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Some results on U-cross Gram matrices by using K-frames

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Abstract
U-cross Gram matrices are produced by frames and Riesz bases. In this paper, we represent bounded operators as matrix operators using K-frames. We study the invertibility matrices respect to K-frames. Moreover, we apply the concept of K-Riesz bases in Hilbert space $\mathcal{H}$ to the concept of matrix induced by U with respect to K-Riesz bases.

Keywords U-cross Gram matrix · Cross Gram matrix · K-frame · K-Riesz basis

Mathematics Subject Classification Primary 41A58 · Secondary 43A35

1 Introduction, notation and motivation

A unitary system is a set of unitary operators $\mathcal{U}$ acting on a Hilbert space $\mathcal{H}$ which contains the identity operator I of $B(\mathcal{H})$. A Bessel generator for $\mathcal{U}$ is a vector $x \in \mathcal{H}$ with the property that $\mathcal{U}x := \{Ux : U \in \mathcal{U}\}$ is Bessel sequence for $\mathcal{H}$. Many useful frames, which play an essential role in both theory and applications, can been considered as unitary systems, group-like unitary systems and atomic systems [16,18]. K-frames were recently introduced by Gavruta to study atomic systems with respect to a bounded operator $K \in B(\mathcal{H})$. It is a generalization of frame theory such that the lower bound is only satisfied for the elements in the range of K [17]. It is shown that an atomic system for K is a K-frame and vice versa. For this reason, K-frames are a useful mathematical tool to study the structure of unitary systems. Another purpose of this paper is to study Gram Matrices. The operator equation $Uf = v$ where $U \in B(\mathcal{H})$ does not have a smooth solution (i.e. have all derivatives continuous) in general. It can be rewritten of the form

$$Ax = b \tag{1.1}$$

where $A_{i,j} = \langle Ue_i, e_j\rangle$ and $\{e_i\}_{i\in I}$ is an orthonormal basis of H. To solve linear systems (1.1) variational method can be applied for example [25]. Recently, frames, Riesz bases and

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g-frames are applied to obtain (1.1) [3,4,12]. In this paper, we apply K-frames to get (1.1) as atomic decompositions of elements in the range of K which may not be closed.

Let $\mathcal{H}$ be a separable Hilbert space and K an operator from $\mathcal{H}$ to $\mathcal{H}$. A sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *K-frame* for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.2}$$

Clearly if $K = I_{\mathcal{H}}$, then F is an ordinary frame. The constants A and B in (1.2) are called lower and upper bounds of F, respectively. We call F a *A-tight K-frame* if $A\|K^*f\|^2 = \sum_{i\in I} |\langle f, f_i\rangle|^2$ and a 1-tight K-frame as *Parseval K-frame*. A K-frame is called an *exact K-frame*, if by removing any element, the reminder sequence is not a K-frame.

Obviously, every K-frame is a Bessel sequence, hence similar to ordinary frames the *synthesis operator* can be defined as $T_F : l^2 \to \mathcal{H}$; $T_F(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i$. It is a bounded operator and its adjoint which is called the *analysis operator* given by $T_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I}$. Finally, the *frame operator* is given by $S_F : \mathcal{H} \to \mathcal{H}$; $S_F f = T_F T_F^* f = \sum_{i\in I}\langle f, f_i\rangle f_i$. Many properties of ordinary frames do not hold for K-frames, for example, the frame operator of a K-frame is not invertible in general. It is worthwhile to mention that if K has close range then S_F from $R(K)$ onto $S_F(R(K))$ is an invertible operator [24]. In particular,

$$B^{-1}\|f\| \leq \|S_F^{-1}f\| \leq A^{-1}\|K^\dagger\|^2\|f\|, \quad (f \in S_F(R(K))), \tag{1.3}$$

where $K^\dagger$ is the pseudo-inverse of K.

Let $\{f_i\}_{i\in I}$ be a Bessel sequence. A Bessel sequence $\{g_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *K-dual* of $\{f_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I}\langle f, g_i\rangle \pi_{R(K)} f_i, \quad (f \in \mathcal{H}). \tag{1.4}$$

In [17], it was shown that for every K-frame of $\mathcal{H}$ there exists at least a Bessel sequence $\{g_i\}_{i\in I}$ which satisfies (1.4).

