Afrika Matematika (2022) 33:64
https://doi.org/10.1007/s13370-022-00992-z

Fixed point results of almost generalized $(\phi, \psi, \theta)_s$-contractive mappings in ordered b-metric spaces

N. Seshagiri Rao[1] · K. Kalyani[2] · Belay Mitiku[1]

Received: 6 October 2020 / Accepted: 15 April 2022 / Published online: 12 May 2022

Abstract
The aim of this article is to establish some fixed point, coupled coincidence point and coupled common fixed point results for mappings satisfying an almost generalized $(\phi, \psi, \theta)_s$-contractive conditions in the frame work of partially ordered b-metric spaces. These results generalize, extend and unify several comparable results in the existing literature. Few examples are illustrated to support our results.

Keywords Partially ordered b-metric space · Fixed point · Coupled coincidence point · Coupled common fixed point · Compatible · Mixed f-monotone mapping

Mathematics Subject Classification Primary 47H10; Secondary 54H25

1 Introduction and preliminaries

The Banach contraction principle is one of the most important results in nonlinear analysis. It plays an important role in many branches of mathematical analysis, and it has many applications in solving nonlinear equations and scientific problems. Later, it has been generalized and improved in many different directions, one of the most influential generalization is a b-metric space, also called metric type space by some authors, introduced and studied by Bakhtin [11] and Czerwik [16]. There after, a large number of articles have been dedicated to the improvement of the fixed point theory for single valued and multivalued operators in b-metric spaces, the readers may refer to [1, 5, 6, 8, 10, 17, 18, 20, 21, 23, 28, 36] and

✉ N. Seshagiri Rao
seshu.namana@gmail.com

K. Kalyani
kalyani.namana@gmail.com

Belay Mitiku
belaymida@yahoo.com

1 Department of Applied Mathematics School of Applied Natural Sciences, Adama Science and Technology University, Post Box No. 1888 Adama, Ethiopia

2 Department of Mathematics, Vignan's Foundation for Science, Technology and Research, Vadlamudi, Andhra Pradesh 522213, India

the references therein. The concept of coupled fixed points of mixed monotone mappings in partially ordered metric spaces was introduced by Bhaskar and Lakshmikantham [13] and applied theiry results to first order differential equation with boundary condition. After that Lakshmikantham and Ćirić [30] have introduced the concept of coupled coincidence and coupled common fixed point for mappings with mixed monotone property and generalized the result of Bhaskar and Lakshmikantham [13]. Then, several authors have obtained fixed point, common fixed point, coupled fixed point and coupled common fixed point results for mappings under various contractive conditions in ordered b-metric spaces, some of which are in [2–4, 7, 9, 14, 15, 19, 22, 31, 32] and the references therein. Recently, some results on fixed point, coincidence point, coupled coincidence point for the self mappings satisfying generalized weak contractions have been discussed by Belay Mituku et al. [12], Seshagiri Rao et al. [33–35] and Kalyani et al. [24–27] in partially ordered b-metric space with necessary topological properties. Some important results of fixed points of distance spaces can be found from Todoŕcević [37] and William Kirk et al. [29].

In this paper, some fixed point, coincidence point, coupled coincidence point and coupled common fixed points for mappings satisfying an almost generalized $(\phi, \psi, \theta)_s$-contraction conditions in complete partially ordered b-metric spaces are proved. These results generalize and extend the results of [13, 30, 33, 34] and several comparable results in the existing literature. Some examples are presented to support our results.

For the sake of convenience some definitions and suitable results are recalled from [2, 19, 30, 32] which will be needed in what follows.

Definition 1.1 [35] A map $d : P \times P \to [0, +\infty)$, where P is a non-empty set is said to be a b-metric, if it satisfies the properties given below for any $\upsilon, \xi, \mu \in P$ and for some real number $s \geq 1$:

(a) $d(\upsilon, \xi) = 0$ if and only if $\upsilon = \xi$,
(b) $d(\upsilon, \xi) = d(\xi, \upsilon)$,
(c) $d(\upsilon, \xi) \leq s\,(d(\upsilon, \mu) + d(\mu, \xi))$.

Then (P, d, s) is known as a b-metric space. If $(P, \preceq)$ is still a partially ordered set, then $(P, d, s, \preceq)$ is called a partially ordered b-metric space.

Definition 1.2 [33] Let (P, d, s) be a b-metric space. Then

(1) a sequence $\{\upsilon_n\}$ is said to converge to υ, if $\lim\limits_{n\to+\infty} d(\upsilon_n, \upsilon) = 0$ and written as $\lim_{n\to+\infty} \upsilon_n = \upsilon$.
(2) $\{\upsilon_n\}$ is said to be a Cauchy sequence in P, if $\lim\limits_{n,m\to+\infty} d(\upsilon_n, \upsilon_m) = 0$.
(3) (P, d, s) is said to be complete, if every Cauchy sequence in it is convergent.

Definition 1.3 If the metric d is complete then $(P, d, s, \preceq)$ is called complete partially ordered b-metric space.

Definition 1.4 [32] Let $(P, \preceq)$ be a partially ordered set and let $f, S : P \to P$ are two mappings. Then

(1) S is called a monotone nondecreasing, if $S(\upsilon) \preceq S(\xi)$ for all $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$.
(2) an element $\upsilon \in P$ is called a coincidence (common fixed) point of f and S, if $f\upsilon = S\upsilon$ $(f\upsilon = S\upsilon = \upsilon)$.
(3) f and S are called commuting, if $fS\upsilon = Sf\upsilon$, for all $\upsilon \in P$.
(4) f and S are called compatible, if any sequence $\{\upsilon_n\}$ with $\lim_{n\to+\infty} f\upsilon_n = \lim_{n\to+\infty} S\upsilon_n = \mu$, for $\mu \in P$ then $\lim_{n\to+\infty} d(Sf\upsilon_n, fS\upsilon_n) = 0$.

(5) a pair of self maps (f, S) is called weakly compatible, if $fS\upsilon = Sf\upsilon$, when $S\upsilon = f\upsilon$ for some $\upsilon \in P$.

(6) S is called monotone f-nondecreasing, if

$$f\upsilon \preceq f\xi \text{ implies } S\upsilon \preceq S\xi, \text{ for any } \upsilon, \xi \in P.$$

(7) a non empty set P is called well ordered set, if very two elements of it are comparable i.e., $\upsilon \preceq \xi$ or $\xi \preceq \upsilon$, for $\upsilon, \xi \in P$.

Definition 1.5 [2, 30] Suppose $(P, \preceq)$ be a partially ordered set and let $S : P \times P \to P$ and $f : P \to P$ be two mappings. Then

(1) S has the mixed f-monotone property, if S is non-decreasing f-monotone in its first argument and is non-increasing f-monotone in its second argument, that is for any $\upsilon, \xi \in P$

$$\upsilon_1, \upsilon_2 \in P, \quad f\upsilon_1 \preceq f\upsilon_2 \text{ implies } S(\upsilon_1, \xi) \preceq S(\upsilon_2, \xi) \text{ and}$$
$$\xi_1, \xi_2 \in P, \quad f\xi_1 \preceq f\xi_2 \text{ implies } S(\upsilon, \xi_1) \succeq S(\upsilon, \xi_2).$$

Suppose, if f is an identity mapping then S is said to have the mixed monotone property.

(2) an element $(\upsilon, \xi) \in P \times P$ is called a coupled coincidence point of S and f, if $S(\upsilon, \xi) = f\upsilon$ and $S(\xi, \upsilon) = f\xi$. Note that, if f is an identity mapping then (υ, ξ) is said to be a coupled fixed point of S.

(3) an element $\upsilon \in P$ is called a common fixed point of S and f, if $S(\upsilon, \upsilon) = f\upsilon = \upsilon$.

(4) S and f are commutative, if for all $\upsilon, \xi \in P$, $S(f\upsilon, f\xi) = f(S\upsilon, S\xi)$.

(5) S and f are said to be compatible, if

$$\lim_{n\to+\infty} d(f(S(\upsilon_n, \xi_n)), S(f\upsilon_n, f\xi_n)) = 0 \text{ and } \lim_{n\to+\infty} d(f(S(\xi_n, \upsilon_n)), S(f\xi_n, f\upsilon_n)) = 0,$$

whenever $\{\upsilon_n\}$ and $\{\xi_n\}$ are any two sequences in P such that $\lim_{n\to+\infty} S(\upsilon_n, \xi_n) = \lim_{n\to+\infty} f\upsilon_n = \upsilon$ and $\lim_{n\to+\infty} S(\xi_n, \upsilon_n) = \lim_{n\to+\infty} f\xi_n = \xi$, for any $\upsilon, \xi \in P$.

We know that b-metric is not continuous, so the following lemma is used frequently in our results for the convergence of sequences in a b-metric spaces.

