{-\infty\leq\theta\leq 0}\chi\Big(\Big\{\Big(z^n\big(t_i+\theta\big)+\tilde{\xi}\big(t_i+\theta\big)\Big)\Big\}_{n=1}^{\infty}\Big)\\
\leq\ & M\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}\sup_{0\leq\delta\leq s}\chi\big(\{z^n(\delta)\}_{n=1}^{\infty}\big)\\
& +M\sum_{i=1}^{n}\sigma_i\sup_{0\leq\delta_i\leq t_i}\chi\big(\{z^n(\delta_i)\}_{n=1}^{\infty}\big).
\end{aligned}
$$

Hence,

$$
\chi_\infty\big(T(D)\big) \leq M\Big(\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}+\sum_{i=1}^{n}\sigma_i\Big)\chi_\infty(D). \tag{12}
$$

On the other hand, using Theorem 1, we obtain

$$
\mathrm{mod}_{\mathcal{C}}\big(T(D)\big) \leq M\Big(\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}+2\sum_{i=1}^{n}\sigma_i\Big)\chi_\infty(D). \tag{13}
$$

Then, Combining (12), and (13), we get

$$
\psi\big(T(D)\big) \leq M\Big(2\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1\left([0,a],\mathbb{R}^+\right)}+3\sum_{i=1}^{n}\sigma_i\Big)\psi(D). \tag{14}
$$

From (11), and condition (4), we see that

$$
\psi(D) \leq \psi\Big(\overline{co}\{z_0\}\cup T(D)\Big) = \psi\big(T(D)\big) \leq l_0\psi(D).
$$

This implies that $\psi(D) = 0$, since $l_0 < 1$, therefore D is relatively compact. From Theorem 2, we conclude that T has a fixed point. Then $u = z+\tilde{\xi}$ is a fixed point of S in Ξ_a, which is a mild solution of (1). □

4 Controllability results

In this section, we deal with the controllability of impulsive stochastic integro-differential equation of the form:

$$
\begin{cases}
du(t) = \big[Au(t)+\displaystyle\int_0^t \Upsilon(t-s)u(s)ds+\Gamma v(t)\big]dt+f(t,u_t)\,d\mathbb{W}(t),\\
\qquad t\neq t_i,\ t\in[0,a],\\
u(t_i^+) = u(t_i^-)+\mathcal{J}_i\big(u(t_i)\big), \qquad i=1,2,\ldots,n,\\
u(t) = \xi(t)\in\mathfrak{B}.
\end{cases} \tag{15}
$$

Where the control function $v(\cdot)$ takes values in $\mathrm{L}^p\big([0,a],\mathrm{U}\big)$ of admissible control function for a separable Hilbert space U, $\Gamma:\mathrm{U}\to\mathbb{H}$ is a bounded linear operator. The rest are defined as in problem (1).

Definition 6 A $\mathcal{F}_t$-adapted stochastic process $\{u(t):t\in(-\infty,a]\}$ is called a mild solution of (15), if $u(t)=\xi(t)\in\mathfrak{B}$, $u\mid_{[0,a]}\in\mathcal{PC}$, and for $t\in[0,a]$

$$u(t)=R(t)\xi(0)+\int_0^t R(t-s)f(s,u_s)\,d\mathbb{W}s+\int_0^t R(t-s)\Gamma v(s)\,ds+\sum_{0<t_i<t}R(t-t_i)\mathcal{J}_i\big(u(t_i^-)\big),$$

Definition 7 The stochastic control system (15) is called controllable on the interval $[0,a]$, if for every initial function $\xi\in\mathfrak{B}$ and $u_T\in\mathbb{H}$, there exists a suitable stochastic control $v(\cdot)\in\mathrm{L}^p([0,a],\mathrm{U})$ such that the mild solution of (15) satisfies $u(a)=u_T$, where u_T and a are preassigned terminal state and time, respectively.

