Essential ideals represented by mod-annihilators of modules

Rameez Raja[1] · Shariefuddin Pirzada[2]

Abstract

Let R be a commutative ring with unity, M be a unitary R-module and G a finite abelian group (viewed as a $\mathbb{Z}$-module). The main objective of this paper is to study properties of *mod-annihilators* of M. For $x \in M$, we study the ideals $[x : M] = \{r \in R \mid rM \subseteq Rx\}$ of R corresponding to *mod-annihilator* of M. We investigate as when $[x : M]$ is an essential ideal of R. We prove that the arbitrary intersection of essential ideals represented by *mod-annihilators* is an essential ideal. We observe that $[x : M]$ is injective if and only if R is non-singular and the radical of $R/[x : M]$ is zero. Moreover, if essential socle of M is non-zero, then we show that $[x : M]$ is the intersection of maximal ideals and $[x : M]^2 = [x : M]$. Finally, we discuss the correspondence of essential ideals of R and vertices of the annihilating graphs realized by M over R.

Keywords Module · Ring · Essential ideal · Annihilator · Graph

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1 Introduction

A nonzero ideal in a commutative ring is called essential if it intersects with every other nonzero ideal nontrivially. The study of essential ideals in a ring R is a classical problem. For instance, Green and Van Wyk in [8] characterized essential ideals in certain classes of commutative and non-commutative rings. The authors in [5, 12] studied essential ideals in $C(X)$, where $C(X)$ denotes the set of continuous functions on X. They topologically characterized the scole and essential ideals. Moreover, essential ideals have been investigated in rings of measurable functions [14] and C^*-algebras [11]. For more on essential ideals, see [4, 9, 10, 21].

✉ Shariefuddin Pirzada
pirzadasd@kashmiruniversity.ac.in

Rameez Raja
rameeznaqash@nitsri.ac.in

1   Department of Mathematics, National Institute of Technology Srinagar, Srinagar, Kashmir, India

2   Department of Mathematics, University of Kashmir, Hazratbal, Srinagar, India

Throughout, R is a commutative ring (with $1 \neq 0$) and all modules are unitary unless otherwise stated. $[N : M] = \{r \in R \mid rM \subseteq N\}$ denotes an ideal of R. The symbols $\subseteq$ and $\subset$ have usual set theoretic meaning as containment and proper containment. We will denote the ring of integers by $\mathbb{Z}$, positive integers by $\mathbb{N}$ and the ring of integers modulo n by $\mathbb{Z}_n$. For basic definitions from ring and module theory we refer to [7, 25].

For a R-module M and $x \in M$, set $[x : M] = \{r \in R \mid rM \subseteq Rx\}$, which clearly is an ideal of R and an annihilator of the factor module M/Rx, whereas the annihilator of M denoted by $ann(M)$ is $[0 : M]$.

Recently in [17], the elements of a module M have been classified into *full-annihilators*, *semi-annihilators* and *star-annihilators*. We recall a definition concerning full-annihilators, semi-annihilators and star-annihilators of a module M.

Definition 1.1 An element $x \in M$ is a

  (i) full-annihilator, if either $x = 0$ or $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $[y : M] \neq R$,
 (ii) semi-annihilator, if either $x = 0$ or $[x : M] \neq 0$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $0 \neq [y : M] \neq R$,
(iii) star-annihilator, if either $x = 0$ or $ann(M) \subset [x : M]$ and $[x : M][y : M]M = 0$, for some nonzero $y \in M$ with $ann(M) \subset [y : M] \neq R$.

We denote by $A_f(M)$, $A_s(M)$ and $A_t(M)$, respectively, the sets of full-annihilators, semi-annihilators and star-annihilators for any module M over R and call these annihilators as *mod-annihilators*. We set $\widehat{A_f(M)} = A_f(M)/\{0\}$, $\widehat{A_s(M)} = A_s(M)/\{0\}$ and $\widehat{A_t(M)} = A_t(M)/\{0\}$.

