Afrika Matematika (2022) 33:36
https://doi.org/10.1007/s13370-022-00969-y

The Jacobson Property in Banach algebras

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Received: 25 March 2021 / Accepted: 2 March 2022 / Published online: 21 March 2022

Abstract
In a Banach algebra A it is well known that the usual spectrum has the following property:

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$$

for elements $a, b \in A$. In this note we are interested in subsets of A that have the Jacobson Property, i.e. $X \subset A$ such that for $a, b \in A$:

$$1 - ab \in X \implies 1 - ba \in X.$$

Keywords Banach algebra · Regularities · Semiregularities · Spectral theory

Mathematics Subject Classification 46H05

1 Preliminaries

All algebras in this paper are complex and unital. Denote by A^{-1} the group of invertible elements in a Banach algebra A and by $\sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$ the ordinary spectrum of $a \in A$. When no confusion can arise we write simply $\sigma(a)$. Denote by A_ℓ^{-1} and by A_r^{-1} the left and right invertibles respectively. If $K \subset \mathbb{C}$, we use the symbol acc K to indicate the set of accumulation points of K and the symbol iso K for the set of isolated points of K. The topological boundary is denoted by ∂K and the closure by $\overline{K}$. If $X \subset A$ we say that X has the Jacobson Property, if for $a, b \in A$:

$$1 - ab \in X \implies 1 - ba \in X. \tag{1}$$

By an ideal in A we mean a two-sided ideal. An ideal J in A is said to be *inessential* [1], p. 106 if

$$a \in J \implies \operatorname{acc} \sigma(a) \subset \{0\},$$

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so that the spectrum of an element of J is either finite or a sequence converging to zero. The collection of all inessential elements in A will be denoted by $\mathcal{I}(A)$. Note that $\mathcal{I}(A) = \{a \in A : \operatorname{acc}\sigma(a) \subset \{0\}\}$. An element $a \in A$ is *quasinilpotent* if $\sigma(a) = \{0\}$. The collection of these elements will be denoted by $\mathrm{QN}(A)$. We say $a \in A$ is *almost invertible* if $0 \notin \operatorname{acc}\sigma(a)$. The set of almost invertible elements in A will be denoted by A_{ai}.

If A is a ring or an algebra, we call an element $a \in A$ *regular* if it has a *generalised inverse*, $b \in A$ for which $a = aba$, and we write

$$\widehat{A} = \{a \in A : a \in aAa\} \tag{2}$$

for the collection of regular elements. These include both the left and right invertible elements,

$$A_\ell^{-1} \cup A_r^{-1} \subset \widehat{A} \tag{3}$$

as well as the idempotents,

$$A^\bullet = \left\{a \in A : a = a^2\right\}. \tag{4}$$

The *decomposably regular* elements are those which admit invertible generalised inverses; they are precisely those elements which can be written as a product of an invertible and an idempotent:

$$A^{-1}A^\bullet = A^\bullet A^{-1} = \left\{a \in A : a \in aA^{-1}a\right\} \subset \widehat{A}. \tag{5}$$

Notice that

$$A_\ell^{-1} \cap A_r^{-1} = A_\ell^{-1} \cap A^{-1}A^\bullet = A_r^{-1} \cap A^{-1}A^\bullet = A^{-1}. \tag{6}$$

For our next observation one needs completeness. So, if A is a Banach algebra, then by [7, Theorem 7.3.4]

$$A^{-1}A^\bullet = \widehat{A} \cap \overline{A^{-1}}. \tag{7}$$

The following notion finds most of its application in the setting of commutative rings and algebras. As the author of [11] comments in that article, it is fine to use it in a noncommutative setting if needed. We will illustrate one such use later in this article.

Definition 1.1 Let I be an ideal in a Banach algebra A. I is *subprime* if $ab \in I \implies a \in I$ or $b \in I$.

The notion of a subprime ideal defined above coincides with that of a *prime* ideal in a commutative ring or algebra. The generalization of the notion of prime ideal to a (possibly noncommutative) ring or algebra is different from the notion of prime in a commutative ring or algebra. As one would expect, the more general definition of a prime ideal is equivalent to the idea of a prime ideal if the ring or algebra happens to be commutative.

