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Coefficient bounds and Fekete–Szegö problem for qualitative subclass of bi-univalent functions

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Abstract
In this paper, we introduce new and qualitative subclasses $\mathbf{B}^{\varepsilon}(\kappa, \alpha, \sigma)$, $\mathbf{B}^{\gamma}(\kappa, \alpha, \sigma)$ and $\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$ of bi-univalent functions. The coefficient bounds and Fekete-Szegö inequalities for functions belonging to these subclasses are obtained. Also, we will get a variety of new results through special cases of our main results.

Keywords Bi-univalent functions · Analytic function · Univalent functions · Coefficient inequalities · Fekete–Szegö problems

Mathematics Subject Classification 30C45

1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ with conditions $f(0) = 0$ and $f'(0) = 1$ having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathcal{U}). \tag{1.1}$$

Further, all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$ we will denote it by $\mathcal{S}$. So, every function $f \in \mathcal{S}$ has an inverse f^{-1}, such that

$$f^{-1}(f(z)) = z \text{ and } f(f^{-1}(w)) = w \qquad \left(z \in \mathcal{U},\ |w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ given by (1.1) is in the class Σ of all bi-univalent functions in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$ (see [1–3,15,18]).

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The class $\mathcal{S}^*(\varepsilon)$ of starlike functions of order ε in $\mathcal{U}$ is well-studied and subset of the function class $\mathcal{S}$. By definition, we have

$$\mathcal{S}^*(\varepsilon) := \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \varepsilon, \quad (z \in \mathcal{U}; 0 \leq \varepsilon < 1) \right\}. \tag{1.3}$$

Ezrohi [7] introduced the class

$$\mathcal{U}(\varepsilon) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{f'(z)\right\} > \varepsilon, \quad (z \in \mathcal{U}; 0 \leq \varepsilon < 1) \right\}$$

Also, Chen [6] introduced the class

$$\mathcal{ST}(\varepsilon) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re}\left\{\frac{f(z)}{z}\right\} > \varepsilon, \quad (z \in \mathcal{U}; 0 \leq \varepsilon < 1) \right\}.$$

It is stated in [4] that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma[\varepsilon]$ of strongly bi-starlike functions of order $\varepsilon (0 < \varepsilon \leq 1)$ if each of the following requirements is met

$$f \in \Sigma \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\varepsilon\pi}{2}, \ (z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\varepsilon\pi}{2} \qquad (w \in \mathcal{U}),$$

where $g = f^{-1}$ and *given by* (1.2).

Also, a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma(\gamma)$ of bi-starlike functions of order $\gamma (0 \leq \gamma < 1)$ if each of the following requirements is met

$$f \in \Sigma \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma \qquad (z \in \mathcal{U})$$

and

$$\operatorname{Re}\left(\frac{wg'(w)}{g(w)}\right) > \gamma \qquad (w \in \mathcal{U}),$$

where $g = f^{-1}$and *given by* (1.2).

Now, we introduce the new and Comprehensive subclasses $\mathbf{B}^\varepsilon(\kappa, \alpha, \sigma)$, $\mathbf{B}^\gamma(\kappa, \alpha, \sigma)$ and $\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$.

Definition 1.1 A function $f(z)$ given by (1.1) is said to be in the class $\mathbf{B}^\varepsilon(\kappa, \alpha, \sigma)$ *where* $\alpha, \kappa \geq 1$, $\sigma \in \mathbb{C}$, $\operatorname{Re}(\sigma) \geq 0$, and $0 < \varepsilon \leq 1$, if the following inequalities are satisfied:

$$f \in \Sigma \text{ and } \left|\arg\left((1-\kappa)f'(z) + \kappa\left(f'(z)\right)^\alpha \left(\frac{f(z)}{z}\right)^{\sigma-1}\right)\right| < \frac{\varepsilon\pi}{2} \qquad (z \in \mathcal{U}) \tag{1.4}$$

and

$$\left|\arg\left((1-\kappa)g'(w) + \kappa\left(g'(w)\right)^\alpha \left(\frac{g(w)}{w}\right)^{\sigma-1}\right)\right| < \frac{\varepsilon\pi}{2} \qquad (w \in \mathcal{U}), \tag{1.5}$$

where g is given by (1.2).

