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Some properties for certain subclasses of multivalent functions associated with the $q-$difference linear operator

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Abstract
Making use of the $q-$difference operator $L_{p,q}(a, c)$, we introduce a new two subclasses of $p-$valent analytic functions in the open unit disk. The main objective of the present paper is to investigate the various important properties and characteristics of each of these subclasses. Furthermore, several properties involving neighborhoods and modified Hadamard products of functions in these subclasses are obtained.

Keywords Analytic functions · Multivalent functions · Hadamard product · $q-$difference operator · Subordination · Neighborhoods of analytic functions · $q-$Jack's lemma; Cauchy$-$Schwarz inequality

Mathematics Subject Classification 30C45 · 30C50

1 Introduction

Let $\mathcal{A}(p)$ denote the class of functions normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, 3 \ldots\}), \tag{1.1}$$

which are analytic and $p-$valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as $f \prec g$ or $f(z) \prec g(z)\ (z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ such that $f(z) = g(w(z))\ (z \in \mathbb{U})$(see [7,12]).

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For functions $f \in \mathcal{A}(p)$, given by (1.1), and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} \quad (z \in \mathbb{U}). \tag{1.3}$$

Recently, $q-$derivative has played a crucial role in the theory of univalent and multivalent functions especially in estimating the sharp inequalities bound for various subclasses of univalent functions (see [1,18,21,30,31]). For $0 < q < 1$, Jackson [19,20] (see also [11,14] and [35]) defined the $q-$ derivative of f as follows:

$$D_{p,q} f(z) := \begin{cases} f'(0) & \text{if } z = 0, \\ \dfrac{f(z) - f(qz)}{(1-q)z} & \text{if } z \neq 0, \end{cases} \tag{1.4}$$

provided that $f'(0)$ exists. For $f \in \mathcal{A}(p)$ given by (1.1), we deduce that

$$D_{p,q} f(z) = [p]_q \, z^{p-1} + \sum_{k=1}^{\infty} [k+p]_q \, a_{k+p} \, z^{k+p-1} \; (z \neq 0), \tag{1.5}$$

where

$$[i]_q = \frac{1-q^i}{1-q} = 1 + q + q^2 + \cdots + q^{i-1}, \tag{1.6}$$

and

$$\lim_{q \to 1^-} D_{p,q} f(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. We note that $D_{1,q} f(z) = D_q f(z)$ and

1. $D_{p,q}(c) = 0$, where c is constant;
2. $D_{p,q}(f(z) \pm g(z)) = D_{p,q} f(z) + D_{p,q} g(z)$;
3. $D_{p,q}(f(z) g(z)) = g(z) D_{p,q} f(z) + f(qz) D_{p,q} g(z)$;
4. $D_{p,q}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z) D_{p,q} f(z) - f(z) D_{p,q} g(z)}{g(qz)\, g(z)}$.

As a right inverse, Jackson [20] introduced the $q-$integral of a function $f \in \mathcal{A}(p)$ given by (1.1) as follows:

$$\int_0^z f(t)\, d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f\left(zq^k\right) = \frac{z^{p+1}}{[p+1]_q} + \sum_{k=1}^{\infty} a_{k+p} \frac{z^{k+p+1}}{[k+p+1]_q},$$

provided that the series converges. We observe that

$$\lim_{q \to 1^-} \int_0^z f(t)\, d_q t = \int_0^z f(t)\, dt = \frac{z^{p+1}}{p+1} + \sum_{k=1}^{\infty} a_{k+p} \frac{z^{k+p+1}}{k+p+1},$$

where $\int_0^z f(t)\, dt$ is the ordinary integral a function f.

Next, in terms of the q-Pochhammer symbol $\left([v]_q\right)_n$ given by

$$\left([v]_q\right)_n = \begin{cases} 1 & (n=0)\,, \\ [v]_q\,[v+1]_q\,[v+2]_q \cdots [v+n-1]_q & (p \in \mathbb{N})\,, \end{cases}$$

we define the function $\phi_{p,q}(a, c; z)$ by

$$\phi_{p,q}(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} z^{k+p} \quad (a > 0; c > 0; z \in \mathbb{U})\,. \tag{1.7}$$

Corresponding to the function $\phi_{p,q}(a, c; z)$, we consider a linear operator $L_{p,q}(a, c) : \mathcal{A}(p) \to \mathcal{A}(p)$ which is defined by means of the following Hadamard product (or convolution):

$$L_{p,q}(a, c) f(z) = \phi_{p,q}(a, c; z) * f(z) = z^p + \sum_{k=1}^{\infty} \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} a_{k+p} z^{k+p}. \tag{1.8}$$

It is easily verified from (1.8) that

$$q^{a-p} z D_{p,q}\left(L_{p,q}(a, c) f(z)\right) = [a]_q\, L_{p,q}(a+1, c) f(z) - [a-p]_q\, L_{p,q}(a, c) f(z)\,. \tag{1.9}$$