Definition 1.2 [13, Definition 6.1] A nonempty subset R of a Banach algebra A is called a *regularity* if it satisfies the following two conditions:

(i) if $a \in A, n \in \mathbb{N}$ then $a \in R \iff a^n \in R$;
(ii) if a, b, c, d are mutually commuting elements of A satisfying $ac + bd = 1$ then $a, b \in R \iff ab \in R$.

It is easy to see that the following theorem holds.

Theorem 1.1 [13, Theorem 6.4] Let R be a nonempty subset of a Banach algebra A satisfying:

$$ab \in R \iff a, b \in R \tag{P1}$$

for all commuting elements $a, b \in A$. Then R is a regularity.

A regularity that satisfies the condition of the above theorem will be called a $(P1)$ regularity. The definition of a regularity can be divided into two parts as follows:

Definition 1.3 [13, Definition 23.1] A nonempty subset R in a Banach algebra A will be called a *lower semiregularity* if it satisfies the following conditions:

(i) $a \in A$, $n \in \mathbb{N}$ and $a^n \in R \Rightarrow a \in R$,
(ii) if a, b, c, d are commuting elements of A and $ac + bd = 1$, then $ab \in R \Rightarrow a \in R$ and $b \in R$.

If a nonempty subset R in A satisfies

$$a, b \in A, ab = ba, ab \in R \Rightarrow a \in R \text{ and } b \in R \tag{P1}$$

then clearly it is a lower semiregularity, see [13, Remark 23.3]

Definition 1.4 [13, Definition 23.10] A nonempty subset R in a Banach algebra A is called an *upper semiregularity* if it satisfies the following conditions:

(i) $a \in R$, $n \in \mathbb{N} \Rightarrow a^n \in R$,
(ii) if a, b, c, d are commuting elements of A and $ac + bd = 1$, then $a, b \in R \Rightarrow ab \in R$,
(iii) R contains a neighbourhood of the unit element 1.

It is not difficult to see that any semigroup containing a neighbourhood of the identity is an upper semiregularity, see [13, Remark 23.11]. Any subset R in A assigns to each $a \in A$ a subset of $\mathbb{C}$ defined by

$$\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda - a \notin R\}.$$

This mapping will be called the *spectrum corresponding* to R. If R is a regularity or a semiregularity then the spectrum σ_R has interesting properties, see [13, sections 6, 23]. Our interest in the Jacobson Property is the following: Let R be a nonempty subset of a Banach algebra A. Then

$$R \text{ has the Jacobson Property} \quad \Longleftrightarrow \quad \sigma_R(ab)\backslash\{0\} = \sigma_R(ba)\backslash\{0\}$$

for all $a, b \in A$

The fact that the invertible group A^{-1} in A has the Jacobson Property follows from [1, Lemma 3.1.2]. In [13] there are many examples of subsets in a Banach algebra possessing the Jacobson Property.

Let A be a Banach algebra and I a closed ideal in A. We denote the quotient Banach algebra by A/I. The canonical homomorphism $\pi : A \to A/I$ is defined by $\pi(a) = a + I$ for $a \in A$. We define the sets:

$$\Phi_\ell(I) = \left\{a \in A : a + I \in (A/I)_\ell^{-1}\right\},$$
$$\Phi_r(I) = \left\{a \in A : a + I \in (A/I)_r^{-1}\right\}$$

and

$$\Phi(I) = \left\{a \in A : a + I \in (A/I)^{-1}\right\}$$

If $a \in \Phi_\ell(I)$ we call *a left Fredholm relative to I*. If $a \in \Phi_r(I)$ we call *a right Fredholm relative to I*. Finally, if $a \in \Phi(I)$ we call *a Fredholm relative to I*.

Denote the ideal $\pi^{-1}(\mathrm{Rad}(A/I))$ by $\mathrm{kh}(I)$. This is the largest ideal consisting of Riesz elements relative to the ideal I. We say that $a \in A$ is *Riesz relative to I* if $a + I \in \mathrm{QN}(A/I)$.