Let $F = \{f_i\}_{i\in I}$ be a K-frame. The Bessel sequence $\{K^*S_F^{-1}\pi_{S_F(R(K))} f_i\}_{i\in I}$ can be considered as *the canonical K-dual* of F [1]. For simplicity, the canonical K-dual is denoted by $F^\ddagger = \{f_i^\ddagger\}_{i\in I}$. In the sequel, we show that for each $f \in \mathcal{H}$, the sequence $\{\langle f, f_i^\ddagger\rangle\}_{i\in I}$ has minimal ℓ^2-norm among all sequences representating Kf.

The next proposition is important in K-frame theory.

Proposition 1.1 *[14] Let $L_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in B(\mathcal{H}_2, \mathcal{H})$ be two bounded operators. The following statements are equivalent:*

(1) $R(L_1) \subseteq R(L_2)$.
(2) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ for some $\lambda \geq 0$.
(3) there exists a bounded operator $X \in B(\mathcal{H}_1, \mathcal{H}_2)$ so that $L_1 = L_2X$

In this paper, we establish the notion of K-Riesz bases and show that, similar to ordinary frames, a K-Riesz basis has a unique K-dual. Also, try to state an operator as a matrix operator induced by K-frames and K-Riesz bases. More precisely, every K-frame is a Bessel sequence, and therefore we can induce matrix representations (3.1) for operators by K-frames. The inverse of such matrices are computed if they are exist. Moreover, we investigate sufficient conditions such that a matrix operator induced by K-frames is invertible. For more similar information see [5].

2 K-Riesz bases

In this section, we present K-Riesz sequences in $\mathcal{H}$ and investigate their properties. Also, we state K-Riesz bases and give some characterizations of this concept such as we prove that they are a unique K-dual. Throughout this paper we suppose K is a bounded operator with closed range.

Definition 2.1 A family $F := \{f_i\}_{i\in I}$ is called a *K-Riesz sequence* for $\mathcal{H}$ if there exists an injective bounded operator $U : \mathcal{H} \to \mathcal{H}$ such that $\left\{\pi_{R(K)} f_i\right\}_{i\in I} = \{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$. In addition, if F is a K-frame, then $\{f_i\}_{i\in I}$ is called a *K-Riesz basis.*

The next theorem, which used frequently throughout the paper, gives an equivalent condition for K-Riesz sequences.

Theorem 2.2 *For a K-frame $F = \{f_i\}_{i\in I}$ in $\mathcal{H}$, the following are equivalent:*

(1) $\{f_i\}_{i\in I}$ is a K-Riesz basis for $\mathcal{H}$.
(2) There exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_i\}_{i\in I}$,

$$A\sum_{i\in I}|c_i|^2 \le \left\|\sum_{i\in I} c_i\pi_{R(K)} f_i\right\|^2 \le B\sum_{i\in I}|c_i|^2. \tag{2.1}$$

Proof (1) $\Rightarrow$ (2) Let $\{f_i\}_{i\in I}$ be a K-Riesz sequence. Then there exists an injective bounded operator $U : \mathcal{H} \to \mathcal{H}$ such that $Ue_i = \pi_{R(K)} f_i$. Moreover, applying the lower K-frame condition and Theorem 1.1 we have

$$\begin{aligned} R(U) &= \pi_{R(K)}\pi_{R(T_F)}\mathcal{H} \\ &= \pi_{R(T_F)}\pi_{R(K)}\mathcal{H} \\ &= \pi_{R(K)}\mathcal{H} = R(K). \end{aligned}$$

In particular, $R(U)$ is closed, and so U has a bounded left inverse denoted by L. Hence,

$$\left\|\sum_{i\in I} c_i\pi_{R(K)} f_i\right\|^2 = \left\|\sum_{i\in I} c_i Ue_i\right\|^2 \le \|U\|^2\sum_{i\in I}|c_i|^2$$

and

$$\sum_{i\in I}|c_i|^2 = \left\|LU\sum_{i\in I} c_i e_i\right\|^2 \le \|L\|^2\left\|\sum_{i\in I} c_i\pi_{R(K)} f_i\right\|^2,$$

for every finite scalar sequence $\{c_i\}_{i\in I}$.

