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Time asymptotic behavior of the solution to the linear Boltzmann equation in finite bodies

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Abstract
In this paper we discuss time asymptotic behavior of the solution to the Cauchy problem governed by the transport operator in bounded geometry in the case where the boundary conditions are dissipative and modeled by the bounce-back boundary operator plus a compact in L^1-spaces. The case of multiplying compact boundary operator is considered in the last subsection.

Keywords Transport operator · dissipative (multiplying)boundary conditions · C_0-semigroup · Dyson-Phillips expansion

Mathematics Subject Classification 47A10 · 47A55 · 35Q20

1 Introduction

The purpose of this paper is to discuss the time asymptotic behavior of the solution of the following Cauchy problem governed by a transport operator

$$\begin{cases} \dfrac{\partial \psi}{\partial t}(x, v, t) = A_H \psi(x, v, t) := T_H \psi(x, v, t) + K \psi(x, v, t) \\ \qquad\qquad = -v \cdot \nabla_x \psi(x, v, t) - \sigma(v)\psi(x, v, t) + \displaystyle\int_{\mathbb{R}^N} \kappa(x, v, v')\psi(x, v', t)dv', \\ \psi(x, v, 0) = \psi_0(x, v), \end{cases} \tag{1}$$

where $(x, v) \in \Omega \times \mathbb{R}^N$ and K is the partial integral part of A_H and called the collisional operator. Here Ω is a smooth bounded open convex subset of $\mathbb{R}^N$. The function $\psi(x, v, t)$ represents the number (or probability) density of gas particles having the position x and the

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velocity v at time t. The functions $\sigma(\cdot)$ and $\kappa(\cdot,\cdot,\cdot)$ are called, respectively, the collision frequency and the scattering kernel.
The boundary conditions are modeled by:

$$\psi_- = H(\psi_+), \tag{2}$$

where ψ_-(resp. ψ_+) is the restriction of ψ to Γ_- (resp. Γ_+), with Γ_- (resp. Γ_+) is the incoming (resp. the outgoing) part of the phase space and H is a linear bounded operator from a suitable space on Γ_+ to a similar one on Γ_-.
Let S be the infinitesimal generator of a C_0-semigroup $(U(t))_{t\geq 0}$ on a Banach space X and let $\mathcal{L}(X)$ denote the set of all bounded linear operators on X. If $B \in \mathcal{L}(X)$, then by the classical perturbation theorem, $S+B$ generates a strongly continuous semigroup $(V(t))_{t\geq 0}$ given by the Dyson-Phyllips expansion, that is

$$V(t) = \sum_{j=0}^{n-1} U_j(t) + R_n(t), \tag{3}$$

where $U_0(t) = U(t)$, $U_j(t) = \int_0^t U(s)BU_{j-1}(t-s)ds$ $(j \geq 1)$ and the series (3) converges in $\mathcal{L}(X)$ uniformly in bounded times and the n^{th} remainder term $R_n(t)$ is given by

$$R_n(t) = \sum_{j=n}^{\infty} U_j(t) = \int_{s_1+s_2+\cdots+s_n \leq t, s_i \geq 0} U(s_1)B\cdots U(s_n)BV\left(t - \sum_{i=1}^{n} s_i\right) ds_1 \cdots ds_n. \tag{4}$$

So the Cauchy problem

$$\frac{d\psi}{dt} = (S+B)\psi(t), \quad \psi(0) = \psi_0 \tag{5}$$

has a unique classical solution given by $\psi(t) = V(t)\psi_0$ provided that ψ_0 belongs to $D(S)$. In general this results follows from the Hille–Yosida–Phillips theorem (see, for example [9]). This procedure is not constructive, so in order to get more information on the solution, in particular, its behavior for large times, the knowledge of the spectrum of $S+B$ or $(V(t))_{t\geq 0}$ plays a central role.
Let $W \in \mathcal{L}(X)$. The essential spectral radius of the operator W is defined by (see [34])

$$r_e(W) := \sup\left\{ |\lambda| ;\ \lambda \in \sigma(W) \text{ but } \lambda \text{ is not an isolated eigenvalue of finite algebraic multiplicity} \right\}.$$

Let ω be the type of the semigroup $(U(t))_{t\geq 0}$. It follows from Lemma 2.1 in [34] that there exists $\omega_e \in [-\infty, \omega]$ such that

$$r_e(U(t)) = e^{\omega_e t}, \quad \text{for all } t \in [0, +\infty).$$

The number ω_e is called the essential type of $(U(t))_{t\geq 0}$.
It is well known that, if some remainder term of the Dyson-Phillips expansion $R_n(t)$ is compact, then the operators $U(t)$ and $V(t)$ have the same essential type [26,34,35] and therefore, for all $t \geq 0$ we have $r_e(U(t)) = r_e(V(t))$. Thus, for $\nu > 0$, $\sigma(V(t)) \cap \mathbb{C} \backslash B(0, r_e(U(t)) + \nu)$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities. Assuming the existence of such eigenvalues, the semigroup $(V(t))_{t\geq 0}$ can be decomposed into two parts: the first containing the time development of finitely many eigenmodes, the second being of faster decay. Using the spectral mapping theorem for the point spectrum, we infer that, for any $\eta > \omega$, $\sigma(S+B) \cap \{\text{Re}\lambda \geq \eta\}$ consists of finitely many isolated eigenvalues say $\{\lambda_1, \ldots, \lambda_q\}$.