Moreover, for $f(z) \in \mathcal{A}(p)$, we observe that

1. $L_{p,q}(a, a) f(z) = f(z)\,;$
2. $L_{p,q}(p+1, p) f(z) = \frac{z D_{p,q} f(z)}{[p]_q}$ and $\lim_{q \to 1^-} L_{p,q}(p+1, p) f(z) = \frac{z f'(z)}{p}$;
3. The operator $\lim_{q \to 1^-} L_{p,q}(a, c) = L_p(a, c)$ was introduced by Saitoh [28] and studied by Srivastava and Patel [34];
4. $L_{p,q}(n+p, 1) f(z) = R_q^{n+p-1} f(z)\,(n > -p)$, where $R_q^{n+p-1} f(z)$ denotes the Ruscheweyh $q-$derivative of a function $f \in \mathcal{A}(p)$ of order $n+p-1$ (see [1], [21] and [32]) and $\lim_{q \to 1^-} L_{p,q}(n+p, 1) f(z) = D^{n+p-1} f(z)\,(n > -p)$, where $D^{n+p-1} f(z)$ denotes the Ruscheweyh derivative of a function $f(z) \in \mathcal{A}(p)$ of order $n+p-1$ (see [22,23]).

For $p \in \mathbb{N}$, $0 < q < 1$, $a > 0$ and $c > 0$, and for the parameters λ, A and B such that $-1 \leq A < B \leq 1$, $0 < B \leq 1$ and $0 \leq \lambda < [p]_q$, we say that a function $f \in \mathcal{A}(p)$ is in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$ if it satisfies the following subordination condition:

$$\frac{D_{p,q}\left(L_{p,q}(a, c) f(z)\right)}{z^{p-1}} \prec \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} z}{1 + Bz} \quad (z \in \mathbb{U})\,, \tag{1.10}$$

or, equivalently, if the following inequality holds true:

$$\left| \frac{\dfrac{D_{p,q}\left(L_{p,q}(a, c) f(z)\right)}{z^{p-1}} - [p]_q}{B \dfrac{D_{p,q}\left(L_{p,q}(a, c) f(z)\right)}{z^{p-1}} - \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\}} \right| < 1 \quad (z \in \mathbb{U})\,. \tag{1.11}$$

By specializing the parameters a, c, A, B, p, q and λ involved in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$, we obtain the following subclasses which were studied in many earlier works:

1. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(a, c, A, B, \lambda) = \mathcal{P}_{a,c}(A, B; \lambda, p)$ (Aouf et al. [9]);
2. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(a, a, A, B, \lambda) = \mathcal{S}_p(A, B, \lambda)$ (Aouf [4]);
3. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(a, a, -1, 1, \lambda) = \mathcal{S}_p(\lambda)$ (Owa [26]);
4. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(a, a, A, B, 0) = \mathcal{S}_p(A, B)$ (Chen [13]);
5. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(n + p, 1, -1, 1, \lambda) = \mathcal{T}_{n+p-1}(\lambda)$ $(n > -p)$ (Goel and Sohi [15]);
6. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(n + p, 1, -A, -B, 0) = \mathcal{V}_{n+p}(A, B)$ $(n > -p)$ (Kumar and Shukla [23]);
7. $\lim_{q\to 1^-} \mathcal{T}_{p,q}(n + p, 1, -A, -B, \lambda) = \mathcal{V}_{n+p}(A, B, \lambda)$ $(n > -p)$ (Aouf [5]);
8. $\lim_{q\to 1^-} \mathcal{T}_{1,q}(a, a, -A, -B, 0) = \mathcal{R}(A, B)$ (Mehrok [25]).

Furthermore, we say that a function $f \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$ is in the subclass $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{k=1}^{\infty} \left|a_{k+p}\right| z^{k+p} \quad (p \in \mathbb{N}). \tag{1.12}$$

Thus, by specializing the parameters a, c, A, B, p, q and λ, we obtain the following familiar subclasses of analytic functions in $\mathbb{U}$ with negative coefficients:

1. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(a, c, A, B, \lambda) = \mathcal{P}^+_{a,c}(A, B; \lambda, p)$ (Aouf et al. [9]);
2. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(a, a, A, B, \lambda) = \mathcal{P}^*(p, A, B, \lambda)$ (Aouf [2]);
3. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(a, a, -\beta, \beta, \lambda) = \mathcal{P}^*_p(\lambda, \beta)$ $(0 < \beta \le 1)$ (Aouf [3]);
4. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(a, a, A, B, 0) = \mathcal{P}^*(p, A, B)$(Shukla and Dashrath [33]);
5. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(a, a, -1, 1, \lambda) = \mathcal{F}_p(1, \lambda)$ (Lee et al. [24]);
6. $\lim_{q\to 1^-} \mathcal{T}^*_{1,q}(a, a, -\beta, \beta, \lambda) = \mathcal{P}^*(\lambda, \beta)$ (Gupta and Jain [17]);
7. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(n + p, 1, -1, 1, \lambda) = \mathcal{Q}_{n+p-1}(\lambda)$ $(n > -p)$ (Aouf and Darwish [6]);
8. $\lim_{q\to 1^-} \mathcal{T}^*_{p,q}(n + 1, 1, -1, 1, \lambda) = \mathcal{Q}_n(\lambda)$ $(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ (Uralegaddi and Sarangi [37]).