Lemma 1.6 [2] *Let $(P, d, s, \preceq)$ be a b-metric space with $s > 1$ and suppose that $\{\upsilon_n\}$ and $\{\xi_n\}$ are b-convergent to υ and ξ respectively. Then*

$$\frac{1}{s^2} d(\upsilon, \xi) \le \lim_{n\to+\infty} \inf d(\upsilon_n, \xi_n) \le \lim_{n\to+\infty} \sup d(\upsilon_n, \xi_n) \le s^2 d(\upsilon, \xi).$$

In particular, if $\upsilon = \xi$, then $\lim_{n\to+\infty} d(\upsilon_n, \xi_n) = 0$. Moreover, for each $\tau \in P$, we have

$$\frac{1}{s} d(\upsilon, \tau) \le \lim_{n\to+\infty} \inf d(\upsilon_n, \tau) \le \lim_{n\to+\infty} \sup d(\upsilon_n, \tau) \le s d(\upsilon, \tau).$$

2 Main results

Throughout this paper, we use the following denotations of the distances functions.

A self mapping ϕ defined on $[0, +\infty)$ is said to be an altering distance function, if it satisfies the following conditions:

(i) ϕ is continuous,

(ii) ϕ is nondecreasing,

(iii) $\phi(t) = 0$ if and only if $t = 0$.

Let us denote the set of all altering distance functions on $[0, +\infty)$ by Φ.

Similarly, Ψ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(i) ψ is lower semi-continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

and Θ denote the set of all continuous functions $\theta : [0, +\infty) \to [0, +\infty)$ with $\theta(t) = 0$ if and only if $t = 0$.

Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s > 1$ and, let $S : P \to P$ be a mapping. Set

$$M(\upsilon, \xi) = \max\left\{\frac{d(\xi, S\xi)\,[1 + d(\upsilon, S\upsilon)]}{1 + d(\upsilon, \xi)}, \frac{d(\upsilon, S\xi) + d(\xi, S\upsilon)}{2s}, d(\upsilon, S\upsilon), d(\xi, S\xi), d(\upsilon, \xi)\right\} \quad (1)$$

and

$$N(\upsilon, \xi) = \min\{d(\upsilon, S\upsilon), d(\xi, S\xi), d(\xi, S\upsilon), d(\upsilon, S\xi)\}. \quad (2)$$

Let $\phi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$. The mapping S is called an almost generalized $(\phi, \psi, \theta)_s$-contraction mapping if it satisfies the following condition

$$\phi(sd(S\upsilon, S\xi)) \leq \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)) + L\theta(N(\upsilon, \xi)), \quad (3)$$

for all $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$ and $L \geq 0$.

Now in this paper, we start with the following fixed point theorem in the context of partially ordered b-metric space.

Theorem 2.1 *Suppose that $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with parameter $s > 1$. Let $S : P \to P$ be an almost generalized $(\phi, \psi, \theta)_s$-contractive mapping, and be continuous, nondecreasing mapping with regards to $\preceq$. If there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq S\upsilon_0$, then S has a fixed point in P.*

Proof For some $\upsilon_0 \in P$ such that $S\upsilon_0 = \upsilon_0$, then the proof is finished. Assume that $\upsilon_0 \prec S\upsilon_0$, then construct a sequence $\{\upsilon_n\} \subset P$ by $\upsilon_{n+1} = S\upsilon_n$ for $n \geq 0$. Since S is nondecreasing, then by induction we obtain that

$$\upsilon_0 \prec S\upsilon_0 = \upsilon_1 \preceq \cdots \preceq \upsilon_n \preceq S\upsilon_n = \upsilon_{n+1} \preceq \cdots . \quad (4)$$

If for some $n_0 \in \mathbb{N}$ such that $\upsilon_{n_0} = \upsilon_{n_0+1}$ then from (4), υ_{n_0} is a fixed point of S and we have nothing to prove. Suppose that $\upsilon_n \neq \upsilon_{n+1}$ for all $n \geq 1$. Since $\upsilon_n > \upsilon_{n-1}$ for any $n \geq 1$ and then by contraction condition (3), we have

$$\begin{aligned}\phi(d(\upsilon_n, \upsilon_{n+1})) &= \phi(d(S\upsilon_{n-1}, S\upsilon_n)) \leq \phi(sd(S\upsilon_{n-1}, S\upsilon_n)) \\ &\leq \phi(M(\upsilon_{n-1}, \upsilon_n)) - \psi(M(\upsilon_{n-1}, \upsilon_n)) + L\theta(N(\upsilon_{n-1}, \upsilon_n)),\end{aligned} \quad (5)$$

where

$$M(\upsilon_{n-1},\upsilon_n)=\max\left\{\frac{d(\upsilon_n,S\upsilon_n)\left[1+d(\upsilon_{n-1},S\upsilon_{n-1})\right]}{1+d(\upsilon_{n-1},\upsilon_n)},\frac{d(\upsilon_{n-1},S\upsilon_n)+d(\upsilon_n,S\upsilon_{n-1})}{2s},\right.$$
$$\left.\times\, d(\upsilon_{n-1},S\upsilon_{n-1}),d(\upsilon_n,S\upsilon_n),d(\upsilon_{n-1},\upsilon_n)\right\}$$
$$=\max\left\{d(\upsilon_n,\upsilon_{n+1}),\frac{d(\upsilon_{n-1},\upsilon_{n+1})+d(\upsilon_n,\upsilon_n)}{2s},d(\upsilon_{n-1},\upsilon_n)\right\}$$
$$\leq\max\left\{d(\upsilon_n,\upsilon_{n+1}),\frac{d(\upsilon_{n-1},\upsilon_n)+d(\upsilon_n,\upsilon_{n+1})}{2},d(\upsilon_{n-1},\upsilon_n)\right\}$$
$$\leq\max\{d(\upsilon_n,\upsilon_{n+1}),d(\upsilon_{n-1},\upsilon_n)\}$$

and

$$N(\upsilon_{n-1},\upsilon_n)=\min\{d(\upsilon_{n-1},S\upsilon_{n-1}),d(\upsilon_n,S\upsilon_n),d(\upsilon_n,S\upsilon_{n-1}),d(\upsilon_{n-1},S\upsilon_n)\}$$
$$=\min\{d(\upsilon_{n-1},\upsilon_n),d(\upsilon_n,\upsilon_{n+1}),d(\upsilon_n,\upsilon_n),d(\upsilon_{n-1},\upsilon_{n+1})\}=0.$$

From (5), we get

$$d(\upsilon_n,\upsilon_{n+1})=d(S\upsilon_{n-1},S\upsilon_n)\leq\frac{1}{s}M(\upsilon_{n-1},\upsilon_n).\tag{6}$$

If $\max\{d(\upsilon_n,\upsilon_{n+1}),d(\upsilon_{n-1},\upsilon_n)\}=d(\upsilon_n,\upsilon_{n+1})$ for some $n\geq 1$, then from (6) follows

$$d(\upsilon_n,\upsilon_{n+1})\leq\frac{1}{s}d(\upsilon_n,\upsilon_{n+1}),\tag{7}$$

which is a contradiction. This means that $\max\{d(\upsilon_n,\upsilon_{n+1}),d(\upsilon_{n-1},\upsilon_n)\}=d(\upsilon_{n-1},\upsilon_n)$ for $n\geq 1$. Hence, we obtain from (6) that

$$d(\upsilon_n,\upsilon_{n+1})\leq\frac{1}{s}d(\upsilon_{n-1},\upsilon_n).\tag{8}$$

Since, $\frac{1}{s}\in(0,1)$ then the sequence $\{\upsilon_n\}$ is a Cauchy sequence by [1, 5, 10, 18]. But P is complete, then there exists some $\mu\in P$ such that $\upsilon_n\to\mu$.

From the continuity of S implies that

$$S\mu=S\left(\lim_{n\to+\infty}\upsilon_n\right)=\lim_{n\to+\infty}S\upsilon_n=\lim_{n\to+\infty}\upsilon_{n+1}=\mu.\tag{9}$$

Therefore, μ is a fixed point of S in P. □

By relaxing the continuity criteria of a map S in Theorem 2.1, we have the following result.

Theorem 2.2 *In Theorem 2.1, assume that P satisfies*

if a nondecreasing sequence $\{\upsilon_n\}\to\mu$ in P, then $\upsilon_n\preceq\mu$ for all $n\in\mathbb{N}$, i.e., $\mu=\sup\upsilon_n$.

Then a nondecreasing mapping S has a fixed point in P.

Proof From Theorem 2.1, we construct a nondecreasing Cauchy sequence $\{\upsilon_n\}$ in P such that $\upsilon_n\to\mu\in P$. Therefore from the hypotheses, we have $\upsilon_n\preceq\mu$ for any $n\in\mathbb{N}$, implies that $\mu=\sup\upsilon_n$.