To establish our result, we need to state the following condition

H(5) The linear operator $\mathcal{W}:\mathrm{L}^p([0,a],\mathrm{U})\to\mathbb{H}$ defined by

$$\mathcal{W}v=\int_0^a R(a-s)\Gamma v(s)\,ds,$$

has a bounded invertible operator $\mathcal{W}^{-1}$ which takes values in $\mathrm{L}^p([0,a],\mathrm{U})/\mathrm{Ker}\mathcal{W}$, and

1. there exist two positive constants Θ_1,Θ_2 such that

$$\|\Gamma\|\leq\Theta_1,\qquad\|\mathcal{W}^{-1}\|\leq\Theta_2,$$

4. there exists $L_{\mathcal{W}}(t)\in\mathrm{L}^1([0,a],\mathbb{R}^+)$ such that for any bounded set $D\subset\mathbb{H}$

$$\chi\Big(\mathcal{W}^{-1}(D)(t)\Big)\leq L_{\mathcal{W}}(t)\chi\big(D\big).$$

Theorem 5 *Assume that the hypotheses H(1) - H(5) are satisfied. Then the control function of stochastic integro-differential system is controllable on* $[0,a]$, *provided that*

$$l_0=M\Big(2\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1}+3\sum_{i=1}^n\sigma_i\Big)\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)<1.$$

Proof Using H(5), we define for an arbitrary function $u(\cdot)$, the following control

$$v_u(t)=\mathcal{W}^{-1}\Big[u_T-R(a)\xi(0)-\int_0^a R(a-s)f(s,u_s)d\mathbb{W}s-\sum_{0<t_i<a}R(a-t_i)\mathcal{J}_i\big(u(t_i^-)\big)\Big](t).$$

Then, we shall show that using this control function, the operator $\digamma:\Xi_a\to\Xi_a$ defined by

$$(\digamma u)(t)=\begin{cases}\xi(t), & t\in(-\infty,0],\\ R(t)\xi(0)+\int_0^t R(t-s)f(s,u_s)\,d\mathbb{W}s & \\ +\int_0^t R(t-s)\Gamma v(s)\,ds+\sum_{0<t_i<t}R(t-t_i)\mathcal{J}_i\big(u(t_i^-)\big), & t\in[0,a],\end{cases}$$

has a fixed point. This fixed point is the mild solution of (15).

Obviously, $u(a) = (\digamma u)(a) = u_T$, which means that the system (15) is controllable. Let $G : \Xi_a^0 \to \Xi_a^0$ be the operator defined as follows

$$(Gz)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \displaystyle\int_0^t R(t-s) f\big(s, z_s + \tilde{\xi}_s\big)\, d\mathbb{W}s + \int_0^t R(t-s)\Gamma v_z(s)\, ds & \\ \quad + \displaystyle\sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\big(z(t_i^-) + \tilde{\xi}(t_i^-)\big), & t \in [0, a], \end{cases} \tag{16}$$

where

$$\begin{aligned} v_z(s) = \mathcal{W}^{-1}\Bigg[& u_T - R(a)\xi(0) - \int_0^a R(a-s) f\Big(s, z_s + \tilde{\xi}_s\Big) d\mathbb{W}s \\ & - \sum_{0<t_i<a} R(a-t_i)\mathcal{J}_i\Big(z(t_i^-) + \tilde{\xi}(t_i^-)\Big)\Bigg](s). \end{aligned}$$

It turns out that the operator $\digamma$ has a fixed point is equivalent to G has one. All we need is to prove that G has one.

Step 01: We claim that there exists $r > 0$, such that $G(\Omega_r^0) \subseteq \Omega_r^0$. If it is not true, then for each positive number r, there exists a function $z^r \in \Omega_r^0$, such that $E\|G(z^r)(t)\|^p > r$, then for any $t \in [0, a]$, we have

$$\begin{aligned} r < E\| \, G(z^r)(t) \, \|^p \leq \; & 3^{p-1} E\left\| \int_0^t R(t-s) f\Big(s, z_s^r + \tilde{\xi}_s\Big) d\mathbb{W}s \right\|^p \\ & + 3^{p-1} E\left\| \int_0^t R(t-s)\Gamma v_{z^r}(s)\, ds \right\|^p \\ & + 3^{p-1} E\left\| \sum_{0<t_i<t} R(t-t_i)\mathcal{J}_i\Big(z^r(t_i^-) + \tilde{\xi}(t_i^-)\Big) \right\|^p \\ =: \; & 3^{p-1} \sum_{i=1}^{3} E\| \, \Delta_i(t) \, \|^p. \end{aligned}$$