If $\mathcal{R}(I)$ denotes the collection of Riesz elements relative to I, then $\mathcal{R}(I) = \pi^{-1}(\mathrm{QN}(A/I))$. It is easy to see that

$$I \subset \mathrm{kh}(I) \subset \mathcal{R}(I)$$

and if I is a closed inessential ideal in A, then one can prove, [1,Corollary 5.7.5], that

$$I \subset \mathrm{kh}(I) \subset \mathcal{R}(I) \subset \mathcal{I}(A).$$

Let I be an ideal in a Banach algebra A. A function $\tau : I \to \mathbb{C}$ is called a *trace* on I if τ is linear, $\tau(p) = 1$ for every rank one idempotent $p \in I$ and $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in A$. We shall refer to an ideal on which a trace is defined as a *trace ideal*. If I is a trace ideal in A, then it is possible to define an index function ι on $\Phi(I)$, see [4,Definition 3.3]. This abstract index function has all the properties that the index function defined on the collection of Fredholm operators defined on a Banach space has, see [4, 5].

The left socle of A is the sum of all the minimal left ideals of A. The right socle of A is the sum of all minimal right ideals of A. If the left and right socle of A coincide then their common value is called the socle of A. We denote the socle of A by $\mathrm{Soc}(A)$.

2 Algebraic and topological results

In this section we exhibit basic properties of sets having the Jacobson Property.

Lemma 2.1 *Suppose that $X \subset Y \subset A$ and suppose that X, Y have the Jacobson Property. Then $Y \setminus X$ has the Jacobson Property.*

Proof Assume that X and Y are as described, and that $1 - ab \in Y \setminus X$. Then $1 - ab \in Y$ and $1 - ab \notin X$. Since Y has the Jacobson Property, $1 - ab \in Y \implies 1 - ba \in Y$. Similarly, since X has the Jacobson Property, $1 - ab \notin X \implies 1 - ba \notin X$. Hence we have that $1 - ba \in Y \setminus X$ and we have shown that $Y \setminus X$ has the Jacobson Property.

Remark 2.1 If we let $Y = A$ in the above lemma we obtain the following important special case:

If $X \subset A$ and X has the Jacobson Property, then $A \setminus X$ has the Jacobson Property.

Lemma 2.2 *Every proper subprime ideal has the Jacobson property.*

Proof Suppose $1 - ab \in I$, and I is a subprime ideal. Then since I is a proper ideal $ab \notin I$. Since I is an ideal, we also have that $a \notin I$ and $b \notin I$. Since I is an ideal $b(1 - ab) \in I$. Hence $b - bab \in I \implies (1 - ba)b \in I$. Since I is subprime and we know that $b \notin I$ we have that $1 - ba \in I$ as required.

Theorem 2.1 *Let I be a proper subprime ideal in a Banach algebra A. Then $A \setminus I$ is a $(P1)$ regularity that has the Jacobson Property.*

Proof Suppose $ab \in A \setminus I$. Then $ab \notin I$. Since I is a two sided ideal, we must have that $a \notin I$ and $b \notin I$. Hence $a, b \in A \setminus I$. Conversely, suppose $a, b \in A \setminus I$. Then $a \notin I$ and $b \notin I$. Since I is subprime, we must have $ab \notin I$, or $ab \in A \setminus I$.

To see that $A \setminus I$ has the Jacobson property, we note that since I is subprime, by Lemma 2.2 we have that I has the Jacobson Property. By Lemma 2.1 and the remark following it $A \setminus I$ also has the Jacobson Property.

Given the fact that the concept of *subprime* is usually used in a commutative context, in Corollary 4.2 we take care to construct a nontrivial, proper, two-sided subprime ideal in a non commutative Banach algebra, and in that way justify the use of the concept.

Theorem 2.1 immediately raises the question as to whether every $(P1)$ regularity is the complement of a proper subprime ideal. The question is easy to answer as A^{-1} is a $(P1)$ regularity but its complement is not an ideal in A.