In this paper, we investigate the various important properties and characteristics of $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$ and $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$. Furthermore, several properties involving neighborhoods of functions in these subclasses are investigated. We also derive many results for the modified Hadamard products of functions belonging to the class $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$.

2 Inclusion properties of the function class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbb{N}$, $0 < q < 1$, $a > 0$, $c > 0$, $-1 \le A < B \le 1$, $0 < B \le 1$, $0 \le \lambda < [p]_q$, $[i]_q$ is given by (1.6) and $z \in \mathbb{U}$.

For proving our first inclusion result, we shall make use of the following lemma.

Lemma 1 (see [10] and [36]) *Let the nonconstant function $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 w(z_0) = \gamma D_{p,q} w(z), \tag{2.1}$$

where γ is a real number and $\gamma \ge 1$.

Theorem 1 *If $a > 0$, then*

$$\mathcal{T}_{p,q}(a + 1, c, A, B, \lambda) \subset \mathcal{T}_{p,q}(a, c, A, B, \lambda).$$

Proof If $f \in \mathcal{T}_{p,q}(a+1, c, A, B, \lambda)$, then we find from (1.10) that

$$\frac{D_{p,q}\left(L_{p,q}(a+1,c)\, f(z)\right)}{z^{p-1}} = \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} w_1(z)}{1 + Bw_1(z)}, \quad (2.2)$$

where $w_1(z)$ is a Schwarz function. To prove that $f(z)$ is in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$, we write

$$\frac{D_{p,q}\left(L_{p,q}(a,c)\, f(z)\right)}{z^{p-1}} = \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} w(z)}{1 + Bw(z)}. \quad (2.3)$$

It now suffices to show that $|w(z)| < 1$. Indeed, by using (1.9) and (2.3), we have

$$\frac{D_{p,q}\left(L_{p,q}(a+1,c)\, f(z)\right)}{z^{p-1}} = \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} w(z)}{1 + Bw(z)} + \frac{q^a (A-B)\left([p]_q - \lambda\right) zD_{p,q}w(z)}{[a]_q\, [1 + Bw(qz)][1 + Bw(z)]}. \quad (2.4)$$

We claim that

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

Otherwise, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)|.$$

Applying Lemma 1, we have

$$z_0 D_{p,q} w(z_0) = \gamma w(z_0) \quad (\gamma \geq 1).$$

Now, upon setting

$$w(z_0) = e^{i\theta} \quad (0 \leq \theta \leq 2\pi),$$

if we put $z = z_0$ in (2.4), we get

$$\left| \frac{\dfrac{D_{p,q}\left(L_{p,q}(a+1,c)\, f(z_0)\right)}{z_0^{p-1}} - [p]_q}{B\dfrac{D_{p,q}\left(L_{p,q}(a+1,c)\, f(z_0)\right)}{z_0^{p-1}} - \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\}} \right|^2 - 1$$

$$= \frac{\left|\left([a]_q + q^a\gamma\right) + [a]_q Be^{i\theta}\right|^2 - \left|[a]_q + \left([a]_q - q^a\gamma\right) Be^{i\theta}\right|^2}{\left|[a]_q + \left([a]_q - q^a\gamma\right) Be^{i\theta}\right|^2}$$

$$= \frac{q^{2a}\gamma^2\left(1 - B^2\right) + 2[a]_q q^a \gamma \left(1 + B^2 + 2B\cos\theta\right)}{\left|[a]_q + \left([a]_q - q^a\gamma\right) Be^{i\theta}\right|^2} \geq 0,$$

which, in view of (1.11), contradicts our hypothesis that $f \in \mathcal{T}_{p,q}(a+1, c, A, B, \lambda)$. Thus we must have

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

So, by applying (2.3), we conclude that $f \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$. This completes the proof of Theorem 1. □

Theorem 2 *If $f \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$, then the function $F(z)$ given by*

$$F(z) = \frac{[p+v]_q}{z^v} \int_0^z t^{v-1} f(t)\, d_q t \quad (f \in \mathcal{A}(p);\, v > -p;\, p \in \mathbb{N}) \tag{2.5}$$

is also in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$.

