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On 3-Lie algebras with a derivation

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Abstract
In this paper, we study 3-Lie algebras with derivations. We call the pair consisting of a 3-Lie algebra and a distinguished derivation by the 3-LieDer pair. We define a cohomology theory for 3-LieDer pair with coefficients in a representation. We study central extensions of a 3-LieDer pair and show that central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation. We generalize Gerstenhaber's formal deformation theory to 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivation.

Keywords 3-Lie algebra · Derivation · Representation · Cohomology · Central extension · Deformation

Mathematics Subject Classification 17A42 · 17B10 · 17B40 · 17B56

1 Introduction

3-Lie algebras are special types of n-Lie algebras and have close relationships with many important fields in mathematics and mathematical physics [4, 5]. The structure of 3-Lie algebras is closely linked to the supersymmetry and gauge symmetry transformations of the world-volume theory of multiple coincident $M2$-branes and is applied to the study of the Bagger-Lambert theory. Moreover, the n-Jacobi identity can be regarded as a generalized Plucker relation in the physics literature. In particular, the metric 3-Lie algebras, or more generally, the 3-Lie algebras with invariant symmetric bilinear forms attract even more attention in physics. Recently, many more properties and structures of 3-Lie algebras have been developed, see [6, 8, 16, 22, 27, 31, 32] and references cited therein.

Derivations on algebraic structures were first started by Ritt [25] for commutative algebras and field. The structure is called a differential (commutative) algebra. For the notion of differential n-Lie algebras and related structures, see [7, 10, 21]. Derivations of types of algebra provide many important aspects of the algebraic structure. For example, Coll, Gertstenhaber,

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and Giaquinto [11] described explicitly a deformation formula for algebras whose Lie algebra of derivations contains the unique non-abelian Lie algebra of dimension two. Amitsur [1, 2] studied derivations of central simple algebras. Derivations are also used to construct homotopy Lie algebras [30] and play an important role in the study of differential Galois theory [24]. One may also look at some interesting roles played by derivations in control theory and gauge theory in quantum field theory [3]. In [15, 23], the authors studied algebras with derivations from an operadic point of view. Recently, Lie algebras with derivations (called LieDer pairs) are studied from a cohomological point of view [29] and extensions, deformations of LieDer pairs are considered. The results of [29] have been extended to associative algebras and Leibniz algebras with derivations in [12] and [13].

The deformation is a tool to study a mathematical object by deforming it into a family of the same kind of objects depending on a certain parameter. The deformation theory was introduced by Gerstenhaber for rings and algebras [18, 19], and by Zhang for 3-Lie color algebras [32]. They studied 1-parameter formal deformations and established the connection between the cohomology groups and infinitesimal deformations. Motivated by Tang's [29] terminology of LieDer pairs. Due to the importance of 3-Lie algebras, cohomology, and deformation theories, Our main objective of this paper is to study the cohomology and deformation theory of 3-Lie algebra with a derivation.

The paper is organized as follows. In Sect. 2, we define a cohomology theory for 3-LieDer pair with coefficients in a representation. In Sect. 3, we study central extensions of a 3-LieDer pair and show that isomorphic classes of central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation. In Sect. 4, we study formal one-parameter deformations of 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivations.

Throughout this paper, we work over the field $\mathbb{F}$ of characteristics 0.

2 Cohomology of 3-LieDer pairs

In this section, we define a cohomology theory for 3-LieDer pair with coefficients in a representation.

Definition 2.1 [17] A 3-Lie algebra is a tuple $(L, [\cdot, \cdot, \cdot])$ consisting of a vector space L, a 3-ary skew-symmetric operation $[\cdot, \cdot, \cdot] : \wedge^3 L \to L$ satisfying the following Jacobi identity

$$[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]], \tag{2.1}$$

for any $x, y, u, v, w \in L$.

Definition 2.2 [20] A representation of a 3-Lie algebra $(L, [\cdot, \cdot, \cdot])$ on the vector space M is a linear map $\rho : L \wedge L \to \mathfrak{gl}(M)$, such that for any $x, y, z, u \in L$, the following equalities are satisfied

$$\rho([x, y, z], u) = \rho(y, z)\rho(x, u) + \rho(z, x)\rho(y, u) + \rho(x, y)\rho(z, u),$$
$$\rho(x, y)\rho(z, u) = \rho(z, u)\rho(x, y) + \rho([x, y, z], u) + \rho(z, [x, y, u]).$$

Then (M, ρ) is called a representation of L, or M is an L-module.

Definition 2.3 [17] Let $(L, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. A derivation on L is given by a linear map $\phi_L : L \to L$ satisfying

$$\phi_L([x, y, z]) = [\phi_L(x), y, z] + [x, \phi_L(y), z] + [x, y, \phi_L(z)], \quad \forall x, y, z \in L.$$

We call the pair (L, ϕ_L) of a 3-Lie algebra and a derivation by a 3-LieDer pair.

Remark 2.4 Let $(L, [\cdot, \cdot, \cdot])$ be a 3-Lie algebra. For all $x_1, x_2 \in L$, the map defined by

$$ad_{x_1,x_2}x := [x_1, x_2, x], \text{ for all } x \in L,$$

is called the adjoint map. From the Eq. 2.1, it is clear that ad_{x_1,x_2} is a derivation. The linear map $ad : L \wedge L \to \mathfrak{gl}(L)$ defines a representation of $(L, [\cdot, \cdot, \cdot])$ on itself. This representation is called the adjoint representation.

Definition 2.5 Let (L, ϕ_L) be a 3-LieDer pair. A representation of (L, ϕ_L) is given by (M, ϕ_M) in which M is a representation of L and $\phi_M : M \to M$ is a linear map satisfying

$$\phi_M(\rho(x, y)(m)) = \rho(\phi_L(x), y)(m) + \rho(x, \phi_L(y))(m) + \rho(x, y)(\phi_M(m)),$$

for all $x, y \in L$ and $m \in M$.