(2) $\Rightarrow$ (1) Given

$$U : \mathcal{H} \to \overline{span}\left\{\pi_{R(K)} f_i\right\}_{i\in I}, (e_i \mapsto \pi_{R(K)} f_i).$$

Then (2.1) yields

$$\begin{aligned} A\sum_{i\in I}|c_i|^2 &= \left\|\sum_{i\in I} c_i\pi_{R(K)} f_i\right\|^2 \\ &= \left\|U\sum_{i\in I} c_i e_i\right\|^2 \le B\sum_{i\in I}|c_i|^2. \end{aligned}$$

So, U is bounded and injective. □

The next corollary gives equivalent conditions for a Bessel sequence being a K-Riesz basis.

Corollary 2.3 *Let $F = \{f_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$. The following are equivalent:*

(1) F is a K-Riesz basis.
(2) F is a K-frame and

$$A\sum_{i\in I}|c_i|^2 \leq \left\|\sum_{i\in I} c_i \pi_{R(K)} f_i\right\|^2 \leq B\sum_{i\in I}|c_i|^2.$$

(3) $\pi_{R(K)}T_F$ is invertible from ℓ^2 onto $R(K)$.

An *overcomplete* or *redundant K-frame* is a K-frame $\{f_i\}_{i\in I}$ such that $\{f_i\}_{i\in I}$ is not a K-Riesz basis. In other word, a K^*-frame $\{f_i\}_{i\in I}$ is redundant, if there exist coefficients $\{c_i\}_{i\in I} \in \ell^2 \setminus \{0\}$ for which $\sum_{i\in I} c_i\pi_{R(K)} f_i = 0$. In fact, a K-frame $\{f_i\}_{i\in I}$ is a K-Riesz basis if the elements of $\{\pi_{R(K)} f_i\}_{i\in I}$ are independent.

Proposition 2.4 *Let $\{f_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$. The following are equivalent:*

(1) $\{f_i\}_{i\in I}$ is K-Riesz sequence for $\mathcal{H}$.
(2) $\{\pi_{R(K)} f_i\}_{i\in I}$ is a Riesz sequence.
(3) $\{\pi_{R(K)} f_i\}_{i\in I}$ is $\omega-$independent.

Moreover, let $\{f_i\}_{i\in I}$ be a K-frame. Then $\{f_i\}_{i\in I}$ is a K-Riesz basis if and only if $\{\pi_{R(K)} f_i\}_{i\in I}$ is $\omega-$independent.

The relationship between K-Riesz bases and exact K^*-frames is discussed on the following proposition, see Theorem 3.3.2 of [11] for the ordinary case.

Proposition 2.5 *Let $F = \{f_i\}_{i\in I}$ be a K-frame in $\mathcal{H}$. The following are equivalent.*

(1) F is a K-Riesz basis.
(2) F has a unique K-dual in $\mathcal{H}$.

Proof (1) $\Rightarrow$ (2) Assume that F is a K-Riesz basis of the form of $\{Ue_i\}_{i\in I}$, where $U \in B(\mathcal{H})$ is injective. If $\{g_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ are K-dual of F, then

$$\begin{aligned} U\sum_{i\in I}\langle f, g_i\rangle e_i &= \sum_{i\in I}\langle f, g_i\rangle Ue_i \\ &= \sum_{i\in I}\langle f, g_i\rangle \pi_{R(K)} f_i \\ &= Kf \\ &= \sum_{i\in I}\langle f, h_i\rangle \pi_{R(K)} f_i \\ &= \sum_{i\in I}\langle f, h_i\rangle Ue_i = U\sum_{i\in I}\langle f, h_i\rangle e_i, \end{aligned}$$

for every $f \in \mathcal{H}$. The injectivity U induces that $\{g_i\}_{i\in I} = \{h_i\}_{i\in I}$.
(2) $\Rightarrow$ (1) Assume that F has a unique K-dual in $\mathcal{H}$. On the contrary, suppose that F is not a K-Riesz basis. Using Proposition 2.4 follows that $\{\pi_{R(K)_{f_i}}\}$ is not a Riesz basis, or equivalently, $\pi_{R(K)}T_F$ is not injective. Choose $0 \neq \{c_i\}_{i\in I} \in l^2$ such that

$$\pi_{R(K)}T_F\{c_i\}_{i\in I} = 0. \tag{2.2}$$

Defining the sequence $\{g_i\}_{i\in I}$ in $\mathcal{H}$ weakly by

$$\langle f, g_i\rangle = \langle f, f_i^{\ddagger}\rangle + c_i, \quad (i \in I, f \in \mathcal{H}). \tag{2.3}$$

