Afrika Matematika (2022) 33:69
https://doi.org/10.1007/s13370-022-01004-w

General weighted class of quaternion-valued functions with lacunary series expansions

Alaa Kamal[1,2] · **T. I. Yassen**[3]

Received: 31 August 2019 / Accepted: 15 April 2022 / Published online: 1 June 2022

Abstract
The purpose of this article is to define a new general weighted class of hyperholomorphic functions, the so called $\mathbf{B}^q_{\alpha,\omega}(G)$ Spaces. For this class we obtain characterizations by weighted Bloch $\mathcal{B}^\alpha_\omega$ spaces. Moreover, we characterize the hyperholomorphic $\mathbf{B}^q_{\alpha,\omega}(G)$ functions by the coefficients of certain lacunary series expansions in Clifford analysis.

Keywords Quaternionic analysis · $\mathbf{B}^q_{\alpha,\omega}(G)$ spaces · Hyperholomorphic functions · Clifford analysis

Mathematics Subject Classification 32Axx · 32Hxx

1 Introduction

Quaternion analysis is the generalizations of the theory of holomorphic functions in one complex variable to Euclidean space. The concept of the hyperholomorphic functions based on the consideration of functions in the kernel of the generalized Cauchy–Riemann operator. The algebraic structure represents the measure difference between the theory of hyperholomorphic functions and the classical theory of analytic functions in the complex plane $\mathbb{C}$. Analytic functions in $\mathbb{C}$ form an algebra while the same is not true in the sense of hyperholomorphic functions. The study of hyperholomorphic function spaces began with the interesting papers (see [5, 8, 11, 14]) and others.

Our objective in this article is twofold. First, we introduce a new generalized quaternion space $\mathbf{B}^q_{\alpha,\omega}(G)$ and study relations to the quaternion $\mathcal{B}^\alpha_\omega$ space. Furthermore, we will consider

Alaa Kamal and T. I. Yassen contributed equally to this work.

✉ Alaa Kamal
alaa_kamal@Sci.psu.edu.eg; ak.ahmed@qu.edu.sa

T. I. Yassen
taha_hmour@yahoo.com

1 Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said 42521, Egypt

2 Department of Mathematics, College of Science and Arts AlMithnab, Qassim University, Buridah 51931, Saudi Arabia

3 Department of Basic Science, The Higher Engineering Institute in Al-Minya, El Minya 61511, Egypt

some essential properties of $\mathbf{B}^q_{\alpha,\omega}(G)$ spaces of quaternion-valued function as basic scale properties. Second, characterizations of the hyperholomorphic $\mathbf{B}^q_{\alpha,\omega}(G)$ functions by the coefficients of certain lacunary series expansions in clifford analysis are obtained.

2 Preliminaries

2.1 Analytic function spaces

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disk. The well known Bloch space is defined by:

$$\mathcal{B} = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \mathcal{B}(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)| < \infty \right\}.$$

Composing the Möbius transform $\varphi_a(z)$, which maps the unit disk $\mathbb{D}$ onto itself, and the fundamental solution of the two-dimensional real Laplacian on $\mathbb{D}$, we have the Green's function $g(z,a) = \ln|\frac{1-\bar{a}z}{a-z}|$ with logarithmic singularity at $a \in \mathbb{D}$. Here, φ_a always stands for the Möbius transformation $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. Stroethoff [26] gave the following definition:

Definition 1 Let f be an analytic function in $\mathbb{D}$ and let $0 < q < \infty$. If

$$\|f\|^q_{\mathbf{B}^q} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-2} \left(1-|\varphi_a(x)|^2\right)^2 dA(z) < \infty,$$

then $f \in \mathbf{B}^q$.

Definition 2 (See [4, 16]) Let aright-continuous and nondecreasing function $\omega : (0,1] \to (0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$ is defined as the set of all analytic functions f on $\mathbb{D}$ satisfying

$$(1-|z|)|f'(z)| \leq C\omega(1-|z|), \quad z \in \mathbb{D},$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1$, $\mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$.

Definition 3 (See [17, 23]) Let $0 < \alpha < \infty$ and $\omega : (0,1] \to (0,\infty)$. For an analytic function f in $\mathbb{D}$, we define the weighted α-Bloch space $\mathcal{B}^\alpha_\omega$, as follows:

$$\mathcal{B}^\alpha_\omega = \left\{ f : f \text{ analytic in } \mathbb{D} \text{ and } \|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^\alpha \left|f'(z)\right|}{\omega(1-|z|)} < \infty \right\}.$$

Also, the little weighted α-Bloch space $\mathcal{B}^\alpha_{\omega,0}$ is a subspace of $\mathcal{B}^\alpha_\omega$ consisting of all $f \in \mathcal{B}^\alpha_\omega$, such that

$$\lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha \left|f'(z)\right|}{\omega(1-|z|)} = 0.$$

2.2 Quaternion function spaces

To introduce the meaning of hyperholomorphic functions, let $\mathbb{H}$ be the skew field of quaternions. This means we can write each element $w \in \mathbb{H}$ in the form

$$w = w_0 + w_1 i + w_2 j + w_3 k, \quad w_0, w_1, w_2, w_3 \in \mathbb{R},$$

where $1, i, j, k$ are the basis elements of $\mathbb{H}$. For these elements we have the multiplication rules

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, kj = -jk = i, ki = -ik = j.$$

The conjugate element $\bar{w}$ is given by $\bar{w} = w_0 - w_1 i - w_2 j - w_3 k$, and we have the property

$$w\bar{w} = \bar{w}w = \|w\|^2 = w_0^2 + w_1^2 + w_2^2 + w_3^2.$$

Moreover, we can identify each vector $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3$ with a quaternion x of the form

$$x = x_0 + x_1 i + x_2 j.$$

In what follows we will work in $\mathbb{B}_1(0) \subset \mathbb{R}^3$, the unit ball in the real three-dimensional space. We will consider functions f defined on $\mathbb{B}_1(0)$ with values in $\mathbb{H}$. We define a generalized Cauchy–Riemann operator D by

$$Df = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2},$$

and it's conjugate operator by

$$\overline{D}f = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2}.$$

For these operators, we have that

$$D\overline{D} = \overline{D}D = \Delta_3,$$

where Δ_3 is the Laplacian for functions defined over domains in $\mathbb{R}^3$. For $|a| < 1$, we will denote by

$$\varphi_a(x) = (a - x)(1 - \bar{a}x)^{-1},$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x, a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right),$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. Especially, we denote for all $p \geq 0$

$$g^p(x, a) = \frac{1}{4^p \pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p.$$

Let $f : \mathbb{B} \mapsto \mathbb{H}$ be a hyperholomorphic function. Then from [11], we have the seminorms

- $\mathcal{B}(f) = \sup_{x\in\mathbb{B}}(1 - |x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D}f(x)|^2 g^p(x, a) d\mathbb{B}_x$,