Let $\beta_1 = \sup\{\mathrm{Re}\lambda,\ \lambda \in \sigma(S+B),\ \mathrm{Re}\lambda < \omega\}$, and $\beta_2 = \min\{\mathrm{Re}\lambda_j,\ 1 \leq j \leq q\}$. The solution of the problem (5) satisfies

$$\left\| \psi(t) - \sum_{j=1}^{q} e^{\lambda_j t}\, e^{D_j t} P_j \psi_0 \right\| = o(e^{\beta^* t}) \text{ with } \beta_1 < \beta^* < \beta_2, \tag{6}$$

where $\psi_0 \in D(S)$, P_j and D_j denote, respectively, the spectral projection and the nilpotent operator associated with $\lambda_j,\ j = 1, 2, \ldots, q$.
In the context of neutron transport theory, these ideas were initiated by Vidav [33] and developed afterwards by Voigt [34], Weis [35], Mokhtar-Kharroubi [25–27] and others (see the reference therein). The compactness of some order remainder term of the Dyson Phillips expansion in L^p-spaces, $1 \leq p < +\infty$, was established only for no-reentry boundary conditions (i.e. with zero incoming flux in the spacial domain) [25,27,32,34,35]. It is clear that the success of this method is related to the possibility of computing some remainder order term of the Dyson-Phillips expansion and to the possibility of discussing their compactness properties. Unfortunately, when dealing with reentry boundary conditions, except the one-dimensional case with reflective or periodic boundary conditions [6,7], the semigroup generated by the advection operator is not explicit (see, for example, [16–18,20]) and therefore it is difficult to compute $R_n(t)$ because its expression involves the boundary operator. So, except some simple cases, this approach does not work.
An alternative way to discuss the time structure of $\psi(t)$ is the so called resolvent approach.

It is based on the following assumption:

$$\begin{cases} \exists m \in \mathbb{N} \text{ such that } [(\lambda - S)^{-1}B]^m \text{ is compact for all } \lambda \text{ such that } \mathrm{Re}\lambda > \omega, \\ \lim\limits_{|\mathrm{Im}\lambda| \to +\infty} \left\| [(\lambda - S)^{-1}B]^m \right\| = 0 \text{ uniformly on } \left\{\mathrm{Re}\,\lambda \geq \eta,\ \eta > \omega\right\}. \end{cases} \tag{7}$$

In [25, Theorem 1.1], it is proved that under the condition (7), the part of the spectrum of $A := S + B$ lying in the half plane $\{\mathrm{Re}\lambda \geq \eta\}$ consists of, at most, a finite number of isolated eigenvalues with finite algebraic multiplicity, say, $\{\lambda_1, \cdots, \lambda_n\}$. Further, the Cauchy problem (5) fulfills

$$\| \psi(t) - \sum_{i=1}^{n} e^{\lambda_i t}\, e^{D_i t} P_i \psi_0 \| = o(e^{\beta^* t}) \text{ with } \beta_1 < \beta^* < \beta_2 \tag{8}$$

provided that $\psi_0 \in D(A^2)$. Here P_i, D_i, β_1, β_2 and β^* have the same meaning as above.
Let us recall the streaming operator T_H (where $\|H\| \neq 1$) generates a strongly continuous semigroup $(U_H(t))_{t\geq 0}$ on $L^p(\Omega \times \mathbb{R}^N, dxdv)$, $1 \leq p < +\infty$ (in fact, when $\|H\| < 1$ the generation of the C_0-semigroup is an immediate consequence of Lumer–Phillips's theorem while the case of the multiplying boundary condition has been investigated in [22,24] and sufficient condition on H guaranteeing the generation of the C_0-semigroup were provided). Since the collision operator K is bounded, $A_H = T_H + K$ generates also a strongly continuous semigroup $(V_H(t))_{t\geq 0}$ on $L^p(\Omega \times \mathbb{R}^N, dxdv)$, $1 \leq p < +\infty$ given by

$$V_H(t) = \sum_{j=0}^{n-1} U_j^H(t) + R_n^H(t).$$

For the meaning of the different terms appearing in the last equation, we refer to (3).
The resolvent approach was already applied to transport equations with vacuum boundary conditions ($H = 0$) in bounded geometry [25]. In [16], it is shown that this method works also

for one-dimensional transport equation for a large class of boundary conditions. The drawback of this method lies in the fact that, unlike (6), the quantity $\psi(t) - \sum_{i=1}^{n} e^{\lambda_i t} e^{D_i t} P_i \psi_0$ can be evaluated only for initial data ψ_0 belonging to $D(A_H^2)$ (see [30, Theorem 2.1] or [25, Lemma 2.1]). Note however that, using of Wrobel's result [36, Theorem 3.3] about the asymptotic behavior of C_0-semigroups on B-convex space and observing that L^p-spaces, for $p \in (1, +\infty)$, are B-convex, we see that the condition $\psi_0 \in D(A_H^2)$ can be relaxed and therefore the estimate (8) holds true for all initial data ψ_0 belonging to $D(A_H)$ (we refer to [36] for general B-convex spaces, to [31] for vacuum boundary conditions and to [18] for one dimensional transport equation with particular noreentry boundary conditions). Unfortunately, except some particular boundary conditions in slab geometry [20], the results available in L^1-spaces for reentry boundary conditions assume that ψ_0 belongs to $D(A_H^2)$.