Proposition 2.6 *Let (L, ϕ_L) be a 3-LieDer pair and (M, ϕ_M) be a representation of it. Then $(L \oplus M, \phi_L \oplus \phi_M)$ is a 3-LieDer pair where the 3-Lie algebra bracket on $L \oplus M$ is given by the semi-direct product*

$$[(x, m), (y, n), (z, p)] = ([x, y, z], \rho(y, z)(m) + \rho(z, x)(n) + \rho(x, y)(p)),$$

for any $x, y, z \in L$ and $m, n, p \in M$.

Proof It is known that $L \oplus M$ equipped with the above product is a 3-Lie algebra. Moreover, we have

$$\begin{aligned}
&(\phi_L \oplus \phi_M)([(x, m), (y, n), (z, p)]) \\
&\quad = (\phi_L([x, y, z]), \phi_M(\rho(y, z)(m)) + \phi_M(\rho(z, x)(n)) + \phi_M(\rho(x, y)(p))) \\
&\quad = ([\phi_L(x), y, z], \rho(y, z)(\phi_M(m)) + \rho(\phi_L(x), z)(n) + \rho(\phi_L(x), y)(p)) \\
&\qquad +([x, \phi_L(y), z], \rho(\phi_L(y), z)(m) + \rho(z, x)(\phi_M(n)) + \rho(x, \phi_L(y))(p)) \\
&\qquad +([x, y, \phi_T(z)], \rho(y, \phi_L(z))(m) + \rho(z, \phi_L(x))(n) + \rho(x, y)(\phi_M(p))) \\
&\quad = [(\phi_L \oplus \phi_M)(x, m), (y, n), (z, p)] + [(x, m), (\phi_L \oplus \phi_M)(y, n), (z, p)] \\
&\qquad +[(x, m), (y, n), (\phi_L \oplus \phi_M)(z, p)].
\end{aligned}$$

Hence the proof is finished. □

Recall from [28] that let ρ be a representation of $(L, [\cdot, \cdot, \cdot])$ on M. Denote by $C^n(L, M)$ the set of all n-cochains and defined as

$$C^n(L, M) = \mathrm{Hom}((\wedge^2 L)^{\otimes n-1}, M), \ n \geq 1.$$

Let $d^n : C^n(L, M) \to C^{n+1}(L, M)$ be defined by

$$\begin{aligned}
&d^n f(X_1, \ldots, X_n, x_{n+1}) \\
&\quad = (-1)^{n+1}\rho(y_n, x_{n+1}) f(X_1, \ldots, X_{n-1}, x_n) \\
&\qquad + (-1)^{n+1}\rho(x_{n+1}, x_n) f(X_1, \ldots, X_{n-1}, y_n) \\
&\qquad + \sum_{j=1}^{n}(-1)^{j+1}\rho(x_j, y_j) f(X_1, \ldots, \hat{X}_j, \ldots, X_n, x_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}(-1)^j f(X_1,\dots,\hat{X}_j,\dots,X_n,[x_j,y_j,x_{n+1}]),\\
&+\sum_{1\le j<k\le n}(-1)^j f(X_1,\dots,\hat{X}_j,\dots,X_{k-1},[x_j,y_j,x_k]\wedge y_k\\
&+x_k\wedge[x_j,y_j,x_k],X_{k+1},\dots,X_n,x_{n+1}),
\end{aligned}
$$

for all $X_i = x_i \wedge y_i \in \otimes^2 L, i = 1, 2, \dots, n$ and $x_{n+1} \in L$, it was proved that $d^{n+1} \circ d^n = 0$. Therefore, $(C^*(L, M), d^*)$ is a cochain complex.

Observe that for trivial representation coboundary maps d^1 and d^2 are explicitly given as follows:

$$
\begin{aligned}
d^1(f)(a,b,c) &= [f(a),b,c]+[a,f(b),c]+[a,b,f(c)]-f([a,b,c]),\ \ f\in C^1(L,M).\\
d^2(f)(a,b,c,d,e) &= [a,b,f(c,d,e)]-f([a,b,c],d,e)+f(a,b,[c,d,e])\\
&\quad -[f(a,b,c),d,e],\ \ f\in C^2(L,M).
\end{aligned}
$$

In [26], the graded space $C^*(L,L) = \bigoplus_{n\ge 0} C^{n+1}(L,L)$ of cochain groups carries a degree -1 graded Lie bracket given by $[f,g] = f\circ g-(-1)^{mn} g\circ f$, for $f \in C^{m+1}(L,L)$, $g \in C^{n+1}(L,L)$, where $f \circ g \in C^{m+n+1}(L,L)$, and defined as follows:

$$
\begin{aligned}
&f\circ g(X_1,\dots,X_{m+n},x)\\
&=\sum_{k=1}^{m}(-1)^{(k-1)n}\sum_{\sigma\in\mathbb{S}(k-1,n)} f(X_{\sigma(1)},\dots,X_{\sigma(k-1)},g(X_{\sigma(k)},\cdots,X_{\sigma(k+n-1)},x_{k+n})\\
&\quad\wedge y_{k+n},X_{\sigma(k+n+1)},\dots,X_{\sigma(m+n)},x)\\
&\quad+\sum_{k=1}^{m}(-1)^{(k-1)n}\sum_{\sigma\in\mathbb{S}(k-1,n)}(-1)^{\sigma} f(X_{\sigma(1)},\dots,X_{\sigma(k-1)},x_{k+n}\\
&\quad\wedge g(X_{\sigma(k)},\dots,X_{\sigma(k+n-1)},y_{k+n}),X_{k+n+1},\dots,X_{m+n},x)\\
&\quad\sum_{\sigma\in\mathbb{S}(m,n)}(-1)^{mn}(-1)^{\sigma} f(X_{\sigma(1)},\dots,X_{\sigma(m)},g(X_{\sigma(m+1)},\dots,X_{\sigma(m+n-1)},X_{\sigma(m+n)},x)),
\end{aligned}
$$

for all $X_i = x_i \wedge y_i \in \otimes^2 L, i = 1, 2, \dots, m+n$ and $x \in L$. Here $\mathbb{S}(k-1, n)$ denotes the set of all $(k-1, n)$-shuffles. Moreover, $\mu : \otimes^3 L \to L$ is a 3-Lie bracket if and only if $[\mu, \mu] = 0$, i.e. μ is a Maurer–Cartan element of the graded Lie algebra $(C^*(L,L), [\cdot,\cdot]$. where μ is considered as an element in $C^2(L,L)$. With this notation, the differential (with coefficients in L) is given by

$$
df = (-1)^n[\mu, f], \quad \text{for all } f \in C^n(L,L).
$$

In the next, we introduce cohomology for a 3-LieDer pair with coefficients in a representation.