Definition 4 (See [5]) Let $0 < \alpha < \infty$. Recall that the hyperholomorphic α-Bloch space is defined as follows:

$$\mathcal{B}^\alpha = \left\{ f \in \ker D(\mathbb{B}_1(0)) : \sup_{x\in\mathbb{B}}(1 - |x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| < \infty \right\},$$

the little α-Bloch type space $\mathcal{B}_0^\alpha$ is a subspace of $\mathcal{B}$ consisting of all $f \in \mathcal{B}^\alpha$ such that

$$\lim_{|x|\to 1^-}(1 - |x|^2)^{\frac{3\alpha}{2}}|\overline{D}f(x)| = 0.$$

Quite recently, El-Sayed Ahmed in [5], gave the following definition:

Definition 5 Let f be quaternion-valued function in $\mathbb{B}$. For $0 < q < \infty$, and $1 \le \alpha < \infty$. If

$$\|f\|^q_{\mathbf{B}^q_\alpha} = \sup_{a\in\mathbb{B}} \int_{\mathbb{B}} |\overline{D} f(x)|^q (1-|x|^2)^{\frac{3\alpha q}{2}-3} \Big(1-|\varphi_a(x)|^2\Big)^3 d\mathbb{B}_x < \infty,$$

then $f \in \mathbf{B}^q_\alpha$. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{B}} |\overline{D} f(x)|^q (1-|x|^2)^{\frac{3\alpha q}{2}-3} \Big(1-|\varphi_a(x)|^2\Big)^3 d\mathbb{B}_x = 0,$$

then $f \in \mathbf{B}^q_{\alpha,0}$.

Ahmed and Asiri in [8], gave the following definition:

Definition 6 Let aright-continuous and nondecreasing function $\omega : (0, 1] \to (0, \infty)$, and $1 < \alpha < \infty$. A quaternion-valued function f on $\mathbb{B}_1(0)$ is said to belong to the weighted α-Bloch space $\mathcal{B}^\alpha_\omega$, if

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{x\in\mathbb{B}_1(0)} \frac{(1-|x|^2)^{\frac{3\alpha}{2}}}{\omega(1-|x|)} |\overline{D} f(x)| < \infty.$$

Moreover, a quaternion-valued function f on $\mathbb{B}_1(0)$ is said to belong to the weighted α-Bloch space $\mathcal{B}^\alpha_{\omega,0}$, if

$$\|f\|_{\mathcal{B}^\alpha_{\omega,0}} = \lim_{|x|\to 1^-} \frac{(1-|x|^2)^{\frac{3\alpha}{2}}}{\omega(1-|x|)} |\overline{D} f(x)| < \infty.$$

Now, we use the definition of Green function in $\mathbb{R}^3$ (see [2])

$$G(x, a) = \frac{1-|\varphi_a(x)|^2}{|1-\overline{a}x|}.$$

Then, we introduce the following new definition of the so called the hyperholomorphic $\mathbf{B}^q_{\alpha,\omega}(G)$ spaces.

Definition 7 Let $1 \le \alpha < \infty$, $0 < q < \infty$, and $\omega : (0, 1] \to (0, \infty)$. Assume that f be hyperholomorphic function in the unit ball $\mathbb{B}_1(0)$. Then, $f \in \mathbf{B}^q_{\alpha,\omega}(G)$, if

$$\mathbf{B}^q_{\alpha,\omega}(G) = \left\{ f \in ker D : \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D} f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \big(G(x, a)\big)^3 d\mathbb{B}_x < \infty \right\}.$$

The space $\mathbf{B}^q_{\alpha,\omega,0}(G)$ is subspace of $\mathbf{B}^q_{\alpha,\omega}(G)$ consisting of all functions $f \in \mathbf{B}^q_{\alpha,\omega}(G)$, such that

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_1(0)} |\overline{D} f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \big(G(x, a)\big)^3 d\mathbb{B}_x = 0.$$

The following lemma, we will need in the sequel:

Lemma 1 (See [24]) *Let* $f : \mathbb{B}_1(0) \longrightarrow \mathbb{H}$ *be a hyperholomorphic function. Let* $0 < R < 1$, $1 < q$. *Then for every* $a \in \mathbb{B}_1(0)$

$$|\overline{D} f(a)|^q \le \frac{3\cdot 4^{2+q}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3} \int_{\mathcal{M}(a,R)} \big|\overline{D} f(x)\big|^q d\mathbb{B}_x.$$

3 Characterization of $\mathbf{B}^{q}_{\alpha,\omega}(G)$ spaces in Clifford analysis

The relations between $\mathbf{B}^{q}_{\alpha,\omega}(G)$ space and $\mathcal{B}^{\alpha}_{\omega}$ spaces are given in quaternion sense. The results in this section are extensions and generalization of the results (see [13]).

Proposition 1 *Let* $1 \le \alpha < \infty$, $0 \le p < \infty$, $0 < q < \infty$, *and* $\omega : (0,1] \to (0,\infty)$. *Assume that* f *be hyperholomorphic function in the unit ball* $\mathbb{B}_1(0)$. *Then*

$$\frac{(1-|a|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|a|)}|\overline{D}f(a)|^q \le \frac{48(2)^{2q}(1+R)^3}{\pi R^3(1-R^2)^{2q+3}(1-|a|^2)^3}\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}\big(G(x,a)\big)^3 d\mathbb{B}_x.$$

Proof Let $\mathcal{M}(a,R)=\{x\in\mathbb{B}_1(0):|\varphi_a(x)|=\frac{|a-x|}{|1-\overline{a}x|}<R\}$ be pseudo-hyperbolic ball with center a and radius R. Then

$$\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}\big(G(x,a)\big)^3 d\mathbb{B}_x \ge \int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}\big(G(x,a)\big)^3 d\mathbb{B}_x.$$

Since

$$|\varphi_a(x)| < R, \quad \forall x\in\mathcal{M}(a,R),$$

and

$$G(x,a)=\frac{1-|\varphi_a(x)|^2}{|1-\overline{a}x|}.$$

Then, we have

$$G(x,a)=\frac{1-R^2}{1+R}, \quad where\ 1-R\le |1-\overline{a}x| \le 1+R.$$

Now, for fixed $R\in(0,1)$ and $a\in\mathbb{B}_1(0)$.
Let $\mathcal{E}(a,R)\subset\mathcal{M}(a,R)$, such that

$$\mathcal{E}(a,R)=\{x\in\mathbb{B}_1(0): |x-a| < R|1-a|\}.$$

Then, we deduce that

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}\big(G(x,a)\big)^3 d\mathbb{B}_x\\
&\ge\left(\frac{1-R^2}{1+R}\right)^3\int_{\mathcal{M}(a,R)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}d\mathbb{B}_x\\
&\ge\left(\frac{1-R^2}{1+R}\right)^3\int_{\mathcal{E}(a,R)}|\overline{D}f(x)|^q\frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)}d\mathbb{B}_x\\
&\ge\left(\frac{1-R^2}{1+R}\right)^3\frac{(1-|a|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|a|)}\int_{\mathcal{E}(a,R)}|\overline{D}f(x)|^q d\mathbb{B}_x.
\end{aligned}$$