Proof From (2.5), we have

$$[p+v]_q L_{p,q}(a,c) f(z) = [v]_q L_{p,q}(a,c) F(z) + q^v z D_{p,q}\left(L_{p,q}(a,c) F(z)\right). \tag{2.6}$$

Let

$$\frac{D_{p,q}\left(L_{p,q}(a,c) F(z)\right)}{z^{p-1}} = \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} w(z)}{1 + Bw(z)}, \tag{2.7}$$

where $w(z)$ is either analytic or meromorphic in $\mathbb{U}$ with $w(0) = 0$. Then, by differentiating (2.7) and using (2.6), we obtain

$$\frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}} = \frac{[p]_q + \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} w(z)}{1 + Bw(z)} + \frac{q^{v-p}(A-B)\left([p]_q - \lambda\right) z D_{p,q} w(z)}{[p+v]_q [1 + Bw(qz)][1 + Bw(z)]}. \tag{2.8}$$

The remaining part of the proof of Theorem 2 is much akin to that of Theorem 1, and so it is being omitted here. □

Theorem 3 *The function $f \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$ if and only if the function $g(z)$ given by*

$$g(z) = \frac{[a]_q}{z^{a-p}} \int_0^z t^{a-p-1} f(t)\, d_q t \tag{2.9}$$

is in the class $\mathcal{T}_{p,q}(a+1, c, A, B, \lambda)$.

Proof Making use of (2.9), we have

$$[a]_q f(z) = [a-p]_q\ g(z) + q^{a-p} z D_{p,q} g(z), \tag{2.10}$$

which, in the light of (1.9), yields

$$\begin{aligned} [a]_q L_{p,q}(a,c) f(z) &= [a-p]_q\ L_{p,q}(a,c) g(z) + q^{a-p} z D_{p,q} L_{p,q}(a,c) g(z) \\ &= [a]_q L_{p,q}(a+1,c) g(z). \end{aligned}$$

Therefore, we have

$$L_{p,q}(a,c) f(z) = L_{p,q}(a+1,c) g(z),$$

and the desired result follows at once. □

3 Basic properties of the function class $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$

We first determine a necessary and sufficient condition for a function $f \in \mathcal{A}(p)$ of the form (1.12) to be in the class $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$.

Theorem 4 *Let the function* $f \in \mathcal{A}(p)$ *be given by* (1.12)*. Then* $f \in \mathcal{T}^{*}_{p,q}(a, c, A, B, \lambda)$ *if and only if*

$$\sum_{k=1}^{\infty} [k+p]_q (1+B) \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| \leq (B-A)\left([p]_q - \lambda\right). \tag{3.1}$$

Proof If the condition (3.1) holds true, we find from (1.12) and (3.1) that

$$\begin{aligned}
&\left|\frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}} - [p]_q\right| \\
&\quad - \left|B \frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}} - \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\}\right| \\
&= \left|-\sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k\right| \\
&\quad - \left|(B-A)\left([p]_q - \lambda\right) - B \sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k\right| \\
&= \sum_{k=1}^{\infty} [k+p]_q (1+B) \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| - (B-A)\left([p]_q - \lambda\right) \leq 0 \\
&\left(z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}\right).
\end{aligned}$$

Hence, by the Maximum Modulus Theorem, we have $f \in \mathcal{T}^{*}_{p,q}(a, c, A, B, \lambda)$.

Conversely, let $f \in \mathcal{T}^{*}_{p,q}(a, c, A, B, \lambda)$ be given by (1.12). Then, from (1.11) and (1.12), we find that

$$\begin{aligned}
&\left|\frac{\frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}} - [p]_q}{B \frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}} - \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\}}\right| \\
&= \left|\frac{-\sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k}{(B-A)\left([p]_q - \lambda\right) - B \sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k}\right| < 1 \quad (z \in \mathbb{U}).
\end{aligned} \tag{3.2}$$

Now, since $|\Re(z)| \leq |z|$ for all z, we have

$$= \Re\left(\frac{-\sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k}{(B-A)\left([p]_q - \lambda\right) - B \sum_{k=1}^{\infty} [k+p]_q \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| z^k}\right) < 1. \tag{3.3}$$

We choose values of z on the real axis so that $\frac{D_{p,q}\left(L_{p,q}(a,c) f(z)\right)}{z^{p-1}}$ is real. Then, upon clearing the denominator in (3.3) and letting $z \to 1^{-}$ through real values, we get

$$\sum_{k=1}^{\infty} [k+p]_q (1+B) \frac{\left([a]_q\right)_k}{\left([c]_q\right)_k} \left|a_{k+p}\right| \leq (B-A)\left([p]_q - \lambda\right).$$

This completes the proof of Theorem 4. □

Remark 1 Since $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$ is contained in the function class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$, a sufficient condition for $f(z)$ defined by (1.1) to be in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$ is that it satisfies the condition (3.1) of Theorem 4.