The purpose of this paper is twofold. First, we shall pursue the investigation started in the work [19] concerning the time asymptotic behavior of the solution to the Cauchy problem (1) and (2) in L^1-spaces for a slightly more general boundary operator than the bounce back one. Our main goal is to show that the estimate (8) is also valid for all initial data ψ_0 lying in $D(A_H)$. Our approach uses the following results.

Proposition 1 [21, Corollary 1.1] *Let T be the generator of a C_0-semigroup $(U(t))_{t\geq 0}$ on a Banach space X and let B be a bounded linear operator on X. Let ω be the type of $(U(t))_{t\geq 0}$. If there exists $m \in \mathbb{N}$ and $\eta > \omega$ such that*

(a) $(\lambda - T)^{-1}[B(\lambda - T)^{-1}]^m$ *is compact for all* λ *such that* $Re\lambda \geq \eta$,
(b) $\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda| \left\|(\lambda - T)^{-1}[B(\lambda - T)^{-1}]^m\right\| = 0$ *for every* λ *such that* $Re\lambda \geq \eta$

then $R_{2m+1}(t)$ is compact on X for each $t > 0$.

Proposition 2 [26, Theorem 2.10] *Let T be the generator of a C_0-semigroup $(U(t))_{t\geq 0}$ on a Banach space X and let B be a bounded linear operator on X. If some remainder term $R_n(t)$ $(n \geq 1)$ of the Dyson-Phillips expansion is compact, then $(U(t))_{t\geq 0}$ and $(V(t))_{t\geq 0}$ have the same essential type.*

We shall establish that all order remainder terms $R_n^H(t)$, with $n \geq 9$, of the Dyson-Phillips expansion of the transport semigroup are compact on $L^1(\Omega \times \mathbb{R}^N, dxdv)$ and therefore, according the semigroup approach, we infer that the estimate (8) holds true for all initial data in $D(A_H)$ which shows that the condition $\psi_0 \in D(A_H^2)$ can be relaxed and replaced by $\psi_0 \in D(A_H)$. It should be noticed that the transport semigroup is not differentiable nor analytic, so the condition $\psi_0 \in D(A_H)$ is optimal and can not be improved.

Secondly, for $p \in (1, +\infty)$, using some arguments due to Brendle [3] (which were refined afterwards by Sbihi [28]) we shall prove that the first order remainder term of the Dyson-Phillips expansion, $R_1^H(t)$ is compact. In our opinion, regardless of its consequences, this result is interesting in itself. The latter implies that, for all $t \geq 0$, we have $\sigma_{ess}(V_H(t)) = \sigma_{ess}(U_H(t))$ and therefore $r_e(V_H(t)) = r_e(U_H(t))$. Accordingly, the estimate (8) holds true for all $\psi_0 \in D(A_H)$.

The layout of this papers is as follows. In Sect. 2, we fix the functional setting of the problem and we derive some preliminary facts concerning the problem. The aim of Sect. 3 is to establish some lemmas required in the proofs of the results presented in Sect. 4. The main result of this paper are given in Sect. 4. Section 4.1 is devoted to the case where the boundary operator H is dissipative (i.e. $\|H\| < 1$) . More precisely, we suppose that H is a sum of the bounce-back boundary operator and a compact one. The main result of this subsection is Theorem 3 which asserts that, even in the space $L^1(\Omega \times \mathbb{R}^N, dxdv)$, the estimate (8) is satisfied for all initial data belonging to $D(A_H)$. In Sect. 4.2 we discuss the case of multiplying compact boundary operators. We show in Theorem 5 that the estimate (8) holds also true for initial data $\psi_0 \in D(A_H)$ without any restriction on the value of p. Finally, in Appendix, we give some results about regular collision operators.

Notation: Let X be a Banach space and let A be a linear operator on X. As usually we denote by $\sigma(A)$ and $\rho(A)$ the spectrum and the resolvent set of A, respectively. If $A \in \mathcal{L}(X)$, we denote by $r_\sigma(A)$ the spectral radius of A. If A is an unbounded linear operator on X, we call the spectral bound of A the real defined by $s(A) := \sup\left\{\mathrm{Re}\lambda \,:\, \lambda \in \sigma(A)\right\}$. And finally, if A is a closed densely defined linear operator on X, then by the essential spectrum of A we mean the set

$$\sigma_{ess}(A) = \bigcap_{C \in \mathcal{K}(X)} \sigma(A + C),$$

where $\mathcal{K}(X)$ stands for the ideal of compact operators on X (cf. [29, p. 172]). This definition of the essential spectrum suffices for our own use.

2 Preliminaries

The goal of this section is to recall some basic definitions and results for the usual neutron transport equation which we shall use in the sequel.