Now, using Lemma 1, we obtain

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \mid \overline{D} f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\geq \left(\frac{1-R^2}{1+R}\right)^3 \frac{(1-\mid a\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid a\mid)} \frac{\pi R^3(1-R^2)^{2q}(1-\mid a\mid^2)^3}{3(4)^{q+2}} |\overline{D} f(a)|^q \\
&= \frac{\pi R^3(1-R^2)^{2q+3}(1-\mid a\mid^2)^{\frac{3}{2}(\alpha q+6)}}{3(4)^{q+2}(1+R)^3\omega^q(1-\mid a\mid)} |\overline{D} f(a)|^q,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\frac{(1-\mid a\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid a\mid)} |\overline{D} f(a)|^q \\
&\leq \frac{48(2)^{2q}(1+R)^3}{\pi R^3(1-R^2)^{2q+3}(1-\mid a\mid^2)^3} \int_{\mathbb{B}_1(0)} \mid \overline{D} f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x.
\end{aligned}$$

This completes the proof. □

Corollary 1 *From Proposition 1, we get for* $1 \leq \alpha < \infty$, $0 \leq p < \infty$, $0 < q < \infty$, *and* $\omega : (0, 1] \to (0, \infty)$ *that*

$$\mathbf{B}^q_{\alpha,\omega}(G) \subset \mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}.$$

Proposition 2 *Let* $1 \leq \alpha < \infty$, $0 \leq p < \infty$, $0 < q < \infty$, *and* $\omega : (0, 1] \to (0, \infty)$. *Let* f *be a hyperholomorphic function in* $B_1(0)$, $\forall a \in B_1(0)$; $|a| < 1$ *and* $f \in \mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}$. *Then, we have that*

$$\int_{\mathbb{B}_1(0)} \mid \overline{D} f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \leq k(\mathcal{B}^\gamma_\omega(f)),$$

where $\gamma = \frac{3}{2q}(\alpha q + 4)$.

Proof

$$\frac{(1-\mid x\mid^2)^{\frac{3\alpha}{2}}}{\omega^q(1-\mid x\mid)} |\overline{D} f(a)|^q \leq \mathcal{B}^\alpha_\omega(f).$$

(See [8, 11]). Then,

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \mid \overline{D} f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\leq (\mathcal{B}^\gamma_\omega(f)) \int_{\mathbb{B}_1(0)} \big(G(x,a)\big)^3 d\mathbb{B}_x.
\end{aligned}$$

Using the equality

$$G(x,a) = \frac{1-\mid \varphi_a(x)\mid^2}{\mid 1-\overline{a}x\mid} = \frac{(1-\mid a\mid^2)(1-\mid x\mid^2)}{\mid 1-\overline{a}x\mid^3}, \tag{1}$$

where

$$1-\mid x\mid \leq \mid 1-\overline{a}x\mid \leq 1+\mid x\mid, \quad 1-\mid a\mid \leq \mid 1-\overline{a}x\mid \leq 1+\mid a\mid \leq 2. \tag{2}$$

Then, we get

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \mid \overline{D}f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\leq (\mathcal{B}^{\gamma}_{\omega}(f)) \int_{\mathbb{B}_1(0)} \frac{(1-\mid a\mid^2)^3(1-\mid x\mid^2)^3}{\mid 1-\overline{a}x\mid^9} d\mathbb{B}_x. \\
&\leq 2^6(\mathcal{B}^{\gamma}_{\omega}(f)) \int_{\mathbb{B}_1(0)} \frac{(1-\mid a\mid)^3(1-\mid x\mid)^3}{(1-\mid x\mid^3)(1-\mid a\mid)^6} d\mathbb{B}_x. \\
&\leq 2^6(\mathcal{B}^{\gamma}_{\omega}(f)) \int_{\mathbb{B}_1(0)} \frac{1}{(1-\mid a\mid)^3} d\mathbb{B}_x. \\
&\leq k(\mathcal{B}^{\gamma}_{\omega}(f)).
\end{aligned}$$

Therefore, the proof of proposition is complete. □

Corollary 2 *From Proposition 2, we get for* $1 \leq \alpha < \infty,\ 0 \leq p < \infty$, $0 < q < \infty$, *and* $\omega : (0,1] \to (0,\infty)$, *that*

$$\mathcal{B}_{\omega}^{\frac{3}{2q}(\alpha q+4)} \subset \mathbf{B}^q_{\alpha,\omega}(G).$$

The results in Corollaries 1 and 2 prove the following theorem, which give to us the characterization for the hyperholomorphic weighted Bloch space by the integral norms of $\mathbf{B}^q_{\alpha,\omega}(G)$ spaces of hyperholomorphic functions.

Theorem 3 *Let f be a hyperholomorphic function in $B_1(0)$. Then for $\alpha,\ p \geq 1$, $0 < q < \infty$, and $\omega : (0,1] \to (0,\infty)$, we have*

$$\mathcal{B}_{\omega}^{\frac{3}{2q}(\alpha q+4)} = \mathbf{B}^q_{\alpha,\omega}(G).$$

For characterization the little hyperholomorphic weighted Bloch space, used the same arguments in the previous theorem to prove the following theorem.

Theorem 4 *Let f be a hyperholomorphic function in $B_1(0)$.Then for $\alpha,\ p \geq 1$, $0 < q < \infty$, and $\omega : (0,1] \to (0,\infty)$, we have*

$$\mathcal{B}_{\omega,0}^{\frac{3}{2q}(\alpha q+4)} = \mathbf{B}^q_{\alpha,\omega,0}(G).$$

Theorem 5 *Let $0 < R < 1$ and $\omega : (0,1] \to (0,\infty)$. for the hyperholomorphic function f in $\mathbb{B}_1(0)$. The following are equivalent*

(a) $f \in \mathcal{B}_{\omega}^{\frac{3}{2q}(\alpha q+4)}$.

(b) *For each* $1 \leq \alpha < \infty$, *and* $0 < q < \infty$

$$\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \mid \overline{D}f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x < \infty.$$

(c) *For each* $1 \leq \alpha < \infty$, *and* $0 < q < \infty$

$$\sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} \mid \overline{D}f(x) \mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} d\mathbb{B}_x < \infty.$$

(d) *For each* $1 \leq \alpha < \infty$, *and* $0 < q < \infty$

$$\sup_{a\in\mathbb{B}_1(0)} \frac{|\mathcal{M}(a,R)|^{\frac{\alpha q+4}{2}}}{\omega^q(1-|a|)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q d\mathbb{B}_x < \infty.$$

Proof To prove (a) implies (b). Using (1) and (2), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\leq \sup_{a\in\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \int_{\mathbb{B}_1(0)} \big(G(x,a)\big)^3 d\mathbb{B}_x.
\end{aligned}$$

Using the same steps as in Proposition 2, we obtain

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\leq 2^6 \|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \int_{\mathbb{B}_1(0)} \frac{1}{(1-|a|)^3} d\mathbb{B}_x \\
&\leq K_1 \|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \\
&< \infty.
\end{aligned}$$

(b) implies (c), using the same steps as in Proposition 1, we deduce that

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\geq \left(\frac{1-R^2}{2}\right)^3 \sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} d\mathbb{B}_x.
\end{aligned}$$

For (c) implies (d), we use the fact $(1-|x|^2)^3 \approx |\mathcal{M}(a,R)|, \forall\, x \in \mathcal{M}(a,R)$ (see [15]). Then

$$\begin{aligned}
&\sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} d\mathbb{B}_x \\
&\approx \frac{|\mathcal{M}(a,R)|^{\frac{\alpha q+4}{2}}}{\omega^q(1-|a|)} \sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q d\mathbb{B}_x.
\end{aligned}$$