This paper is organized as follows. In Sect. 2, we study the correspondence of essential ideals in R and submodules of M represented by mod-annihilators. For some finite abelian group G (viewed as a $\mathbb{Z}$-module), we determine the value of n such that $[x : G] = n\mathbb{Z}$, where $x \in G$. We characterize all $\mathbb{Z}$-module M such that $[x : M]$ is an essential ideal of R. Furthermore, we discuss when $[x : M]$ as a R-module is injective. Also, if essential socle of M is non-zero, then we prove that $[x : M]$ is the intersection of maximal ideals and $[x : M]^2 = [x : M]$. In Sect. 3, we discuss the correspondence of essential ideals of R and the vertices of the annihilating graphs realized by modules over commutative rings. We conclude this paper with a discussion on some problems in this area of research.

2 Essential ideals represented by mod-annihilators

In this section, we discuss the correspondence of essential ideals in R represented by elements of $\widehat{A_f(M)}$, and submodules of M generated by elements of $\widehat{A_f(M)}$. We characterize essential ideals corresponding to $\mathbb{Z}$-modules. We discuss the cases of finite abelian groups where essential ideals which are represented by elements of $\widehat{A_f(M)}$ corresponding to submodules of M are isomorphic. If M is a non-simple R-module, then for $x \in \widehat{A_f(M)}$, we show that an ideal $[x : M]$ considered as an R-module is injective. We also study essential ideals represented by mod-annihilators over hereditary and regular rings.

By Definition 1.1, we see that there is a correspondence of ideals in R represented by elements of $\widehat{A_f(M)}$, $\widehat{A_s(M)}$, and $\widehat{A_t(M)}$ and cyclic submodules of M generated by elements of sets $\widehat{A_f(M)}$, $\widehat{A_s(M)}$, and $\widehat{A_t(M)}$. Furthermore, the containment $A_t(M) \subseteq A_s(M) \subseteq$

$A_f(M)$ is clear, so our main emphasis is on the set $\widehat{A_f(M)}$. However, one can study these sets separately for any module M.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition of n denoted by $\lambda \vdash n$. For any $\mu \vdash n$, we have an abelian group of order p^n and conversely every abelian group corresponds to some partition of n. In fact, if $H_{\mu, p} = \mathbb{Z}/p^{\mu_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\mu_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\mu_r}\mathbb{Z}$ is a subgroup of $G_{\lambda, p} = \mathbb{Z}/p^{\lambda_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\lambda_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\lambda_r}\mathbb{Z}$, then $\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2, \ldots, \mu_r \leq \lambda_r$. If these inequalities hold, we write $\mu \subset \lambda$, that is a "containment order" on partitions. For example, a p-group $\mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ is of type $\lambda = (5, 1, 1)$. The possible types for its subgroup are $(5, 1, 1), (4, 1, 1), (3, 1, 1), (2, 1, 1), (1, 1, 1), 2(5, 1), 2(4, 1), 2(3, 1), 2(2, 1), 2(1, 1), (5),$ $(4), (3), (2), 2(1)$. Note that the types $(5, 1), (4, 1), (3, 1), (2, 1), (1, 1)$ are appearing twice in the sequence of partitions for a subgroup.

Let $\lambda = (1, 1, \ldots, 1) = (1^n)$. A group of type λ is nothing but the $\mathbb{Z}/p\mathbb{Z}$-vector space $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$. Its subgroups are of type (1^r), where $0 \leq r \leq n$. The essential ideals corresponding to subspaces of vector space $\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}$ (represented by elements of the set $A_f(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z})$ are same. In fact, $[x : \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}] = ann(\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p\mathbb{Z}) = p\mathbb{Z}$.

More generally, for a finite abelian p-group of the type $\mathbb{Z}/p^\alpha\mathbb{Z} \oplus \mathbb{Z}/p^\alpha\mathbb{Z} \cdots \oplus \mathbb{Z}/p^\alpha\mathbb{Z}$, where $\alpha \geq 2$, the essential ideals represented by elements of the set $A_f(\mathbb{Z}/p^\alpha\mathbb{Z} \oplus \mathbb{Z}/p^\alpha\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^\alpha\mathbb{Z}) = p^\alpha\mathbb{Z}$.

A finite abelian p-group is isomorphic to the group of the form $\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z}$ whereas a finitely generated abelian p-group with Betti number n is of the from $\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. It is very difficult to determine the exact ideals represented by mod-annihilators of sets $A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z})$ and $A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$. However, it is clear from the definition of mod-annihilators that for some $x \in A_f(\mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z})$, $[x : \mathbb{Z}/p^{\alpha_1}\mathbb{Z} \oplus \mathbb{Z}/p^{\alpha_2}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^{\alpha_n}\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}]$ is some ideal in $\mathbb{Z}$.