Let (L, ϕ_L) be a 3-LieDer pair and (M, ϕ_M) be a representation of it. For any $n \ge 2$, we define cochain groups for 3-LieDer pair as follows:

$$
C^n_{\text{3-LieDer}}(L,M) := C^n(L,M) \oplus C^{n-1}(L,M).
$$

Define the space $C^0_{\text{3-LieDer}}(L,M)$ of 0-cochains to be 0 and the space $C^1_{\text{3-LieDer}}(L,M)$ of 1-cochains to be $\mathrm{Hom}(L,M)$. Note that $\mu = [\cdot,\cdot,\cdot] \in C^2(L,L)$ and derivation $\phi_L \in C^1(L,L)$. Thus, the pair $(\mu, \phi_L) \in C^2_{\text{3-LieDer}}(L,L)$. To define the coboundary map for 3-LieDer pair, we need following map $\delta : C^n(L, M) \to C^n(L, M)$ by

$$\delta f = \sum_{i=1}^{n} f \circ (Id_L \otimes \cdots \otimes \phi_L \otimes \cdots \otimes Id_L) - \phi_M \circ f.$$

The following lemma shows maps ∂ and δ commute, and is useful to define the coboundary operator of the cohomology of 3-LieDer pair.

Lemma 2.7 *The map δ commute with d, i.e, $d \circ \delta = \delta \circ d$.*

Proof Note that in case of self representation, that is, when $(M, \phi_M) = (L, \phi_L)$, we have

$$\delta(f) = -[\phi_L, f], \text{ for all } f \in C^n(L, L).$$

Therefore, we have

$$\begin{aligned}(d \circ \delta)(f) &= -d[\phi_L, f]\\ &= (-1)^n[\mu, [\phi_L, f]]\\ &= (-1)^n[[\mu, \phi_L], f] + (-1)^n[\phi_L, [\mu, f]]\\ &= (-1)^n[\phi_L, [\mu, f]]\\ &= (\delta \circ \delta)(f)\end{aligned}$$

□

We are now in a position to define the cohomology of the 3-LieDer pair. We define a map $\partial : C^n_{\text{3-LieDer}}(L, M) \to C^{n+1}_{\text{3-LieDer}}(L, M)$ by

$$\partial f = (df, -\delta f), \quad \text{for all} \quad f \in C^1_{\text{3-LieDer}}(L, M),$$
$$\partial(f_n, \overline{f_n}) = (df_n, d\overline{f}_n + (-1)^n \delta f_n), \quad \text{for all} \quad (f_n, \overline{f}_n) \in C^n_{\text{3-LieDer}}(L, M).$$

Proposition 2.8 *The map ∂ satisfies $\partial \circ \partial = 0$.*

Proof For any $f \in C^1_{\text{3-LieDer}}(L, M)$, we have

$$(\partial \circ \partial)f = \partial(df, -\delta f) = ((d \circ d)f, -(d \circ \delta)f + (\delta \circ d)f) = 0.$$

Similarly, for any $(f_n, \overline{f}_n) \in C^n_{\text{3-LieDer}}(L, M)$, we have

$$\begin{aligned}(\partial \circ \partial)(f_n, \overline{f_n}) &= \partial(df_n, df_n + (-1)^n f_n)\\ &= (d^2 f_n, d^2\overline{f_n} + (-1)^n d\delta f_n + (-1)^{n+1}\delta df_n)\\ &= 0.\end{aligned}$$ □

Therefore, $(C^*_{\text{3-LieDer}}(L, M), \partial)$ forms a cochain complex. We denote the corresponding cohomology groups by $H^*_{\text{3-LieDer}}(L, M)$.

3 Central extensions of 3-LieDer pairs

In this section, we study central extensions of a 3-LieDer pair. Similar to the classical cases, we show that isomorphic classes of central extensions are classified by the second cohomology of the 3-LieDer pair with coefficients in the trivial representation.

Let (L, ϕ_L) be a 3-LieDer pair and (M, ϕ_M) be an abelian 3-LieDer pair i.e, the 3-Lie algebra bracket of M is trivial.

Definition 3.1 A central extension of (L, ϕ_L) by (M, ϕ_M) is an exact sequence of 3-LieDer pairs

$$0 \longrightarrow (M, \phi_M) \xrightarrow{\quad i \quad} (\hat{L}, \phi_{\hat{L}}) \xrightarrow{\quad p \quad} (L, \phi_L) \longrightarrow 0 \tag{3.1}$$

such that $[i(m), \hat{x}, \hat{y}] = 0$, for all $m \in M$ and $\hat{x}, \hat{y} \in \hat{L}$.

In a central extension, using the map i we can identify M with the corresponding subalgebra of $\hat{L}$ and with this $\phi_M = \phi_{\hat{L}}|_M$.