Definition 1.2 A function $f(z)$ given by (1.1) is said to be in the class $\mathbf{B}^{\gamma}(\kappa, \alpha, \sigma)$ *where* $\alpha, \kappa \geq 1$, $\sigma \in \mathbb{C}$, Re $(\sigma) \geq 0$, and $0 \leq \gamma < 1$, if the following inequalities are satisfied:

$$f \in \Sigma \text{ and } \mathrm{Re}\left((1-\kappa)f'(z) + \kappa\left(f'(z)\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{\sigma-1}\right) > \gamma \qquad (z \in \mathcal{U}) \tag{1.6}$$

and

$$\mathrm{Re}\left((1-\kappa)g'(w) + \kappa\left(g'(w)\right)^{\alpha}\left(\frac{g(w)}{w}\right)^{\sigma-1}\right) > \gamma \qquad (w \in \mathcal{U}), \tag{1.7}$$

where g is given by (1.2).

Definition 1.3 Let the functions $s, t : \mathcal{U} \to \mathbb{C}$ such that

$$\min\{\mathrm{Re}\,(s(z)), \mathrm{Re}\,(t(z))\} > 0 \ (z \in \mathcal{U}) \ \text{ and } s(0) = t(0) = 1.$$

Also let $f \in \mathcal{A}$, defined by (1.1). We say that $f \in \mathbf{B}_{s,t}(\kappa, \delta, \mu)$ *where* $\alpha, \kappa \geq 1$, $\sigma \in \mathbb{C}$, Re $(\sigma) \geq 0$ if the following inequalities are satisfied:

$$f \in \Sigma \text{ and } \mathrm{Re}\left((1-\kappa)f'(z) + \kappa\left(f'(z)\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{\sigma-1}\right) \in t(\mathcal{U}) \qquad (z \in \mathcal{U}) \tag{1.8}$$

and

$$\mathrm{Re}\left((1-\kappa)g'(w) + \kappa\left(g'(w)\right)^{\alpha}\left(\frac{g(w)}{w}\right)^{\sigma-1}\right) \in s(\mathcal{U}) \qquad (w \in \mathcal{U}), \tag{1.9}$$

where g is given by (1.2).

Remark 1.4 By taking specific values of the functions $s(z)$ and $t(z)$ in Definition 1.3 we get various well known subclasses of $\mathcal{A}$, for example, if

$$s(z) = t(z) = \left(\frac{1+z}{1-z}\right)^{\varepsilon} \qquad (0 < \varepsilon \leq 1; z \in \mathcal{U})$$

or

$$s(z) = t(z) = \frac{1 + (1-2\gamma)z}{1-z} \qquad (0 \leq \gamma < 1; z \in \mathcal{U})$$

it is simple to verify that $s(z)$ and $t(z)$ satisfy the Definition 1.3. If $f \in \mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$, then the function f satisfied the inequalities (1.4) and (1.5) or (1.6) and (1.7), where g *is given by* (1.2). This means that, $f \in \mathbf{B}^{\varepsilon}(\kappa, \alpha, \sigma)$ or $f \in \mathbf{B}^{\gamma}(\kappa, \alpha, \sigma)$, where $\alpha, \kappa \geq 1$, $0 < \varepsilon \leq 1$, $0 \leq \gamma < 1$, $\sigma \in \mathbb{C}$ and Re $(\sigma) \geq 0$.