Now, we prove that μ is a fixed point of S. Suppose that $S\mu \neq \mu$. Let

$$M(\upsilon_n, \mu) = \max\left\{\frac{d(\mu, S\mu)\,[1 + d(\upsilon_n, S\upsilon_n)]}{1 + d(\upsilon_n, \mu)}, \frac{d(\upsilon_n, S\mu) + d(\mu, S\upsilon_n)}{2s}, d(\upsilon_n, S\upsilon_n),\right.$$
$$\left. \times\, d(\mu, S\mu), d(\upsilon_n, \mu)\right\},$$

and

$$N(\upsilon_n, \mu) = \min\{d(\upsilon_n, S\upsilon_n), d(\mu, S\mu), d(\mu, S\upsilon_n), d(\upsilon_n, S\mu)\}.$$

Letting $n \to +\infty$ and use of $\lim_{n\to+\infty} \upsilon_n = \mu$, we get

$$\lim_{n\to+\infty} M(\upsilon_n, \mu) = \max\left\{d(\mu, S\mu), \frac{d(\mu, S\mu)}{2s}, 0\right\} = d(\mu, S\mu), \tag{10}$$

and

$$\lim_{n\to+\infty} N(\upsilon_n, \mu) = \min\{0, d(\mu, S\mu)\} = 0. \tag{11}$$

We know that $\upsilon_n \preceq \mu$, for all n then from contraction condition (3), we get

$$\phi(d(\upsilon_{n+1}, S\mu)) = \phi(d(S\upsilon_n, S\mu)) \leq \phi(sd(S\upsilon_n, S\mu)) \leq \phi(M(\upsilon_n, \mu)) - \psi(M(\upsilon_n, \mu)). \tag{12}$$

Letting $n \to +\infty$ and use of (10) and (11), we get

$$\phi(d(\mu, S\mu)) \leq \phi(d(\mu, S\mu)) - \psi(d(\mu, S\mu)) < \phi(d(\mu, S\mu)), \tag{13}$$

which is a contradiction under (13). Thus, $S\mu = \mu$, that is S has a fixed point μ in P. □

Now we give the sufficient condition for the uniqueness of the fixed point exists in Theorem 2.1 and Theorem 2.2.

every pair of elements has a lower bound or an upper bound. (14)

This condition is equivalent to,

for every $\upsilon, \xi \in P$, there exists $w \in P$ which is comparable to υ and ξ.

Theorem 2.3 *In addition to the hypotheses of Theorem 2.1 (or Theorem 2.2), condition (14) provides uniqueness of a fixed point of S in P.*

Proof From Theorem 2.1 (or Theorem 2.2), we conclude that S has a nonempty set of fixed points. Suppose that υ^* and ξ^* be two fixed points of S then, we claim that $\upsilon^* = \xi^*$. Suppose that $\upsilon^* \neq \xi^*$, then from the hypotheses we have

$$\phi(d(S\upsilon^*, S\xi^*)) \leq \phi(sd(S\upsilon^*, S\xi^*)) \leq \phi(M(\upsilon^*, \xi^*)) - \psi(M(\upsilon^*, \xi^*)) + L\theta(N(\upsilon^*, \xi^*)), \tag{15}$$

where

$$\begin{aligned} M(\upsilon^*, \xi^*) &= \max\left\{\frac{d(\xi^*, S\xi^*)\left[1+d(\upsilon^*, S\upsilon^*)\right]}{1+d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, S\xi^*)+d(\xi^*, S\upsilon^*)}{2s}, d(\upsilon^*, S\upsilon^*),\right.\\ &\quad \left.\times\, d(\xi^*, S\xi^*), d(\upsilon^*, \xi^*)\right\}\\ &= \max\left\{\frac{d(\xi^*, \xi^*)\left[1+d(\upsilon^*, \upsilon^*)\right]}{1+d(\upsilon^*, \xi^*)}, \frac{d(\upsilon^*, \xi^*)+d(\xi^*, \upsilon^*)}{2s}, d(\upsilon^*, \upsilon^*),\right.\\ &\quad \left.\times\, d(\xi^*, \xi^*), d(\upsilon^*, \xi^*)\right\}\\ &= \max\left\{0, \frac{d(\upsilon^*, \xi^*)}{s}, d(\upsilon^*, \xi^*)\right\}\\ &= d(\upsilon^*, \xi^*), \end{aligned}$$

and

$$N(\upsilon^*, \xi^*) = \min\{d(\upsilon^*, S\upsilon^*), d(\xi^*, S\xi^*), d(\xi^*, S\upsilon^*), d(\upsilon^*, S\xi^*)\} = 0.$$

Consequently, we get

$$d(\upsilon^*, \xi^*) = d(S\upsilon^*, S\xi^*) \leq \frac{1}{s}M(\upsilon^*, \xi^*). \tag{16}$$

Therefore from (16), we obtain that

$$d(\upsilon^*, \xi^*) \leq \frac{1}{s}d(\upsilon^*, \xi^*) < d(\upsilon^*, \xi^*), \tag{17}$$

which is a contradiction. Hence, $\upsilon^* = \xi^*$. This completes the proof. □

Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with parameter $s > 1$ and let $S, f : P \to P$ be two mappings. Set

$$\begin{aligned} M_f(\upsilon, \xi) = \max\{&\frac{d(f\xi, S\xi)\left[1+d(f\upsilon, S\upsilon)\right]}{1+d(f\upsilon, f\xi)}, \frac{d(f\upsilon, S\xi)+d(f\xi, S\upsilon)}{2s},\\ &\times\, d(f\upsilon, S\upsilon), d(f\xi, S\xi), d(f\upsilon, f\xi)\}, \end{aligned} \tag{18}$$

and

$$N_f(\upsilon, \xi) = \min\{d(f\upsilon, S\upsilon), d(f\xi, S\xi), d(f\xi, S\upsilon), d(f\upsilon, S\xi)\}. \tag{19}$$

Now, we introduce the following definition.

Definition 2.4 Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with $s > 1$. The mapping $S : P \to P$ is called an almost generalized $(\phi, \psi, \theta)_s$-contraction mapping with respect to $f : P \to P$ for some $\phi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$, if

$$\phi(sd(S\upsilon, S\xi)) \leq \phi(M_f(\upsilon, \xi)) - \psi(M_f(\upsilon, \xi)) + L\theta(N_f(\upsilon, \xi)), \tag{20}$$

for any $\upsilon, \xi \in P$ with $f\upsilon \preceq f\xi$, $L \geq 0$ and where $M_f(\upsilon, \xi)$ and $N_f(\upsilon, \xi)$ are given by (18) and (19) respectively.

Theorem 2.5 *Suppose that $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Let $S : P \to P$ be an almost generalized $(\phi, \psi, \theta)_s$-contractive mapping with respect to $f : P \to P$ and, S and f are continuous such that S is a monotone f-non decreasing mapping, compatible with f and $SP \subseteq fP$. If for some $\upsilon_0 \in P$ such that $f\upsilon_0 \preceq S\upsilon_0$, then S and f have a coincidence point in P.*

Proof By following the proof of a Theorem 2.2 in [7], we construct two sequences $\{\upsilon_n\}$ and $\{\xi_n\}$ in P such that

$$\xi_n = S\upsilon_n = f\upsilon_{n+1} \text{ for all } n \geq 0, \tag{21}$$

for which

$$f\upsilon_0 \preceq f\upsilon_1 \preceq \cdots \preceq f\upsilon_n \preceq f\upsilon_{n+1} \preceq \cdots. \tag{22}$$

Again from [7], we have to show that

$$d(\xi_n, \xi_{n+1}) \leq \lambda d(\xi_{n-1}, \xi_n), \tag{23}$$

for all $n \geq 1$ and where $\lambda \in [0, \frac{1}{s})$. Now from (20) and use of (21) and (22), we have

$$\begin{aligned} \phi(sd(\xi_n, \xi_{n+1})) &= \phi(sd(S\upsilon_n, S\upsilon_{n+1})) \\ &\leq \phi(M_f(\upsilon_n, \upsilon_{n+1})) - \psi(M_f(\upsilon_n, \upsilon_{n+1})) + L\theta(N_f(\upsilon_n, \upsilon_{n+1})), \end{aligned} \tag{24}$$

where

$$\begin{aligned} M_f(\upsilon_n, \upsilon_{n+1}) &= \max\left\{\frac{d(f\upsilon_{n+1}, S\upsilon_{n+1})\,[1 + d(f\upsilon_n, S\upsilon_n)]}{1 + d(f\upsilon_n, f\upsilon_{n+1})}, \frac{d(f\upsilon_n, S\upsilon_{n+1}) + d(f\upsilon_{n+1}, S\upsilon_n)}{2s},\right. \\ &\quad \left.\times\, d(f\upsilon_n, S\upsilon_n), d(f\upsilon_{n+1}, S\upsilon_{n+1}), d(f\upsilon_n, f\upsilon_{n+1})\right\} \\ &= \max\left\{\frac{d(\xi_n, \xi_{n+1})\left[1 + d(\xi_{n-1}, \xi_n)\right]}{1 + d(\xi_{n-1}, \xi_n)}, \frac{d(\xi_{n-1}, \xi_{n+1}) + d(\xi_n, \xi_n)}{2s}, d(\xi_{n-1}, \xi_n),\right. \\ &\quad \left.\times\, d(\xi_n, \xi_{n+1}), d(\xi_{n-1}, \xi_n)\right\} \\ &= \max\left\{d(\xi_n, \xi_{n+1}), \frac{d(\xi_{n-1}, \xi_n) + d(\xi_n, \xi_{n+1})}{2s}, d(\xi_{n-1}, \xi_n)\right\} \\ &\leq \max\{d(\xi_n, \xi_{n+1}), d(\xi_{n-1}, \xi_n)\} \end{aligned}$$

and

$$\begin{aligned} N_f(\upsilon_n, \upsilon_{n+1}) &= \min\{d(f\upsilon_n, S\upsilon_n), d(f\upsilon_{n+1}, S\upsilon_{n+1}), d(f\upsilon_{n+1}, S\upsilon_n), d(f\upsilon_n, S\upsilon_{n+1})\} \\ &= \min\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1}), d(\xi_n, \xi_n), d(\xi_{n-1}, \xi_{n+1})\} = 0. \end{aligned}$$