Definition 3.2 Two central extensions $(\hat{L}, \phi_{\hat{T}})$ and $(\hat{L}', \phi_{\hat{L}'})$ are said to be isomorphic if there is an isomorphism $\eta : (\hat{L}, \phi_{\hat{L}}) \to (\hat{L}', \phi_{\hat{L}'})$ of 3-LieDer pairs that makes the following diagram commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (M, \phi_M) & \xrightarrow{\quad i \quad} & (\hat{L}, \phi_{\hat{L}}) & \xrightarrow{\quad p \quad} & (L, \phi_L) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle Id_M} & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle Id_L} & & \\
0 & \longrightarrow & (M, \phi_M) & \xrightarrow{\quad i' \quad} & (\hat{L}', \phi_{\hat{L}'}) & \xrightarrow{\quad q \quad} & (L, \phi_L) & \longrightarrow & 0.
\end{array}$$

Let Eq. (3.1) be a central extension of (L, ϕ_L). A section of the map p is given by a linear map $s : L \to \hat{L}$ such that $p \circ s = Id_L$.

For any section s, we define linear maps $\psi : L \wedge L \wedge L \to M$ and $\chi : L \to M$ by

$$\psi(x, y, z) := [s(x), s(y), s(z)] - s([x, y, z]), \qquad \chi(x) = \phi_{\hat{L}}(s(x)) - s(\phi_L(x)), \quad \text{for all } x, y, z \in L.$$

Note that the vector space $\hat{L}$ is isomorphic to the direct sum $L \oplus M$ via the section s. Therefore, we may transfer the structures of $\hat{L}$ to $L \oplus M$. The product and linear maps on $L \oplus M$ are given by

$$[(x, m), (y, n), (z, p)]_\psi = ([x, y, z], \psi(x, y, z)),$$
$$\phi_{L \oplus M}(x, m) = (\phi_L(x), \phi_M(m) + \chi(x)).$$

Proposition 3.3 *The vector space $L \oplus M$ equipped with the above product and linear maps $\phi_{L \oplus M}$ forms a 3-LieDer pair if and only if (ψ, χ) is a 2-cocycle in the cohomology of the 3-LieDer pair (L, ϕ_L) with coefficients in the trivial representation M. Moreover, the cohomology class of (ψ, χ) does not depend on the choice of the section s.*

Proof The tuple $(L \oplus M, \phi_{L\oplus M})$ is a 3-LieDer pair if and only if the following equations holds:

$$[(x,m),(y,n),[(z,p),(v,k),(w,l)]_\psi]_\psi$$
$$= [[(x,m),(y,n),(z,p)]_\psi,(v,k),(w,l)]_\psi + [(z,p),[(x,m),(y,n),(v,k)]_\psi,(w,l)]_\psi$$
$$+[(z,p),(v,k),[(x,m),(y,n),(w,l)]_\psi]_\psi, \quad (3.2)$$

and, (3.3)

$$\phi_{L\oplus M}[(x,m),(y,n),(z,p)]_\psi$$
$$= [\phi_{L\oplus M}(x,m),(y,n),(z,p)]_\psi + [(x,m),\phi_{L\oplus M}(y,n),(z,p)]_\psi$$
$$+[(x,m),(y,n),\phi_{L\oplus M}(z,p)]_\psi, \quad (3.4)$$

for all $x\oplus m, y\oplus n, z\oplus p, v\oplus k, w\oplus l \in L\oplus M$. The condition Eq. (3.2) is equivalent to

$$\psi(x,y,[z,v,w]) = \psi([x,y,z],v,w) + \psi(z,[x,y,v],w) + \psi(z,v,[x,y,w]),$$

or, equivalently, $d(\psi) = 0$, as we are considering only trivial representation. The condition Eq. (3.3) is equivalent to

$$\phi_M(\psi(x,y,z)) + \chi([x,y,z]) = \psi(\phi_L(x),y,z) + \psi(x,\phi_L(y),z) + \psi(x,y,\phi_L(z)).$$

This is same as $d(\chi) + \delta\psi = 0$. This implies (ψ,χ) is a 2-cocycle.

Let s_1, s_2 be two sections of p. Define a map $u : L \to M$ by $u(x) := s_1(x) - s_2(x)$. Observe that

$$\begin{aligned}\psi(x,y,z) &= [s_1(x),s_1(y),s_1(z)] - s_1([x,y,z])\\ &= [s_2(x)+u(x),s_2(y)+u(y),s_2(z)+u(z)] - s_2([x,y,z]) - u([x,y,z])\\ &= \psi'(x,y,z) - u([x,y,z]),\end{aligned}$$

as $u(x), u(y), u(z) \in M$ and (M,ϕ_M) be an abelian 3-LieDer pair.

Also note that

$$\begin{aligned}\chi(x) &= \phi_{\hat{L}}(s_1(x)) - s_1(\phi_L(x))\\ &= \phi_{\hat{L}}(s_2(x)+u(x)) - s_2(\phi_L(x)) - u(\phi_L(x))\\ &= \chi'(x) + \phi_M(u(x)) - u(\phi_L(x)).\end{aligned}$$

This shows that $(\psi,\chi) - (\psi',\chi') = \partial u$. Hence they correspond to the same cohomology class. □

Theorem 3.4 *Let (L,ϕ_L) be a 3-LieDer pair and (M,ϕ_M) be an abelian 3-LieDer pair. Then the isomorphism classes of central extensions of L by M are classified by the second cohomology group $H^2_{3\text{-}LieDer}(L,M)$.*

Proof Let $(\hat{L},\phi_{\hat{L}})$ and $(\hat{L}',\phi_{\hat{L}'})$ be two isomorphic central extensions and the isomorphism is given by $\eta : \hat{L} \to \hat{L}'$. Let $s : L \to \hat{L}$ be a section of p. Then

$$p' \circ (\eta \circ s) = (p' \circ \eta) \circ s = p \circ s = Id_L.$$

This shows that $s' := \eta \circ s$ is a section of p'. Since η is a morphism of 3-LieDer pairs, we have $\eta|_M = Id_M$. Thus,

$$\begin{aligned}\psi'(x, y, z) &= [s'(x), s'(y), s'(z)] - s'([x, y, z]) \\ &= \eta([s(x), s(y), s(z)] - [x, y, z]) \\ &= \psi(x, y, z),\end{aligned}$$

and

$$\begin{aligned}\chi'(x) &= \phi_{\hat{L}'}(s'(x)) - s'(\phi_L(x)) \\ &= \phi_{\hat{L}'}(\eta \circ s(x)) - \eta \circ s(\phi_L(x)) \\ &= \phi_{\hat{L}}(s(x)) - s(\phi_L(x)) \\ &= \chi(x).\end{aligned}$$

Therefore, isomorphic central extensions give rise to the same 2-cocycle, hence, correspond to the same element in $H^2_{3-LieDer}(L, M)$.