Theorem 2.2 *Let A be a Banach algebra and let I be a closed ideal in A. Suppose $\pi : A \to A/I$ is the canonical homomorphism and $X \subset A/I$.*

(i) *If X has the Jacobson Property, then $\pi^{-1}(X)$ in A has the Jacobson Property.*
(ii) *If X does not have the Jacobson Property, then $\pi^{-1}(X)$ does not have the Jacobson Property.*

Proof (i) Let $a, b \in A$ and $1 - ab \in \pi^{-1}(X)$. Since π is a homomorphism, $\pi(1 - ab) = 1 - \pi(a)\pi(b) \in X$. Because X has the Jacobson Property $1 - \pi(b)\pi(a) \in X$. Again, since π is a homomorphism $\pi(1 - ba) \in X$, i.e., $1 - ba \in \pi^{-1}(X)$.

(ii) If X does not have the Jacobson Property, there are $a, b \in A$ with $1 - ab + I \in X$, but $1 - ba + I \notin X$. Hence, $1 - ab \in \pi^{-1}(X)$ and $1 - ba \notin \pi^{-1}(X)$.

This completes the proof.

Theorem 2.3 *Let A be a Banach algebra and let $X \subset A$ be a semigroup that contains a neignbourhood of the identity. If X has the Jacobson Property, then $\overline{X}$ and ∂X have the Jacobson Property.*

Proof We show first that $\overline{X}$ has the Jacobson Property. Let $a, b \in A$ and suppose that $1-ab \in \overline{X}$. Then there is a sequence (x_n) in X such that $x_n \to 1-ab$. Hence, $c_n = x_n+ab \to 1$. If U is a neighbourhood of 1 with $U \subset X$, then $c_n \in U$ for n large enough. Also, $c_n^{-1} \to 1$. Hence, for n large enough $c_n^{-1} \in X$. Note that $c_n - ab \in X$ and so $c_n - ab = c_n(1 - c_n^{-1}ab) \in X$. Since X is a semigroup this implies that $1 - c_n^{-1}ab \in X$. Since X has the Jacobson Property, $1 - b(c_n^{-1}a) \in X$ and so $1 - ba \in \overline{X}$. One can use the same proof to show that ∂X has the Jacobson Property.

3 Examples

In this section we provide examples of subsets of a Banach algebra that satisfy the Jacobson Property.

In [13,Theorem 1.29] and [1,Lemma 3.1.2] it is shown that in a Banach algebra A the sets A_ℓ^{-1}, A_r^{-1} and A^{-1} have the Jacobson Property. If I is a closed ideal in A, then in the quotient Banach algebra A/I, the sets $(A/I)_\ell^{-1}$, $(A/I)_r^{-1}$ and $(A/I)^{-1}$ have the Jacobson Property. In view of Theorem 2.2 we get that $\Phi_\ell(I)$, $\Phi_r(I)$ and $\Phi(I)$ have the Jacobson Property.

Corollary 3.1 *Let A be a Banach algebra and let I be a closed ideal in A. Then $\overline{X}$ and ∂X have the Jacobson Property if*

$$X \in \{A_\ell^{-1}, A_r^{-1}, A_\ell^{-1} \cup A_r^{-1}, \Phi_\ell(I), \Phi_r(I), \Phi(I), \Phi_\ell(I) \cup \Phi_r(I)\}$$

Proof This follows from Theorem 2.3 since all the sets above are semigroups containing a neighbourhood of the identity.

In [12], the author introduces the *boundary spectrum* and discusses some of its properties. We now show that the set $R = A \setminus \partial A^{-1}$ which generates the boundary spectrum, has the Jacobson Property.

Corollary 3.2 *Let A be a Banach algebra. The sets* $\overline{A^{-1}}$, ∂A^{-1}, $A \setminus \partial A^{-1}$ *and* $A \setminus \overline{A^{-1}}$ *all possess the Jacobson Property.*

Proof It is well known that A^{-1} has the Jacobson Property. Next we use Lemma 2.3 to get that $\overline{A^{-1}}$ and ∂A^{-1} have the Jacobson Property. Then we use the remark following Lemma 2.1 to prove the rest of the statement.

In [9, 10] the authors prove that the set A_{ai} of almost invertible elements in A is a regularity.

Theorem 3.1 *The regularity* A_{ai} *has the Jacobson Property.*

Proof Suppose $1 - ab \in A_{\text{ai}}$. Then $0 \notin \sigma(1 - ab)$ or $0 \in \operatorname{iso} \sigma(1 - ab)$. Suppose $0 \notin \sigma(1 - ab)$. Then $1 - ab \in A^{-1}$ and since A^{-1} satisfies the Jacobson Property we have that $1 - ba \in A^{-1}$, hence $0 \notin \sigma(1 - ba)$ and $1 - ba \in A_{\text{ai}}$ as required.