For (d) implies (a). From Lemma 1, we have

$$\begin{aligned}
&|\overline{D}f(a)|^q \frac{(1-|a|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|a|)} \\
&\leq \frac{3\cdot 4^{2+q}(1-|a|^2)^{\frac{3}{2}(\alpha q+4)}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3\omega^q(1-|a|)} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x \\
&= \frac{3\cdot 4^{2+q}(1-|a|^2)^{\frac{3}{2}(\alpha q+4)}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3\omega^q(1-|a|)} \\
&\quad \times \frac{(1-R^2|a|^2)^{\frac{3}{2}(\alpha q+4)} R^{\frac{3}{2}(\alpha q+4)}}{(1-R^2|a|^2)^{\frac{3}{2}(\alpha q+4)} R^{\frac{3}{2}(\alpha q+4)}} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x.
\end{aligned}$$

Now, since

$$| \mathcal{M}(a,R) | = \frac{(1- | a |^2)^3}{(1-R^2 | a |^2)^3} R^3.$$

Also, we used the following inequalities

$$1-R^2 \le 1-R^2 | a |^2 \le 1+R^2 \quad and \quad 1- | a |^2 \le 1-R^2 | a |^2 \le 1+ | a |^2.$$

Then, we have

$$\begin{aligned}
&|\overline{D}f(a)|^q \frac{(1- | a |^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1- | a |)} \\
&\le \frac{3\cdot 4^{2+q} | \mathcal{M}(a,R) |^{\frac{\alpha q+4}{2}}}{\pi R^3(1-R^2)^{2q}(1-|a|^2)^3\omega^q(1- | a |)} \times \frac{(1-R^2 | a |^2)^{\frac{3}{2}(\alpha q+4)}}{R^{\frac{3}{2}(\alpha q+4)}} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x \\
&\le \frac{3\cdot 4^{2+q} | \mathcal{M}(a,R) |^{\frac{(\alpha q+4)}{2}}}{\pi R^3(1-R^2)^{2q}(1-R^2)^3\omega^q(1- | a |)} \times \frac{(1+R^2)^{\frac{3}{2}(\alpha q+4)}}{R^{\frac{3}{2}(\alpha q+4)}} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x \\
&= \frac{3\cdot 4^{2+q}(1+R^2)^{\frac{3}{2}(\alpha q+4)}}{\pi R^{\frac{3}{2}(\alpha q+6)}(1-R^2)^{2q+3}} \times \frac{| \mathcal{M}(a,R) |^{\frac{(\alpha q+4)}{2}}}{\omega^q(1- | a |)} \int_{\mathcal{M}(a,R)} \left|\overline{D}f(x)\right|^q d\mathbb{B}_x.
\end{aligned}$$

Therefore, our theorem is proved. □

From Theorem 5, using the same arguments, we directly obtain the following theorem.

Theorem 6 *Let* $0 < R < 1$ *and* $\omega : (0,1] \to (0,\infty)$. *Then for the hyperholomorphic function* f *in* $\mathbb{B}_1(0)$, *the following are equivalent*

(a) $f \in \mathcal{B}_{\omega,0}^{\frac{3}{2q}(\alpha q+4)}$.
(b) *For each* $1 \le \alpha < \infty$, *and* $0 < q < \infty$

$$\lim_{|a|\to 1^-} \int_{\mathbb{B}_1(0)} | \overline{D}f(x) |^q \frac{(1- | x |^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1- | x |)} \big(G(x,a)\big)^3 d\mathbb{B}_x = 0.$$

(c) *For each* $1 \le \alpha < \infty$, *and* $0 < q < \infty$

$$\lim_{|a|\to 1^-} \int_{\mathcal{M}(a,R)} | \overline{D}f(x) |^q \frac{(1- | x |^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1- | x |)} d\mathbb{B}_x = 0.$$

(d) *For each* $1 \le \alpha < \infty$, *and* $0 < q < \infty$

$$\lim_{|a|\to 1^-} \frac{|\mathcal{M}(a,R)|^{\frac{\alpha q+4}{2}}}{\omega^q(1- | a |)} \int_{\mathcal{M}(a,R)} | \overline{D}f(x) |^q d\mathbb{B}_x = 0.$$

The following theorem gives another relation between the quaternion $\mathcal{B}_\omega^\alpha$ space and the quaternion valued-functions space $\mathbf{B}_{\alpha,\omega}^q(G)$.

Theorem 7 *Let* f *be a hyperholomorphic function in* $B_1(0)$. *Then for* $1 \le< \alpha < \infty$, $0 < q < \infty$, *and* $\omega : (0,1] \to (0,\infty)$, *we have*

$$\|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \approx \int_{\mathbb{B}_1(0)} | \overline{D}f(x) |^q \frac{(1- | x |^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1- | x |)} (1- | \varphi_a(x) |^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x.$$

Proof From Theorem 5, we have

$$\|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \approx \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} d\mathbb{B}_x.$$

Let the constant

$$C(R) = (1-R^2)^\beta \left(\frac{1-R^2}{1+R}\right)^3,$$

since $C(R)$ depending on R is finite, then

$$\begin{aligned} &\|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \\ &\approx \sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-R^2)^\beta \left(\frac{1-R^2}{1+R}\right)^3 d\mathbb{B}_x. \end{aligned} \tag{3}$$

Since $x \in \mathcal{M}(a, R)$, then $|\varphi_a(x)| < R$, $|\varphi_a(x)|^2 < R^2$, and $1-|\varphi_a(x)|^2 > 1-R^2$.
Then, we have

$$\begin{aligned} &\|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \\ &\approx \sup_{a\in\mathbb{B}_1(0)} \int_{\mathcal{M}(a,R)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-R^2)^\beta \left(\frac{1-R^2}{1+R}\right)^3 d\mathbb{B}_x \\ &\leq \lambda \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-|\varphi_a(x)|^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x. \end{aligned} \tag{4}$$

Conversely, we have

$$\begin{aligned} &\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-|\varphi_a(x)|^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x \\ &\leq \|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2}(\alpha q+4)}} \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} (1-|\varphi_a(x)|^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x. \end{aligned} \tag{5}$$

Using (1) and (2), we obtain

$$\begin{aligned} &\sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} |\overline{D}f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-|\varphi_a(x)|^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x \\ &\leq \|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \frac{(1-|x|^2)^\beta(1-|a|^2)^\beta}{(1-|x|)^\beta} \frac{(1-|x|^2)^3(1-|a|^2)^3}{(1-|a|)^6(1-|x|)^3} d\mathbb{B}_x. \\ &\leq 2^{2(\beta+3)} \|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} (1-|a|)^{\beta-3} d\mathbb{B}_x. \end{aligned} \tag{6}$$

Then, we obtain

$$\begin{aligned} &\|f\|^q_{\mathcal{B}_\omega^{\frac{3}{2q}(\alpha q+4)}} \\ &\geq \lambda \sup_{a\in\mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \overline{D}|f(x)|^q \frac{(1-|x|^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-|x|)} (1-|\varphi_a(x)|^2)^\beta \big(G(x,a)\big)^3 d\mathbb{B}_x. \end{aligned} \tag{7}$$

From (4) and (7), the proof is complete. □

4 Power series expansions of hyperholomorphic functions in $\mathbb{R}^3$

The major difference to power series in the complex case consists in the absence of regularity of the basic variable $x = x_0 + x_1 i + x_2 j$ and of all of its natural powers x^n, $n = 2, 3, \ldots$. This means that we should expect other types of terms, which could be designated as generalized powers. We use a pair $\underline{y} = (y_1, y_2)$ of two regular variables given by

$$y_1 = x_1 - i x_0 \quad and \quad y_2 = x_2 - j x_0,$$

and a multi-index $\nu = (\nu_1, \nu_2)$, $|\nu| = (\nu_1 + \nu_2)$ to define the ν-power of $\underline{y}$ by a $|\nu|$-ary product (see [10, 14, 21]).