The purpose of this paper is to introduce qualitative subclasses $\mathbf{B}^{\varepsilon}(\kappa, \alpha, \sigma)$, $\mathbf{B}^{\gamma}(\kappa, \alpha, \sigma)$ and $\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$ of the function class Σ. Motivated by the earlier work of, Bulut [5], Frasin et al. [8–10], Li and Wang [11], Siregar and Raman [14], and Yousef et al. [17,19–21], we find estimates on the coefficients $|a_2|$, $|a_3|$ and $\left|a_3 - \varsigma a_2^2\right|$. Furthermore, variety of new results will follow by specializing cases in our main results.

To proof our theorem we will need the following lemma:

Lemma 1.5 [12] *If* $h \in \mathcal{H}$, *then* $|h_i| \leq 2$ *for each* i, *where* $\mathcal{H}$ *is the family of all functions* h *analytic in* $\mathcal{U}$ *for which*

$$Re\,(h(z)) > 0, h(z) = 1 + h_1 z + h_2 z^2 + \cdots \ (z \in \mathcal{U}).$$

2 Coefficient bounds for subclass $\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$

In this section we state and prove the main results for subclass$\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$ given by Definition 1.3.

Theorem 2.1 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(\kappa, \alpha, \sigma)$, where $\alpha, \kappa \geq 1$, $\sigma \in \mathbb{C}$ and* $\operatorname{Re}(\sigma) \geq 0$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|s'(0)|^2+|t'(0)|^2}{2\,|\kappa\,(2\alpha+\sigma-3)+2|^2}}, \sqrt{\frac{|s''(0)|+|t''(0)|}{2\,|\kappa\,((\sigma+2)\,(\sigma-3)+4\alpha\,(\sigma-1)+2\alpha(2\alpha+1))+6|}}\right\},$$

$$|a_3| \leq \min\left\{\frac{|s'(0)|^2+|t'(0)|^2}{2\,|\kappa\,(2\alpha+\sigma-3)+2|^2}+\frac{|s''(0)|+|t''(0)|}{4\,|\kappa\,(3\alpha+\sigma-4)+3|},\right.$$
$$\left.\frac{\left|\kappa\left(\sigma^2+\sigma+4\alpha\,(\sigma-1)+4\alpha(\alpha+2)-14\right)+12\right|\left|s''(0)\right| + |\kappa\,((\sigma-1)\,(\sigma-2)+4\alpha\,(\sigma-1)+4\alpha(\alpha-1))|\left|t''(0)\right|}{4\,|\kappa\,((\sigma+2)\,(\sigma-3)+4\alpha\,(\sigma-1)+2\alpha(2\alpha+1))+6|\,|\kappa\,(3\alpha+\sigma-4)+3|}\right\}, \tag{2.1}$$

and

$$\left|a_3-\varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{|\kappa\,(3\alpha+\sigma-4)+3|},$$

where

$$\varsigma = \frac{\kappa\left(\frac{(\sigma-2)(\sigma+3)}{2}+2\alpha\,(\sigma-1)+2\alpha(\alpha+2)-4\right)+6}{\kappa\,(3\alpha+\sigma-4)+3}.$$

Proof First, we write the equivalent forms for inequalities (1.6) and (1.7) as follows:

$$(1-\kappa)f'(z)+\kappa\left(f'(z)\right)^{\alpha}\left(\frac{f(z)}{z}\right)^{\sigma-1}=s(z) \tag{2.2}$$

and

$$(1-\kappa)g'(w)+\kappa\left(g'(w)\right)^{\alpha}\left(\frac{g(w)}{w}\right)^{\sigma-1}=t(w) \tag{2.3}$$

where $s(z)$ and $t(w)$ are in $\mathcal{H}$ and satisfy the conditions of Definition 1.3 and have the forms

$$s(z)=1+s_1z+s_2z^2+s_3z^3+\cdots \quad \text{and} \quad t(w)=1+t_1w+t_2w^2+t_3w^3+\cdots.$$