Conversely, let (ψ, χ) and (ψ', χ') be two cohomologous 2-cocycles. Therefore, there exists a map $v : L \to M$ such that

$$(\psi, \chi) - (\psi', \chi') = \partial v.$$

The 3-LieDer pair structures on $L \oplus M$ corresponding to the above 2-cocycles are isomorphic via the map $\eta : L \oplus M \to L \oplus M$ given by $\eta(x, m) = (x, m + v(x))$. This proves our theorem. □

4 Extensions of a pair of derivations

It is well-known that derivations are infinitesimals of automorphisms, and a study [9] has been done on extensions of a pair of automorphisms of Lie-algebras. In this section, we study extensions of a pair of derivations and see how it is related to the cohomology of the 3-LieDer pair.

Let

$$0 \longrightarrow M \xrightarrow{\quad i \quad} \hat{L} \xrightarrow{\quad p \quad} L \longrightarrow 0 \tag{4.1}$$

be a fixed central extensions of 3-Lie algebras. Given a pair of derivations $(\phi_L, \phi_M) \in Der(L) \times Der(M)$, here we study extensions of them to a derivation $\phi_{\hat{L}} \in Der(\hat{L})$ which makes

$$0 \longrightarrow (M, \phi_M) \xrightarrow{\quad i \quad} (\hat{L}, \phi_{\hat{L}}) \xrightarrow{\quad p \quad} (L, \phi_L) \longrightarrow 0 \tag{4.2}$$

into an exact sequence of 3-LieDer pairs. In such a case, the pair $(\phi_L, \phi_M) \in Der(L) \times Der(M)$ is said to be extensible.

Let $s : L \to \hat{L}$ be a section of Eq. (4.1), we define a map $\psi : L \otimes L \otimes L \to M$ by

$$\psi(x, y, z) := [s(x), s(y), s(z)] - s([x, y, z]), \qquad \chi(x) = \phi_{\hat{L}}(s(x)) - s(\phi_L(x)), \quad \forall x, y, z \in L.$$

Given a pair of derivations $(\phi_L, \phi_M) \in Der(L) \times Der(M)$, we define another map $Ob^{\hat{L}}_{(\phi_L,\phi_M)} : L \otimes L \otimes L \to M$ by

$$Ob^{M}_{(\phi_L,\phi_M)}(x, y, z) := \phi_M(\psi(x, y, z)) - \psi(\phi_L(x), y, z) - \psi(x, \phi_L(y), z) - \psi(x, y, \phi_L(z)).$$

Proposition 4.1 *The map $Ob^{\hat{L}}_{(\phi_L,\phi_M)} : L \otimes L \otimes L \to M$ is a 2-cocycle in the cohomology of the 3-Lie algebra L with coefficients in the trivial representation a. Moreover, the cohomology class $[Ob^{\hat{L}}_{(\phi_L,\phi_M)}] \in H^2(L, M)$ does not depend on the choice of sections.*

Proof First observe that ψ is a 1-cocycle in the cohomology of the 3-Lie algebra L with coefficients in the trivial representation M. Thus, we have

$$\begin{aligned}
&(dOb^{M}_{(\phi_L,\phi_M)})(x, y, u, v, w)\\
&= -Ob^{M}_{(\phi_L,\phi_M)}(x, y, [u, v, w]) + Ob^{M}_{(\phi_L,\phi_M)}([x, y, u], v, w)\\
&\quad + Ob^{M}_{(\phi_L,\phi_M)}(u, [x, y, v], w) + Ob^{M}_{(\phi_L,\phi_M)}(u, v, [x, y, w])\\
&= -\phi_M(\psi(x, y, [u, v, w])) + \psi(\phi_L(x), y, [u, v, w]) + \psi(x, \phi_L(y), [u, v, w])\\
&\quad + \psi(x, y, \phi_L([u, v, w])) + \phi_M(\psi([x, y, u], v, w)) - \psi(\phi_L([x, y, u]), v, w)\\
&\quad - \psi([x, y, u], \phi_L(v), w) - \psi([x, y, u], v, \phi_L(w)) + \phi_M(\psi(u, [x, y, v], w))\\
&\quad - \psi(\phi_L(u), [x, y, v], w) - \psi(u, \phi_L([x, y, v]), w) - \psi(u, [x, y, v], \phi_L(w))\\
&\quad + \phi_M(\psi(u, v, [x, y, w])) - \psi(\phi_L(u), v, [x, y, w]) - \psi(u, \phi_L(v), [x, y, w])\\
&\quad - \psi(u, v, \phi_L([x, y, w]))\\
&= \psi(\phi_L(x), y, [u, v, w]) + \psi(x, \phi_L(y), [u, v, w]) + \psi(x, y, \phi_L([u, v, w]))\\
&\quad - \psi(\phi_L([x, y, u]), v, w) - \psi([x, y, u], \phi_L(v), w) - \psi([x, y, u], v, \phi_L(w))\\
&\quad - \psi(\phi_L(u), [x, y, v], w) - \psi(u, \phi_L([x, y, v]), w) - \psi(u, [x, y, v], \phi_L(w))\\
&\quad - \psi(\phi_L(u), v, [x, y, w]) - \psi(u, \phi_L(v), [x, y, w]) - \psi(u, v, \phi_L([x, y, w]))\\
&= 0.
\end{aligned}$$