Using the description given above, we now characterize all essential ideals represented by elements of $\widehat{A_f(M)}$ and corresponding to $\mathbb{Z}$-modules.

Lemma 2.1 *If M is any $\mathbb{Z}$-module, then $[x : M]$ is an essential ideal if and only if $[x : M]$ is non-zero for all $x \in \widehat{A_f(M)}$.*

Proof Let M be a $\mathbb{Z}$-module. Clearly, M is an abelian group in a unique way. For all $x \in \widehat{A_f(M)}$, we have $[x : M] = n\mathbb{Z}, n \in \mathbb{N}$. The ideal $n\mathbb{Z}$ intersects non-trivially with any ideal $m\mathbb{Z}, m \in \mathbb{N}$ in $\mathbb{Z}$. So, if M is a non-simple $\mathbb{Z}$-module, then for every $x \in M$, it follows that $[x : M]$ is an essential ideal. Note that M is simple if and only if $\widehat{A_f(G)} = \emptyset$.

If possible, suppose $[x : M] = \{0\}$, then $[x : M]$ does not intersect non-trivially with non-trivial ideals of $\mathbb{Z}$, a contradiction. □

Since it is possible to have some finitely generated $\mathbb{Z}$-modules such that the set of mod-annihilators is equal to zero only which of course by definition is not an essential ideal. Consider a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, which is a direct sum of n copies of $\mathbb{Z}$. It is easy to verify that $\widehat{A_f(M)} = \widehat{M}$ with $[x : M][y : M]M = 0$ for all $x, y \in M$. The cyclic submodules generated by elements of $\widehat{A_f(M)}$ are simply lines with integral coordinates passing through the origin in the hyperplane $\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ and these lines intersect at the origin only. Thus, for each $x \in M$, it follows that $[x : M]$ is not an essential ideal in $\mathbb{Z}$. In fact $[x : M]$ is a zero-ideal in $\mathbb{Z}$.

For any R-module M and $x \in \widehat{A_f(M)}$, it would be interesting to characterize essential ideals $[x : M]$ represented by elements of $\widehat{A_f(M)}$ such that the intersection of all essential

ideals is again an essential ideal. It is easy to see that a finite intersection of essential ideals in any commutative ring is an essential ideal. But an infinite intersection of essential ideals need not to be an essential ideal, even a countable intersection of essential ideals in general is not an essential ideal, as can be seen in [5]. If the cardinality of M is finite over R, then the submodules determined by elements of $\widehat{A_f(M)}$ are finite and therefore the ideals corresponding to submodules are finite in number. Thus, we conclude that for every $x \in \widehat{A_f(M)}$, the intersection of essential ideals $[x : M]$ in R is an essential ideal. For the other case, that is, if the cardinality of M is infinite over R, we have the following result. Note that, a nonzero submodule of a module M is said to be an essential submodule of M if it intersects non-trivially with other nonzero submodules of M.

Theorem 2.2 *Let M be an R-module such that every proper submodule of M is cyclic over R. For $x \in \widehat{A_f(M)}$, if the submodule generated by x intersects non-trivially with every other nonzero submodule of M, then $[x : M]$ is an essential ideal in R.*

Proof Assume that $\bigcap_{0 \neq x \in M} Rx \neq 0$. If $\widehat{A_f(M)} = \phi$, then M is simple, a contradiction. Let $x \in \widehat{A_f(M)}$ and let Rx be the submodule generated by x. Since Rx intersects non-trivially with every other submodule, so there exists $y \in \widehat{A_f(M)}$ such that $Rx \cap Ry \neq 0$. It suffices to prove the result for $Rx \cap Ry$. Let $z \in Rx \cap Ry$ and let $[x : M]$, $[y : M]$, $[z : M]$ be ideals of R corresponding to submodules Rx, Ry and Rz. Then $[z : M] \subseteq [x : M] \cap [y : M] \neq 0$, which implies $[x : M]$ intersects non-trivially with every nonzero ideal corresponding to the submodule generated by an element of $\widehat{A_f(M)}$. For any other ideal I of R, it is clear that $IM = \{\sum_{finite} am : a \in I, \ m \in M\} = Ra$ for some $a \in M$. Thus I corresponds to the cyclic submodule generated by $a \in M$. It follows that $[x : M] \cap I \neq 0$, for every nonzero ideal of R and we conclude that $[x : M]$ is an essential ideal for each $x \in \widehat{A_f(M)}$.   □