From (H3), (2), and Lemma 3, we obtain

$$\begin{aligned} E\| \, \Delta_1(t) \, \|^p &= E\left\| \int_0^t R(t-s) f\Big(s, z_s^r + \tilde{\xi}_s\Big) d\mathbb{W}s \right\|^p \\ &\leq M^p C_p \left[\int_0^t \left(E\left\| \, f\Big(s, z_s^r + \tilde{\xi}_s\Big) \, \right\|_{\mathrm{L}_2^0}^p \right)^{\frac{2}{p}} ds \right]^{\frac{p}{2}} \\ &\leq M^p C_p a^{p/2-1} \int_0^t m(s)\varphi_f(\| \, z_s^r + \tilde{\xi}_s \, \|_{\mathfrak{B}}^p)\, ds \\ &\leq M^p C_p a^{p/2-1} \| \, m \, \|_{\mathrm{L}^1} \varphi_f(r^*). \end{aligned} \tag{17}$$

By (H3)-H(5), (2), (7), and Lemma 3, we have

$$\begin{aligned} E\| \, \Delta_2(t) \, \|^p &= E\left\| \int_0^t R(t-s)\Gamma v_{z^r}(s)\, ds \right\|^p \\ &\leq M^p \Theta_1^p \Theta_2^p \int_0^t \Big[E\| \, z_T \, \|^p + M^p \, E\| \, \xi(0) \, \|^p \end{aligned}$$

$$
\begin{aligned}
&\quad +M^p C_p a^{p/2-1} \| m \|_{L^1} \varphi_f (\| z_s^r + \tilde{\xi}_s \|_{\mathfrak{B}}^p) \\
&\quad + M^p \sum_{i=1}^{n} N_i (\| z_s^r + \tilde{\xi}_s \|_{\mathfrak{B}}^p) \Big] ds \\
&\leq M^p \Theta_1^p \Theta_2^p a \Big[E \| z_T \|^p + M^p H^p \| \xi \|_{\mathfrak{B}}^p \\
&\quad + M^p C_p a^{p/2-1} \| m \|_{L^1} \varphi_f (r^*) + M^p \sum_{i=1}^{n} N_i (r^*) \Big]. \qquad (18)
\end{aligned}
$$

And from H(4), we obtain

$$
\begin{aligned}
E \| \Delta_3(t) \|^p &= E \Big\| \sum_{0<t_i<t} R(t-t_i) \mathcal{J}_i \Big(z^r(t_i^-) + \tilde{\xi}(t_i^-) \Big) \Big\|^p \\
&\leq M^p \sum_{i=1}^{n} N_i (\| z_s^r + \tilde{\xi}_s \|_{\mathfrak{B}}^p) \\
&\leq M^p \sum_{i=1}^{n} N_i (r^*). \qquad (19)
\end{aligned}
$$

Combining (17), (18), and (19) yields us to

$$
\begin{aligned}
r < E \Big\| G(z^r)(t) \Big\|^p &\leq 3^{p-1} M^p C_p a^{p/2-1} \| m \|_{L^1} \varphi_f (r^*) \\
&\quad + 12^{p-1} M^p \Theta_1^p \Theta_2^p a \Big[E \| z_T \|^p + M^p H^p \| \xi \|_{\mathfrak{B}}^p \\
&\quad + M^p C_p a^{p/2-1} \| m \|_{L^1} \varphi_f (r^*) + M^p \sum_{i=1}^{n} N_i (r^*) \Big] \\
&\quad + 3^{p-1} M^p \sum_{i=1}^{n} N_i (r^*).
\end{aligned}
$$

Dividing both sides by r, and taking the lower limit as $r \to \infty$, we get that $1 \leq 0$, which is a contradiction. Hence, $G(\Omega_r^0) \subseteq \Omega_r^0$.