Therefore from Eq. (24), we get

$$\phi(sd(\xi_n, \xi_{n+1})) \leq \phi(\max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}) - \psi(\max\{d(\xi_{n-1}, \xi_n), d(\xi_n, \xi_{n+1})\}). \tag{25}$$

If $0 < d(\xi_{n-1}, \xi_n) \leq d(\xi_n, \xi_{n+1})$ for some $n \in \mathbb{N}$, then from (25) we get

$$\phi(sd(\xi_n, \xi_{n+1})) \leq \phi(d(\xi_n, \xi_{n+1})) - \psi(d(\xi_n, \xi_{n+1})) < \phi(d(\xi_n, \xi_{n+1})), \tag{26}$$

or equivalently

$$sd(\xi_n, \xi_{n+1}) \leq d(\xi_n, \xi_{n+1}). \tag{27}$$

This is a contradiction. Hence from (25) we obtain that

$$sd(\xi_n, \xi_{n+1}) \leq d(\xi_{n-1}, \xi_n). \tag{28}$$

Thus Eq. (23) holds, where $\lambda \in [0, \frac{1}{s})$. Therefore from (23) and Lemma 3.1 of [23], we conclude that $\{\xi_n\} = \{S\upsilon_n\} = \{f\upsilon_{n+1}\}$ is a Cauchy sequence in P and then converges to some $\mu \in P$ as P is complete such that

$$\lim_{n\to+\infty} S\upsilon_n = \lim_{n\to+\infty} f\upsilon_{n+1} = \mu.$$

Thus by the compatibility of S and f, we obtain that

$$\lim_{n\to+\infty} d(f(S\upsilon_n), S(f\upsilon_n)) = 0, \tag{29}$$

and from the continuity of S and f, we have

$$\lim_{n\to+\infty} f(S\upsilon_n) = f\mu, \quad \lim_{n\to+\infty} S(f\upsilon_n) = S\mu. \tag{30}$$

Further by use of triangular inequality and from Eqs. (29) and (30), we get

$$\frac{1}{s}d(S\mu, f\mu) \leq d(S\mu, S(f\upsilon_n)) + sd(S(f\upsilon_n), f(S\upsilon_n)) + sd(f(S\upsilon_n), f\mu). \tag{31}$$

Finally, we arrive at $d(S\upsilon, f\upsilon) = 0$ as $n \to +\infty$ in (31). Therefore, υ is a coincidence point of S and f in P. □

Relaxing the continuity criteria of f and S in Theorem 2.5, we obtain the following result.

Theorem 2.6 *In Theorem 2.5, assume that P satisfies*

for any nondecreasing sequence $\{f\upsilon_n\} \subset P$ with $\lim_{n\to+\infty} f\upsilon_n = f\upsilon$ in fP, where fP is a closed subset of P implies that $f\upsilon_n \preceq f\upsilon$, $f\upsilon \preceq f(f\upsilon)$ for $n \in \mathbb{N}$.

If there exists $\upsilon_0 \in P$ such that $f\upsilon_0 \preceq S\upsilon_0$, then the weakly compatible mappings S and f have a coincidence point in P. Moreover, S and f have a common fixed point, if S and f commute at their coincidence points.

Proof The sequence, $\{\xi_n\} = \{S\upsilon_n\} = \{f\upsilon_{n+1}\}$ is a Cauchy sequence from the proof of Theorem 2.5. Since fP is closed, then there is some $\mu \in P$ such that

$$\lim_{n\to+\infty} S\upsilon_n = \lim_{n\to+\infty} f\upsilon_{n+1} = f\mu.$$

Thus from the hypotheses, we have $f\upsilon_n \preceq f\mu$ for all $n \in \mathbb{N}$. Now, we have to prove that μ is a coincidence point of S and f.

From equation (20), we have

$$\phi(sd(S\upsilon_n, S\upsilon)) \leq \phi(M_f(\upsilon_n, \upsilon)) - \psi(M_f(\upsilon_n, \upsilon)) + L\theta(N_f(\upsilon_n, \upsilon)), \tag{32}$$

where

$$\begin{aligned} M_f(\upsilon_n, \mu) = \max\left\{\frac{d(f\mu, S\mu)\,[1 + d(f\upsilon_n, S\upsilon_n)]}{1 + d(f\upsilon_n, f\mu)}, \frac{d(f\upsilon_n, S\mu) + d(f\mu, S\upsilon_n)}{2s},\right. \\ \left. \times\, d(f\upsilon_n, S\upsilon_n), d(f\mu, S\mu), d(f\upsilon_n, f\mu)\right\} \\ \to \max\left\{d(f\mu, S\mu), \frac{d(f\mu, S\mu)}{2s}, 0, d(f\mu, S\mu), 0\right\} \\ = d(f\mu, S\mu) \text{ as } n \to +\infty, \end{aligned}$$

and

$$\begin{aligned} N_f(\upsilon_n, \mu) &= \min\{d(f\upsilon_n, S\upsilon_n), d(f\mu, S\mu), d(f\mu, S\upsilon_n), d(f\upsilon_n, S\mu)\} \\ &\to \min\{0, d(f\mu, S\mu), 0, d(f\mu, S\mu)\} \\ &= 0 \text{ as } n \to +\infty. \end{aligned}$$

Therefore Eq. (32) becomes

$$\phi\left(s \lim_{n\to+\infty} d(S\upsilon_n, S\upsilon)\right) \leq \phi(d(f\mu, S\mu)) - \psi(d(f\mu, S\mu)) < \phi(d(f\mu, S\mu)).$$

Consequently, we get

$$\lim_{n\to+\infty} d(S\upsilon_n, S\upsilon) < \frac{1}{s} d(f\mu, S\mu). \tag{33}$$

Further by triangular inequality, we have

$$\frac{1}{s} d(f\mu, S\mu) \leq d(f\mu, S\upsilon_n) + d(S\upsilon_n, S\mu), \tag{34}$$

then (33) and (34) lead to contradiction, if $f\mu \neq S\mu$. Hence, $f\mu = S\mu$.

Let $f\mu = S\mu = \rho$, that is S and f commute at ρ, then $S\rho = S(f\mu) = f(S\mu) = f\rho$. Since $f\mu = f(f\mu) = f\rho$, then by Eq. (32) with $f\mu = S\mu$ and $f\rho = S\rho$, we get

$$\phi(sd(S\mu, S\rho)) \leq \phi(M_f(\mu, \rho)) - \psi(M_f(\mu, \rho)) < \phi(d(S\mu, S\rho)), \tag{35}$$

or equivalently,

$$sd(S\mu, S\rho) \leq d(S\mu, S\rho),$$

which is a contradiction, if $S\mu \neq S\rho$. Thus, $S\mu = S\rho = \rho$. Hence, $S\mu = f\rho = \rho$, that is ρ is a common fixed point of S and f. □

Definition 2.7 Let $(P, d, s, \preceq)$ be a partially ordered b-metric space with $s > 1$, $\phi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$. A mapping $S : P \times P \to P$ is said to be an almost generalized $(\phi, \psi, \theta)_s$-contractive mapping with respect to $f : P \to P$ such that

$$\phi(s^k d(S(\upsilon, \xi), S(\rho, \tau))) \leq \phi(M_f(\upsilon, \xi, \rho, \tau)) - \psi(M_f(\upsilon, \xi, \rho, \tau)) + L\theta(N_f(\upsilon, \xi, \rho, \tau)), \tag{36}$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $f\upsilon \preceq f\rho$ and $f\xi \succeq f\tau$, $k > 2$, $L \geq 0$ and where

$$\begin{aligned} M_f(\upsilon, \xi, \rho, \tau) = \max\Bigg\{ &\frac{d(f\rho, S(\rho, \tau))\,[1 + d(f\upsilon, S(\upsilon, \xi))]}{1 + d(f\upsilon, f\rho)}, \\ &\frac{d(f\upsilon, S(\rho, \tau)) + d(f\rho, S(\upsilon, \xi))}{2s}, \\ &\times d(f\upsilon, S(\upsilon, \xi)), d(f\rho, S(\rho, \tau)), d(f\upsilon, f\rho)\Bigg\}, \end{aligned}$$

and

$$N_f(\upsilon, \xi, \rho, \tau) = \min\{d(f\upsilon, S(\upsilon, \xi)), d(f\rho, S(\rho, \tau)), d(f\rho, S(\upsilon, \xi)), d(f\upsilon, S(\rho, \tau))\}.$$