Definition 8 Let ν_1 elements of the set $a_1, \ldots, a_{|\nu|}$ be equal to y_1 and ν_2 elements be equal to y_2. Then the ν-power of $\underline{y}$ is defined by

$$\underline{y} := \frac{1}{|\nu|!} \sum_{(i_1,\ldots,i_{|\nu|}) \in \pi(1,\ldots|\nu|)} a_{i1} a_{i2} \ldots a_{i_{|\nu|}}, \tag{8}$$

where the sum runs over all permutations of $(1, \ldots, |\nu|)$.

It was shown in [21], that the general form of the Taylor series of left monogenic functions in the neighborhood of the origin is given by

$$P(\underline{y}) := \sum_{n=0}^{\infty} \left(\sum_{|\nu|=n} \underline{y}^{\nu} c_{\nu} \right), \quad with\ c_{\nu} \in \mathbb{H}. \tag{9}$$

The following results, we will need in the following section:

Theorem 8 (See [12, 14]) *Let $g(x)$ be left hyperholomorphic in a neighborhood of the origin with the Taylor series given in the form (9). Then there holds*

$$\left| \frac{1}{2} \overline{D} g(x) \right| \leq \sum_{n=1}^{\infty} n \left(\sum_{|\nu|=n} |c_{\nu}| \right) |x|^{n-1}. \tag{10}$$

In order to formulate the next theorem we added the abbreviated notation $\mathbf{H}_n(x) := \sum_{|\nu|=n} \underline{y}^{|\nu|} c_{\nu}$ for such a homogeneous monogenic polynomial of degree n and consider monogenic functions composed by $\mathbf{H}_n(x)$ in the following form:

$$f(x) = \sum_{n=0}^{\infty} \mathbf{H}_n(x) b_n, \quad b_n \in \mathbb{H}.$$

Taking into account formula (10), we see that

$$\left| \frac{1}{2} \overline{D} f(x) \right| \leq \sum_{n=1}^{\infty} n \left(\sum_{|\nu|=n} |c_{\nu}| \right) |b_n| \, |x|^{n-1}. \tag{11}$$

This is the motivation for another shorthand notation, namely,

$$a_n := \left(\sum_{|\nu|=n} |c_{\nu}| \right) |b_n| \quad (a_n \geq 0),$$

finally, we have

$$\left| \frac{1}{2} \overline{D} f(x) \right| \leq \sum_{n=1}^{\infty} n a_n |x|^{n-1}. \tag{12}$$

5 Lacunary series expansions in $\mathbf{B}^{q}_{\alpha,\omega}(G)$ spaces

In this section, we obtain a sufficient and necessary condition for any hyperholomorphic function f on the unite ball $\mathbb{B}_1(0)$ of $\mathbb{R}^3$ with Hadamard gaps to belong to the weighted hyperholomorphic $\mathbf{B}^{q}_{\alpha,\omega}(G)$ spaces. The function

$$f(r) = \sum_{k}^{\infty} a_k r^{n_k} \quad (n_k \in \mathbb{N};\ \forall\, k \in \mathbb{N}) \tag{13}$$

is said to belong to the Hadamard gap class (Lacunary series) if there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geq \lambda,\ \forall\, k \in \mathbb{N}$. In the past few decades both Taylor and Fourier series expansions were studied by the help of lacunary series (see [6, 7, 18, 20, 25, 27]). On the other hand there are some characterizations in higher dimensions using several complex variables and quaternion sense (see [1, 9, 19]).

Theorem 9 *Let*

$$f(r) = \sum_{n=1}^{\infty} a_n r^n,$$

with $a_n \geq 0$. *If* $\alpha > 0,\ p > 0$ *and* $\omega : (0, 1] \to (0, \infty)$, *then*

$$\int_0^1 (1-r)^{\alpha-1} (f(r))^p \frac{1}{\omega^p(\log \frac{1}{r})}\, dr \approx \sum_{n=0}^{\infty} 2^{-n\alpha}\, t_n^p \frac{1}{\omega^p(\frac{1}{2^n})}, \tag{14}$$

where $t_n = \sum_{k \in I_n} a_k,\ n \in \mathbb{N},\ I_n = \{k : 2^n \leq k < 2^{n+1};\ \ k \in \mathbb{N}\}$.

Proof The proof of this theorem can be obtained easily from Theorem 2.1 in [7] with the same steps, so it will be omitted.

It should be remarked that using simple computations will allow that Theorem 9 is still satisfying for the function $f(r) = \sum_{n=1}^{\infty} a_n r^{n-1}$. □

Theorem 10 *Let* $1 \leq \alpha < \infty$, $1 < q < \infty$, $\omega : (0, 1] \to (0, \infty)$ *and* $I_n = \{k : 2^n \leq k < 2^{n+1}; k \in \mathbb{N}\}$. *Suppose that* $f(x) = \sum_{n=0}^{\infty} H_n(x) b_n,\ b_n \in \mathbb{H}$, *where* $H_n(x)$ *be homogenous hyperholomorphic polynomials of degree* n, *and let* a_n *be defined as before. If*

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q - q + 4)} \left(\frac{\sum_{k \in I_n} |\, a_k\,|}{\omega(\frac{1}{2^n})} \right)^q < \infty, \tag{15}$$

then

$$\sup_{a \in \mathbb{B}_1(0)} \int_{\mathbb{B}_1(0)} \left| \frac{1}{2} \overline{D} f(x) \right|^q \frac{(1 - |\, x\,|^2)^{\frac{3}{2}(\alpha q + 4)}}{\omega^q(1 - |\, x\,|)} \big(G(x, a)\big)^3 d\mathbb{B}_x < \infty, \tag{16}$$

and $f \in \mathbf{B}^{q}_{\alpha,\omega}(G)$.