Now, equating coefficients in (2.2) and (2.3), yields

$$(\kappa\,(2\alpha+\sigma-3)+2)\,a_2=s_1, \tag{2.4}$$

$$\left[\kappa\left(\frac{(\sigma-1)\,(\sigma-2)}{2}+2\alpha\,(\sigma-1)+2\alpha(\alpha-1)\right)\right]a_2^2+[\kappa\,(3\alpha+\sigma-4)+3]\,a_3=s_2 \tag{2.5}$$

$$-(\kappa\,(2\alpha+\sigma-3)+2)\,a_2=t_1 \tag{2.6}$$

and

$$\left[\kappa\left(\frac{(\sigma-2)\,(\sigma+3)}{2}+2\alpha\,(\sigma-1)+2\alpha(\alpha+2)-4\right)+6\right]a_2^2-[\kappa\,(3\alpha+\sigma-4)+3]\,a_3$$
$$=t_2. \tag{2.7}$$

From (2.4) and (2.6), we get

$$s_1 = -t_1 \tag{2.8}$$

and

$$2\left(\kappa\left(2\alpha+\sigma-3\right)+2\right)^2 a_2^2 = s_1^2 + t_1^2. \tag{2.9}$$

Also, adding (2.5) to (2.7), we find that

$$\left[\kappa\left(\left(\sigma+2\right)\left(\sigma-3\right)+4\alpha\left(\sigma-1\right)+2\alpha(2\alpha+1)\right)+6\right] a_2^2 = s_2 + t_2. \tag{2.10}$$

From Eqs. (2.9) and (2.10), we have

$$\left|a_2^2\right| \leq \frac{\left|s'(0)\right|^2 + \left|t'(0)\right|^2}{2\left|\kappa\left(2\alpha+\sigma-3\right)+2\right|^2} \tag{2.11}$$

and

$$\left|a_2^2\right| \leq \frac{\left|s''(0)\right| + \left|t''(0)\right|}{2\left|\kappa\left(\left(\sigma+2\right)\left(\sigma-3\right)+4\alpha\left(\sigma-1\right)+2\alpha(2\alpha+1)\right)+6\right|}, \tag{2.12}$$

respectively. So we get the inequality (2.1).

Next, to find the bound on $|a_3|$, by subtracting (2.7) from (2.5), we get

$$2\left[\kappa\left(3\alpha+\sigma-4\right)+3\right]\left(a_3 - a_2^2\right) = s_2 - t_2. \tag{2.13}$$

Further, in view of (2.9) in Eq. (2.13), it follows that

$$a_3 = \frac{s_1^2 + t_1^2}{2(\kappa\left(2\alpha+\sigma-3\right)+2)^2} + \frac{s_2 - t_2}{2\kappa\left(3\alpha+\sigma-4\right)+6}. \tag{2.14}$$

We thus find that

$$|a_3| \leq \frac{\left|s'(0)\right|^2 + \left|t'(0)\right|^2}{2\left|\kappa\left(2\alpha+\sigma-3\right)+2\right|^2} + \frac{\left|s''(0)\right| + \left|t''(0)\right|}{4\left|\kappa\left(3\alpha+\sigma-4\right)+3\right|}.$$

On other hand, by using (2.10) in (2.13), we get

$$a_3 = \frac{\begin{array}{c}\left[\kappa\left(\sigma^2+\sigma+4\alpha\left(\sigma-1\right)+4\alpha(\alpha+2)-14\right)+12\right] s_2 \\ -\left[\kappa\left(\left(\sigma-1\right)\left(\sigma-2\right)+4\alpha\left(\sigma-1\right)+4\alpha(\alpha-1)\right)\right] t_2\end{array}}{\left[\kappa\left(\left(\sigma+2\right)\left(\sigma-3\right)+4\alpha\left(\sigma-1\right)+2\alpha(2\alpha+1)\right)+6\right]\left[2\kappa\left(3\alpha+\sigma-4\right)+6\right]}. \tag{2.15}$$