Step 02: We show that the operator $G : \Omega_r^0 \to \Omega_r^0$ is continuous. For that purpose, let us consider a sequence $\{z^n\}_{n=1}^{\infty} \subseteq \Omega_r^0$, such that $\lim_{n\to\infty} z^n = z \in \Omega_r^0$

Then, employing the Lebesgue dominated convergence theorem, combined with (2), (8), (9), (10), and Lemma 3, we obtain

$$
\begin{aligned}
&E \Big\| (Gz^n)(t) - (Gz)(t) \Big\|^p \\
&\leq 3^{p-1} E \Big\| \int_0^t R(t-s) f\Big(s, z_s^n + \tilde{\xi}_s\Big) - f\Big(s, z_s + \tilde{\xi}_s\Big) d\mathbb{W}s \Big\|^p \\
&+ 3^{p-1} E \Big\| \int_0^t R(t-s) \Gamma \Big(v_{z^n}(s) - v_z(s) \Big) \, ds \Big\|^p \\
&+ 3^{p-1} E \Big\| \sum_{0<t_i<t} R(t-t_i) \mathcal{J}_i \Big(z^n(t_i) + \tilde{\xi}(t_i) \Big) - \mathcal{J}_i \Big(z(t_i) + \tilde{\xi}(t_i) \Big) \Big\|^p
\end{aligned}
$$

$$
\begin{aligned}
&\leq 3^{p-1}M^{p}C_{p}\bigg[\int_{0}^{t}\Big(E\Big\| f\Big(s,z_{s}^{n}+\tilde{\xi}_{s}\Big)-f\Big(s,z_{s}+\tilde{\xi}_{s}\Big)\Big\|_{\mathrm{L}_{2}^{0}}^{p}\Big)^{\frac{2}{p}}ds\bigg]^{\frac{p}{2}}\\
&+3^{p-1}M^{p}\Theta_{1}^{p}\int_{0}^{t}E\Big\|v_{z^{n}}(s)-v_{z}(s)\Big\|^{p}ds\\
&+3^{p-1}M^{p}\sum_{0<t_{i}<t}E\Big\|\mathcal{J}_{i}\Big(z^{n}(t_{i})+\tilde{\xi}(t_{i})\Big)-\mathcal{J}_{i}\Big(z(t_{i})+\tilde{\xi}(t_{i})\Big)\Big\|^{p}\\
&\longrightarrow 0 \ \text{ as } \ n\longrightarrow\infty.
\end{aligned}
$$

where

$$
\begin{aligned}
E\Big\|v_{z^{n}}(s)-v_{z}(s)\Big\|^{p} \leq \Theta_{2}^{p}\bigg[&M^{p}C_{p}\Big(\int_{0}^{a}\Big(E\Big\| f\Big(s,z_{s}^{n}+\tilde{\xi}_{s}\Big)-f\Big(s,z_{s}+\tilde{\xi}_{s}\Big)\Big\|_{\mathrm{L}_{2}^{0}}^{p}\Big)^{\frac{2}{p}}ds\Big)^{\frac{p}{2}}\\
&+M^{p}\sum_{0<t_{i}<a}E\Big\|\mathcal{J}_{i}\Big(z^{n}(t_{i})+\tilde{\xi}(t_{i})\Big)-\mathcal{J}_{i}\Big(z(t_{i})+\tilde{\xi}(t_{i})\Big)\Big\|^{p}\bigg]\\
&\longrightarrow 0 \ \text{ as } \ n\longrightarrow\infty.
\end{aligned}
$$

Then, we conclude that

$$
\Big\| (Gz^{n})-(Gz)\Big\|_{a}^{p}\longrightarrow 0,\ \text{ as } n\to\infty.
$$

Therefore G is continuous.
Step 03: Let $D=\{z^{n}\}_{n=1}^{\infty}$ be a countable subset of Ω_{r}^{0}, such that

$$
D\subseteq\Big(\overline{co}\{z_{0}\}\cup G(D)\Big). \tag{20}
$$

We show that the set D is relatively compact.
By (H3)-H(5), using Lemma 4, we have