Then $\{g_i\}_{i\in I}$ is a Bessel sequence. Moreover, applying (2.2) and (2.3) we obtain

$$\begin{aligned}\sum_{i\in I}\langle f, g_i\rangle \pi_{R(K)} f_i &= \sum_{i\in I}\left(\langle f, f_i^{\ddagger}\rangle + c_i\right)\pi_{R(K)} f_i \\ &= \sum_{i\in I}\langle f, f_i^{\ddagger}\rangle \pi_{R(K)} f_i + \pi_{R(K)} T_F\{c_i\}_{i\in I} = Kf\end{aligned}$$

Hence, $\{g_i\}_{i\in I}$ is a K-dual of F and so $g_i = f_i^{\ddagger}$, for all $i \in I$ by the assumption. This easily follows that $\{c_i\} = 0$ which is impossible. □

The question that may involve is that the relationship between Riesz bases and K-Riesz bases.

Corollary 2.6 *Let $\{f_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}$. If $\{f_i\}_{i\in I}$ is a Riesz basis, then $\{Kf_i\}_{i\in I}$ is a K-Riesz basis in $\mathcal{H}$. If $\{Kf_i\}_{i\in I}$ is a Riesz basis, then $\{f_i\}_{i\in I}$ is a K-Riesz sequence in $\mathcal{H}$.*

3 *U*-Gram matrix with respect to *K*-frames

The standard matrix description of an operator U, using an orthonormal basis $\{e_i\}_{i\in I}$, is the matrix M defined by

$$(Mc)_j = \sum_k M_{jk} c_k, \qquad (c = \{c_k\}_k \in \ell^2),$$

where $M_{jk} = \left\langle Ue_k, e_j\right\rangle$. The same can be constructed with frames and their duals. More precisely, assume that $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ are a pair of dual frames and $Uf = v$ is an operator equation, then

$$\begin{aligned}\sum_j \langle U\psi_j, \phi_i\rangle\langle f, \phi_j\rangle &= \left\langle \sum_j \langle f, \phi_j\rangle \psi_j, U^*\phi_i\right\rangle \\ &= \langle Uf, \phi_i\rangle = \langle v, \phi_i\rangle .\end{aligned}$$

Thus, the operator equation can be reduced to the linear system

$$\left(\langle U\psi_j, \phi_i\rangle\right)_{i,j}\left(\langle f, \phi_j\rangle\right) = \left(\langle v, \phi_i\rangle\right).$$

We say that $\left(\langle U\psi_j, \phi_i\rangle\right)_{i,j}$ is the matrix representation of U by using dual pairs Φ and Ψ. In [22], it is shown that operators can be described as the form of matrices by using fusion frames.

In this section, we represent an operator in $B(\mathcal{H})$ as the form of a matrix in the base of K-frames. Also, we investigate its inverse if there exists.

Definition 3.1 Let $\Psi = \{\psi_i\}_{i\in I}$ be a Bessel sequence in $\mathcal{H}_1$ and $\Phi = \{\phi_i\}_{i\in I}$ a Bessel sequence in $\mathcal{H}_2$. For $U \in B(\mathcal{H}_1, \mathcal{H}_2)$, the matrix induced by operator U with respect to the Bessel sequences $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$, denoted by $\mathbf{G}_{U,\Phi,\Psi}$, is given by

$$\left(\mathbf{G}_{U,\Phi,\Psi}\right)_{i,j} = \left\langle U\psi_j, \phi_i\right\rangle, \qquad (i, j \in I), \tag{3.1}$$