Suppose $0 \in \operatorname{iso} \sigma(1 - ab)$. Then $1 \in \operatorname{iso} \sigma(ab)$ and so $1 \in \operatorname{iso} \sigma(ba)$ since $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. Hence, $0 \in \operatorname{iso} \sigma(1 - ab)$ and we are done.

Theorem 3.2 *The set* $\mathcal{I}(A)$ *of inessential elements in a Banach algebra A has the Jacobson Property.*

Proof Let $1 - ab \in \mathcal{I}(A)$. Then $\sigma(1 - ab)$ is finite or a sequence converging to zero. Suppose that $\sigma(1 - ab)$ is finite. Then $1 - \sigma(ab)$ is finite, and since $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$ we know that $\sigma(1 - ba)$ must be finite as well.

Alternatively, suppose that $\sigma(1 - ab)$ is a sequence converging to zero. Then $\sigma(ab)$ is a sequence converging to 1, hence $\sigma(ba)$ is also a sequence converging to 1, and so $\sigma(1 - ab)$ is a sequence converging to 0, as required.

Theorem 3.3 *The set* $\{a \in A : \sigma(a) = \{1\}\}$ *has the Jacobson Property.*

Proof Suppose $\sigma(1 - ab) = \{1\}$. Then $1 - \sigma(ab) = \{1\}$, hence $\sigma(ab) = \{0\}$. Hence $\sigma(ba) \setminus \{0\} = \sigma(ab) \setminus \{0\} = \emptyset$. Now, $\sigma(ba)$ cannot be empty, and so $\sigma(ba) = \{0\}$, hence $\sigma(1 - ba) = \{1\}$ as required.

If A is a Banach algebra, then it is well known that the set $\widehat{A}$ of generalised invertible elements in A has the Jacobson Property, [13,Proposition 13.11]. This together with the fact that $\overline{A^{-1}}$ has the Jacobson Property, Corollary 3.1, implies that the collection $A^{-1}A^{\bullet}$ of decomposably regular elements has the Jacobson Property, see (7).

We show with Theorem 5.1 that the (Jacobson) radical does not have the Jacobson Property.

Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\operatorname{Soc} A \subset I \subset \operatorname{kh}(I)$. In [4, 5] an abstract index function ι was defined on the set $\Phi(I)$ of Fredholm elements relative to I. It was shown that this abstract index function has all the desirable properties of the classical index for Fredholm operators defined on a Banach space. One can extend the domain of the abstract index fuction ι to the set $\Phi_\ell(I) \cup \Phi_r(I)$ by defining

$$\iota(a) = \begin{cases} -\infty & \text{if } a \in \Phi_\ell(I) \setminus \Phi(I) \\ \infty & \text{if } a \in \Phi_r(I) \setminus \Phi(I) \end{cases}$$

If $Z \subset \mathbb{Z}$, let

$$\Phi_Z(I) = \{a \in \Phi_\ell(I) \cup \Phi_r(I) \, : \, \iota(a) \in Z\}.$$

Corollary 3.3 *Let A be a semisimple Banach algebra and let I be a closed trace ideal in A with $\operatorname{Soc} A \subset I \subset \operatorname{kh}(I)$. If $Z \subset \mathbb{Z}$ and $\Phi_Z(I)$ is nonempty, then $\Phi_Z(I)$ has the Jacobson Property.*

Proof Let $a, b \in A$ with $1-ab \in \Phi_Z(I)$. By definition, $\iota(1-ab) \in Z$. In view of [4,Theorem 3.20], $\iota(1-ab) = \iota(1-ba)$ and so $1-ba \in \Phi_Z(I)$.

By the above Corollary, if $Z = \{k\}$ with $k \in \mathbb{Z}$, then $\Phi_k(I)$ has the Jacobson Property. In particular, if $k = 0$, then $\Phi_0(I)$ is an upper semiregularity with the Jacobson property, see [5,Theorem 3.6]. Also, $\Phi_0 = \mathcal{W}(I)$ with $\mathcal{W}(I)$ the collection of Weyl elements in A relative to I.