$$
\begin{aligned}
&\chi\Big(\Big\{v_{z^{n}}(s)\Big\}_{n=1}^{\infty}\Big)\\
&\quad\leq L_{\mathcal{W}}(s)\bigg[M\chi\Big(\Big\{\int_{0}^{a}f\Big(s,z_{s}^{n}+\tilde{\xi}_{s}\Big)d\mathbb{W}s\Big\}_{n=1}^{\infty}\Big)\\
&\qquad+M\sum_{0<t_{i}<a}\chi\Big(\Big\{\mathcal{J}_{i}\big(z^{n}(t_{i})+\tilde{\xi}(t_{i})\big)\Big\}_{n=1}^{\infty}\Big)\bigg]\\
&\quad\leq L_{\mathcal{W}}(s)\bigg[M\sqrt{a\frac{p}{2}(p-1)}\zeta_{f}(t)\sup_{-\infty\leq\theta\leq 0}\chi\bigg(\bigg\{z^{n}(s+\theta)+\tilde{\xi}(s+\theta)\bigg\}_{n=1}^{\infty}\bigg)\\
&\qquad+M\sum_{i=1}^{n}\sigma_{i}\sup_{-\infty\leq\theta\leq 0}\chi\Big(\Big\{\Big(z^{n}\big(t_{i}+\theta\big)+\tilde{\xi}\big(t_{i}+\theta\big)\Big)\Big\}_{n=1}^{\infty}\Big)\bigg]\\
&\quad\leq L_{\mathcal{W}}(s)\bigg[M\sqrt{a\frac{p}{2}(p-1)}\zeta_{f}(t)\sup_{0\leq\delta\leq s}\chi\big(\big\{z^{n}(\delta)\big\}_{n=1}^{\infty}\big)\\
&\qquad+M\sum_{i=1}^{n}\sigma_{i}\sup_{0\leq\delta_{i}\leq t_{i}}\chi\big(\{z^{n}(\delta_{i})\}_{n=1}^{\infty}\big)\bigg].
\end{aligned} \tag{21}
$$

Then, from (H3)-H(5), and (21), for each $t \in [0, a]$, we obtain that

$$\begin{aligned}
&\chi\big(G(D)(t)\big) \\
&\quad \leq M\chi\left(\left\{\int_0^t f\left(s, z_s^n + \tilde{\xi}_s\right)d\mathbb{W}s\right\}_{n=1}^{\infty}\right) \\
&\qquad + M\int_0^t \Theta_1\chi\left(\left\{v_{z^n}(s)\right\}_{n=1}^{\infty}\right) ds \\
&\qquad + M\sum_{0<t_i<a}\chi\left(\left\{\mathcal{J}_i\big(z^n(t_i)+\tilde{\xi}(t_i)\big)\right\}_{n=1}^{\infty}\right) \\
&\quad \leq M\sqrt{a\frac{p}{2}(p-1)\zeta_f(t)}\sup_{0\leq\delta\leq s}\chi\left(\left\{z^n(\delta)\right\}_{n=1}^{\infty}\right) \\
&\qquad + M\int_0^t \Theta_1 L_{\mathcal{W}}(s)\Bigg[M\sqrt{a\frac{p}{2}(p-1)}\zeta_f(s)\sup_{0\leq\delta\leq s}\chi\left(\left\{z^n(\delta)\right\}_{n=1}^{\infty}\right) \\
&\qquad + M\sum_{i=1}^{n}\sigma_i\sup_{0\leq\delta_i\leq t_i}\chi\left(\{z^n(\delta_i)\}_{n=1}^{\infty}\right)\Bigg] ds \\
&\qquad + M\sum_{i=1}^{n}\sigma_i\sup_{0\leq\delta_i\leq t_i}\chi\left(\{z^n(\delta_i)\}_{n=1}^{\infty}\right) \\
&\quad \leq M\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1}\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)\chi_\infty(D) \\
&\qquad + M\sum_{i=1}^{n}\sigma_i\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)\chi_\infty(D).
\end{aligned}$$

Hence

$$\chi_\infty\big(G(D)\big) \leq M\left(\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1} + \sum_{i=1}^{n}\sigma_i\right)\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)\chi_\infty(D). \tag{22}$$

On the other hand, using Theorem 1, we obtain

$$\begin{aligned}
\mathrm{mod}_C\big(G(D)\big) \leq M\left(\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1} + 2\sum_{i=1}^{n}\sigma_i\right) \times \\
\times\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)\chi_\infty(D).
\end{aligned} \tag{23}$$

Then, by combining (22), and (23), we get

$$\psi\big(G(D)\big) \leq M\left(2\sqrt{a\frac{p}{2}(p-1)}\|\zeta_f\|_{\mathrm{L}^1} + 3\sum_{i=1}^{n}\sigma_i\right)\Big(1+M\Theta_1\|L_{\mathcal{W}}\|_{\mathrm{L}^1}\Big)\psi(D). \tag{24}$$