Consequently, we have

$$|a_3| \leq \frac{\begin{array}{c}\left|\kappa\left(\sigma^2+\sigma+4\alpha\left(\sigma-1\right)+4\alpha(\alpha+2)-14\right)+12\right|\left|s''(0)\right| \\ +\left|\kappa\left(\left(\sigma-1\right)\left(\sigma-2\right)+4\alpha\left(\sigma-1\right)+4\alpha(\alpha-1)\right)\right|\left|t''(0)\right|\end{array}}{4\left|\kappa\left(\left(\sigma+2\right)\left(\sigma-3\right)+4\alpha\left(\sigma-1\right)+2\alpha(2\alpha+1)\right)+6\right|\left|\kappa\left(3\alpha+\sigma-4\right)+3\right|}.$$

Also, from (2.7) we find that

$$\frac{\kappa\left(\frac{(\sigma-2)(\sigma+3)}{2}+2\alpha\left(\sigma-1\right)+2\alpha(\alpha+2)-4\right)+6}{\kappa\left(3\alpha+\sigma-4\right)+3} a_2^2 - a_3 = \frac{t_2}{\kappa\left(3\alpha+\sigma-4\right)+3}.$$

Consequently, we have

$$\left|a_3 - \varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{\left|\kappa\left(3\alpha+\sigma-4\right)+3\right|},$$

where

$$\varsigma = \frac{\kappa\left(\frac{(\sigma-2)(\sigma+3)}{2} + 2\alpha\,(\sigma-1) + 2\alpha(\alpha+2) - 4\right) + 6}{\kappa\,(3\alpha+\sigma-4)+3}.$$

Which completes the proof. □

3 Corollaries and consequences

Choosing $\kappa = 1$ in Theorem 2.1, we obtain the following Corollary:

Corollary 3.1 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(1,\alpha,\sigma)$, where $\alpha \geq 1$, $\sigma \in \mathbb{C}$ and* $\operatorname{Re}(\sigma) \geq 0$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|s'(0)|^2+|t'(0)|^2}{2\,|2\alpha+\sigma-1|^2}}, \sqrt{\frac{|s''(0)|+|t''(0)|}{2\,|(\sigma+2)\,(\sigma-3)+4\alpha\,(\sigma-1)+2\alpha(2\alpha+1)+6|}}\right\},$$

$$|a_3| \leq \min\left\{\frac{\left|s'(0)\right|^2+\left|t'(0)\right|^2}{2\,|2\alpha+\sigma-1|^2} + \frac{\left|s''(0)\right|+\left|t''(0)\right|}{4\,|3\alpha+\sigma-1|},\right.$$
$$\left.\frac{\left|\sigma^2+\sigma+4\alpha\,(\sigma-1)+4\alpha(\alpha+2)-2\right|\left|s''(0)\right| + |(\sigma-1)\,(\sigma-2)+4\alpha\,(\sigma-1)+4\alpha(\alpha-1)|\left|t''(0)\right|}{4\,|(\sigma+2)\,(\sigma-3)+4\alpha\,(\sigma-1)+2\alpha(2\alpha+1)+6|\,|3\alpha+\sigma-1|}\right\},$$

and

$$\left|a_3-\varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{|3\alpha+\sigma-1|},$$

where

$$\varsigma = \frac{\frac{(\sigma-2)(\sigma+3)}{2} + 2\alpha\,(\sigma-1) + 2\alpha(\alpha+2) + 2}{3\alpha+\sigma-1}.$$

Putting $\alpha = 1$ in Corollary 3.1, we obtain the following Corollary:

Corollary 3.2 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(1,1,\sigma)$, where $\sigma \in \mathbb{C}$ and* $\operatorname{Re}(\sigma) \geq 0$. *Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{\left|s'(0)\right|^2+\left|t'(0)\right|^2}{2\,|\sigma+1|^2}}, \sqrt{\frac{\left|s''(0)\right|+\left|t''(0)\right|}{2\,|(\sigma+1)\,(\sigma+2)|}}\right\},$$

$$|a_3| \leq \min\left\{\frac{\left|s'(0)\right|^2+\left|t'(0)\right|^2}{2\,|\sigma+1|^2} + \frac{\left|s''(0)\right|+\left|t''(0)\right|}{4\,|\sigma+2|}, \frac{|(\sigma+3)|\left|s''(0)\right|+|(\sigma-1)|\left|t''(0)\right|}{4\,|(\sigma+1)\,(\sigma+2)|}\right\},$$

and

$$\left|a_3-\varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{|\sigma+2|},$$

where

$$\varsigma = \frac{\sigma + 3}{2}.$$

Putting $\sigma = 0$ in Corollary 3.1, we obtain the following Corollary:

Corollary 3.3 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(1, \alpha, 0)$, where $\alpha \geq 1$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|s'(0)|^2 + |t'(0)|^2}{2\,|2\alpha - 1|^2}}, \sqrt{\frac{|s''(0)| + |t''(0)|}{2\,|2\alpha(2\alpha - 1)|}}\right\},$$

$$|a_3| \leq \min\left\{\frac{\left|s'(0)\right|^2 + \left|t'(0)\right|^2}{2\,|2\alpha - 1|^2} + \frac{\left|s''(0)\right| + \left|t''(0)\right|}{4\,|3\alpha - 1|},\right.$$
$$\left.\frac{|4\alpha(\alpha + 1) - 2|\left|s''(0)\right| + |4\alpha(\alpha - 2) + 2|\left|t''(0)\right|}{4\,|2\alpha(2\alpha - 1)|\,|3\alpha - 1|}\right\},$$

and

$$\left|a_3 - \varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{|3\alpha - 1|},$$

where

$$\varsigma = \frac{2\alpha^2 + 2\alpha - 1}{3\alpha - 1}.$$

Putting $\alpha = 1$ in Corollary 3.3, we obtain the following Corollary:

Corollary 3.4 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(1, 1, 0)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|s'(0)|^2 + |t'(0)|^2}{2}}, \sqrt{\frac{|s''(0)| + |t''(0)|}{4}}\right\},$$

$$|a_3| \leq \min\left\{\frac{\left|s'(0)\right|^2 + \left|t'(0)\right|^2}{2} + \frac{\left|s''(0)\right| + \left|t''(0)\right|}{8}, \frac{3\left|s''(0)\right| + \left|t''(0)\right|}{8}\right\},$$

and

$$\left|a_3 - \varsigma a_2^2\right| \leq \frac{\left|t''(0)\right|}{2},$$

where

$$\varsigma = \frac{3}{2}.$$

Remark 3.5 The estimates for coefficients $|a_2|$ and $|a_3|$ in Corollary 3.4 obtained by Bulut [5]

Putting $\sigma = 1$ in Corollary 3.2, we obtain the following Corollary:

Corollary 3.6 *Let $f(z)$ given by (1.1) be in the class $\mathbf{B}_{s,t}(1,1,1)$. Then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|s'(0)|^2+|t'(0)|^2}{8}}, \sqrt{\frac{|s''(0)|+|t''(0)|}{12}}\right\},$$

$$|a_3| \leq \min\left\{\frac{\left|s'(0)\right|^2+\left|t'(0)\right|^2}{8}+\frac{\left|s''(0)\right|+\left|t''(0)\right|}{12}, \frac{\left|s''(0)\right|}{6}\right\},$$

and

$$\left|a_3-2a_2^2\right| \leq \frac{\left|t''(0)\right|}{3}.$$

Remark 3.7 The estimates for coefficients $|a_2|$ and $|a_3|$ in Corollary 3.6 obtained by Xu et al. [16]