For $a, b \in A$, we call $ab - ba$ the *commutator* of a and b, denoted by $[a, b]$. In [14], p305, the author defines the set A_D as the smallest ideal that contains all the commutators. We call A_D the *commutator ideal* of A.

Theorem 3.4 *If A is a Banach algebra then A_D and $A \setminus A_D$ have the Jacobson Property.*

Proof Suppose $1-ab \in A_D$. Then $1-ba = 1-ab+ab-ba = 1-ab+[a, b] \in A_D$. Next we use the remark following Lemma 2.1 to conclude the remaining part of the statement.

4 Multiplicative linear functionals

The concept of a multiplicative linear functional on A gives us a means to generate a large number of these sets that satisfy the Jacobson Property. While multiplicative linear functionals find their application mostly in commutative settings it is interesting that there are noncommutative Banach algebras that admit nonzero multiplicative linear functionals. The authors of [3] discuss sufficient conditions for a Banach algebra to admit such nonzero multiplicative linear functionals, and discuss two classes of such Banach algebras as examples. Below, wherever we refer to a Banach algebra on which is defined a multiplicative linear functional, we will tacitly assume that the linear functional is nonzero.

Theorem 4.1 *Let A be a complex unital Banach algebra and let f be a multiplicative linear functional on A. Then the set $P = \{f^{-1}(\lambda) : \lambda \in \mathbb{C}\}$ is a partition of A and every member of P has the Jacobson Property.*

Proof Since f is a multiplicative linear functional, it is onto $\mathbb{C}$. Hence we have that $\emptyset \notin P$. Next, f is defined for each $a \in A$, hence for each a there is a part of P that contains it. Suppose that $a \in f^{-1}(\lambda_1) \cap f^{-1}(\lambda_2)$ where $\lambda_1 \neq \lambda_2$. Then $f(a) = \lambda_1$ and $f(a) = \lambda_2$ which is impossible since f is a function, hence well-defined. We have shown that P is a partition of A.

To see that each part of P has the Jacobson Property, suppose $1 - ab \in f^{-1}(\lambda)$, where $\lambda \in \mathbb{C}$. Then $f(1-ab) = \lambda$, which means $1 - f(a)f(b) = \lambda$, so that $1 - f(b)f(a) = \lambda$. This means $f(1-ba) = \lambda$, hence $1 - ba \in f^{-1}(\lambda)$, as required.

The following corollary is a simple application of Theorem 4.1 and is stated without proof.

Corollary 4.1 *Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ has the Jacobson Property.*

We can generalize Theorem 4.1 above as follows:

Theorem 4.2 *Let f be a multiplicative linear functional on a complex, unital Banach algebra A, and $B \subset \mathbb{C}$, $B \neq \emptyset$. Then $f^{-1}(B)$ has the Jacobson Property.*

Proof The proof is similar to the proof of the previous theorem.

For f a multiplicative linear functional, the set $\ker(f)$ gives us an interesting application of one of the previous results. We note that for the next result, the existence of a nonzero multiplicative linear functional is a necessary condition, since there are many noncommutative Banach algebras on which no nonzero multiplicative linear functional can be defined.

Corollary 4.2 *Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $\ker(f)$ is a proper, non-trivial subprime ideal of A.*

Proof It is well known that $\ker(f)$ is an ideal. To see that it is subprime, suppose $ab \in \ker(f)$. Then $f(ab) = f(a)f(b) = 0$, hence $f(a) = 0$ or $f(b) = 0$, hence $a \in \ker(f)$ or $b \in \ker(f)$. Since f is assumed to be nonzero, $\ker(f) = A$ is not possible, hence $\ker(f)$ is a proper ideal. Finally, since A is not commutative, find $a, b \in A$ such that $ab \neq ba$. Then $0 \neq ab - ba \in \ker(f)$, hence $\ker(f)$ is not trivial.

Corollary 4.3 *Let A be a noncommutative Banach algebra, and f a multiplicative linear functional on A. Then $A \setminus \ker(f)$ is a $(P1)$ regularity in A which has the Jacobson Property.*

Proof The proof is a simple application of Corollary 4.2 and Theorem 2.1.