Let Ω be a smooth bounded open convex subset of $\mathbb{R}^N$. The boundary of the phase space $\Omega \times \mathbb{R}^n$ writes as $\partial\Omega \times \mathbb{R}^N := \Gamma_- \cup \Gamma_+ \cup \Gamma_0$ where

$$\Gamma_\pm = \left\{(x, v) \in \partial\Omega \times \mathbb{R}^N, \pm v.\nu_x \geq 0\right\}$$

and

$$\Gamma_0 = \{(x, v) \in \partial\Omega \times \mathbb{R}^N, \pm v.\nu_x = 0\},$$

with ν_x stands for the outer unit normal vector at $x \in \partial\Omega$. We shall suppose throughout this paper that Γ_0 is of zero measure with respect to $d\gamma_x dv$ (which is a natural hypothesis), $d\gamma_x$ being the Lebesgue measure on $\partial\Omega$.

We denote by d its diameter, that is

$$\mathsf{d} := \sup\left\{ \|x - y\| \,:\, x, y \in \Omega\right\}.$$

Let $p \in [1, +\infty)$ and let X_p be the space

$$X_p := L^p(\Omega \times \mathbb{R}^N; dx\, dv).$$

We define the partial Sobolev space W_p by

$$W_p = \left\{\psi \in X_p \text{ such that } v.\nabla_x\psi \in X_p\right\}.$$

It is well known [4,5,10] that any function in W_p possesses traces $\psi^\pm$ on $\Gamma_\pm$ belonging to $L^\pm_{p,\mathrm{loc}}(\Gamma_\pm; |v.\nu_x|d\gamma_x dv)$. It should be noticed that, in applications, suitable L^p-spaces for the traces are

$$L^\pm_p := L^p(\Gamma_\pm \,:\, |v.\nu_x|\, d\gamma_x dv).$$

Accordingly, we define the set

$$\widetilde{W}_p = \left\{\psi \in W_p \,:\, \psi^- \in L^-_p\right\}.$$

According to [4,5,10], if $\psi \in W_p$, $1 \leq p < +\infty$, and $\psi^- \in L_p^-$, then $\psi^+ \in L_p^+$ and vice versa. More precisely we have the identity

$$\widetilde{W}_p = \left\{\psi \in W_p \;:\; \psi^- \in L_p^-\right\} = \left\{\psi \in W_p \;:\; \psi^+ \in L_p^+\right\}.$$

Definition 1 Let $(x, v) \in \bar{\Omega} \times \mathbb{R}^N$. We set

$$\begin{aligned} t^{\pm}(x, v) &= \sup\{t > 0,\ x \pm sv \in \Omega, 0 < s < t\} \\ &= \inf\{t > 0,\ x \pm tv \notin \Omega\} \end{aligned}$$

and

$$\tau(x, v) := t_-(x, v) + t_+(x, v) \quad \text{for any } (x, v) \in \overline{\Omega} \times \mathbb{R}^N.$$

□

Hence, for $(x, v) \in \Gamma_{\pm}$, one has $t^{\pm}(x, v) = 0$ and in all cases $x \mp t^{\mp}(x, v)v \in \Gamma_{\mp}$. The number $t^{\pm}(x, v)$ is the time required by a particle having the position $x \in \Omega$ and the velocity $\pm v \in \mathbb{R}^N$ to go out Ω.
Let $H \in \mathcal{L}(L_p^+, L_p^-)$ be a boundary operator. The streaming operator T_H is defined by

$$\begin{cases} T_H \;:\; D(T_H) \subseteq X_p \longrightarrow X_p \\ \qquad \psi \longrightarrow T_H\psi(x, v) = -v.\nabla_x\psi(x, v) - \sigma(v)\psi(x, v) \end{cases}$$

with domain

$$D(T_H) = \left\{\psi \in \widetilde{W}_p \text{ such that } \psi^- = H(\psi^+)\right\}.$$

In this paper, we suppose that $\sigma(\cdot)$ is a measurable function belonging to $L^{\infty}(\mathbb{R}^N, dv)$ and satisfies

$$0 \leq \sigma(v) \text{ for almost all } v \in \mathbb{R}^N.$$

Let $\varphi \in X_p$ and consider the resolvent equation for T_H

$$(\lambda - T_H)\psi = \varphi \tag{9}$$

where λ is a complex number and the unknown ψ must be sought in $D(T_H)$. Let λ^* denote the real defined by

$$\lambda^* := \inf_{v \in \mathbb{R}^N} \sigma(v).$$

It is shown in [17] that, for $\mathrm{Re}\lambda > -\lambda^*$, the invertibility of $(\lambda - T_H)$, reduces to the invertibility of the operator $\mathcal{P}_\lambda := I - M_\lambda H$ where M_λ is given by

$$L_p^- \ni u \longrightarrow M_\lambda u = u(x - \tau(x, v)v, v)e^{-\tau(x,v)(\lambda + \sigma(v))} \in L_p^+.$$

More precisely following [17], if $\mathcal{P}_\lambda$ is invertible, then the solution of (9) writes as

$$\psi = B_\lambda H \mathcal{P}_\lambda^{-1} G_\lambda \varphi + C_\lambda \varphi$$

and therefore the resolvent of T_H is given by

$$(\lambda - T_H)^{-1} = B_\lambda H \mathcal{P}_\lambda^{-1} G_\lambda + C_\lambda. \tag{10}$$