The converse of Theorem 2.2 is not true in general. We can easily construct examples from $\mathbb{Z}$-modules such that an ideal corresponding to the submodule generated by some element of $\widehat{A_f(M)}$ is an essential ideal, but the intersection of all submodules determined by elements of $\widehat{A_f(M)}$ is empty. However, if every ideal $[x : M]$, where $x \in \widehat{A_f(M)}$, corresponds to an essential submodule of M, then we have a non-zero intersection.

Corollary 2.3 *Let M be an R-module.*

 (i) *For $x \in \widehat{A_f(M)}$, if the cyclic submodule Rx intersects with every other cyclic nonzero submodule of M non-trivially, then $[x : M]$ is an essential ideal in R.*
 (ii) *The intersection $\bigcap_{x \in \widehat{A_f(M)}} [x : M]$ is an essential ideal in R if and only if every submodule of M is essentially cyclic over R.*

In the preceding results, we proved that "arbitrary intersection of essentials ideals is an essential ideal". We formulated this theory of essential ideals using the concept of mod-annihilators and mainly the theory involves study of cyclic submodules of M. It is interesting to develop a similar theory that would employ the other finitely generated submodules of M. So, motivated by [5], we have the following question regarding essential ideals represented by elements of $\widehat{A_f(M_N)}$, where $\widehat{A_f(M_N)} = \{r \in R \mid rM \subseteq N\}$, N is a finitely generated submodule of M.

Problem 2.4 *Let M be an R-module. For $x \in \widehat{A_f(M_N)}$, characterize essential ideals $[x : M]$ in R such that their intersection is an essential ideal.*

For an R-module M, let $Z(M)$ denote the following.

$$Z(M) = \{m \in M : ann(m) \text{ is an essential ideal in } R\}.$$

If $Z(M) = M$, then M is said to be singular and if $Z(M) = 0$, then M is said to be non-singular. By $rad(M)$, we denote the intersection of all maximal submodules of M. So, $rad(R)$ is the Jacobson radical $J(R)$ of a ring R. The socle of an R-module M denoted by $Soc(M)$ is the sum of simple submodules or equivalently the intersection of all essential submodules. To say that $Soc(M)$ is an essential socle is equivalent to saying that every cyclic submodule of M contains a simple submodule of M. An essential socle of M is denoted by $essoc(M)$.

Lemma 2.5 *Let M be an R-module with $essoc(M) \neq 0$, $\bigcap_{0 \neq x \in M} Rx \neq 0$. Then for $x \in \widehat{A_f(M)}$, $R/[x : M]$ is a singular module.*

Proof Since $\bigcap_{0 \neq x \in M} Rx \neq 0$ and $essoc(M) \neq 0$, therefore, $\widehat{A_f(M)} \neq \emptyset$. Thus, $[x : M]$ is an essential ideal. Moreover, $Z\big(R/[x : M]\big) = R/[x : M]$. Therefore, $R/[x : M]$ is a singular module. □

A ring R is said to be a *regular* ring if for all $a \in R$, $a^2 x = a$ for some $x \in R$.

Lemma 2.6 [24] *A commutative ring R with unity is regular if and only if every simple R-module is injective.*

Now, we consider singular simple R-modules (ideals) which are injective, and obtain some properties of essential ideals corresponding to submodules generated by elements of $\widehat{A_f(M)}$.