From (20), we see that

$$\psi(D) \leq \psi\Big(\overline{co}\{z_0\}\cup G(D)\Big) = \psi\big(G(D)\big) \leq l_0\psi(D).$$

This implies that $\psi(D) = 0$, since $l_0 < 1$, therefore D is relatively compact. From Theorem 2, we conclude that G has a fixed point. Then $u = z + \tilde{\xi}$ is a fixed point of $\mathcal{F}$ in Ξ_a, satisfying $u(a) = u_T$. Hence, the system is controllable on $[0, a]$. □

5 Applications

Example 1 Let us consider the following problem

$$\begin{cases} \dfrac{\partial}{\partial t} w(t,z) = \dfrac{\partial^2}{\partial z^2} w(t,z) + \displaystyle\int_0^t \zeta(t-s) \dfrac{\partial^2}{\partial z^2} w(s,z)\, ds \\ \qquad\qquad + f_1(t) f_2\big(w(t-r,z)\big)\, d\mathbb{W}(t), & \text{for } t \in [0,1],\ z \in [0,\pi], \\ w(t,0) = w(t,\pi) = 0, & \text{for } t \in [0,1], \\ w(t_i^+) = w(t_i^-) + \mathcal{J}_i\big(w(t_i)\big), & i = 1,2,\ldots,n, \\ w(t,z) = \xi(t,z) \in \mathfrak{B} & \text{for } t \in (-\infty,0],\ z \in [0,\pi]. \end{cases} \tag{25}$$

Where $\mathbb{W}(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$.

Let $\mathbb{U} = \mathbb{H} = \mathrm{L}^2([0,\pi],\mathbb{R})$ the space of all square integrable functions on $\mathbb{R}$, and the phase space $\mathfrak{B} = \mathcal{PC} \times \mathrm{L}^p(h,\mathbb{H})$, where $h :]-\infty, -r] \to \mathbb{R}$ is a positive Lebesgue integrable function, as introduced in [16], it is well known that $\mathfrak{B}$ satisfies axioms (A)-(B). Moreover, when $r = 0$, we can take $H = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$, and $K(t) = 1 + (\int_{-t}^{0} h(\tau)d\tau)^{\frac{1}{2}}$.

we define $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ by $Aw = w^{''}$, with

$$D(A) = \{w \in \mathbb{H},\, w,\, w^{'} \text{ are absolutely continuous },\, w^{''} \in \mathbb{H},\, w(0) = w(\pi) = 0\}.$$

Then $Aw = \Sigma_{n=1}^{+\infty} n^2 (w, e_n) e_n$, $w \in D(A)$, where $e_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n \geq 1$ is the orthogonal set of eigenvectors. From [8], it is well known that A is the infinitesimal generator of a strongly continuous semigroup $T(t)(t \geq 0)$ in $\mathbb{H}$, which is not a compact semigroup for $t \geq 0$, then $T(t)w = \Sigma_{n=1}^{+\infty} e^{-n^2 t}(w, e_n)e_n$. Furthermore, we suppose that $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and $\mathcal{C}^1$ continuous function, with $\zeta^{'}$ is bounded and uniformly continuous then (H1)-(H2) are satisfied.

Let $\Upsilon : \mathbb{U} \to \mathbb{H}$ defined as follow

$$\Upsilon(t)w = \zeta(t) A \cdot w, \quad \text{for } t \geq 0,\, w \in D(A).$$

If we take $u(t)(z) = w(t,z)$, then the Equation (25) can be written into the following abstract form

$$\begin{cases} du(t) = \big[Au(t) + \displaystyle\int_0^t \Upsilon(t-s)u(s)ds + f(t,u_t)\, d\mathbb{W}(t), \ t \neq t_i, t \in [0,a], \\ u(t_i^+) = u(t_i^-) + \mathcal{J}_i\big(u(t_i)\big), \qquad i = 1,2,\ldots,n, \\ u(t) = \xi(t) \in \mathfrak{B}. \end{cases}$$