Here

$$X_p \ni \varphi \longrightarrow C_\lambda \varphi := (\lambda - T_0)^{-1}\varphi = \int_0^{t^-(x,v)} e^{-s(\lambda+\sigma(v))}\varphi(x - sv, v)ds \in X_p,$$

$$X_p \ni \varphi \longrightarrow G_\lambda \varphi := C_\lambda \varphi_{|\Gamma^+} = \int_0^{\tau(x,v)} e^{-s(\lambda+\sigma(v))}\varphi(x - sv, v)ds \in L_p^+$$

and

$$L_p^- \ni u \longrightarrow B_\lambda u := u(x - t^-(x,v)v, v)e^{-t^-(x,v)(\lambda+\sigma(v))} \in X_p.$$

Remark 1 These operators are bounded on their respective spaces. In fact, the norms of B_λ and C_λ are bounded above, respectively, by $[p(Re\lambda + \lambda^*)]^{-\frac{1}{p}}$ and $(Re\lambda + \lambda^*)^{-1}$ where q denotes the conjugate of p. Moreover, the operator $M_\lambda u = [B_\lambda u]_{|\Gamma_+}$ is a contraction and $G_\lambda \varphi = [C_\lambda \varphi]_{|\Gamma_+}$ is bounded with norm less than $[q(Re\lambda + \lambda^*)]^{-\frac{1}{q}}$. □

Note that in neutron transport theory, in general, the collision operator K has the form

$$K : \varphi \in X_p \longrightarrow K\varphi(x, v) = \int_{\mathbb{R}^N} \kappa(x, v, v')\varphi(x, v')dv', \tag{11}$$

where $\kappa(\cdot, \cdot, \cdot)$ is a non-negative measurable function. It is a partial integral operator.

Remark 2 In the remainder of the paper, we will assume that K is a regular collision operator in the sense of [23,25] and refer the reader to Appendix for formal definition and mains properties of this class of operator.

We close this section to state the following compactness result which will play a fundamental role in this paper.

Theorem 1 *Let K be a regular operator and let λ be such that $Re\lambda > s(T_H)$.*

(1) *If $p \in (1, \infty)$, then $(\lambda - T_H)^{-1}K$ and $K(\lambda - T_H)^{-1}$ are compact on X_p.*
(2) *If $p = 1$, then $K(\lambda - T_H)^{-1}K$ is weakly compact on X_1.*

Proof The first item of the theorem was established in [17]. So it suffices to prove the second assertion.
Recall that

$$K(\lambda - T_H)^{-1}K = KB_\lambda H(I - M_\lambda)^{-1}G_\lambda K + KC_\lambda K.$$

Note however that, C_λ is nothing else but the resolvent of the streaming operator with vacuum boundary condition T_0. So, we know from [25] that, if K is a regular collision operator on X_1, then the operator $KC_\lambda K$ is weakly compact operator on X_1. Thus, in order to prove the weakly compactness of $K(\lambda - T_H)^{-1}K$, it suffices to show that the operator KB_λ is weakly compact. Since K is a regular collision operator, according to Lemma 11, it suffices to establish the result for a collision operator of the form

$$\varphi \in X_1 \to f(v)\int_{\mathbb{R}^N} \varphi(x, v')dv'$$

where $f(\cdot) \in L^1(\mathbb{R}^N; dv)$. For $\varphi \in L^{1,-}$, one can write

$$KB_\lambda \varphi(x, v) := f(v)\int_{\mathbb{R}^N} B_\lambda \varphi(x, v')dv',$$

where $B_\lambda\varphi(x,v)=\varphi(x-t^-(x,v)v,v)e^{-t^-(x,v)(\lambda+\sigma(v))}$ and $\|B_\lambda\|\le(\mathrm{Re}\lambda+\lambda^*)^{-1}$. This yields that

$$\|KB_\lambda\varphi(x,v)\|_{X_1}\le\int_{\mathbb{R}^N}|f(v)|dv\int_{\Omega\times\mathbb{R}^N}|B_\lambda\varphi(x,v')|dxdv'$$

$$\le\|f(\cdot)\|_{L^1(\mathbb{R}^N)}\|B_\lambda\varphi(\cdot,\cdot)\|_{X_1}$$

$$\le(\mathrm{Re}\lambda+\lambda^*)^{-1}\|f(\cdot)\|_{L^1(\mathbb{R}^N)}\|\varphi\|_{L^{1,-}}.$$

So, we conclude that

$$\|KB_\lambda\|\le(\mathrm{Re}\lambda+\lambda^*)^{-1}\|f(\cdot)\|_{L^1(\mathbb{R}^N)}. \tag{12}$$

The estimate (25) shows that KB_λ depends continuously (for the uniform topology) on $f(\cdot)\in L^1(\mathbb{R}^N)$. So, by approximating $f(\cdot)$ (in the L^1-norm) by bounded functions, KB_λ is a limit (for operator topology) of integral operators with bounded kernel. Hence, KB_λ is weakly compact on X_1 (cf., [9, Corollary 11, p. 294]). □

Remark 3 We point out that the assertion (2) of Theorem 1 is new and, as in slab geometry [15, Theorem 2.1], the weak compactness of $K(\lambda-T_H)^{-1}K$ does not depend on the boundary operator (see also Remarks 3.1 and 3.2 in [17]). We note however that this result is proved only for Lebesgue measure on the velocity space. In fact, the same arguments of the proof of this result work for general positive radon measures on the velocity space. □