Theorem 2.7 *Let M be an R-module with $essoc(M) \neq 0$ and $\bigcap_{0 \neq x \in M} Rx \neq 0$. Then every singular simple R-module $[x : M]$, $x \in \widehat{A_f(M)}$ is injective if and only if $Z(R) = 0$ and $rad(R/[x : M]) = 0$.*

Proof We have $essoc(M) \neq 0$ and $\bigcap_{0 \neq x \in M} Rx \neq 0$, so that $\widehat{A_f(M)} \neq \emptyset$. Therefore corresponding to every cyclic submodule generated by elements of $\widehat{A_f(M)}$, we have an ideal in R. For $x \in \widehat{A_f(M)}$, suppose all singular simple R-modules $[x : M]$ are injective. If for some $z \in \widehat{A_f(M)}$, $I = [z : M] \subseteq Z(R)$ is a simple R-module, then $Z(I) = I$. This implies that I is injective and thus a direct summand of R. However, the set $Z(R)$ is free from nonzero idempotent elements. Therefore, $I = 0$ and so $Z(R) = 0$. For $x \in \widehat{A_f(M)}$, clearly $A = [x : M]$ is an essential ideal of R. Thus, by Lemma 2.5, R/A is a singular module and so is every submodule of R/A. Therefore every simple submodule of R/A is injective, which implies that every simple submodule is excluded by some maximal submodule. Thus we conclude that $rad(R/A) = 0$.

For the converse, we again consider the correspondence of cyclic submodules of M and ideals of R. Let $\tilde{I}$ be a singular simple R-module corresponding to the submodule of M. In order to show that $\tilde{I}$ is injective, we must show that for every essential ideal A in R corresponding to the submodule determined by an element $x \in \widehat{A_f(M)}$, every $\varphi \in Hom_R(A, \tilde{I})$ has a lift $\psi \in Hom_R(R, \tilde{I})$ such that the following diagram commutes.

$$A \xrightarrow{\ i\ } R$$
$$\varphi \searrow \quad \downarrow \psi$$
$$\tilde{I}$$

Let $K = ker(\varphi)$. We claim that K is an essential ideal of R. For, if $K \cap J = \{0\}$, for some nonzero ideal J of R, then $I^* = J \cap A \neq 0$ and $I^* \cap K = \{0\}$. This implies that $I^* \subseteq \varphi(I^*) \subseteq \tilde{I}$, a contradiction, since $\tilde{I}$ is a singular simple submodule and $Z(R) = 0$. For $\mu \neq 0$, it is clear that φ induces an isomorphism $\mu : A/K \longrightarrow \tilde{I}$. So, A/K is a simple R-submodule of R/K. By our assumption, $rad(R/K) = 0$, so there is a maximal submodule M/K such that $R/K = A/K \oplus M/K$. Let $g : R \longrightarrow R/K$ be a canonical map and let $p : R/K \longrightarrow A/K$ be a projection map. Then, we have $pg : R \longrightarrow A/K$. Therefore the composition $\psi = \mu pg : R \longrightarrow \tilde{I}$ is the required lift such that the above diagram commutes. □

Now, we discuss some interesting consequences of the preceding theorem.

Theorem 2.8 *Let M be an R-module with $essoc(M) \neq 0$, $\bigcap_{0 \neq x \in M} Rx \neq 0$ and for $x \in \widehat{A_f(M)}$, let every singular simple R-module $[x : M]$ be injective. Then every ideal $[x : M]$ is an intersection of maximal ideals, $J(R)^2 = 0$ and $[x : M]^2 = [x : M]$.*

Proof For any $x \in \widehat{A_f(M)}$, clearly $[x : M]$ is an essential ideal in R. Therefore, $J(R) \subseteq [x : M]$, since $J(R)$ is contained in every essential ideal of R. On the other hand, intersection of all essential ideals in R is Socle of R, therefore $J(R) \subseteq Soc(R)$. This implies that $J(R)^2 = 0$ and $[x : M]$ is the intersection of maximal ideals in R. Suppose that $[x : M]^2 \neq [x : M]$, for an essential ideal $[x : M]$ of R. By Theorem 2.7, $Z(R) = 0$ and therefore for every essential ideal I, we have $I \subseteq I^2$. In particular, $[x : M] \subseteq [x : M]^2$ for each $x \in \widehat{A_f(M)}$. It follows that $[x : M]^2$ is an essential ideal and is the intersection of maximal ideals in R. Finally, if $y \in [x : M]^2$, $y \notin [x : M]$, there is some maximal ideal P of R such that $[x : M] \subseteq P$, $y \notin P$. Then $R = Ry + P$, that is, $1 = ry + m$. This implies that $y = yry + ym \in P$, a contradiction. Hence we conclude that $[x : M]^2 = [x : M]$. □