Therefore, $Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ is a 2-cocycle. To prove the second part, let s_1 and s_2 be two sections of Eq. (4.1). Consider the map $u : L \to M$ given by $u(x) := s_1(x) - s_2(x)$. Then

$$\psi_1(x, y, z) = \psi_2(x, y, z) - u[x, y, z].$$

If ${}^1Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ and ${}^2Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ denote the one cocycles corresponding to the sections s_1 and s_2, then

$$\begin{aligned}
&{}^1Ob^{M}_{(\phi_L,\phi_M)}(x, y, z)\\
&= \phi_M(\psi_1(x, y, z)) - \psi_1(\phi_L(x), y, z) - \psi_1(x, \phi_L(y), z) - \psi_1(x, y, \phi_L(z))\\
&= \phi_M(\psi_2(x, y, z)) - \phi_M(u(x, y, z)) - \psi_2(\phi_L(x), y, z) + u(\phi_L(x), y, z)\\
&\quad - \psi_2(x, \phi_L(y), z) + u(x, \phi_L(y), z) - \psi_2(x, y, \phi_L(z)) + u(x, y, \phi_L(z))\\
&=^2 Ob^{M}_{(\phi_L,\phi_M)}(x, y, z) + d(\phi_M \circ u - u \circ \phi_L)(x, y, z).
\end{aligned}$$

This shows that the 2-cocycles ${}^1Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ and ${}^2Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ are cohomologous. Hence they correspond to the same cohomology class in $\in H^2(L, M)$. □

The cohomology class $[Ob^{\hat{L}}_{(\phi_L,\phi_M)}] \in H^2(L, M)$ is called the obstruction class to extend the pair of derivations (ϕ_L, ϕ_M).

Theorem 4.2 *Let* Eq. (4.1) *be a central extension of 3-Lie algebras. A pair of derivations* $(\phi_L, \phi_M) \in Der(L) \times Der(M)$ *is extensible if and only if the obstruction class* $[Ob^{\hat{L}}_{(\phi_L,\phi_M)}] \in H^2(L, M)$ *is trivial.*

Proof Suppose there exists a derivations $\phi_{\hat{L}} \in Der(\hat{L})$ such that Eq. (4.2) is an exact sequence of 3-LieDer pairs. For any $x \in L$, we observe that $p(\phi_{\hat{L}}(s(x)) - s(\phi_L(x))) = 0$. Hence $\phi_{\hat{L}}(s(x)) - s(\phi_L(x)) \in ker(p) = im(i)$. We define $\lambda : L \to M$ by

$$\lambda(x) = \phi_{\hat{L}}(s(x)) - s(\phi_L(x)).$$

For any $s(x) + a \in \hat{L}$, we have

$$\phi_{\hat{L}}(s(x) + a) = s(\phi_L(x)) + \lambda(x) + \phi_{\hat{L}}(a).$$

Since $\phi_{\hat{L}}$ is a derivation, for any $s(x) + a, s(y) + b \in \hat{L}$, we have

$$\phi_M(\psi(x, y, z)) - \psi(\phi_L(x), y, z) - \psi(x, \phi_L(y), z) - \psi(x, y, \phi_L(z)) = -\lambda([x, y, z]),$$

or, equivalently, $Ob^{\hat{L}}_{(\phi_L,\phi_M)} = \partial\lambda$ is a coboundary. Hence the obstruction class $[Ob^{\hat{L}}_{(\phi_L,\phi_M)}] \in H^2(L, M)$ is trivial. □

To prove the converse part, suppose $Ob^{\hat{L}}_{(\phi_L,\phi_M)}$ is given by a coboundary, say $Ob^{\hat{L}}_{(\phi_L,\phi_M)} = \partial\lambda$. We define a map $\phi_{\hat{L}} : \hat{L} \to \hat{L}$ by

$$\phi_{\hat{L}}(s(x) + a) = s(\phi_L(x)) + \lambda(x) + \phi_{\hat{L}}(a).$$

Then $\phi_{\hat{L}}$ is a derivation on $\hat{L}$ and Eq. (4.2) is an exact sequence of 3-LieDer pairs. Hence the pair (ϕ_L, ϕ_M) is extensible. Thus, we obtain the following.

Theorem 4.3 *If* $H^2(L, M) = 0$*, then any pair of derivations* $(\phi_L, \phi_M) \in Der(L) \times Der(M)$ *is extensible.*

5 Formal deformations of 3-LieDer pairs

In this section, we study one-parameter formal deformations of 3-LieDer pairs in which we deform both the 3-Lie bracket and the distinguished derivations.

Let (L, ϕ_L) be a 3-LieDer pair. We denote the 3-Lie bracket on L by μ, i.e, $\mu(x, y, z) = [x, y, z]$, for all $x, y, z \in L$. Consider the space $L[[t]]$ of formal power series in t with coefficients from L. Then $L[[t]]$ is a $\mathbb{F}[[t]]$-module.

A formal one-parameter deformation of the 3-LieDer pair (L, ϕ_L) consist of formal power series

$$\mu_t = \sum_{i=0}^{\infty} t^i \mu_i \in \mathrm{Hom}(L^{\otimes 3}, L)[[t]] \text{ with } \mu_0 = \mu,$$

$$\phi_t = \sum_{i=0}^{\infty} t^i \phi_i \in \mathrm{Hom}(L, L)[[t]] \text{ with } \phi_0 = \phi_L,$$

such that $L[[t]]$ together with the bracket μ_t forms a 3-Lie algebra over $\mathbb{F}[[t]]$ and ϕ_t is a derivation on $L[[t]]$.