3 Preparatory results

Our aim in this section is to establish some technical lemmas required in the sequel. We define the full transport operator A_H by $A_H=T_H+K$. Since the collision operator K is bounded, A_H is a bounded perturbation of T_H.
We suppose that the boundary operator H has the form

$$\begin{cases}H\,:\,L_p^+\longrightarrow L_p^-\\ \quad\psi^-\longrightarrow H\psi^+=\alpha I_1\psi^++\beta I_2\psi^+,\quad\alpha,\beta\in[0,+\infty),\end{cases} \tag{13}$$

where I_1 is a compact operator and I_2 is defined by

$$I_2u(x,v)=u(x,-v)$$

(I_2 is the so called bounce-back boundary operator). The constants α and β are chosen so that

$$\|H\|<1. \tag{14}$$

For the sake of simplicity and in order to avoid some technical difficulties, we shall assume in this section that the collision frequency is an even function on $\mathbb{R}^N$, that is,

$$\sigma(-v)=\sigma(v),\quad\forall v\in\mathbb{R}^N.$$

Since $\|H\|<1$, $\mathcal{P}_\lambda$ is invertible and Eq. (10) (see also [13,17]) shows that, for all $\lambda\in\mathbb{C}$ such that $\mathrm{Re}\lambda+\lambda^*>0$, the resolvent of the operator T_H is given by

$$R(\lambda,T_H)=\sum_{n\ge0}B_\lambda H(M_\lambda H)^nG_\lambda\psi+C_\lambda\varphi.$$

This shows that

$$\left\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\lambda^*\right\} \subseteq \rho(T_H)$$

and, for any $\lambda \in \mathbb{C}$ such that $\mathrm{Re}\lambda > -\lambda^*$, we have

$$\|KR(\lambda, T_H)K\| \leq \sum_{n\geq 0} \left\|KB_\lambda H(M_\lambda H)^n G_\lambda K\right\| + \|KC_\lambda K\|.$$

Let $\omega > 0$ and denote by Γ_ω the set

$$\Gamma_\omega = \left\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq -\lambda^* + \omega\right\}.$$

Lemma 1 *If K is a regular collision operator, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda|^r \|KC_\lambda K\| = 0 \quad \textit{uniformly on } \Gamma_\omega.$$

Proof According to Remark 8, we may take K in the form (24). So, $KC_\lambda K$ writes in the form

$$(KC_\lambda K\varphi)(x, v) = \int_{\mathbb{R}^N} f(v)h(v')dv' \int_0^{t^-(x,v')} e^{-(\lambda+\sigma(v'))t}dt \int_{\mathbb{R}^N} g(v'')\varphi(x - tv', v'')dv'',$$

where $h(v') := g(v')f(v') \in L^1(\mathbb{R}^N)$ with $f_i(\cdot) \in L_p(\mathbb{R}^N; dv)$ and $g_i(\cdot) \in L_q(\mathbb{R}^N; dv)$. Here for simplicity we take $\alpha(\cdot) = \chi_\Omega(\cdot)$ (the function characteristic of the set Ω).
This yields the factorization $KC_\lambda K := A_3A_2(\lambda)A_1$ with

$$A_1 : X_p \to L^p(\Omega), \quad \psi \to \int_{\mathbb{R}^N} g(v)\psi(x, v)dv,$$

$$A_2(\lambda) : L^p(\Omega) \to L^p(\Omega), \varphi \to \int_{\mathbb{R}^N} h(v) \int_0^{t^-(x,v)} e^{-(\lambda+\sigma(v))t}\varphi(x - tv)dtdv$$

and

$$A_3 : L^p(\Omega) \to X_p, \quad \varphi \to \varphi(x)f(v).$$

Since A_1 and A_3 are bounded operators and independent of the parameter λ, it suffices to establish the result for $A_2(\lambda)$. To this end, let $\varepsilon > 0$ and define the operator

$$\begin{cases} A_2^\varepsilon : L^p(\Omega) \longrightarrow L^p(\Omega) \\ \varphi \to \displaystyle\int_{\mathbb{R}^N} h(v) \int_\varepsilon^{t^-(x,v)} e^{-(\lambda+\sigma(v))t}\varphi(x - tv)dtdv. \end{cases}$$

Since $\lim_{\varepsilon\to 0} \left\|A_2^\varepsilon(\lambda) - A_2(\lambda)\right\|_{\mathcal{L}(L^p(\Omega))} = 0$ uniformly on Γ_ω, so it suffices to establish the lemma for the operator $A_2^\varepsilon(\lambda)$. Now, using the convexity of Ω, one sees that, for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^N$,

$$t \in (0, t^-(x, v)) \Leftrightarrow y = x - tv \in \Omega.$$

Hence, using the change of variable $y = x - tv$, we get

$$(A_2^\varepsilon(\lambda)\varphi)(x) = \int_\Omega \varphi(y)dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-\left(\lambda+\sigma\left(\frac{x-y}{t}\right)\right)t} \frac{dt}{t^N}.$$

Let us recall that

$$\lambda^* := \inf_{v\in\mathbb{R}^N} \sigma(v).$$

Note that if $\lambda \in \Gamma_\omega$, then

$$\mathrm{Re}\lambda + \lambda^* - \frac{\omega}{2} \geq \frac{\omega}{2} > 0. \tag{15}$$