Theorem 2.8 *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Suppose that $S : P \times P \to P$ be an almost generalized $(\phi, \psi, \theta)_s$-contractive mapping with respect to $f : P \to P$ and, S and f are continuous functions such that S has the mixed f-monotone property and commutes with f. Also assume that $S(P \times P) \subseteq f(P)$. Then S and f have a coupled coincidence point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $f\upsilon_0 \preceq S(\upsilon_0, \xi_0)$ and $f\xi_0 \succeq S(\xi_0, \upsilon_0)$.*

Proof From the hypotheses and following the proof of Theorem 2.2 of [7], we construct two sequences $\{\upsilon_n\}$ and $\{\xi_n\}$ in P such that

$$f\upsilon_{n+1} = S(\upsilon_n, \xi_n), \quad f\xi_{n+1} = S(\xi_n, \upsilon_n), \quad \text{for all } n \geq 0.$$

In particular, $\{f\upsilon_n\}$ is nondecreasing and $\{f\xi_n\}$ is nonincreasing sequences in P. Now from (36) by replacing $\upsilon = \upsilon_n, \xi = \xi_n, \rho = \upsilon_{n+1}, \tau = \xi_{n+1}$, we get

$$\begin{aligned}\phi(s^k d(f\upsilon_{n+1}, f\upsilon_{n+2})) &= \phi(s^k d(S(\upsilon_n, \xi_n), S(\upsilon_{n+1}, \xi_{n+1})))\\ &\leq \phi(M_f(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1})) - \psi(M_f(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1}))\\ &\quad + L\theta(N_f(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1})),\end{aligned} \tag{37}$$

where

$$M_f(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1}) \leq \max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2})\},$$

and

$$\begin{aligned}N_f(\upsilon_n, \xi_n, \upsilon_{n+1}, \xi_{n+1}) = \min\{&d(f\upsilon_n, S(\upsilon_n, \xi_n)), d(f\upsilon_{n+1}, S(\upsilon_{n+1}, \xi_{n+1})),\\ &\times d(f\upsilon_n, S(\upsilon_{n+1}, \xi_{n+1})), d(f\upsilon_{n+1}, S(\upsilon_n, \xi_n))\} = 0.\end{aligned}$$

Therefore from (37), we have

$$\begin{aligned}\phi(s^k d(f\upsilon_{n+1}, f\upsilon_{n+2})) &\leq \phi(\max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2})\})\\ &\quad - \psi(\max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2})\}).\end{aligned} \tag{38}$$

Similarly by taking $\upsilon = \xi_{n+1}, \xi = \upsilon_{n+1}, \rho = \upsilon_n, \tau = \upsilon_n$ in (36), we get

$$\begin{aligned}\phi(s^k d(f\xi_{n+1}, f\xi_{n+2})) &\leq \phi(\max\{d(f\xi_n, f\xi_{n+1}), d(f\xi_{n+1}, f\xi_{n+2})\})\\ &\quad - \psi(\max\{d(f\xi_n, f\xi_{n+1}), d(f\xi_{n+1}, f\xi_{n+2})\}).\end{aligned} \tag{39}$$

From the fact that $\max\{\phi(c), \phi(d)\} = \phi\{\max\{c, d\}\}$ for all $c, d \in [0, +\infty)$. Then combining (38) and (39), we get

$$\begin{aligned}\phi(s^k \delta_n) &\leq \phi(\max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2}), d(f\xi_n, f\xi_{n+1}), d(f\xi_{n+1}, f\xi_{n+2})\})\\ &\quad - \psi(\max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2}), d(f\xi_n, f\xi_{n+1}), d(f\xi_{n+1}, f\xi_{n+2})\})\end{aligned} \tag{40}$$

where

$$\delta_n = \max\{d(f\upsilon_{n+1}, f\upsilon_{n+2}), d(f\xi_{n+1}, f\xi_{n+2})\}. \tag{41}$$

Let us denote,

$$\Delta_n = \max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\upsilon_{n+1}, f\upsilon_{n+2}), d(f\xi_n, f\xi_{n+1}), d(f\xi_{n+1}, f\xi_{n+2})\}. \tag{42}$$

Hence from Eqs. (38)–(41), we obtain

$$s^k \delta_n \leq \Delta_n. \tag{43}$$

Next, we prove that

$$\delta_n \leq \lambda \delta_{n-1}, \tag{44}$$

for all $n \geq 1$ and where $\lambda = \frac{1}{s^k} \in [0, 1)$.

Suppose that if $\Delta_n = \delta_n$ then from (43), we get $s^k \delta_n \leq \delta_n$ which leads to $\delta_n = 0$ as $s > 1$ and hence (44) holds. If $\Delta_n = \max\{d(f\upsilon_n, f\upsilon_{n+1}), d(f\xi_n, f\xi_{n+1})\}$, i.e., $\Delta_n = \delta_{n-1}$ then (43) follows (44).

Now from (43), we obtain that $\delta_n \leq \lambda^n \delta_0$ and hence,

$$d(f\upsilon_{n+1}, f\upsilon_{n+2}) \leq \lambda^n \delta_0 \text{ and } d(f\xi_{n+1}, f\xi_{n+2}) \leq \lambda^n \delta_0. \tag{45}$$

Therefore from Lemma 3.1 of [23], the sequences $\{f\upsilon_n\}$ and $\{f\xi_n\}$ are Cauchy sequences in P. Hence, by following the remaining proof of Theorem 2.2 of [3], we can show that S and f have a coincidence point in P. □

Corollary 2.9 *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $S : P \times P \to P$ be a continuous mapping such that S has a mixed monotone property. Suppose there exists $\phi \in \Phi$, $\psi \in \Psi$ and $\theta \in \Theta$ such that*

$$\phi(s^k d(S(\upsilon, \xi), S(\rho, \tau))) \leq \phi(M_f(\upsilon, \xi, \rho, \tau)) - \psi(M_f(\upsilon, \xi, \rho, \tau)) + L\theta(N_f(\upsilon, \xi, \rho, \tau)),$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $\upsilon \preceq \rho$ and $\xi \succeq \tau$, $k > 2$, $L \geq 0$ and where

$$M_f(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, S(\rho, \tau))\,[1 + d(\upsilon, S(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, \frac{d(\upsilon, S(\rho, \tau)) + d(\rho, S(\upsilon, \xi))}{2s},\right.$$
$$\left. \times\, d(\upsilon, S(\upsilon, \xi)), d(\rho, S(\rho, \tau)), d(\upsilon, \rho)\right\},$$

and

$$N_f(\upsilon, \xi, \rho, \tau) = \min\{d(\upsilon, S(\upsilon, \xi)), d(\rho, S(\rho, \tau)), d(\rho, S(\upsilon, \xi)), d(\upsilon, S(\rho, \tau))\}.$$

Then S has a coupled fixed point in P, if there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $\upsilon_0 \preceq S(\upsilon_0, \xi_0)$ and $\xi_0 \succeq S(\xi_0, \upsilon_0)$.

Proof Set $f = I_P$ in Theorem 2.8. □

Corollary 2.10 *Let $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$, and $S : P \times P \to P$ be a continuous mapping such that S has a mixed monotone property. Suppose there exists $\psi \in \Psi$ such that*

$$d(S(\upsilon, \xi), S(\rho, \tau)) \leq \frac{1}{s^k} M_f(\upsilon, \xi, \rho, \tau) - \frac{1}{s^k} \psi(M_f(\upsilon, \xi, \rho, \tau)),$$

for all $\upsilon, \xi, \rho, \tau \in P$ with $\upsilon \preceq \rho$ and $\xi \succeq \tau$, $k > 2$ where

$$M_f(\upsilon, \xi, \rho, \tau) = \max\left\{\frac{d(\rho, S(\rho, \tau))\,[1 + d(\upsilon, S(\upsilon, \xi))]}{1 + d(\upsilon, \rho)}, \frac{d(\upsilon, S(\rho, \tau)) + d(\rho, S(\upsilon, \xi))}{2s},\right.$$
$$\left. \times\, d(\upsilon, S(\upsilon, \xi)), d(\rho, S(\rho, \tau)), d(\upsilon, \rho)\right\}.$$

If there exists $(\upsilon_0, \xi_0) \in P \times P$ such that $\upsilon_0 \preceq S(\upsilon_0, \xi_0)$ and $\xi_0 \succeq S(\xi_0, \upsilon_0)$, then S has a coupled fixed point in P.

Theorem 2.11 *In addition to Theorem 2.8, if for all* $(\upsilon, \xi), (r, s) \in P \times P$, *there exists* $(c^*, d^*) \in P \times P$ *such that* $(S(c^*, d^*), S(d^*, c^*))$ *is comparable to* $(S(\upsilon, \xi), S(\xi, \upsilon))$ *and to* $(S(r, s), S(s, r))$, *then* S *and* f *have a unique coupled common fixed point in* $P \times P$.

Proof From Theorem 2.8, we know that there exists atleast one coupled coincidence point in P for S and f. Assume that (υ, ξ) and (r, s) are two coupled coincidence points of S and f, i.e., $S(\upsilon, \xi) = f\upsilon$, $S(\xi, \upsilon,) = f\xi$ and $S(r, s) = fr$, $S(s, r) = fs$. Now, we have to prove that $f\upsilon = fr$ and $f\xi = fs$.