Let $f : [0,1] \times \mathfrak{B} \longrightarrow \mathrm{L}_2^0(\mathbb{K},\mathbb{H})$ be given by $f(t,\varphi)(z) = f_1(t) f_2\big(\varphi(-\tau)(z)\big)$, where $f_1 : [0,1] \to \mathbb{R}$ is integrable, and $f_2 : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with Lipschitz constant L_{f_2}. For $t \in [0,1]$ and $\varphi \in \mathfrak{B}$

$$\|f(t,\varphi) - f(t,\psi)\| \leq\mid f_1(t) \mid L_{f_2} \|\varphi(-\tau) - \psi(-\tau)\|,$$

by the property of the Hausdorff measure of noncompactness, we have for any bounded subset $D \in \mathfrak{B}$

$$\chi(f(t,D)) \leq\mid f_1(t) \mid L_{f_2} \sup_{-r \leq \theta \leq 0} \chi\big(D(\theta)\big).$$

Furthermore, we have

$$\begin{aligned} \|f(t,\varphi)\| &\leq \mid f_1(t) \mid \Big(\|f_2\big(\varphi(-\tau)(z)\big) + f_2(0)\|\Big) \\ &\leq \mid f_1(t) \mid \varphi_f(\|q\|), \end{aligned}$$

where $\varphi_f(\|q\|) = L_{f_2}q + \mid f_2(0) \mid$, this mean that (H3) hold.

Assuming $\mathcal{J}_i : \mathbb{H} \to \mathbb{H}$ defined by $\mathcal{J}_i\big(u(t_i)\big)(z) = \mathcal{J}_i\big(w(t_i, z)\big)$ satisfies H(4). Furthermore, if the condition of Theorem 4 is fulfilled, then the problem (25) has at least one mild solution.

Example 2

$$\begin{cases} \dfrac{\partial \varpi(t,\varepsilon)}{\partial t} = A\,\varpi(t,\varepsilon) + \displaystyle\int_0^t \zeta(t-s)A\,\varpi(t,\varepsilon)\,ds & \\ +\mu\eta(t,\varepsilon) + f_1(t)f_2\big(\varpi(t-r,\varepsilon)\big)\,d\mathbb{W}(t), & t\in[0,a],\ \varepsilon\in[0,1], \\ \varpi(t,0) = \varpi(t,1) = 0, & \text{for } t\in[0,a], \\ \varpi(t_i^+) = \varpi(t_i^-) + \mathcal{J}_i\big(\varpi(t_i,z)\big), & i = 1,2,\ldots,n, \\ \varpi(t,\varepsilon) = \xi(t,\varepsilon)\in\mathfrak{B} & t\in(-\infty,0],\ \varepsilon\in[0,1]. \end{cases} \tag{26}$$

Where $\mathbb{W}(t)$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$. $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{C}^1$ function, with ζ' bounded and uniformly continuous, the coefficients $\bar{a}$, $\bar{b}$ and $\bar{c}$ are unbounded, $\mu > 0$, and $\eta : [0,a]\times[0,1] \to \mathbb{R}^+$ is continuous in t, and the phase space $\mathfrak{B} = \mathcal{PC}\times \mathrm{L}^p(h,\mathbb{H})$, where $h :]-\infty, -r] \to \mathbb{R}$ is a positive Lebesgue integrable function, as introduced in [16], it is well known that $\mathfrak{B}$ satisfies axioms (A)-(B). Moreover, when $r = 0$, we can take $H = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$, and $K(t) = 1 + (\int_{-t}^0 h(\tau)d\tau)^{\frac{1}{2}}$.

Let $\mathbb{H} = \mathrm{L}^2([0,1],\mathbb{R})$ the Banach space of square integrable functions, we define the operator A by

$$\begin{cases} D(A) = \big\{\varpi\in\mathbb{H}, \varpi', \varpi''\in\mathbb{H}, \varpi(0) = \varpi(1)\big\} \\ A\,\varpi(t,\varepsilon) = \bar{a}(\varepsilon)\dfrac{\partial^2\varpi(t,\varepsilon)}{\partial\varepsilon^2} + \bar{b}(\varepsilon)\dfrac{\partial\varpi(t,\varepsilon)}{\partial\varepsilon}\varpi(t,\varepsilon) + \bar{c}(\varepsilon)\varpi(t,\varepsilon). \end{cases}$$

From [8], it is well known that A is the infinitesimal generator of a strongly continuous semigroup $T(t)(t\geq 0)$ on $\mathbb{H}$.