Corollary 2.9 *Let M be an R-module, where R is hereditary. For $x \in \widehat{A_f(M)}$, if $[x : M]$ is an essential ideal of R and $J(R)^2 = 0$, then every singular simple R-module $[x : M]$ is injective.*

Proof Let R be hereditary. From [7], the exact sequence

$$0 \longrightarrow ann(x) \longrightarrow R \longrightarrow Rx \longrightarrow 0$$

splits for any $x \in R$. Since $J(R)^2 = 0$ and $R/J(R)$ is an artinian ring, therefore $J(R) \subseteq Soc(R)$. But any essential ideal of R contains $Soc(R)$. So, $J(R) \subseteq [x : M]$. This implies that $R/[x : M]$ is a completely reducible R-module and therefore $rad(R/[x : M]) = 0$. Thus, by Theorem 2.7, every singular simple R-module $[x : M]$ is injective. □

Next, we consider the modules over regular rings.

Theorem 2.10 *Let M be an R-module such that every submodule of M is cyclic over R and $\bigcap_{0 \neq x \in M} Rx \neq 0$. The following are equivalent.*

(i) *R is regular.*
(ii) *$A^2 = A$ for each ideal A of R.*

(iii) $[x : M]^2 = [x : M]$ *for each* $x \in \widehat{A_f(M)}$.

Proof The equivalence of (i) and (ii) is clear and certainly (ii) implies (iii). Thus, we just need to show that (iii) implies (ii). By Theorem 2.7, $[x : M]$ is an essential ideal for each $x \in \widehat{A_f(M)}$. Suppose that $[x : M]^2 = [x : M]$. Choose J to be maximal ideal of R such that $A \cap J = 0$, where A is some non essential ideal of R. Then $A + J$ is an essential ideal of R. Therefore, again by Theorem 2.7, $A + J$ corresponds to some submodule of M and we have $A + J = [z : M]$ for some $z \in M$. So, $(A + J)^2 = A^2 + J^2 = A + J$. If $x \in A$, then $x = \sum_{finite} ab + \sum_{finite} mn$, where $a, b \in A$ and $m, n \in J$. Therefore,

$$x - \sum_{finite} ab = \sum_{finite} mn \in A \cap J = 0.$$

This implies that $x \in A^2$ and we conclude that $A = A^2$. $\qquad \square$

Corollary 2.11 *Let M be an R-module with $essoc(M) \neq 0$ and $\bigcap_{x \in M} Rx \neq 0$. Then every singular simple R-module $[x : M]$, where $x \in \widehat{A_f(M)}$, is injective if and only if R is regular.*

Proof By Theorem 2.8, if every singular simple R-module $[x : M]$ is injective, then for $x \in \widehat{A_f(M)}$, we have $[x : M]^2 = [x : M]$. Therefore, by Theorem 2.10, R is regular. If R is regular, then by Lemma 2.6 every simple R-module is injective. $\qquad \square$

3 Representation of essential ideals by vertices of annihilating graphs

In this section, we give a brief discussion on representation of essential ideals by vertices of graphs realized by modules over commutative rings.

A simple *graph* Γ consists of a *vertex set* $V(\Gamma)$ and an *edge set* $E(\Gamma)$, where an edge is an unordered pair of distinct vertices of Γ. One of the areas in algebraic combinatorics introduced by Beck [6] is to study the interplay between graph theoretical and algebraic properties of an algebraic structure. Continuing the concept of associating a graph to an algebraic structure, another combinatorial approach of studying commutative rings was given by Anderson and Livingston in [2]. They associated a simple graph to a commutative ring R with unity called the zero-divisor graph denoted by $\Gamma(R)$ with vertex set $Z^*(R) = Z(R)/\{0\}$, where two distinct vertices $x, y \in Z^*(R)$ are adjacent in $\Gamma(R)$ if and only if $xy = 0$. The study of graph theoretical parameters and spectral properties in zero-divisor graphs of commutative rings are explored in [1–3, 15, 16, 18, 20, 22]. In [2, 18], authors have discussed chromatic number, clique number and metric dimensions of zero-divisor graphs associated with finite commutative rings whereas [16, 22] are related to eigenvalues and Laplacian eigenvalues of zero-divisor graphs associated to finite commutative rings of type $\mathbb{Z}_n$ for $n = p^{N_1} q^{N_2}$, where $p < q$ are primes and N_1, N_2 are positive integers. The extension of zero-divisor graphs to non-commutative rings and semigroups can be found in [13, 23].