From the hypotheses, there exists $(c^*, d^*) \in P \times P$ such that $(S(c^*, d^*), S(d^*, c^*))$ is comparable to $(S(\upsilon, \xi), S(\xi, \upsilon))$ and to $(S(r, s), S(s, r))$. Suppose that

$(S(\upsilon, \xi), S(\xi, \upsilon)) \leq (S(c^*, d^*), S(d^*, c^*))$ and $(S(r, s), S(s, r)) \leq (S(c^*, d^*), S(d^*, c^*))$.

Let $c_0^* = c^*$ and $d_0^* = d^*$ and then choose $(c_1^*, d_1^*) \in P \times P$ as

$$fc_1^* = S(c_0^*, d_0^*), \quad fd_1^* = S(d_0^*, c_0^*) \ \ (n \geq 1).$$

By repeating the same procedure above, we can obtain two sequences $\{fc_n^*\}$ and $\{fd_n^*\}$ in P such that

$$fc_{n+1}^* = S(c_n^*, d_n^*), \quad fd_{n+1}^* = S(d_n^*, c_n^*) \ \ (n \geq 0).$$

Similarly, define the sequences $\{f\upsilon_n\}$, $\{f\xi_n\}$ and $\{fr_n\}$, $\{fs_n\}$ as above in P by setting $\upsilon_0 = \upsilon$, $\xi_0 = \xi$ and $r_0 = r$, $s_0 = s$. Further, we have that

$$f\upsilon_n \to S(\upsilon, \xi), \ f\xi_n \to S(\xi, \upsilon), \ fr_n \to S(r, s), \ fs_n \to S(s, r) \ \ (n \geq 1). \tag{46}$$

Since, $(S(\upsilon, \xi), S(\xi, \upsilon)) = (f\upsilon, f\xi) = (f\upsilon_1, f\xi_1)$ is comparable to $(S(c^*, d^*), S(d^*, c^*))$ $= (fc^*, fd^*) = (fc_1^*, fd_1^*)$ and hence we get $(f\upsilon_1, f\xi_1) \leq (fc_1^*, fd_1^*)$. Thus, by induction we obtain that

$$(f\upsilon_n, f\xi_n) \leq (fc_n^*, fd_n^*) \ \ (n \geq 0). \tag{47}$$

Therefore from (36), we have

$$\begin{aligned} \phi(d(f\upsilon, fc_{n+1}^*)) &\leq \phi(s^3 d(f\upsilon, fc_{n+1}^*)) = \phi(d(S(\upsilon, \xi), S(c_n^*, d_n^*))) \\ &\leq \phi(M_f(\upsilon, \xi, c_n^*, d_n^*)) - \psi(M_f(\upsilon, \xi, c_n^*, d_n^*)) + L\theta(N_f(\upsilon, \xi, c_n^*, d_n^*)), \end{aligned} \tag{48}$$

where

$$\begin{aligned} M_f(\upsilon, \xi, c_n^*, d_n^*) = \max\Bigg\{ & \frac{d(fc_n^*, S(c_n^*, d_n^*))\,[1 + d(f\upsilon, S(\upsilon, \xi))]}{1 + d(f\upsilon, fc_n^*)}, \\ & \frac{d(f\upsilon, S(c_n^*, d_n^*)) + d(fc_n^*, S(\upsilon, \xi))}{2s}, \\ & \times d(f\upsilon, S(\upsilon, \xi)), d(fc_n^*, S(c_n^*, d_n^*)), d(f\upsilon, fc_n^*) \Bigg\} \\ = \max & \left\{ 0, \frac{d(f\upsilon, fc_n^*)}{s}, 0, 0, d(f\upsilon, fc_n^*) \right\} \\ = d(f & \upsilon, fc_n^*) \end{aligned}$$

and

$$\begin{aligned} N_f(\upsilon, \xi, c_n^*, d_n^*) = \min\{ & d(f\upsilon, S(\upsilon, \xi)), d(fc_n^*, S(c_n^*, d_n^*)), \\ & d(fc_n^*, S(\upsilon, \xi)), d(f\upsilon, S(c_n^*, d_n^*))\} = 0. \end{aligned}$$

Thus from (48),

$$\phi(d(f\upsilon, fc^*_{n+1})) \le \phi(d(f\upsilon, fc^*_n)) - \psi(d(f\upsilon, fc^*_n)). \tag{49}$$

As by the similar process, we can prove that

$$\phi(d(f\xi, fd^*_{n+1})) \le \phi(d(f\xi, fd^*_n)) - \psi(d(f\xi, fd^*_n)). \tag{50}$$

From (49) and (50), we have

$$\begin{aligned} \phi(\max\{d(f\upsilon, fc^*_{n+1}), d(f\xi, fd^*_{n+1})\}) &\le \phi(\max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\}) \\ &\quad - \psi(\max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\}) \\ &< \phi(\max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\}). \end{aligned} \tag{51}$$

Hence by the property of ϕ, we get

$$\max\{d(f\upsilon, fc^*_{n+1}), d(f\xi, fd^*_{n+1})\} < \max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\},$$

which shows that $\max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\}$ is a decreasing sequence and by a result there exists $\gamma \ge 0$ such that

$$\lim_{n\to+\infty} \max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\} = \gamma.$$

From (51) taking upper limit as $n \to +\infty$, we get

$$\phi(\gamma) \le \phi(\gamma) - \psi(\gamma), \tag{52}$$

from which we get $\psi(\gamma) = 0$, implies that $\gamma = 0$. Thus,

$$\lim_{n\to+\infty} \max\{d(f\upsilon, fc^*_n), d(f\xi, fd^*_n)\} = 0.$$

Consequently, we get

$$\lim_{n\to+\infty} d(f\upsilon, fc^*_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(f\xi, fd^*_n) = 0. \tag{53}$$

By similar argument, we get

$$\lim_{n\to+\infty} d(fr, fc^*_n) = 0 \quad \text{and} \quad \lim_{n\to+\infty} d(fs, fd^*_n) = 0. \tag{54}$$

Therefore from (53) and (54), we get $f\upsilon = fr$ and $f\xi = fs$. Since $f\upsilon = S(\upsilon, \xi)$ and $f\xi = S(\xi, \upsilon)$, then by the commutativity of S and f, we have

$$f(f\upsilon) = f(S(\upsilon, \xi)) = S(f\upsilon, f\xi) \quad \text{and} \quad f(f\xi) = f(S(\xi, \upsilon)) = S(f\xi, f\upsilon). \tag{55}$$

Let $f\upsilon = a^*$ and $f\xi = b^*$ then (55) becomes

$$f(a^*) = S(a^*, b^*) \quad \text{and} \quad f(b^*) = S(b^*, a^*), \tag{56}$$

which shows that (a^*, b^*) is a coupled coincidence point of S and f. It follows that $f(a^*) = fr$ and $f(b^*) = fs$ that is $f(a^*) = a^*$ and $f(b^*) = b^*$. Thus from (56), we get $a^* = f(a^*) = S(a^*, b^*)$ and $b^* = f(b^*) = S(b^*, a^*)$. Therefore, (a^*, b^*) is a coupled common fixed point of S and f.

For the uniqueness let (u^*, v^*) be another coupled common fixed point of S and f, then we have $u^* = fu^* = S(u^*, v^*)$ and $v^* = fv^* = S(v^*, u^*)$. Since (u^*, v^*) is a coupled common fixed point of S and f, then we obtain that $fu^* = f\upsilon = a^*$ and $fv^* = f\xi = b^*$. Thus, $u^* = fu^* = fa^* = a^*$ and $v^* = fv^* = fb^* = b^*$. Hence the result. □

Theorem 2.12 *In addition to the hypotheses of Theorem 2.11, if* $f\upsilon_0$ *and* $f\xi_0$ *are comparable, then* S *and* f *have a unique common fixed point in* P.

Proof From Theorem 2.11, S and f have a unique coupled common fixed point $(\upsilon, \xi) \in P$. Now, it is enough to prove that $\upsilon = \xi$. From the hypotheses, we have $f\upsilon_0$ and $f\xi_0$ are comparable then we assume that $f\upsilon_0 \preceq f\xi_0$. Hence by induction we get $f\upsilon_n \preceq f\xi_n$ for all $n \geq 0$, where $\{f\upsilon_n\}$ and $\{f\xi_n\}$ are from Theorem 2.8.