Proof Suppose that (15) is hold. Using equality (1) and inequality (2), we have

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \left| \frac{1}{2}\overline{D} f(x)\right|^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&= \int_{\mathbb{B}_1(0)} \left| \frac{1}{2}\overline{D}\Big(\sum_{n=0}^{\infty} H_n(x) b_n\Big)\right|^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \frac{(1-\mid a\mid^2)^3(1-\mid x\mid^2)^3}{\mid 1-\overline{a}x\mid^9} d\mathbb{B}_x \\
&\le \int_{\mathbb{B}_1(0)} \Big(\sum_{n=0}^{\infty} n a_n x^{n-1}\Big)^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \frac{(1-\mid a\mid^2)^3(1-\mid x\mid^2)^3}{(1-\mid a\mid)^3(1-\mid x\mid)^6} d\mathbb{B}_x \\
&\le 2^{\frac{3}{2}(\alpha q+8)} \int_0^1 \Big(\sum_{n=0}^{\infty} n a_n r^{n-1}\Big)^q \frac{(1-r)^{\frac{3}{2}(\alpha q+2)}}{\omega^q(1-r)} r^2 dr \\
&\le \lambda \int_0^1 \Big(\sum_{n=0}^{\infty} n a_n r^{n-1}\Big)^q \frac{(1-r)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-r)} dr. \qquad (17)
\end{aligned}$$

Using Theorem 9 in (17), we deduced that

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \left| \frac{1}{2}\overline{D} f(x)\right|^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\le \lambda \int_0^1 \Big(\sum_{n=0}^{\infty} n a_n r^{n-1}\Big)^q \frac{(1-r)^{\frac{3}{2}(\alpha q+2)}}{\omega^q(1-r)} dr \\
&\le \lambda \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+4)} t_n^q \frac{1}{\omega^q(\frac{1}{2^n})}. \qquad (18)
\end{aligned}$$

Since,

$$t_n = \sum_{k\in I_n} k a_k < 2^{n+1} \sum_{k\in I_n} a_k,$$

we obtain that,

$$\begin{aligned}
&\int_{\mathbb{B}_1(0)} \left| \frac{1}{2}\overline{D} f(x)\right|^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)} \big(G(x,a)\big)^3 d\mathbb{B}_x \\
&\le \lambda_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}-q+4)} \left(\frac{\sum_{k\in I_n}\mid a_k\mid}{\omega(\frac{1}{2^n})}\right)^q.
\end{aligned}$$

Therefore, we have

$$\|f\|^q_{\mathbf{B}^q_{\alpha,\omega}(G)} \le \lambda_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}-q+4)} \left(\frac{\sum_{k\in I_n}\mid a_k\mid}{\omega(\frac{1}{2^n})}\right)^q < \infty,$$

where λ_1 is constant. Then, the last inequality implies that $f \in \mathbf{B}^q_{\alpha,\omega}(G)$ and the proof of our theorem is completed. □

For the converse direction of Theorem 10, we consider the following theorem. We will restrict ourselves to special monogenic homogeneous polynomials of the form

$$\mathrm{H}_{n,\alpha}(x) = (y_1\alpha_1 + y_2\alpha_2)^n = \sum_{k=0}^{n} \binom{n}{k} y_1^{n-k} \times y_2^k \times \alpha_1^{n-k}\alpha_2^k, \tag{19}$$

where $\alpha_i \in \mathbb{R},\ i = 1, 2$. The hypercomplex derivative is given by

$$\left(-\frac{1}{2}\overline{D}\right)\mathrm{H}_{n,\alpha}(x) = n\mathrm{H}_{n-1,\alpha}(x)(\alpha_1 i + \alpha_2 j). \tag{20}$$

Proposition 11 (See [14]) *Let* $\alpha = (\alpha_1, \alpha_2), \alpha_i \in \mathbb{R},\ i = 1, 2$ *be the vector of real coefficients defining the monogenic homogeneous polynomial* $\mathrm{H}_{n,\alpha}(x) = (y_1\alpha_1 + y_2\alpha_2)^n$. *Suppose that* $\mid \alpha \mid^2 = \alpha_1^2 + \alpha_2^2 \neq 0$. *Then,*

$$\|\mathrm{H}_{n,\alpha}\|^p_{L_p(\partial\mathbb{B}_1)} = 2\pi\sqrt{\pi} \mid \alpha \mid^{np} \frac{\Gamma(\frac{n}{2}p + 1)}{\Gamma(\frac{n}{2}p + \frac{3}{2})}, \quad \text{where } 0 < p < \infty. \tag{21}$$

Using formula (20), we have

$$\frac{\| -\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}\|^p_{L_p(\partial\mathbb{B}_1)}}{\|\mid \mathrm{H}_{n,\alpha} \|^p_{L_p(\partial\mathbb{B}_1)}} = n^p \frac{\mathbf{B}\left(\frac{1}{2}, \frac{n-1}{2}p + 1\right)}{\mathbf{B}\left(\frac{1}{2}, \frac{n}{2}p + 1\right)} \geq \lambda n^p > 0, \tag{22}$$

(see [14]), where,

$$\mathbf{B}\left(\frac{1}{2}, \frac{n-1}{2}p + 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2}p + 1)}{\Gamma(\frac{n-1}{2}p + \frac{3}{2})},$$

and

$$\lim_{n\to\infty} \frac{\mathbf{B}\left(\frac{1}{2}, \frac{n-1}{2}p + 1\right)}{\mathbf{B}\left(\frac{1}{2}, \frac{n}{2}p + 1\right)} = 1.$$

Corollary 3 (See [14]) *Assume that* $p \geq 2$. *Then,*

$$\frac{\| -\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}\|^2_{L_2(\partial\mathbb{B}_1)}}{\|\mathrm{H}_{n,\alpha}\|^2_{L_p(\partial\mathbb{B}_1)}} \geq \lambda n^{\frac{2+3p}{2p}}. \tag{23}$$

Theorem 12 *Let* $1 < \alpha < \infty$, $2 \leq q < \infty$, $\omega : (0, 1] \to (0, \infty)$, *and* $0 < \mid x \mid = r < 1$. *If*

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{\mathrm{H}_{n,\alpha}}{(1 - \mid x \mid^2)^{\frac{24+q}{4q}} \|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}} a_n\right) \in \mathbf{B}^q_{\alpha,\omega}(G). \tag{24}$$

Then,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q - q + 4)} \left(\frac{\sum_{k\in I_n} \mid a_k \mid}{\omega(\frac{1}{2^n})}\right)^q < \infty. \tag{25}$$

Proof

$$
\begin{aligned}
\|f\|_{\mathbf{B}^q_{\alpha,\omega}(G)} &= \sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)} \mid \overline{D}f(x)\mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)}\big(G(x,a)\big)^3 d\mathbb{B}_x \\
&= \sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)} \mid \overline{D}f(x)\mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+4)}}{\omega^q(1-\mid x\mid)}\left(\frac{(1-\mid x\mid^2)(1-\mid a\mid^2)}{\mid 1-\overline{a}x\mid^3}\right)^3 d\mathbb{B}_x \\
&\geq \sup_{a\in\mathbb{B}_1(0)}\int_{\mathbb{B}_1(0)} \mid \overline{D}f(x)\mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-\mid x\mid)} d\mathbb{B}_x \quad (where\ a=0).
\end{aligned}
\tag{26}
$$

Hence, we have

$$
\begin{aligned}
&\|f\|_{\mathbf{B}^q_{\alpha,\omega}(G)} \\
&\geq \int_{\mathbb{B}_1(0)} \mid -\frac{1}{2}\overline{D}f(x)\mid^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-\mid x\mid)} d\mathbb{B}_x \quad (where\ a=0) \\
&= \int_{\mathbb{B}_1(0)} \left|\sum_{n=0}^{\infty}\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{(1-|x|^2)^{\frac{24+q}{4q}}\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}}\right]a_n\right|^q \frac{((1-|x|^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q((1-|x|^2)} d\mathbb{B}_x.
\end{aligned}
\tag{27}
$$