The combinatorial properties of zero-divisors discovered in [6] have also been investigated in module theory. In [17], the authors introduced annihilating graphs realized by modules over commutative rings known as *full-annihilating, semi-annihilating and star-annihilating* graphs, denoted by $ann_f(\Gamma(M))$, $ann_s(\Gamma(M))$ and $ann_t(\Gamma(M))$. The vertices of annihilating graphs are elements of sets $\widehat{A_f(M)}$, $\widehat{A_s(M)}$ and $\widehat{A_t(M)}$ respectively, where two vertices x and y are adjacent if and only if $[x : M][y : M]M = 0$. The three simple graphs: full-annihilating, semi-annihilating and star-annihilating with vertex sets: $\widehat{A_f(M)}$, $\widehat{A_s(M)}$, $\widehat{A_t(M)}$ are natural

generalizations of the zero-divisor graph introduced in [2]. This concept was further studied in [19].

We call a vertex x, an *essential vertex* in $ann_f(\Gamma(M))$ if the ideal represented by x is essential in R. Recall that a graph Γ is said to be a *complete* if there is an edge between every pair of distinct vertices.

By Definition 1.1, we see the containment $ann_t(\Gamma(M)) \subseteq ann_s(\Gamma(M)) \subseteq ann_f(\Gamma(M))$ as induced subgraphs of the graph $ann_f(\Gamma(M))$, since $A_t(M) \subseteq A_s(M) \subseteq A_f(M)$. If $ann_f(\Gamma(M))$ is a finite graph, then by [17,Theorem 3.3 and Example 2.2], $|\widehat{A_f(M)}| = |\widehat{A_s(M)}|$ and annihilating graphs $ann_f(\Gamma(M))$, $ann_s(\Gamma(M))$ coincide, whereas the graph $ann_t(\Gamma(M))$ with vertex set $\widehat{A_t(M)}$ may be different. For a $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, we have by Definition 1.1, $[x : M][y : M]M = 0$ for all $x, y \in \widehat{A_f(M)}$. Therefore $ann_f(\Gamma(M))$ is a complete graph whereas the graph $ann_s(\Gamma(M))$ is an empty graph. Thus for finitely generated infinite modules, graphs $ann_f(\Gamma(M))$ and $ann_s(\Gamma(M))$ are different.

As discussed in Sect. 2, for a module $M = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$, the ideal $[x : M]$ represented by a vertex $x \in \widehat{A_f(M)}$ of the graph $ann_f(\Gamma(M))$ is not an essential ideal. So, x is not an essential vertex of the graph $ann_f(\Gamma(M))$. On the other hand, every vertex of a $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_q$ is an essential vertex of the graph $ann_f(\Gamma(\mathbb{Z}_p \oplus \mathbb{Z}_q))$, where p and q are any two primes.

Finally, Problem 2.4 can be restated in the graph theoretical version as follows.

Problem 3.1 *Characterize all annihilating graphs realized by a module M such that every vertex $x \in \widehat{A_f(M_N)}$ of an annihilating graph is an essential vertex.*

4 Conclusion

In this paper, we formulated a new approach of recognition of essential ideals in a commutative ring R. This formulation of essential ideals corresponds to mod-annihilators of a R-module M. It is interesting to characterize essential ideals such that their arbitrary intersection is an essential ideal, since it is specified in [5] that an arbitrary intersection of essential ideals may not be an essential ideal. Furthermore, we obtained the results related to ideals $[x : M]$ of R, where x is a mod-annihilator of M and discussed the representation of vertices of annihilating graphs by essential ideals of R. Apart from the research problems which we mentioned in Sects. 2 and 3, the following problems could be investigated for the future work.
1. Let G be a finite abelian p-group (viewed as a finite $\mathbb{Z}$-module) of rank at least 3. Determine the value of n for the essential ideal $[x : G] = n\mathbb{Z}$, where $x \in G$.
2. Let G be any finite abelian group (viewed as a finite $\mathbb{Z}$-module). Determine the value of n for the essential ideal $[x : G] = n\mathbb{Z}$, where $x \in G$.

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