Now by use of Lemma 1.6, we get

$$\begin{aligned}
\phi(s^{k-2}d(\upsilon,\xi)) = \phi(s^k\frac{1}{s^2}d(\upsilon,\xi)) &\leq \lim_{n\to+\infty}\sup\phi(s^k d(\upsilon_{n+1},\xi_{n+1}))\\
&= \lim_{n\to+\infty}\sup\phi(s^k d(S(\upsilon_n,\xi_n),S(\xi_n,\upsilon_n)))\\
&\leq \lim_{n\to+\infty}\sup\phi(M_f(\upsilon_n,\xi_n,\xi_n,\upsilon_n)) - \lim_{n\to+\infty}\inf\psi(M_f(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\
&\quad + \lim_{n\to+\infty}\sup L\theta(N_f(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\
&\leq \phi(d(\upsilon,\xi)) - \lim_{n\to+\infty}\inf\psi(N_f(\upsilon_n,\xi_n,\xi_n,\upsilon_n))\\
&< \phi(d(\upsilon,\xi)),
\end{aligned}$$

which is a contradiction. Thus, $\upsilon = \xi$, i.e., S and f have a common fixed point in P. □

Remark 2.13 It is well known that b-metric space is a metric space when $s = 1$. So, from the result of Jachymski [22], the condition

$$\begin{aligned}
\phi(d(S(\upsilon,\xi),S(\rho,\tau))) &\leq \phi(\max\{d(f\upsilon,f\rho),d(f\xi,f\tau)\})\\
&\quad -\psi(\max\{d(f\upsilon,f\rho),d(f\xi,f\tau)\})
\end{aligned}$$

is equivalent to,

$$d(S(\upsilon,\xi),S(\rho,\tau)) \leq \varphi(\max\{d(f\upsilon,f\rho),d(f\xi,f\tau)\}),$$

where $\phi \in \Phi$, $\psi \in \Psi$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is continuous, $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$. So, in view of above our results generalize and extend the results of [2, 30, 33, 34] and several other comparable results.

Corollary 2.14 *Suppose* $(P, d, s, \preceq)$ *be a complete partially ordered b-metric space with parameter* $s > 1$. *Let* $S : P \to P$ *be a continuous, nondecreasing map with regards to* $\preceq$ *such that there exists* $\upsilon_0 \in P$ *with* $\upsilon_0 \preceq S\upsilon_0$. *Suppose that*

$$\phi(sd(S\upsilon,S\xi)) \leq \phi(M(\upsilon,\xi)) - \psi(M(\upsilon,\xi)), \tag{57}$$

where $M(\upsilon,\xi)$ *and the conditions upon* ϕ, ψ *are same as in Theorem 2.1. Then* S *has a fixed point in* P.

Proof Set $L = 0$ in a contraction condition (3) and apply Theorem 2.1, we have the required proof. □

Note 1 Besides, if P satisfies the assumptions in Theorem 2.2, then a nondecreasing mapping S has a fixed point in P. Also, if P satisfies the hypothesis (14), then one obtains uniqueness of the fixed point.

Note 2 Setting $L = 0$ and following the proofs of Theorems 2.5 and 2.6, we can find the coincidence point for S and f in P. Similarly, from Theorem 2.8, 2.11 and 2.12, one can obtain a coupled coincidence point and its uniqueness, and a unique common fixed point for mappings S and f in $P \times P$ and P satisfying a generalized contraction condition (57), where $M_f(\upsilon, \xi)$, $M_f(\upsilon, \xi, \rho, \tau)$ and the conditions upon ϕ, ψ are same as above defined.

Corollary 2.15 *Suppose that $(P, d, s, \preceq)$ be a complete partially ordered b-metric space with $s > 1$. Let $S : P \to P$ be a continuous, nondecreasing mapping with regards to $\preceq$. If there exists $k \in [0, 1)$ and for any $\upsilon, \xi \in P$ with $\upsilon \preceq \xi$ such that*

$$d(S\upsilon, S\xi) \leq \frac{k}{s} \max \left\{ \frac{d(\xi, S\xi)\,[1 + d(\upsilon, S\upsilon)]}{1 + d(\upsilon, \xi)}, \frac{d(\upsilon, S\xi) + d(\xi, S\upsilon)}{2s}, \right. \\ \left. d(\upsilon, S\upsilon), d(\xi, S\xi), d(\upsilon, \xi) \right\}. \tag{58}$$

If there exists $\upsilon_0 \in P$ with $\upsilon_0 \preceq S\upsilon_0$, then S has a fixed point in P.

Proof Set $\phi(t) = t$ and $\psi(t) = (1 - k)t$, for all $t \in (0, +\infty)$ in Corollary 2.14. □

Note 3 Relaxing the continuity of a map S in Corollary 2.15, one can obtains a fixed point for S on taking a nondecreasing sequence $\{\upsilon_n\}$ in P by following the proof of Theorem 2.2.

We illustrate the usefulness of the obtained results in different cases such as continuity and discontinuity of a metric d in a space P.

Example 2.16 Define a metric $d : P \times P \to P$ as below and $\leq$ is an usual order on P, where $P = \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned}
&d(\upsilon, \xi) = d(\xi, \upsilon) = 0, \text{ if } \upsilon, \xi = 1, 2, 3, 4, 5, 6 \text{ and } \upsilon = \xi, \\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 3, \; if \; \upsilon, \xi = 1, 2, 3, 4, 5 \text{ and } \upsilon \neq \xi, \\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 12, \; if \; \upsilon = 1, 2, 3, 4 \text{ and } \xi = 6, \\
&d(\upsilon, \xi) = d(\xi, \upsilon) = 20, \; if \; \upsilon = 5 \text{ and } \xi = 6.
\end{aligned}$$

Define a map $S : P \to P$ by $S1 = S2 = S3 = S4 = S5 = 1$, $S6 = 2$ and let $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$ for $t \in [0, +\infty)$. Then S has a fixed point in P.

Proof It is apparent that, $(P, d, s, \preceq)$ is a complete partially ordered b-metric space for $s = 2$. Consider the possible cases for υ, ξ in P:
Case 1. Suppose $\upsilon, \xi \in \{1, 2, 3, 4, 5\}$, $\upsilon < \xi$ then $d(S\upsilon, S\xi) = d(1, 1) = 0$. Hence,

$$\phi(2d(S\upsilon, S\xi)) = 0 \leq \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)).$$

Case 2. Suppose that $\upsilon \in \{1, 2, 3, 4, 5\}$ and $\xi = 6$, then $d(S\upsilon, S\xi) = d(1, 2) = 3$, $M(6, 5) = 20$ and $M(\upsilon, 6) = 12$, for $\upsilon \in \{1, 2, 3, 4\}$. Therefore, we have the following inequality,

$$\phi(2d(S\upsilon, S\xi)) \leq \frac{M(\upsilon, \xi)}{4} = \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)).$$

Thus, condition (57) of Corollary 2.14 holds. Furthermore, the remaining assumptions in Corollary 2.14 are fulfilled. Hence, S has a fixed point in P as Corollary 2.14 is appropriate to S, ϕ, ψ and $(P, d, s, \preceq)$. □

Example 2.17 A metric $d : P \times P \to P$, where $P = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \frac{1}{n}, \ldots\}$ with usual order $\leq$ is as follows

$$d(\upsilon, \xi) = \begin{cases} 0, & if\ \upsilon = \xi \\ 1, & if\ \upsilon \neq \xi \in \{0, 1\} \\ |\upsilon - \xi|, & if\ \upsilon, \xi \in \{0, \frac{1}{2n}, \frac{1}{2m} : n \neq m \geq 1\} \\ 2, & otherwise. \end{cases}$$

A map $S : P \to P$ be such that $S0 = 0$, $S\frac{1}{n} = \frac{1}{12n}$ for all $n \geq 1$ and let $\phi(t) = t$, $\psi(t) = \frac{4t}{5}$ for $t \in [0, +\infty)$. Then, S has a fixed point in P.

Proof It is obvious that for $s = \frac{12}{5}$, $(P, d, s, \preceq)$ is a complete partially ordered b-metric space and also by definition, d is discontinuous b-metric space. Now for $\upsilon, \xi \in P$ with $\upsilon < \xi$, we have the following cases:
Case 1. If $\upsilon = 0$ and $\xi = \frac{1}{n}$, $n \geq 1$, then $d(S\upsilon, S\xi) = d(0, \frac{1}{12n}) = \frac{1}{12n}$ and $M(\upsilon, \xi) = \frac{1}{n}$ or $M(\upsilon, \xi) = \{1, 2\}$. Therefore, we have

$$\phi\left(\frac{12}{5} d(S\upsilon, S\xi)\right) \leq \frac{M(\upsilon, \xi)}{5} = \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)).$$

Case 2. If $\upsilon = \frac{1}{m}$ and $\xi = \frac{1}{n}$ with $m > n \geq 1$, then

$$d(S\upsilon, S\xi) = d(\frac{1}{12m}, \frac{1}{12n}) \text{ and } M(\upsilon, \xi) \geq \frac{1}{n} - \frac{1}{m} \text{ or } M(\upsilon, \xi) = 2.$$

Therefore,

$$\phi\left(\frac{12}{5} d(S\upsilon, S\xi)\right) \leq \frac{M(\upsilon, \xi)}{5} = \phi(M(\upsilon, \xi)) - \psi(M(\upsilon, \xi)).$$

Hence, condition (57) of Corollary 2.14 and remaining assumptions are satisfied. Thus, S has a fixed point in P. □

Example 2.18 Let $P = C[a, b]$ be the set of all continuous functions. Let us define a b-metric d on P by

$$d(\theta_1, \theta_2) = \sup_{t \in C[a,b]} \{|\theta_1(t) - \theta_2(t)|^2\}$$

for all $\theta_1, \theta_2 \in P$ with partial order $\preceq$ defined by $\theta_1 \preceq \theta_2$ if $a \leq \theta_1(t) \leq \theta_2(t) \leq b$, for all $t \in [a, b]$, $0 \leq a < b$. Let $S : P \to P$ be a mapping defined by $S\theta = \frac{\theta}{5}$, $\theta \in P$ and the two altering distance functions by $\phi(t) = t$, $\psi(t) = \frac{t}{3}$, for any $t \in [0, +\infty]$. Then S has a unique fixed point in P.