for more details see [4]. It is straightforward to see that

$$\mathbf{G}_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi. \tag{3.2}$$

Because of the operator representation (3.2), we call $\mathbf{G}_{U,\Phi,\Psi}$ the U-cross Gram matrix of Φ and Ψ, respectively. In other word, $\mathbf{G}_{U,\Phi,\Psi}$ is a bounded operator on ℓ^2 with $\left\|\mathbf{G}_{U,\Phi,\Psi}\right\| \leq \sqrt{B_\Phi B_\Psi}\|U\|$ and $\left(\mathbf{G}_{U,\Phi,\Psi}\right)^* = \mathbf{G}_{U^*,\Psi,\Phi}$. If $\mathcal{H}_1 = \mathcal{H}_2$ and $U = I_{\mathcal{H}_1}$ it is called the *cross Gram matrix* and denoted by $\mathbf{G}_{\Phi,\Psi}$. We use $\mathbf{G}_\Psi$ for $\mathbf{G}_{\Psi,\Psi}$; the so called the *Gram matrix* [11].

An operator $U \in B(\mathcal{H})$ has a K-right inverse (K-left inverse) if there exists an operator $\mathcal{R} \in B(\mathcal{H})$ (resp. $\mathcal{L} \in B(\mathcal{H})$), so that

$$U\mathcal{R} = K, \qquad (resp. \quad \mathcal{L}U = K),$$

for $K \in B(\mathcal{H})$. If $\mathcal{R} = \mathcal{L}$, then $\mathcal{R}$ is the K-inverse of U.

Example 3.2 Let $K \in B(\mathcal{H})$ and $\Phi = \{\phi_i\}_{i\in I}$ be a K-frame in $\mathcal{H}$. Then

(1) $\mathbf{G}_{S_\Phi(K^\dagger)^*,\Phi,\Phi^\ddagger} = \mathbf{G}_\Phi$, when K is a closed range operator in $B(\mathcal{H})$ and $\Phi \subseteq S_\Phi(R(K))$. Indeed, since $R(S_\Phi^{-1}) \subseteq R(K)$ on $S_\Phi(R(K))$ and $KK^\dagger = I\mid_{R(K)}$ we have

$$\begin{aligned}\mathbf{G}_{S_\Phi(K^\dagger)^*,\Phi,\Phi^\ddagger} &= T_\Phi^* S_\Phi \left(K^\dagger\right)^* T_{\Phi^\ddagger} \\ &= T_\Phi^* T_{S_\Phi(K^\dagger)^* K^* S_\Phi^{-1}\pi_{S_\Phi(R(K))}\Phi} \\ &= T_\Phi^* T_\Phi = \mathbf{G}_\Phi.\end{aligned}$$

(2) If $R(U) \subseteq R(K)$ and $\mathbf{G}_{\left(S_\Phi^{-1}\right)^* U\pi_{R(K)},\Phi,\Phi} = I_{\ell^2}$, then U is K-right invertible. According to the fact that $\Phi^\ddagger$ is a K-dual of Φ, we have

$$\begin{aligned}K &= \pi_{R(K)} T_\Phi T_{\Phi^\ddagger}^* \\ &= \pi_{R(K)} T_\Phi \mathbf{G}_{\left(S_\Phi^{-1}\right)^* U\pi_{R(K)},\Phi,\Phi} T_{\Phi^\ddagger}^* \\ &= \pi_{R(K)} T_\Phi T_\Phi^* \left(S_\Phi^{-1}\right)^* U\pi_{R(K)} T_\Phi T_{\Phi^\ddagger}^* \\ &= \pi_{R(K)} S_\Phi \left(S_\Phi^{-1}\right)^* UK \\ &= \pi_{R(K)} UK = UK.\end{aligned}$$

In the following we state a sufficient condition such that $\mathbf{G}_{U,\Phi,\Phi^\ddagger} = I_{\ell^2}$.