For all $v \in \mathbb{R}^N$, we have

$$\sigma(v) - \lambda^* + \frac{\omega}{2} \geq \frac{\omega}{2} > 0. \tag{16}$$

Without loss of generality we may suppose that $h(\cdot)$ is a simple measurable function. For $x \in \Omega$, consider the function $\psi_x(\cdot)$ defined by

$$(\varepsilon, +\infty) \ni t \longmapsto \psi_x(t) = \frac{1}{t^N} e^{-(\sigma(\frac{x}{t})-\lambda^*+\frac{\omega}{2})t}.$$

It is clear that (16) implies $0 \leq \psi_x(\cdot) \in L^1(\varepsilon, +\infty)$. We denote by $(l_{x,i}(\cdot))_{i\in\mathbb{N}}$ a sequence of non negative step functions which converges to $\psi_x(\cdot)$ almost everywhere and satisfying

$$0 \leq l_{x,i}(\cdot) \leq \psi_x(\cdot) \leq \frac{1}{\varepsilon^N} \quad \text{for all } i \in \mathbb{N}. \tag{17}$$

Let $A_{2,i}^\varepsilon(\lambda)$ be the sequence of operators defined, for all $i \in \mathbb{N}$, by

$$\begin{cases} A_{2,i}^\varepsilon(\lambda) \,:\, L^p(\Omega) \longrightarrow L^p(\Omega) \\ \varphi \longmapsto \displaystyle\int_\Omega \varphi(y)dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-(\lambda+\lambda^*-\frac{\omega}{2})t} l_{x-y,i}(t)dt \end{cases}$$

We claim that, for any $\varepsilon > 0$, $(A_{2,i}^\varepsilon(\lambda))_{i\in\mathbb{N}}$ converges uniformly on Γ_ω to $A_2^\varepsilon(\lambda)$ in $\mathcal{L}(L^p(\Omega))$. Indeed, for $\varphi \in L^p(\Omega)$, we have

$$\begin{aligned} &\left\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\right\|_{L^p(\Omega)}^p \\ &= \int_\Omega dx \left|\int_\Omega \varphi(y)dy \int_\varepsilon^\infty h\left(\frac{x-y}{t}\right) e^{-(\lambda+\lambda^*-\frac{\omega}{2})t}\{l_{x-y,i}(t) - \psi_{x-y}(t)\}dt\right|^p. \end{aligned}$$

Applying the Hölder inequality, we get

$$\begin{aligned} &\left\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\right\|_{L^p(\Omega)}^p \\ &\leq |\Omega|^{\frac{p}{q}} \int_\Omega dx \int_\Omega dy \left|\int_\varepsilon^\infty \varphi(y) h\left(\frac{x-y}{t}\right) e^{-(\lambda+\lambda^*-\frac{\omega}{2})t}\{l_{x-y,i}(t) - \psi_{x-y}(t)\}dt\right|^p \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. It follows from Fubini's theorem and the change of variable $x' = x - y$, that

$$\begin{aligned} &\left\|(A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda))\varphi\right\|_{L^p(\Omega)}^p \\ &\leq \sup |h(\cdot)|^p \, |\Omega|^{\frac{p}{q}} \int_\Omega dx' \Big(\int_\varepsilon^\infty e^{-\left(\lambda+\lambda^*-\frac{\omega}{2}\right)t}\{l_{x',i}(t) - \psi_{x'}(t)\}dt\Big)^p \, \|\varphi\|_{L^p(\Omega)}^p. \end{aligned}$$

Hence

$$\left\|A_{2,i}^\varepsilon(\lambda) - A_2^\varepsilon(\lambda)\right\|^p \leq \sup |h(\cdot)|^p \, |\Omega|^{\frac{p}{q}} \int_\Omega dx' \Big(\int_\varepsilon^\infty e^{-(\mathrm{Re}\lambda+\lambda^*-\frac{\omega}{2})t}\{l_{x',i}(t) - \psi_{x'}(t)\}dt\Big)^p.$$

Now, using Eqs. (15)–(17), we get

$$e^{-(\mathrm{Re}\lambda+\lambda^*-\frac{\omega}{2})t}\{l_{x',i}(t)-\psi_{x'}(t)\}\leq\frac{1}{\varepsilon}e^{-\frac{\omega}{2}t}.$$

By the continuity of the norm and the Lebesgue dominated convergence theorem, we conclude $\lim_{i\to\infty}\left\|A_{2,i}^{\varepsilon}(\lambda)-A_2^{\varepsilon}(\lambda)\right\|=0$ uniformly on Γ_ω. This proves our claim.
Now we have only to prove the lemma for $A_{2,i}^{\varepsilon}(\lambda)$. Note that easy calculations show that

$$\|A_{2,i}^{\varepsilon}\|^p\leq|\Omega|^{\frac{p}{q}}\int_{\Omega}dx\left|\int_{\varepsilon}^{\infty}e^{-(\lambda+\lambda^*-\frac{\omega}{2})t}h\left(\frac{x}{t}\right)l_{x,i}(t)dt\right|^p.$$