Corollary 1 *Let the function $f \in \mathcal{A}(p)$ be given by (1.12). If $f \in \mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$, then*

$$\left|a_{k+p}\right| \le \frac{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k}{[k+p]_q(1+B)\left([a]_q\right)_k} \quad (p, k \in \mathbb{N}). \tag{3.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k}{[k+p]_q(1+B)\left([a]_q\right)_k} z^{k+p} \quad (p, k \in \mathbb{N}). \tag{3.5}$$

We next prove the following growth and distortion properties for the class $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$.

Theorem 5 *If a function $f(z)$ defined by (1.12) is in the class $\mathcal{T}^*_{p,q}(a, c, A, B, \lambda)$, then*

$$\left([p]_q - \frac{(B-A)\left([p]_q-\lambda\right)[c]_q|z|}{(1+B)[a]_q}\right)|z|^{p-1} \le \left|D_{p,q}f(z)\right| \le \left([p]_q + \frac{(B-A)\left([p]_q-\lambda\right)[c]_q|z|}{(1+B)[a]_q}\right)|z|^{p-1}. \tag{3.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)\left([p]_q-\lambda\right)[c]_q}{[1+p]_q(1+B)[a]_q} z^{p+1} \quad (p \in \mathbb{N}). \tag{3.7}$$

Proof In view of Theorem 4, we have

$$\frac{(1+B)\ [a]_q}{(B-A)\left([p]_q-\lambda\right)[c]_q}\sum_{k=1}^{\infty}[k+p]_q\left|a_{k+p}\right| \le \sum_{k=1}^{\infty}\frac{[k+p]_q\ (1+B)}{(B-A)\left([p]_q-\lambda\right)}\frac{\left([a]_q\right)_k}{\left([c]_q\right)_k}\left|a_{k+p}\right| \le 1,$$

which readily yields

$$\sum_{k=1}^{\infty}[k+p]_q\left|a_{k+p}\right| \le \frac{(B-A)\left([p]_q-\lambda\right)[c]_q}{(1+B)\ [a]_q} \quad (p \in \mathbb{N}). \tag{3.8}$$

Now, by $q-$differentiating both sides of (1.12) with respect to z, we obtain

$$D_{p,q}f(z) = [p]_q z^{p-1} - \sum_{k=1}^{\infty}[k+p]_q\left|a_{k+p}\right| z^{k+p-1} \quad (p \in \mathbb{N}). \tag{3.9}$$

Theorem 5 follows readily from (3.8) and (3.9).

Finally, it is easy to see that the bounds in (3.6) are attained for the function $f(z)$ given by (3.7). □

4 Properties involving neighborhoods

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [16], Ruscheweyh [27] and Aouf [8], we begin by introducing here the $\delta-$ neighborhood of a function $f \in \mathcal{A}(p)$ of the form (1.1) by means of Definition 1 below.

Definition 1 For $\delta > 0$, $a > 0$, $c > 0$ and a non-negative sequence $T = \{t_k\}_{k=1}^{\infty}$, where

$$t_k = \frac{[k+p]_q (1+B) \left([a]_q\right)_k}{(B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \quad (k \in \mathbb{N}),$$

the $\delta-$neighborhood of a function $f \in \mathcal{A}(p)$ of the form (1.1) is defined as follows:

$$\mathcal{N}_\delta(f) = \left\{ g : g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \in \mathcal{A}(p) \ \text{and} \ \sum_{k=1}^{\infty} t_k \left| b_{k+p} - a_{k+p} \right| \le \delta \right\}. \tag{4.1}$$

We now prove our first result based upon the familiar concept of neighborhood defined by (4.1).

Theorem 6 *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{T}_{p,q}(a, c, A, B, \lambda)$. If f satisfies the following condition:*

$$\frac{f(z) + \varepsilon z^p}{1+\varepsilon} \in \mathcal{T}_{p,q}(a, c, A, B, \lambda) \quad (\varepsilon \in C; |\varepsilon| < \delta; \delta > 0), \tag{4.2}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{T}_{p,q}(a, c, A, B, \lambda). \tag{4.3}$$

Proof It is easily seen from (1.11) that $g \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$ if and only if, for any complex σ $(|\sigma| = 1)$,

$$\frac{zD_{p,q}\left(L_{p,q}(a,c)\, g(z)\right) - [p]_q z^p}{BzD_{p,q}\left(L_{p,q}(a,c)\, g(z)\right) - \left\{[p]_q B + (A-B)\left([p]_q - \lambda\right)\right\} z^p} \neq \sigma, \tag{4.4}$$

which is equivalent to the following inequality:

$$\frac{(g * h)(z)}{z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{4.5}$$

where, for convenience,

$$h(z) = z^p + \sum_{k=1}^{\infty} c_{k+p} z^{k+p} = z^p + \sum_{k=1}^{\infty} \frac{[k+p]_q (1+\sigma B)\left([a]_q\right)_k}{\sigma (B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} z^{k+p}. \tag{4.6}$$