We also define the operator $\Upsilon : \mathbb{U} \to \mathbb{H}$ as follows

$$\Upsilon(t)\varpi = \zeta(t)A\cdot\varpi, \quad \text{for } t\geq 0, \varpi\in D(A).$$

And let $\Gamma : \mathrm{U} \to \mathbb{H}$ be defined as follows

$$(\Gamma v(t))(\varepsilon) = \mu\eta(t,\varepsilon),\ \varepsilon\in[0,1], v\in \mathrm{L}^p\big([0,a],\mathrm{U}\big).$$

If we take $u(t)(\varepsilon) = \varpi(t,\varepsilon)$, then the Equation (26) can be written into the following abstract form

$$\begin{cases} du(t) = \big[Au(t) + \displaystyle\int_0^t \Upsilon(t-s)u(s)ds + \Gamma v(t)\big]dt \\ \qquad +f(t,u_t)\,d\mathbb{W}(t),\ t\neq t_i, t\in[0,a], \\ u(t_i^+) = u(t_i^-) + \mathcal{J}_i\big(u(t_i)\big), \qquad i = 1,2,\ldots,n, \\ u(t) = \xi(t)\in\mathfrak{B}. \end{cases}$$

Clearly

$$\|\Upsilon(t)y\|_{\mathbb{H}} \leq \|\zeta(t)A\cdot y\|_{\mathbb{U}} \leq \zeta(t)\|y\|_{\mathbb{U}},$$

and

$$\|\frac{d}{dt}\Upsilon(t)y\|_{\mathbb{H}} \leq\mid \zeta'(t)\mid \|A\cdot y\|_{\mathbb{H}} \leq \zeta(t)\|y\|_{\mathbb{U}},$$

for all $y\in\mathbb{U}$, and all $t\in\mathbb{R}^+$, accordingly (H1) and (H2) are satisfied.

Let $f : [0, a] \times \mathfrak{B} \longrightarrow \mathrm{L}_2^0(\mathbb{K}, \mathbb{H})$ be given by $f(t, \varphi)(\varepsilon) = f_1(t) f_2\big(\varphi(-\tau)(\varepsilon)\big)$, where $f_1 : [0, a] \to \mathbb{R}$ is square integrable, and $f_2 : \mathbb{R} \to \mathbb{R}$ is Lipschitzian with Lipschitz constant L_{f_2}. For $t \in [0, a]$ and $\varphi \in \mathfrak{B}$

$$\| f(t, \varphi) - f(t, \psi) \| \leq \mid f_1(t) \mid L_{f_2} \|\varphi(-\tau) - \psi(-\tau)\|,$$

by the property of the Hausdorff measure of noncompactness, we have for any bounded subset $D \in \mathfrak{B}$

$$\chi(f(t, D)) \leq \mid f_1(t) \mid L_{f_2} \sup_{-r \leq \theta \leq 0} \chi\big(D(\theta)\big).$$

Furthermore, we have

$$\begin{aligned} \| f(t, \varphi)\| &\leq \mid f_1(t) \mid \Big(\| f_2\big(\varphi(-\tau)(z)\big) + f_2(0) \| \Big) \\ &\leq \mid f_1(t) \mid \varphi_f(\|q\|), \end{aligned}$$

where $\varphi_f(\|q\|) = L_{f_2} q + \mid f_2(0) \mid$, this mean that (H3) hold.

Assuming $\mathcal{J}_i : \mathbb{H} \to \mathbb{H}$ defined by $\mathcal{J}_i\big(u(t_i)\big)(\varepsilon) = \mathcal{J}_i\big(\varpi(t_i, \varepsilon)\big)$ satisfies H(4). and for $\varepsilon \in [0, 1]$, the operator $\mathcal{W}$ defined by

$$\mathcal{W}(\varepsilon) v = \mu \int_0^1 R(b - s) v(s)\, ds,$$

Assuming that $\mathcal{W}$ satisfies H(5). Furthermore, if the condition of Theorem 5 is fulfilled, then the problem (26) has at least one mild solution, which is controllable on [0, a].

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