Theorem 3.3 *Let $U, K \in B(\mathcal{H})$ and $R(U) \subseteq R(K)$. If Φ is a K-frame such that $\mathbf{G}_{U,\Phi,\Phi^\ddagger} = I_{\ell^2}$, then UK^* is a biorthogonal projection on $R(K)$. The converse is true if Φ is a K-Riesz basis.*

Proof Using the duality formula we have

$$T_{\Phi^\ddagger} T_\Phi^* \pi_{R(K)} = K^*.$$

Then

$$\begin{aligned}S_\Phi \pi_{R(K)} &= T_\Phi T_\Phi^* \pi_{R(K)} \\ &= T_\Phi \mathbf{G}_{U,\Phi,\Phi^\ddagger} T_\Phi^* \pi_{R(K)} \\ &= T_\Phi T_\Phi^* U T_{\Phi^\ddagger} T_\Phi^* \pi_{R(K)} = S_\Phi U K^*.\end{aligned}$$

Applying $R(U) \subseteq R(K)$ implies that S_Φ is an invertible on $R(U)$ and then $UK^* = \pi_{R(K)}$. Conversely, it is easy to see that

$$\begin{aligned}\mathbf{G}_{U,\Phi,\Phi^\ddagger} T_\Phi^* \pi_{R(K)} &= T_\Phi^* U T_{\Phi^\ddagger} T_\Phi^* \pi_{R(K)} \\ &= T_\Phi^* U K^* = T_\Phi^* \pi_{R(K)}.\end{aligned}$$

So, $\mathbf{G}_{U,\Phi,\Phi^\ddagger} = I_{\ell^2}$ if $T_\Phi^* \pi_{R(K)}$ is invertible and so by Proposition 2.3 if Φ is a K-Riesz basis. □

It is worthwhile to mention that from the one sided invertibility of Gram matrix induced by $K \in B(\mathcal{H})$ with respect to Bessel sequences Ψ and Φ, respectively, it follows that the Bessel sequences $K\Psi$ and $K^*\Phi$ are K-Riesz sequence and K^*-Riesz sequence, respectively.

Theorem 3.4 *Let $\Psi = \{\psi_i\}_{i \in I}$ be a Bessel sequence in $\mathcal{H}$.*

(1) If $\mathbf{G}_{K,\Phi,\Psi}$ has a left inverse, then $K\Psi$ is a K-Riesz sequence in $\mathcal{H}$.
(2) If $\mathbf{G}_{K,\Phi,\Psi}$ has a right inverse, then $K^*\Phi$ is a K^*-Riesz sequence in $\mathcal{H}$.

Proof Let $\mathcal{L}$ be a left inverse of $\mathbf{G}_{K,\Phi,\Psi}$. Then

$$\begin{aligned}I_{\ell^2} &= \mathcal{L}\mathbf{G}_{K,\Phi,\Psi} \\ &= \mathcal{L} T_\Phi^* K T_\Psi = \mathcal{L} T_\Phi^* T_{K\Psi}.\end{aligned}$$

The above computations show that $T_{K\Psi}$ has a left inverse and so $T_{K\Psi}$ is an injective operator. Hence, by applying Lemma 2.4.1 of [11] there exists $A > 0$ such that

$$A \sum_{i \in I} |c_i|^2 \le \left\| \sum_{i \in I} c_i K \psi_i \right\|^2 = \left\| \sum_{i \in I} c_i \pi_{R(K)} K \psi_i \right\|^2.$$

Therefore, $K\Psi$ is a K-Riesz sequence in $\mathcal{H}$ by Theorem 2.2.

The proof of (2) is similar. □

Now, we study several K-duals of a K-frame by its Gram matrix.

Theorem 3.5 *Let $\Phi = \{\phi_i\}_{i \in I}$ be a K-frame in $\mathcal{H}$. Then $\mathbf{G}_\Phi = I_{\ell^2}$ on $R(T_\Phi^* K)$ if and only if $S_\Phi = I_\mathcal{H}$ on $R(K)$. In this case $\phi_i^\ddagger = K^* \pi_{R(K)} \phi_i$, for all $i \in I$.*

Proof We first claim that $\mathbf{G}_\Phi T_\Phi^* K = T_\Phi^* K$ if and only if $S_\Phi K = K$. Assume that $\mathbf{G}_\Phi T_\Phi^* K = T_\Phi^* K$. Then

$$\begin{aligned}S_\Phi S_\Phi K &= T_\Phi T_\Phi^* T_\Phi T_\Phi^* K \\ &= T_\Phi \mathbf{G}_\Phi T_\Phi^* K \\ &= T_\Phi T_\Phi^* K = S_\Phi K.\end{aligned}$$