It follows from (4.6) that

$$\left| c_{k+p} \right| \le \frac{[k+p]_q (1+\sigma B)\left([a]_q\right)_k}{(B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \quad (p \in \mathbb{N}). \tag{4.7}$$

Now, if $f \in \mathcal{A}(p)$, given by (1.1), satisfies the condition (4.2), then (4.5) yields

$$\frac{\left(\frac{f(z)+\varepsilon z^p}{1+\varepsilon}\right) * h(z)}{z^p} \neq 0 \quad (z \in \mathbb{U})$$

or

$$\frac{f(z) * h(z)}{z^p} \neq -\varepsilon \quad (z \in \mathbb{U}),$$

which is equivalent to the following inequality:

$$\left|\frac{f(z) * h(z)}{z^p}\right| \geq \delta \quad (z \in \mathbb{U}; \delta > 0). \tag{4.8}$$

By letting

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \in \mathcal{N}_\delta(f), \tag{4.9}$$

we deduce that

$$\begin{aligned}\left|\frac{[f(z) - g(z)] * h(z)}{z^p}\right| &= \left|\sum_{k=1}^{\infty} \left(a_{k+p} - b_{k+p}\right) c_{k+p} z^k\right| \\ &\leq \sum_{k=1}^{\infty} \frac{[k+p]_q (1 + \sigma B) \left([a]_q\right)_k}{(B - A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \left|b_{k+p} - a_{k+p}\right| \\ &< \delta \quad (z \in \mathbb{U}; \delta > 0),\end{aligned} \tag{4.10}$$

which leads us to (4.5), and hence also to (4.4) for any complex number σ ($|\sigma| = 1$). This implies that $g(z) \in \mathcal{T}_{p,q}(a, c, A, B, \lambda)$, which completes the proof of the assertion (4.3) of Theorem 6. □

We now define the δ−neighborhood of a function $f \in \mathcal{A}(p)$ of the form (1.12) as follows.

Definition 2 For $\min\{\delta, a, c\} > 0$, the δ− neighborhood of a function $f \in \mathcal{A}(p)$ of the form (1.12) is given by

$$\mathcal{N}_\delta^*(f) = \left\{ g : g(z) = z^p - \sum_{k=1}^{\infty} \left|b_{k+p}\right| z^{k+p} \in \mathcal{A}(p) \text{ and } \right. \\ \left. \sum_{k=1}^{\infty} \frac{[k+p]_q (1 + B) \left([a]_q\right)_k}{(B - A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \left|\left|b_{k+p}\right| - \left|a_{k+p}\right|\right| \leq \delta \right\}. \tag{4.11}$$

Theorem 7 *If the function $f(z)$ defined by (1.12) is in the class $\mathcal{T}_{p,q}^*(a + 1, c, A, B, \lambda)$, then*

$$\mathcal{N}_\delta^*(f) \subset \mathcal{T}_{p,q}^*(a + 1, c, A, B, \lambda) \quad \left(\delta = \frac{q^a}{[a + 1]_q}\right). \tag{4.12}$$

The result is the best possible in the sense that δ cannot be increased.

Proof Let $f \in \mathcal{T}_{p,q}^*(a + 1, c, A, B, \lambda)$ be given by (1.12). Then, by Theorem 4, we have

$$\sum_{k=1}^{\infty} \frac{[k+p]_q (1 + B) \left([a+1]_q\right)_k}{(B - A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \left|a_{k+p}\right| \leq \frac{[a]_q}{[a + 1]_q}. \tag{4.13}$$

Similarly, by taking

$$g(z) = z^p - \sum_{k=1}^{\infty} \left|b_{k+p}\right| z^{k+p} \in \mathcal{N}_\delta^*(f) \quad \left(\delta = \frac{q^a}{[a + 1]_q}\right), \tag{4.14}$$

we find from the definition (4.11) that

$$\sum_{k=1}^{\infty} \frac{[k+p]_q (1+B) \left([a]_q\right)_k}{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k} \left|\left|b_{k+p}\right|-\left|a_{k+p}\right|\right| \leq \delta \quad (\delta>0). \tag{4.15}$$

With the help of (4.13) and (4.15), we get

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{[k+p]_q (1+B) \left([a]_q\right)_k}{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k} \left|b_{k+p}\right| \\ &\leq \sum_{k=1}^{\infty} \frac{[k+p]_q (1+B) \left([a]_q\right)_k}{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k} \left|a_{k+p}\right| \\ &\quad + \sum_{k=1}^{\infty} \frac{[k+p]_q (1+B) \left([a]_q\right)_k}{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k} \left|\left|b_{k+p}\right|-\left|a_{k+p}\right|\right| \\ &\leq \frac{[a]_q}{[a+1]_q}+\delta=1. \end{aligned}$$