Therefore, in a formal one-parameter deformation of 3-LieDer pair, the following relations hold:

$$\mu_t(x, y, \mu_t(z, v, w)) = \mu_t(\mu_t(x, y, z), v, w) + \mu_t(z, \mu_t(x, y, v), w) + \mu_t(z, v, \mu_t(x, y, w)), \tag{5.1}$$

$$\phi_t(\mu_t(x, y, z)) = \mu_t(\phi_t(x), y, z) + \mu_t(x, \phi_t(y), z) + \mu_t(x, y, \phi_t(z)). \tag{5.2}$$

Conditions Eqs.(5.1)–(5.2) are equivalent to the following equations:

$$\sum_{i+j=n} \mu_i(x, y, \mu_j(z, v, w)) = \sum_{i+j=n} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)), \tag{5.3}$$

and,

$$\sum_{i+j=n} \phi_i(\mu_j(x, y, z)) = \sum_{i+j=n} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)). \tag{5.4}$$

For $n = 0$ we simply get (L, ϕ_L) is a 3-LieDer pair. For $n = 1$, we have

$$\begin{aligned} &\mu_1(x, y, [z, v, w]) + [x, y, \mu_1(z, v, w)] \\ &= \mu_1([x, y, z], v, w) + [\mu_1(x, y, z), v, w] + [z, \mu_1(x, y, v), w] \\ &\quad + \mu_1(z, [x, y, v], w) + [z, v, \mu_1(x, y, w)] + \mu_1(z, v, [x, y, w]), \end{aligned} \tag{5.5}$$

and,

$$\begin{aligned} &\phi_1([x, y, z]) + \phi_L(\mu_1(x, y, z)) \\ &= \mu_1(\phi_L(x), y, z) + [\phi_1(x), y, z] + \mu_1(x, \phi_L(y), z) + [x, \phi_1(y), z] \\ &\quad + \mu_1(x, y, \phi_L(z)) + [x, y, \phi_1(z)]. \end{aligned} \tag{5.6}$$

The condition Eq. (5.5) is equivalent to $d(\mu_1) = 0$ whereas the condition Eq. (5.6) is equivalent to $d(\phi_1) + \delta(\mu_1) = 0$. Therefore, we have

$$\partial(\mu_1, \phi_1) = 0.$$

Definition 5.1 Let (μ_t, ϕ_t) be a one-parameter formal deformation of 3-LieDer pair (L, ϕ_L). Suppose (μ_n, ϕ_n) is the first non-zero term of (μ_t, ϕ_t) after (μ_0, ϕ_0), then such (μ_n, ϕ_n) is called the infinitesimal of the deformation of (L, ϕ_L).

Hence, from the above observations, we have the following proposition.

Proposition 5.2 *Let (μ_t, ϕ_t) be a formal one-parameter deformation of a 3-LieDer pair (L, ϕ_L). Then the linear term (μ_1, ϕ_1) is a 1-cocycle in the cohomology of the 3-LieDer pair L with coefficients in itself.*

Proof We have showed that

$$\partial(\mu_1, \phi_1) = 0.$$

If (μ_1, ϕ_1) be the first non-zero term, then we are done. If (μ_n, ϕ_n) be the first non-zero term after (μ_0, ϕ_0), then exactly the same way, one can show that

$$\partial(\mu_n, \phi_n) = 0.$$

□

Next, we define a notion of equivalence between formal deformations of 3-LieDer pairs.

Definition 5.3 Two deformations (μ_t, ϕ_t) and (μ'_t, ϕ'_t) of a 3-LieDer pair (L, ϕ_L) are said to be equivalent if there exists a formal isomorphism $\Phi_t = \sum_{i=0}^{\infty} t^i \phi_i : L[[t]] \to L[[t]]$ with $\Phi_0 = Id_L$ such that

$$\Phi_t \circ \mu_t = \mu'_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \qquad \Phi_t \circ \phi_t = \phi'_t \circ \Phi_t.$$

By comparing coefficients of t^n from both the sides, we have

$$\sum_{i+j=n} \phi_i \circ \mu_j = \sum_{p+q+r+l=n} \mu'_p \circ (\phi_q \otimes \phi_r \otimes \phi_l),$$
$$\sum_{i+j=n} \phi'_i \circ \phi_j = \sum_{p+q=n} \phi_p \circ \phi_q.$$

Easy to see that the above identities hold for $n = 0$. For $n = 1$, we get

$$\mu_1 + \phi_1 \circ \mu = \mu'_1 + \mu \circ (\phi_1 \otimes Id \otimes Id) + \mu \circ (Id \otimes Id \otimes \phi_1), \tag{5.7}$$
$$\phi_L \circ \Phi_1 + \phi'_1 = \phi_1 + \phi_1 \circ \phi_L. \tag{5.8}$$

These two identities together imply that

$$(\mu_1, \phi_1) - (\mu'_1, \phi'_1) = \partial \phi_1.$$

Thus, we have the following.

Proposition 5.4 *The infinitesimals corresponding to equivalent deformations of the 3-LieDer pair (L, ϕ_L) are cohomologous.*

Definition 5.5 A deformation (μ_t, ϕ_t) of a 3-LieDer pair is said to be trivial if it is equivalent to the undeformed deformation $(\mu'_t = \mu, \phi'_t = \phi_L)$.

Definition 5.6 A 3-LieDer pair (L, ϕ_L) is called rigid, if every 1-parameter formal deformation μ_t is equivalent to the trivial deformation.

Theorem 5.7 *Every formal deformation of the 3-LieDer pair (L, ϕ_L) is rigid if the second cohomology group of the 3-LieDer pair vanishes, that is, $H^2_{3\text{-}LieDer}(L, L) = 0$.*

Proof Let (μ_t, ϕ_t) be a deformation of the 3-LieDer pair (L, ϕ_L). From the Proposition 5.2, the linear term (μ_1, ϕ_1) is a 2-cocycle. Therefore, $(\mu_1, \phi_1) = \partial \Phi_1$ for some $\phi_1 \in C^1_{\text{3-LieDer}}(L, L) = \text{Hom}(L, L)$.