References

1. Amourah, A., Frasin, B.A., Abdeljawad, T.: Fekete-Szegö Inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials. J. Funct. Sp. **Article ID 5574673**, 7 (2021)
2. Amourah, A., Frasin, B.A., Murugusundaramoorthy, G., Al-Hawary, T.: Bi-Bazilevič functions of order $\vartheta + i\delta$ associated with $(p, q)-$ Lucas polynomials. AIMS Math. **6**(5), 4296–4305 (2021)
3. Amourah, A., Illafe, M.: A comprehensive subclass of analytic and Bi-univalent functions associated with subordination. Pal. J. Math. **9**, 187–193 (2020)
4. Brannan, D.A., Taha, T.S.: On some classes of bi-univalent functions. Stud. Univ. Babeş-Bolyai Math. **31**(2), 70–77 (1986)
5. Bulut, S.: Coefficient estimates for a class of analytic and bi-univalent functions. Novi Sad J. Math. **43**(2), 59–65 (2013)
6. Chen, M.P.: On functions satisfying $Re\left\{\frac{f(z)}{z}\right\} > \alpha$. Tamk. J. Math. **5**, 231–234 (1974)
7. Ezrohi, T.G.: Certain estimates in special classes of univalent functions in the unitcircle. Doporidi Akademii Nauk Ukrains, RSR **2**, 984–988 (1965)
8. Frasin, B.A., Al-Hawary, T., Yousef, F.: Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions. Afr. Mat. **30**, 223–230 (2019)
9. Frasin, B.A., Al-Hawary, T., Yousef, F.: Some properties of a linear operator involving generalized Mittag–Leffler function. Stud. Univ. Babeş-Bolyai Math. **65**(1), 67–75 (2020)
10. Frasin, B.A., Yousef, F., Al-Hawary, T., Aldawish, I.: Application of generalized Bessel functions to classes of analytic functions. Afr. Mat. **32**(3), 431–439 (2021)
11. Li, X.F., Wang, A.P.: Two new subclasses of bi-univalent functions. Int. Math. Forum **7**(30), 1495–1504 (2012)
12. Pommerenke, Ch.: Univalent Functions. Vandenhoeck and Rupercht, Göttingen (1975)
13. Singh, R.: On Bazilevič functions. Proc. Am. Math. Soc. **38**, 261–271 (1973)
14. Siregar, S., Raman, S.: Certain subclasses of analytic and bi-univalent functions involving double zeta functions. Int. J. Adv. Sci. Eng. Inf. Tech. **2**(5), 16–18 (2012)
15. Srivastava, H.M., Mishra, A.K., Gochhayat, P.: Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. **23**, 1188–1192 (2010)
16. Xu, Q.-H., Gui, Y.-C., Srivastava, H.M.: Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Appl. Math. Lett. **25**, 990–994 (2012)
17. Yousef, F., Alroud, S., Illafe, M.: A comprehensive subclass of bi-univalent functions associated with Chebyshev polynomials of the second kind. Bol. Soc. Mat. Mex. **26**, 329–339 (2020)
18. Yousef, F., Alroud, S., Illafe, M.: New subclasses of analytic and bi-univalent functions endowed with coefficient estimate problems. Anal. Math. Phys. **11**, 58 (2021)
19. Yousef, F., Amourah, A.A., Darus, M.: Differential sandwich theorems for p-valent functions associated with a certain generalized differential operator and integral operator. Ital. J. Pure Appl. Math. **36**, 543–556 (2016)
20. Yousef, F., Frasin, B.A., Al-Hawary, T.: Fekete–Szegö inequality for analytic and bi-univalent functions subordinate to Chebyshev polynomials. Filomat **32**(9), 3229–3236 (2018)

21. Yousef, F., Al-Hawary, T., Murugusundaramoorthy, G.: Fekete–Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator. Afr. Mat. **30**(3–4), 495–503 (2019)