Where $\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}}\right]$ is a homogeneous hyperholomorphic polynomial of degree $n-1$ and it can be written in the form

$$
\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}}\right] = r^{(n-1)}\Phi_n(\phi_1,\phi_2),
\tag{28}
$$

where

$$
\Phi_n(\phi_1,\phi_2) := \left(\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}}\right]\right)_{\partial\mathbb{B}_1}.
\tag{29}
$$

Now, using the quaternion-valued inner product

$$
\langle f,g\rangle_{\partial\mathbb{B}_1(0)} = \int_{\partial\mathbb{B}_1(0)} \overline{f(x)}g(x)d\Gamma_x,
$$

the orthogonality of the spherical monogenic $\Phi_n(\phi_1,\phi_2)$ (see [3]) in $L_2(\partial\mathbb{B}_1(0))$. Then, substituting from (28) and (29) to (27), we obtain

$$
\begin{aligned}
&\int_{\mathbb{B}_1(0)} \left|\sum_{n=0}^{\infty}\left[\frac{-\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha}}{(1-\mid x\mid^2)^{\frac{24+q}{4q}}\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}}\right]a_n\right|^q \frac{(1-\mid x\mid^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-\mid x\mid)} d\mathbb{B}_x \\
&= \int_0^1\int_{\partial\mathbb{B}_1(0)} \left(\left|\sum_{n=0}^{\infty}\frac{r^{n-1}}{(1-r^2)^{\frac{24+p}{4q}}}\Phi_n(\phi_1,\phi_2)a_n\right|^2\right)^{\frac{q}{2}} r^2\frac{(1-r^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-r)} d\Gamma_x dr \\
&= \int_0^1\int_{\partial\mathbb{B}_1(0)} \left(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\overline{a_n}\frac{r^{2n-2}}{(1-r^2)^{\frac{24+q}{2q}}}\overline{\Phi_n}(\phi_1,\phi_2)\Phi_j(\phi_1,\phi_2)a_j\right)^{\frac{q}{2}} \\
&\quad \times r^2\frac{(1-r^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-r)} d\Gamma_x dr = \mathrm{L}.
\end{aligned}
\tag{30}
$$

From Hölder's inequality, we have

$$\int_{\partial\mathbb{B}_1(0)} \mid f(x) \mid^q d\Gamma_x \geq (4\pi)^{1-q} \left| \int_{\partial\mathbb{B}_1(0)} f(x) d\Gamma_x \right|^q, \quad (where\ 1 \leq q < \infty). \tag{31}$$

From (31), for $2 \leq q < \infty$, we have

$$\begin{aligned} \mathrm{L} &\geq (4\pi)^{1-\frac{q}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} \mid a_n \mid^2 \frac{r^{2n-2}}{(1-r^2)^{\frac{24+q}{2q}}} \mid \Phi_n(\phi_1, \phi_2) \mid^2_{L_2(\partial\mathbb{B}_1)} \right)^{\frac{q}{2}} r^2 \frac{(1-r^2)^{\frac{3}{2}(\alpha q+6)}}{\omega^q(1-r)} dr \\ &\geq (4\pi)^{1-\frac{q}{2}} \int_0^1 \left(\sum_{n=0}^{\infty} \mid a_n \mid^2 r^{2n-2} \mid \Phi_n(\phi_1, \phi_2) \mid^2_{L_2(\partial\mathbb{B}_1)} \right)^{\frac{q}{2}} r^3 \frac{(1-r^2)^{\frac{3\alpha q}{2}+3-\frac{p}{4}}}{\omega^q(1-r)} dr. \end{aligned} \tag{32}$$

From Corollary 3, we have

$$\mid \Phi_n(\phi_1, \phi_2) \mid^2_{L_2(\partial\mathbb{B}_1)} = \frac{\| -\frac{1}{2}\overline{D}\mathrm{H}_{n,\alpha} \|_{L_2(\partial\mathbb{B}_1)}}{\|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}} \geq \lambda n^{\frac{2+3q}{2q}} \geq \lambda n^{\frac{3}{2}}.$$

Then, from above we have

$$\begin{aligned} \mathrm{L} &\geq (4\pi)^{1-\frac{q}{2}} \lambda \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} \mid a_n \mid^2 r^{2n-2} \right)^{\frac{q}{2}} r^3 \frac{(1-r^2)^{\frac{3\alpha q}{2}+3-\frac{q}{4}}}{\omega^q(1-r)} dr \\ &= \lambda_1 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} \mid a_n \mid^2 r^{2n-2} \right)^{\frac{q}{2}} r^3 \frac{(1-r^2)^{\frac{3\alpha q}{2}+3-\frac{q}{4}}}{\omega^q(1-r)} dr \\ &= \frac{\lambda_1}{2} \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} \mid a_n \mid^2 \xi^{n-1} \right)^{\frac{q}{2}} \xi \frac{(1-\xi)^{\frac{3\alpha q}{2}+3-\frac{q}{4}}}{\omega^q(1-\sqrt{\xi})} d\xi \\ &\geq \lambda_3 \int_0^1 \left(\sum_{n=0}^{\infty} n^{\frac{3}{2}} \mid a_n \mid^2 \xi^{n-1} \right)^{\frac{q}{2}} \frac{(1-\xi)^{\frac{3\alpha q}{2}+3-\frac{q}{4}}}{\omega^q(1-\xi)} d\xi, \end{aligned} \tag{33}$$

where $\lambda_j,\ j = 1, 2, 3$, are constants not depending on n.

Now, we apply Theorem 9 in Eq. (33), we deduced that

$$\|f\|_{\mathbf{B}^q_{\alpha,\omega}(G)} \geq \mathrm{L} \geq \frac{\lambda_3}{k} \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+4-\frac{q}{4})} \left(\frac{\sum_{k\in I_n} k^{\frac{3}{2}} \mid a_k \mid^2}{\omega^2(\frac{1}{2^n})} \right)^{\frac{q}{2}}. \tag{34}$$

Where

$$\sum_{k\in I_n} k^{\frac{3}{2}} \mid a_k \mid^2 > \left(2^n\right)^{\frac{3}{2}} \left(\sum_{k\in I_n} \mid a_k \mid^2 \right)^{\frac{q}{2}}.$$

Then,

$$\|f\|_{\mathbf{B}^q_{\alpha,\omega}(G)} \geq \mathrm{L} \geq C \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+4-q)} \left(\frac{\sum_{k\in I_n} \mid a_k \mid^2}{\omega^2(\frac{1}{2^n})} \right)^{\frac{q}{2}}, \tag{35}$$

where C be a constant not depending on n.