Thus, in view of Theorem 4 again, we see that $g \in \mathcal{T}_{p,q}^{*}(a, c, A, B, \lambda)$. To show the sharpness of the assertion of Theorem 7, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z)=z^{p}-\frac{(B-A)\left([p]_q-\lambda\right)[c]_q}{[1+p]_q (1+B)[a+1]_q} z^{p+1} \in \mathcal{T}_{p,q}^{*}(a+1, c, A, B, \lambda) \tag{4.16}$$

and

$$g(z)=z^{p}-\left(\frac{(B-A)\left([p]_q-\lambda\right)[c]_q}{[1+p]_q (1+B)[a+1]_q}+\frac{(B-A)\left([p]_q-\lambda\right)[c]_q}{[1+p]_q (1+B)[a]_q} \delta^{'}\right) z^{p+1}, \tag{4.17}$$

where $\delta^{'}>\delta=\frac{q^{a}}{[a+1]_q}$. Clearly, $g \in \mathcal{N}_{\delta}^{*}(f)$. On the other hand, we find from Theorem 4 that $g \notin \mathcal{T}_{p,q}^{*}(a, c, A, B, \lambda)$. The proof of Theorem 7 is thus completed. □

5 Properties associated with modified Hadamard products

For the functions $f_j(z)\,(j=1,2)$ given by

$$f_j(z)=z^{p}-\sum_{k=1}^{\infty}\left|a_{k+p, j}\right| z^{k+p} \quad (j=1,2;\ p \in \mathbb{N}), \tag{5.1}$$

we denote by $(f_1 \bullet f_2)(z)$ the modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 \bullet f_2)(z)=z^{p}-\sum_{k=1}^{\infty}\left|a_{k+p, 1}\right|\left|a_{k+p, 2}\right| z^{k+p}=(f_2 \bullet f_1)(z). \tag{5.2}$$

Theorem 8 *Let the functions $f_j(z)\,(j=1,2)$ defined by (5.1) be in the class $\mathcal{T}_{p,q}^{*}(a, c, A, B, \lambda)$. Then $(f_1 \bullet f_2)(z) \in \mathcal{T}_{p,q}^{*}(a, c, A, B, \gamma)$, where*

$$\gamma=[p]_q-\frac{(B-A)\left([p]_q-\lambda\right)^{2}[c]_q}{[1+p]_q (1+B)[a]_q}. \tag{5.3}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^p - \frac{(B-A)\left([p]_q - \lambda\right)[c]_q}{[1+p]_q(1+B)[a]_q} z^{p+1} \quad (j = 1, 2;\ p \in \mathbb{N}). \tag{5.4}$$

Proof Employing the technique used earlier by Schild and Silverman [29], we need to find the largest γ such that

$$\sum_{k=1}^{\infty} \frac{[k+p]_q(1+B)\left([a]_q\right)_k}{(B-A)\left([p]_q - \gamma\right)\left([c]_q\right)_k} \left|a_{k+p,1}\right| . \left|a_{k+p,2}\right| \leq 1 \tag{5.5}$$

$$\left(f_j \in \mathcal{T}_{p,q}^{*}(a, c, A, B, \lambda)\ (j = 1, 2)\right).$$

Since $f_j \in \mathcal{T}_{p,q}^{*}(a, c, A, B, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=1}^{\infty} \frac{[k+p]_q(1+B)\left([a]_q\right)_k}{(B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \left|a_{k+p,j}\right| \leq 1 \quad (j = 1, 2). \tag{5.6}$$

Therefore, by the Cauchy−Schwarz inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{[k+p]_q(1+B)\left([a]_q\right)_k}{(B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k} \sqrt{\left|a_{k+p,1}\right|} . \sqrt{\left|a_{k+p,2}\right|} \leq 1. \tag{5.7}$$

This implies that we only need to show that

$$\frac{\left|a_{k+p,1}\right| . \left|a_{k+p,2}\right|}{[p]_q - \gamma} \leq \frac{\sqrt{\left|a_{k+p,1}\right|} . \sqrt{\left|a_{k+p,1}\right|}}{[p]_q - \lambda} \quad (k \in \mathbb{N}) \tag{5.8}$$

or, equivalently, that

$$\sqrt{\left|a_{k+p,1}\right|} . \sqrt{\left|a_{k+p,1}\right|} \leq \frac{[p]_q - \gamma}{[p]_q - \lambda} \quad (k \in \mathbb{N}). \tag{5.9}$$

Hence, by making use of the inequality (5.7), it is sufficient to prove that

$$\frac{(B-A)\left([p]_q - \lambda\right)\left([c]_q\right)_k}{[k+p]_q(1+B)\left([a]_q\right)_k} \leq \frac{[p]_q - \gamma}{[p]_q - \lambda} \quad (k \in \mathbb{N}), \tag{5.10}$$

that is, that

$$\gamma \leq [p]_q - \frac{(B-A)\left([p]_q - \lambda\right)^2\left([c]_q\right)_k}{[k+p]_q(1+B)\left([a]_q\right)_k} \quad (k \in \mathbb{N}). \tag{5.11}$$

Now, defining the function $\Phi(k)$ by

$$\Phi(k) = [p]_q - \frac{(B-A)\left([p]_q - \lambda\right)^2\left([c]_q\right)_k}{[k+p]_q(1+B)\left([a]_q\right)_k} \quad (k \in \mathbb{N}), \tag{5.12}$$

we see that $\Phi(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \leq \Phi(1) = [p]_q - \frac{(B-A)\left([p]_q - \lambda\right)^2[c]_q}{[k+p]_q(1+B)[a]_q}, \tag{5.13}$$

which completes the proof of Theorem 8. □

By using arguments similar to those in the proof of Theorem 8, we can derive the following result.