The invertibility S_Φ on $R(K)$ implies that $S_\Phi K = K$. For the reverse,

$$\mathbf{G}_\Phi T_\Phi^* K = T_\Phi^* T_\Phi T_\Phi^* K = T_\Phi^* S_\Phi K = T_\Phi^* K.$$

Moreover,

$$\begin{aligned}\phi_i^\ddagger &= K^* S_\Phi^{-1} \pi_{S_\Phi(R(K))} \phi_i \\ &= K^* S_\Phi S_\Phi^{-1} \pi_{S_\Phi R(K)} \phi_i \\ &= K^* \pi_{S_\Phi R(K)} \phi_i = K^* \pi_{R(K)} \phi_i,\end{aligned}$$

for all $i \in I$. It completes the proof. □

We are going to construct several K-duals for some Bessel sequences in which their Gram matrices are invertible from the left or from the right.

Theorem 3.6 *Let $\Phi = \{\phi_i\}_{i\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$ be a K-frame and a Bessel sequence in $\mathcal{H}$, respectively. Also, $U \in B(\mathcal{H})$ and $\mathbf{G}_{U,\Phi,\Psi}$ has a right inverse as $\mathcal{R}$. Then*

(1) $\left\{S_\Phi^{-1}\pi_{S_\Phi(R(K))}\phi_i\right\}_{i\in I}$ is a K-frame. Moreover, $\left\{K^*T_\Phi\mathcal{R}^*T_\Psi^*U^*\phi_i\right\}_{i\in I}$ is a K-dual of $\left\{S_\Phi^{-1}\pi_{S_\Phi(R(K))}\phi_i\right\}_{i\in I}$.
(2) for every K-dual $\Phi^d = \left\{\phi_i^d\right\}_{i\in I}$, the K^*-frame $\left\{T_{\Phi^d}\mathcal{R}^*T_\Psi^*U^*\phi_i\right\}_{i\in I}$ is a K-dual of Φ.

Proof Notice that $S_\Phi^{-1}S_\Phi = I$ on $R(K)$. So, for all $f \in \mathcal{H}$, we obtain

(1)

$$\begin{aligned} Kf &= S_\Phi^{-1}S_\Phi Kf \\ &= S_\Phi^{-1}T_\Phi\mathbf{G}_{U,\Phi,\Psi}\mathcal{R}T_\Phi^*Kf \\ &= S_\Phi^{-1}T_\Phi T_\Phi^*UT_\Psi\mathcal{R}T_\Phi^*Kf \\ &= \sum_{i\in I}\left\langle f, K^*T_\Phi\mathcal{R}^*T_\Psi^*U^*\phi_i\right\rangle \pi_{R(K)}S_\Phi^{-1}\pi_{S_\Phi(R(K))}\phi_i. \end{aligned}$$

The existence of K-dual for $\left\{S_\Phi^{-1}\pi_{S_\Phi(R(K))}\phi_i\right\}_{i\in I}$ proves the considered sequence is a K-frame.

(2) Using (1.4) we obtain

$$\begin{aligned} Kf &= \pi_{R(K)}T_\Phi T_{\Phi^d}f \\ &= \pi_{R(K)}T_\Phi\mathbf{G}_{U,\Phi,\Psi}\mathcal{R}T_{\Phi^d}f \\ &= \pi_{R(K)}T_\Phi T_\Phi^*UT_\Psi\mathcal{R}T_{\Phi^d}f \\ &= \sum_{i\in I}\left\langle f, T_{\Phi^d}\mathcal{R}^*T_\Psi^*U^*\phi_i\right\rangle \pi_{R(K)}\phi_i. \end{aligned}$$

□

In the following, we present pairs of K-duals by means of the one sided invertibility of Gram matrices.