Proof From the hypotheses, it is clear that $(P, d, s, \preceq)$ is a complete partially ordered b-metric space with parameter $s = 2$ and fulfill all conditions of Corollary 2.14 and Note 1. Furthermore, for any $\theta_1, \theta_2 \in P$, the function $\min(\theta_1, \theta_2)(t) = \min\{\theta_1(t), \theta_2(t)\}$ is also continuous and the conditions of Corollary 2.14 and Note 1 are satisfied. Hence, S has a unique fixed point $\theta = 0$ in P. □

Acknowledgements The authors do thankful to the editor and anonymous referees for their valuable suggestions and comments to improved the content of the paper.

Author Contributions All authors contributed equally to write of this paper. All authors have read and approved the final manuscript.

Funding Not applicable

Data Availability Not applicable.

Declarations

Conflict of interest None (authors' have no complicit of interest)

Competing interests The authors declare that they have no competing interests.

References

1. Abbas, M., Ali, B., Bin-Mohsin, B., Dedović, N., Nazir, T., Radenović, S.: Solutions and Ulam–Hyers stability of differential inclusions involving Suzuki type multivalued mappings on b-metric spaces. Vojnotehnički glasnik/Mil. Tech. Cour. **68**(3), 438–487 (2020)
2. Aghajani, A., Arab, R.: Fixed points of (ψ, ϕ, θ)-contractive mappings in partially ordered b-metric spaces and applications to quadratic integral equations. Fixed Point Theory Appl. (2013). https://doi.org/10.1186/1687-1812-2013-245. (**article id 245, 20 pages**)
3. Aghajani, A., Abbas, M., Roshan, J.R.: Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces. Math. Slov. **64**(4), 941–960 (2014)
4. Akkouchi, M.: Common fixed point theorems for two self mappings of a b-metric space under an implicit relation. Hacet. J. Math. Stat. **40**(6), 805–810 (2011)
5. Aleksić, S., Huang, H., Mitrović, Z.D., Radenović, S.: Remarks on some fixed point results in b-metric spaces. J. Fixed Point Theory Appl. **20**, 147 (2018). https://doi.org/10.1007/s11784-018-2
6. Aleksić, S., Mitrović, Z., Radenović, S.: On some recent fixed point results for single and multi-valued mappings in b-metric spaces. Fasciculi Mathematici (2018). https://doi.org/10.1515/fascmath.2018-0013
7. Allahyari, R., Arab, R., Haghighi, A.S.: A generalization on weak contractions in partially ordered b-metric spaces and its applications to quadratic integral equations. J. Inequal. Appl. **2014**, 355 (2014). https://doi.org/10.1186/1029-242X-2014-355
8. Amini-Harandi, A.: Fixed point theory for quasi-contraction maps in b-metric spaces. Fixed Point Theory **15**(2), 351–358 (2014)
9. Aydi, H., Felhi, A., Sahmim, S.: On common fixed points for (α, ψ)-contractions and generalized cyclic contractions in b-metric-like spaces and consequences. J. Nonlinear Sci. Appl. **9**, 2492–2510 (2016)
10. Aydi, H., Dedović, N., Bin-Mohsin, B., Filipović, M., Radenović, S.: Some new observations on Geraghty and Ćirić type results in b-metric spaces. Mathematics **7**, 643 (2019). https://doi.org/10.3390/math707064
11. Bakhtin, I.A.: The contraction principle in quasimetric spaces. Func. An. Ulianowsk Gos. Fed. Ins. **30**, 26–37 (1989)
12. Belay Mitiku., Kalyani, K., Rao, N.S.: Some fixed point results of generalized (ϕ, ψ)-contractive mappings in ordered b-metric spaces. BMC Res. Notes **13**, 537 (2020). https://doi.org/10.1186/s13104-020-05354-1
13. Bhaskar, T.G., Lakshmikantham, V.: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379–1393 (2006). https://doi.org/10.1016/j.na.2005.10.017
14. Ćirić, L.: Some Recent Results in Metrical Fixed Point Theory. University of Belgrade, Belgrade (2003)
15. Ćirić, L.J., Cakic, N., Rajović, M., Ume, J.S.: Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. (2008). https://doi.org/10.1155/2008/131294. (**article id 131294, 11 pages**)
16. Czerwik, S.: Contraction mappings in b-metric spaces. Acta Math. Univ. Ostrav. **1**, 5–11 (1993)
17. Debnath, P., Konwar, N., Radenović, S.: Metric Fixed Point Theory: Applications in Science. Engineering and Behavioural Sciences. Springer Nature Singapore, Singapore (2021)
18. Faraji, H., Savić, D., Radenović, S.: Fixed point theorems for Geraghty contraction type mappings in b-metric spaces and applications. Aximos **8**, 34 (2019)
19. Huang, H., Radenović, S., Vujaković, J.: On some recent coincidence and immediate consequences in partially ordered b-metric spaces. Fixed Point Theory Appl. **2015**, 63 (2015). https://doi.org/10.1186/s13663-015-0308-3
20. Huang, H., Xu, S.: Fixed point theorems of contractive mappings in cone b-metric spaces and applications. Fixed Point Theory Appl. **2012**, 1–8 (2012)

21. Hussain, N.Z., Mitrović, D., Radenović, S.: A common fixed point theorem of Fisher in *b*-metric spaces. Revista de la Real Academia de Ciencias Exactas Físicas y Naturales. Serie A. Matemáticas **113**, 949–956 (2019). https://doi.org/10.1007/s13398-018-0524-x
22. Jachymski, J.: Equivalent conditions for generalized contractions on (ordered) metric spaces. Nonlinear Anal. **74**, 768–774 (2011)
23. Jovanović, M., Kadelburg, Z., Radenović, S.: Common fixed point results in metric-type spaces. Fixed Point Theory Appl. (2010). https://doi.org/10.1155/2010/978121. (**article id 978121, 15 pages**)
24. Kalyani, K., N. Seshagiri Rao., Mitiku, B.: Some fixed point results in ordered *b*-metric space with auxiliary function. Adv. Theory Nonlinear Anal. Appl. **5**(3), 421–432 (2021). https://doi.org/10.31197/atnaa.758962
25. Kalyani, K., N. Seshagiri Rao., Mitiku, B.: Fixed point results of contractive mappings with altering distance functions in ordered *b*-metric spaces. Inf. Sci. Lett. **10**(2), 267–275 (2021). https://doi.org/10.18576/isl/100211
26. Kalyani, K., Seshagiri Rao, N., Mishra, L.N.: Coupled fixed points theorems for generalized weak contractions in ordered *b*-metric spaces. Asian Eur. J. Math. **15**(3), (2022). https://doi.org/10.1142/S1793557122500504
27. Kalyani, K., Seshagiri Rao, N.: Fixed point results of weakly contraction mappings in partially ordered *b*-metric spaces. BMC Res. Notes **15**, 29 (2022). https://doi.org/10.1186/s13104-022-05914-7
28. Kirk, W.-A., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer, Berlin (2014)
29. Kirk, W., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer International Publishing, Cham (2014)
30. Lakshmikantham, V., Ćirić, L.J.: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341–4349 (2009). https://doi.org/10.1016/j.na.2008.09.020
31. Roshan, J.R., Parvaneh, V., Sedghi, S., Shobkolaei, N., Shatanawi, W.: Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered *b*-metric spaces. Fixed Point Theory Appl. **2013**(159), 1–23 (2013)
32. Roshan, J.R., Parvaneh, V., Altun, I.: Some coincidence point results in ordered *b*-metric spaces and applications in a system of integral equations. Appl. Math. Comput. **226**, 725–737 (2014)
33. Seshagiri Rao, N., Kalyani, K.: Fixed point theorems for nonlinear contractive mappings in ordered *b*-metric space with auxiliary function. BMC Res. Notes **13**, 451 (2020). https://doi.org/10.1186/s13104-020-05273-1
34. Seshagiri Rao, N., Kalyani, K., Prasad, K.: Fixed point results for weak contractions in partially ordered *b*-metric space. BMC Res. Notes **14**, 263 (2021). https://doi.org/10.1186/s13104-021-05649-x
35. Seshagiri Rao, N., Kalyani, K.: Some fixed point results of (ϕ, ψ, θ)-contractive mappings in ordered *b*-metric spaces. Math. Sci. (2021). https://doi.org/10.1007/s40096-021-00408-2
36. Shatanawi, W., Pitea, A., Lazović, R.: Contraction conditions using comparison functions on *b*-metric spaces. Fixed Point Theory Appl. **2014**(135), 1–11 (2014)
37. Todořcević, V.: Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics. Springer Nature Switzerland AG, Cham (2019)