From [22], we have

$$\sum_{n=0}^{N} a_n^q \leq \left(\sum_{n=0}^{N} a_n^q \right)^q \leq N^{q-1} \sum_{n=0}^{N} a_n^q.$$

Then, we have

$$\| f \|_{\mathbf{B}^q_{\alpha,\omega}(G)} \geq \mathrm{L} \geq C_1 \sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+4-q)} \left(\frac{\sum_{k\in I_n} \mid a_k \mid}{\omega(\frac{1}{2^n})} \right)^q, \tag{36}$$

where C_1 be a constant not depending on n. Then, we deduced that

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3\alpha q}{2}+4-q)} \left(\frac{\sum_{k\in I_n} \mid a_k \mid}{\omega(\frac{1}{2^n})} \right)^q < \infty. \tag{37}$$

This completes the proof of theorem. □

Theorem 13 *Let* $1 < \alpha < \infty$, $2 \leq q < \infty$, $\omega : (0, 1] \rightarrow (0, \infty)$, *and* $0 < \mid x \mid = r < 1$, *we have that*

$$f(x) = \left(\sum_{n=0}^{\infty} \frac{\mathrm{H}_{n,\alpha}}{(1 - \mid x \mid^2)^{\frac{24+q}{4q}} \|\mathrm{H}_{n,\alpha}\|_{L_q(\partial\mathbb{B}_1)}} a_n \right) \in \mathbf{B}^q_{\alpha,\omega}(G), \tag{38}$$

if and only if,

$$\sum_{n=0}^{\infty} 2^{-n(\frac{3}{2}\alpha q+4-q)} \left(\frac{\sum_{k\in I_n} \mid a_k \mid}{\omega(\frac{1}{2^n})} \right)^q < \infty. \tag{39}$$

Proof This theorem can be proved directly from Theorems 10 and 12. □

6 Conclusion

The aim of this article was to introduce a new generalized quaternion space $\mathbf{B}^q_{\alpha,\omega}(G)$ and study relations to the quaternion $\mathcal{B}^\alpha_\omega$ space. Furthermore, we considered some essential properties of $\mathbf{B}^q_{\alpha,\omega}(G)$ spaces of quaternion-valued function as basic scale properties. Second, characterizations of the hyperholomorphic $\mathbf{B}^q_{\alpha,\omega}(G)$ functions by the coefficients of certain lacunary series expansions in clifford analysis are obtained.

Acknowledgements We would like to express our gratitude to the anonymous referees for their helpful suggestions and corrections.The authors declare that they have no competing interests.

Author Contributions The authors contributed equally in writing the final version of this article. All authors read and approved the final manuscript. Please refer to Journal-level guidance for any specific requirements.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Avetsiyan, K.L.: Hardy–Bloch type spaces and lacunary series in the polydisk. Glasg. Math. J. **49**(2), 345–356 (2007)
2. Bernstein, S.: Harmonic Q_p spaces. Comput. Methods Funct. Theory **9**(1), 285–304 (2009)
3. Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis, Research Notes in Mathematics, vol. 76. Pitman Advanced Publishing Program, London (1982)

4. Dyakonov, K.M.: Weighted Bloch spaces, H^p and $BMOA$. J. Lond. Math. Soc. II. Ser. **65**(2), 411–417 (2002)
5. El-Sayed Ahmed, A.: On weighted α-Besov spaces and α-Bloch spaces of quaternion-valued functions. Numer. Funct. Anal. Optim. **29**(9–10), 1064–1081 (2008)
6. El-Sayed Ahmed, A., Kamal, A.: Series expansions some analytic function spaces. J. Comput. Theor. Nanosci. **12**(8), 1586–1593 (2015)
7. El-Sayed Ahmed, A., Kamal, A., Yassen, T.I.: Characterizations for $Q_{K,\omega}(p,q)$ type functions by series expansions with Hadamard gaps. CUBO Math. J. **01**, 81–93 (2014)
8. El-Sayed Ahmed, A., Asiri, F.: Characterizations of weighted Bloch space by $Q_{p,\omega}$-type spaces of quaternion-valued functions. J. Comput. Theor. Nanosci. **12**, 4250–4255 (2015)
9. Furdui, O.: On a class of lacunary series in BMOA. J. Math. Anal. Appl. **342**(2), 773–779 (2008)
10. Gürlebeck, K., Sprössig, W.: Quaternionic Analysis and Elliptic Boundary Value Problems. International Series of Numerical Mathematics, vol. 89. Birkhäuser, Basel (1990)
11. Gürlebeck, K., Kähler, U., Shapiro, M., Tovar, L.M.: On $\mathbf{Q}_p$ spaces of quaternion-valued functions. Complex Var. **39**, 115–135 (1999)
12. Gürlebeck, K., Malonek, H.R.: On strict inclusions of weighted Dirichlet spaces of monogenic functions. Bull. Aust. Math. Soc. **64**, 33–50 (2001)
13. Gürlebeck, K., El-Sayed Ahmed, A.: Integral norms for hyperholomorphic Bloch-functions in the unit ball of $\mathbb{R}^3$. In: Begehr et al. (eds.) Progress in Analysis. Kluwer Academic, Dordrecht, pp. 253–263 (2003)
14. Gürlebeck, K., El-Sayed Ahmed, A.: On series expansions of hyperholomorphic B^q functions. Trends Math. Adv. Anal. Geom. 113–129 (2004)
15. Gürlebeck, K., El-Sayed Ahmed, A.: On B^q spaces of hyperholomorphic functions and the Bloch space in R^3. In: Finite or Infinity Dimensional, Complex Analysis and Its Applications, pp. 269–286 (2004)
16. Kamal, A., Ahmed, A., Yassen, T.I.: Quasi-metric spaces and composition operators on $\mathcal{B}^*_{\alpha,\log}$ and $Q*_{p,\log}$ spaces. J. Comput. Theor. Nanosci. **2**(8), 1795–1801 (2015)
17. Kamal, A., Yassen, T.I.: Existence solutions of the complex linear differential equations in $Q_{K,\omega}(p,q)$ spaces. Gen. Lett. Math. **3**(1), 47–56 (2017)
18. Kamal, A., Yassen, T.I.: A property of hyperbolic general family of function spaces with Hadamard gap series. J. Adv. St. Topol. 1–9 (2018)
19. Li, S., Stević, S.: Weighted-Hardy functions with Hadamard gaps on the unit ball. Appl. Math. Comput. **212**(1), 229–233 (2009)
20. Meng, X.: Some sufficient conditions for analytic functions to belong to $Q_{K,0}(p,q)$ space1. Abstr. Appl. Anal. **2008**, 404636 (2008)
21. Malonek, M.: Power series representation for monogenic functions in R^{m+1} based on a permutational product. Complex Var. Theory Appl. **15**, 181–191 (1990)
22. Miao, J.: A property of analytic functions with Hadamard gaps. Bull. Aust. Math. Soc. **45**, 105–112 (1992)
23. Rashwan, R.A., El-Sayed Ahmed, A., Kamal, A.: Integral characterizations of weighted Bloch spaces and $Q_{K,\omega}(p,q)$ spaces. Math. Tome **51**(74)(2), 63–76 (2009)
24. Reséndis, L. F., Tovar, L. M.: Besov-type characterizations for quaternionic Bloch functions. In: Son, L.H., et al. (eds.) Finite or Infinite Complex Analysis and Its Applications. Advances in Complex Analysis and Applications, pp. 207–220. Kluwer Academic Publishers, Boston (2004)
25. Stević, S.: Bloch-type functions with Hadamard gaps. App. Math. Comput. **208**, 416–422 (2009)
26. Stroethoff, K.: Besov-type characterizations for the Bloch space. Bull. Aust. Math. Soc. **39**, 405–420 (1989)
27. Wulan, H., Zhu, K.: Lacunary series in Q_K spaces. Stud. Math. **178**, 217–230 (2007)