Corollary 3.7 *Let $\Phi = \{\phi_i\}_{i\in I}$ be a Bessel sequence and $\Psi = \{\psi_i\}_{i\in I}$ a K-frame in $\mathcal{H}$. Also, let $U \in B(\mathcal{H})$, and $\mathcal{L}$ be a left inverse of $\mathbf{G}_{U,\Phi,\Psi}$. Then*

(1) $\{K^*\psi_i\}_{i\in I}$ is a K-dual of $\left\{S_\Psi^{-1}\pi_{S_\Psi(R(K))}T_\Psi\mathcal{L}T_\Phi^*U\psi_i\right\}_{i\in I}$.
(2) $\left\{K^*S_\Psi^{-1}\pi_{S_\Psi(R(K))}\psi_i\right\}_{i\in I}$ is a K-dual of $\left\{T_\Psi\mathcal{L}T_\Phi^*U\psi_i\right\}_{i\in I}$.
(3) $\Psi^d = \{\psi_i^d\}_{i\in I}$ is a K-dual of $\left\{T_\Psi\mathcal{L}T_\Phi^*U\psi_i\right\}_{i\in I}$, where Ψ^d is a K-dual of Ψ.

Proof (1) Let $\mathcal{L} \in B(\mathcal{H})$ be a left inverse of $\mathbf{G}_{U,\Phi,\Psi}$. It is clear to see that $\{K^*\psi_i\}_{i\in I}$ and $\left\{S_\Psi^{-1}\pi_{S_\Psi(R(K))}T_\Psi \mathcal{L} T_\Phi^* U\psi_i\right\}_{i\in I}$ are Bessel sequences in $\mathcal{H}$. Moreover,

$$\begin{aligned}
Kf &= S_\Psi^{-1} S_\Psi K f \\
&= \pi_{R(K)} S_\Psi^{-1} S_\Psi K f \\
&= \pi_{R(K)} S_\Psi^{-1} \pi_{S_\Psi R(K)} T_\Psi T_\Psi^* K f \\
&= \pi_{R(K)} S_\Psi^{-1} \pi_{S_\Psi R(K)} T_\Psi \mathcal{L} \mathbf{G}_{U,\Phi,\Psi} T_\Psi^* K f \\
&= \sum_{i\in I} \left\langle f, K^*\psi_i \right\rangle \pi_{R(K)} S_\Psi^{-1} \pi_{S_\Psi R(K)} T_\Psi \mathcal{L} T_\phi^* U \psi_i,
\end{aligned}$$

for all $f \in \mathcal{H}$. The rest is similar. □

In the following we present a K-dual for a K-frame by some its K-duals.

Proposition 3.8 *Assume that $\Psi = \{\psi_i\}_{i\in I}$ is a K-dual of a K-frame $\Phi = \{\phi_i\}_{i\in I}$. If $U \in B(\mathcal{H})$ such that $\mathbf{G}_{U,\Phi,\Psi} = I_{\ell^2}$, then $S_\Psi^* U^* \Phi$ is a K-dual of Φ. In particular, if Ψ is the canonical K-dual of Φ and $KU^* = I_\mathcal{H}$, then $S_\Psi^* U^*\Phi = \Psi$.*

Proof For all $f \in \mathcal{H}$ we have

$$\begin{aligned}
Kf &= \pi_{R(K)} T_\Phi T_\Psi^* f \\
&= \pi_{R(K)} T_\Phi \mathbf{G}_{U,\Phi,\Psi} T_\Psi^* f \\
&= \pi_{R(K)} T_\Phi T_\Phi^* U T_\Psi T_\Psi^* f \\
&= \pi_{R(K)} S_\Phi U S_\Psi f \\
&= \sum_{i\in I} \left\langle f, S_\Psi^* U^* \phi_i \right\rangle \pi_{R(K)} \phi_i.
\end{aligned}$$

In particular, if Ψ is the canonical K-dual of Φ, then we have

$$\begin{aligned}
S_\Psi f &= \sum_{i\in I} \left\langle f, K^* S_\Phi^{-1} \pi_{S_\Phi R(K)} \phi_i \right\rangle K^* S_\Phi^{-1} \pi_{S_\Phi R(K)} \phi_i \\
&= K^* S_\Phi^{-1} \pi_{S_\Phi R(K)} S_\Phi (S_\Phi^{-1})^* K f.
\end{aligned}$$

Now, by using $KU^* = I_\mathcal{H}$ we obtain

$$\begin{aligned}
S_\Psi^* U^* \Phi &= K^* S_\Phi^{-1} \pi_{S_\Phi R(K)} S_\Phi^* (S_\Phi^{-1})^* K U^* \Phi \\
&= K^* S_\Phi^{-1} \pi_{S_\Phi R(K)} \Phi = \Psi.
\end{aligned}$$

□

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