Theorem 9 *Let the functions* $f_j(z)\,(j=1,2)$ *defined by* (5.1) *be in the class* $\mathcal{T}^*_{p,q}\left(a,c,A,B,\lambda_j\right)(j=1,2)$. *Then* $(f_1\bullet f_2)(z)\in\mathcal{T}^*_{p,q}(a,c,A,B,\zeta)$, *where*

$$\zeta=[p]_q-\frac{(B-A)\left([p]_q-\lambda_1\right)\left([p]_q-\lambda_2\right)[c]_q}{[1+p]_q(1+B)[a]_q}. \tag{5.14}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ *given by*

$$f_j(z)=z^p-\frac{(B-A)\left([p]_q-\lambda_j\right)[c]_q}{[1+p]_q(1+B)[a]_q}z^{p+1}\quad(j=1,2;\ p\in\mathbb{N}). \tag{5.15}$$

Theorem 10 *Let the functions* $f_j(z)\,(j=1,2)$ *defined by* (5.1) *be in the class* $\mathcal{T}^*_{p,q}(a,c,A,B,\lambda)$. *Then the function* $h(z)$ *defined by*

$$h(z)=z^p-\sum_{k=1}^{\infty}\left(\left|a_{k+p,1}\right|^2+\left|a_{k+p,2}\right|^2\right)z^{k+p} \tag{5.16}$$

belongs to the class $\mathcal{T}^*_{p,q}(a,c,A,B,\chi)$, *where*

$$\chi=[p]_q-\frac{2(B-A)\left([p]_q-\lambda\right)^2[c]_q}{[1+p]_q(1+B)[a]_q}. \tag{5.17}$$

This result is sharp for the functions $f_j(z)\,(j=1,2)$ *given by* (5.4).

Proof By noting that

$$\sum_{k=1}^{\infty}\frac{\left([k+p]_q\right)^2(1+B)^2}{2(B-A)^2\left([p]_q-\lambda\right)^2}\left(\frac{\left([a]_q\right)_k}{\left([c]_q\right)_k}\right)^2\left|a_{k+p,j}\right|^2\leq\left(\sum_{k=1}^{\infty}\frac{[k+p]_q(1+B)\left([a]_q\right)_k}{(B-A)\left([p]_q-\lambda\right)\left([c]_q\right)_k}\left|a_{k+p,j}\right|\right)^2\leq 1$$
$$\left(f_j(z)\in\mathcal{T}^*_{p,q}(a,c,A,B,\lambda)\,(j=1,2)\right), \tag{5.18}$$

we have

$$\sum_{k=1}^{\infty}\frac{\left([k+p]_q\right)^2(1+B)^2}{2(B-A)^2\left([p]_q-\lambda\right)^2}\left(\frac{\left([a]_q\right)_k}{\left([c]_q\right)_k}\right)^2\left(\left|a_{k+p,1}\right|^2+\left|a_{k+p,2}\right|^2\right)\leq 1. \tag{5.19}$$

Therefore, we have to find the largest χ such that

$$\frac{1}{[p]_q-\chi}\leq\frac{[k+p]_q(1+B)\left([a]_q\right)_k}{2(B-A)\left([p]_q-\lambda\right)^2\left([c]_q\right)_k}\quad(k\in\mathbb{N}), \tag{5.20}$$

that is, that

$$\chi\leq[p]_q-\frac{2(B-A)\left([p]_q-\lambda\right)^2\left([c]_q\right)_k}{[1+p]_q(1+B)\left([a]_q\right)_k}\quad(k\in\mathbb{N}). \tag{5.21}$$

Now, if we define a function $\Psi(k)$ by

$$\Psi(k)=[p]_q-\frac{2(B-A)\left([p]_q-\lambda\right)^2\left([c]_q\right)_k}{[1+p]_q(1+B)\left([a]_q\right)_k}\quad(k\in\mathbb{N}), \tag{5.22}$$

we observe that $\Psi(k)$ is an increasing function of k. We thus conclude that

$$\chi \leq \Psi(1) = [p]_q - \frac{2(B-A)\left([p]_q - \lambda\right)^2 [c]_q}{[1+p]_q (1+B) [a]_q}, \tag{5.23}$$

which completes the proof of Theorem 10. □

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