We set $\Phi_t = Id_L + t\Phi_1 : L[[t]] \to L[[t]]$ and define

$$\mu'_t = \Phi_t^{-1} \circ \mu_t \circ (\Phi_t \otimes \Phi_t \otimes \Phi_t), \quad \phi'_t = \Phi_t^{-1} \circ \phi_t \circ \Phi_t. \tag{5.9}$$

By definition, (μ'_t, ϕ'_t) is equivalent to (μ_t, ϕ_t). Moreover, it follows from Eq. (5.7) that

$$\mu'_t = \mu + t^2 \mu'_2 + \cdots \quad \text{and} \quad \phi'_t = \phi_L + t^2 \phi'_2 + \cdots.$$

In other words, the linear terms are vanish. By repeating this argument, we get (μ_t, ϕ_t) is equivalent to (μ, ϕ_L). □

Next, we consider finite order deformations of a 3-LieDer pair (L, ϕ_L), and show that how obstructions of extending deformation of order N to deformation of order $(N+1)$ depends on the third cohomology class of the 3-LieDer pair (L, ϕ_L) .

Definition 5.8 A deformation of order N of a 3-LieDer pair (L, ϕ_L) consist of finite sums $\mu_t = \sum_{i=0}^{N} t^i \mu_i$ and $\phi_t = \sum_{i=0}^{N} t^i \phi_i$ such that μ_t defines 3-Lie bracket on $L[[t]]/(t^{N+1})$ and ϕ_t is a derivation on it.

Therefore, we have

$$\sum_{i+j=n} \mu_i(x, y, \mu_j(z, v, w))$$
$$= \sum_{i+j=n} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)),$$

and,

$$\sum_{i+j=n} \phi_i(\mu_j(x, y, z)) = \sum_{i+j=n} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)),$$

for $n = 0, 1, \ldots, N$. These identities are equivalent to

$$[\mu, \mu_n] = -\frac{1}{2} \sum_{i+j=n, i, j>0} [\mu_i, \mu_j], \tag{5.10}$$

$$-[\phi_L, \mu_n] + [\mu, \phi_n] = \sum_{i+j=n, i, j>0} [\phi_i, \mu_j]. \tag{5.11}$$

Definition 5.9 A deformation $(\mu_t = \sum_{i=0}^{N} t^i \mu_i, \phi_t = \sum_{i=0}^{N} t^i \phi_i)$ of order N is said to be extendable if there is an element $(\mu_{N+1}, \phi_{N+1}) \in C^2_{\text{3-LieDer}}(L, L)$ such that $(\mu'_t = \mu_t + t^{N+1}\mu_{N+1}, \phi'_t = \phi_t + t^{N+1}\phi_{N+1})$ is a deformation of order $N+1$.

Thus, the following two equations need to be satisfied-

$$\sum_{i+j=N+1} \mu_i(x, y, \mu_j(z, v, w))$$
$$= \sum_{i+j=N+1} \mu_i(\mu_j(x, y, z), v, w) + \mu_i(z, \mu_j(x, y, v), w) + \mu_i(z, v, \mu_j(x, y, w)), \tag{5.12}$$

and,

$$\sum_{i+j=N+1} \phi_i(\mu_j(x, y, z))$$
$$= \sum_{i+j=N+1} \mu_i(\phi_j(x), y, z) + \mu_i(x, \phi_j(y), z) + \mu_i(x, y, \phi_j(z)). \tag{5.13}$$

The above two equations can be equivalently written as

$$d(\mu_{N+1}) = -\frac{1}{2} \sum_{i+j=N+1, i, j>0} [\mu_i, \mu_j] = Ob^3 \tag{5.14}$$

$$d(\phi_{N+1}) + \delta(\mu_{N+1}) = - \sum_{i+j=N+1, i, j>0} [\phi_i, \mu_j] = Ob^2. \tag{5.15}$$

Using the Eqs. 5.14 and 5.15, it is routine but lengthy work to prove the following proposition. Thus, we choose to omit the proof.

Proposition 5.10 *The pair $(Ob^3, Ob^2) \in C^3_{3\text{-LieDer}}(L, L)$ is a 3-cocycle in the cohomology of the 3-LieDer pair (L, ϕ_L) with coefficients in itself.*

Definition 5.11 Let (μ_t, ϕ_t) be a deformation of order N of a 3-LieDer pair (L, ϕ_L). The cohomology class $[(Ob^3, Ob^2)] \in H^3_{3\text{-LieDer}}(L, L)$ is called the obstruction class of (μ_t, ϕ_t).

Theorem 5.12 *A deformation (μ_t, ϕ_t) of order N is extendable if and only if the obstruction class $[(Ob^3, Ob^2)] \in H^3_{3\text{-LieDer}}(L, L)$ is trivial.*

Proof Suppose that a deformation (μ_t, ϕ_t) of order N of the 3-LieDer pair (L, ϕ_L) extends to a deformation of order $N + 1$. Then we have

$$\partial(\mu_{N+1}, \phi_{N+1}) = (Ob^3, Ob^2).$$

Thus, the obstruction class $[(Ob^3, Ob^2)] \in H^3_{3\text{-LieDer}}(L, L)$ is trivial.

Conversely, if the obstruction class $[(Ob^3, Ob^2)] \in H^3_{3\text{-LieDer}}(L, L)$ is trivial, suppose that

$$(Ob^3, Ob^2) = \partial(\mu_{N+1}, \phi_{N+1}),$$

for some $(\mu_{N+1}, \phi_{N+1}) \in C^2_{3\text{-LieDer}}(L, L)$. Then it follows from the above observation that $(\mu'_t = \mu_t + t^{N+1}\mu_{N+1}, \phi'_t = \phi_t + t^{N+1}\phi_{N+1})$ is a deformation of order $N + 1$, which implies that (μ_t, ϕ_t) is extendable. □

Theorem 5.13 *If $H^3_{3\text{-LieDer}}(L, L)$, then every finite order deformation of (L, ϕ_L) is extendable.*

Corollary 5.14 *If $H^3_{3\text{-LieDer}}(L, L) = 0$, then every 2-cocycle in the cohomology of the 3-LieDer pair (L, ϕ_L) with coefficients in itself is the infinitesimal of a formal deformation of (L, ϕ_L).*

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