5 Sets that do not have the Jacobson Property

In this section we provide examples of subsets in a Banach algebra that do not possess the Jacobson Property. If A is a Banach algebra, then the upper semiregularity Exp$A$ does not have the Jacobson Property, see [8]. We are now ready to give more examples.

Proposition 5.1 *Let A be a Banach algebra. Then the upper semiregularity* $\overline{\text{ExpA}}$ *does not have the Jacobson Property.*

Proof Since Exp$A$ is a closed normal subgroup of A^{-1},

$$\text{Exp}A = \overline{\text{ExpA}}^{(A^{-1})} = \overline{\text{ExpA}} \cap A^{-1}$$

where $\overline{\text{ExpA}}^{(A^{-1})}$ is the closure of Exp A in A^{-1}. In view of Exp$A$ not having the Jacobson Property and A^{-1} having the Jacobson Property, we get that $\overline{\text{ExpA}}$ does not have the Jacobson Property.

Example 5.1 Let $A = \mathcal{L}(\ell^2)$ and let $R = A \backslash \text{QN}(A)$. Then R is a lower semiregularity that does not have the Jacobson Property.

Proof Define operators $S, T \in A$ by $S(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ and $T(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$ for all $(x_1, x_2, \ldots) \in \ell^2$. Then $TS = I$ and $I - ST$ is a rank one operator. This means that $\sigma(I - ST) = \{0, 1\}$. Consequently, $I - ST \in R$ and $I - ST = 0 \notin R$.

Let A be a Banach algebra. The proof of the above example can be adapted to show that $A \backslash \text{Rad}\, A$ does not have the Jacobson Property.

Theorem 5.1 *Let A be a noncommutative Banach algebra with $A_\ell^{-1} \setminus A^{-1} \neq \emptyset$ or $A_r^{-1} \setminus A^{-1} \neq \emptyset$. Then* Rad A *does not have the Jacobson Property.*

Proof We are going to show that $A\backslash\text{Rad } A$ does not have the Jacobson Property. Let $a, b \in A$ and suppose that $ab = 1$ and $ba \neq 1$. Then $1 - ab = 0 \notin A\backslash\text{Rad } A$. Since $ab = 1$, a is right invertible and b is left invertible. By (3) $a, b \in \widehat{A}$. This means that ba is a projection and so $1 - ba$ is also a projection. Hence, $1 - ba \in A\backslash\text{Rad } A$, and so $A\backslash\text{Rad}A$ does not have the Jacobson Property. In view of the remark following Lemma 2.1 we have that Rad A does not have the Jacobson Property.

Since the radical in a Banach algebra A is the intersection of primitive ideals in A, under the assumptions of Theorem 5.1, a noncommutative Banach algebra will have at least one primitive ideal without the Jacobson Property. Let $A_\ell^{-1} \setminus A^{-1} \neq \emptyset$ or $A_r^{-1} \setminus A^{-1} \neq \emptyset$. If A is noncommutative, the same proof as Theorem 5.1 can be used to show that $\text{QN}(A)$ does not have the Jacobson Property if we replace Rad A by $\text{QN}(A)$.

Corollary 5.1 *Let A be a noncommutative Banach algebra and let I be a closed ideal in A. Then the ideal* $\text{kh}(I)$ *does not have the Jacobson Property.*

Proof In view of Theorem 5.1 Rad (A/I) does not have the Jacobson Property. By Theorem 2.2 $\text{kh}(I) = \pi^{-1}(\text{Rad}(A/I))$ does not have the Jacobson Property.

Corollary 5.2 *Let A be a noncommutative Banach algebra and let I be a closed ideal in A. Then the collection $\mathcal{R}(I)$ of Riesz elements relative to I does not have the Jacobson Property.*

Proof Since QN (A/I) does not have the Jacobson Property, see the proof of Theorem 5.1, it follows from Theorem 2.2 that $\mathcal{R}(I) = \pi^{-1}(\text{QN}(A/I))$ does not have the Jacobson Property.

Acknowledgements The authors would like to thank the referee for several helpful comments that improved the quality of the article.

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