Let x be a fixed real in Ω. Clearly, the map $G_x(\cdot):(\varepsilon,\infty)\to\mathbb{R}^N$, $t\longmapsto e^{-\lambda^4 t}h(\frac{x}{t})l_{x,i}(t)$ is a simple function. Let $(t_j)_{1\leq j\leq m}$ denote a subdivision of its support such that

$$\forall j\in\{1,\ldots,m-1\},\quad G_x(t)=G_x(t_j)\quad\forall t\in[t_j,t_{j+1}[.$$

Hence

$$\begin{aligned}\int_{\varepsilon}^{\infty}e^{-(-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t}G_x(t)dt&=\sum_{j=1}^{m-1}G_x(t_j)\int_{t_j}^{t_{j+1}}e^{-(-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t}dt\\&=\frac{1}{-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2}}\\&\quad\times\sum_{j=1}^{m-1}G_x(t_j)\Big(e^{-(-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t_j}-e^{-(-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t_{j+1}}\Big)\end{aligned}$$

and consequently

$$\left|\int_{\varepsilon}^{\infty}e^{-(-\lambda^4+\lambda+\lambda^*-\frac{\omega}{2})t}G_x(t)dt\right|^p\leq\frac{e^{-p\mathrm{Im}\lambda^4\left(1+\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}-\frac{6\mathrm{Re}\lambda^2}{\mathrm{Im}\lambda^2}\right)}\sup|h(\cdot)|^p\,(2(m-1))^p}{|\mathrm{Im}\lambda|^{4p}\,\varepsilon^{Np}}.$$

Therefore

$$|\mathrm{Im}\lambda|^r\,\|A_{2,i}^{\varepsilon}\|\leq\frac{2(m-1)\,|\Omega|^p\,e^{-\mathrm{Im}\lambda^4\left(1+\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}-\frac{6\mathrm{Re}\lambda^2}{\mathrm{Im}\lambda^2}\right)}\sup|h(\cdot)|}{|\mathrm{Im}\lambda|^{4-r}\,\varepsilon^N}.$$

This ends the proof because, for any $r\in[0,1]$, we have

$$\lim_{|\mathrm{Im}|\lambda\to\infty}\frac{2(m-1)\,|\Omega|^p\,e^{-\mathrm{Im}\lambda^4\left(1+\frac{\mathrm{Re}\lambda}{\mathrm{Im}\lambda}-\frac{6\mathrm{Re}\lambda^2}{\mathrm{Im}\lambda^2}\right)}\sup|h(\cdot)|}{|\mathrm{Im}|^{4-r}\,\varepsilon^N}=0.$$

□

Our next task is to estimate the quantity $|\mathrm{Im}\lambda|^r\left\|KB_\lambda H(I-M_\lambda H)^{-1}G_\lambda K\right\|$ as $|\mathrm{Im}\lambda|$ goes to $+\infty$ for all $r\in[0,1]$. Set $H(\lambda):=M_\lambda H$. Since $\|H\|<1$, the use of Remark 1 and Eq. (10) imply that

$$\left\|KB_\lambda H(I-H(\lambda))^{-1}G_\lambda K\right\|\leq\sum_{n\geq0}\left\|KB_\lambda H(H(\lambda))^n G_\lambda K\right\|.$$

Further, according to (13), one sees that the operator $H(\lambda)$ may be expressed as

$$H(\lambda)=H^1(\lambda)+H^2(\lambda),$$

where $H^1(\lambda) := \alpha M_\lambda I_1$ et $H^2(\lambda) := \beta M_\lambda I_2$. Since the operators $H^i(\lambda)$, $i = 1, 2$, do not commute, the operator $(H(\lambda))^n = (H^1(\lambda) + H^2(\lambda))^n$ is a sum of 2^n different terms, that is, $(H(\lambda))^n = \sum_{j=1}^{2^n} P_j$ where each P_j is a product of n factors formed from the operators $H^i(\lambda)$, $i = 1, 2$. These 2^n terms P_j may be divided into two classes C_1 and C_2. The class C_i consists of those P_j, $1 \leq j \leq (2)^n$, for which the last factor (on the right hand side) is $H^i(\lambda)$, $i = 1, 2$. It follows that, for each integer n, we have

$$\left\| K B_\lambda H (H(\lambda))^n G_\lambda K \right\| \leq \sum_{j=1}^{2^n} \left\| K B_\lambda H P_j G_\lambda K \right\|,$$

where j is an integer belonging to the set $\left\{1, 2, \ldots, 2^n\right\}$.

Lemma 2 *Let K be a regular collision operator. If the boundary operator H is in the form* (13) *and satisfies* (14)*, then*

$$\lim_{|\mathrm{Im}\lambda| \to \infty} |\mathrm{Im}\lambda|^r \left\| K B_\lambda H (I - H(\lambda))^{-1} G_\lambda K \right\| = 0, \quad \textit{uniformly on } \Gamma_\omega,$$

for all $r \in [0, 1]$.

Before proving this lemma, let us establish two preliminary results.

Lemma 3 *If the hypotheses of the Lemma 2 hold true, then, for all $r \in [0, 1]$, we have*

$$\lim_{|\mathrm{Im}\lambda| \to \infty} |\mathrm{Im}\lambda|^r \left\| I_1 G_\lambda K \right\| = 0 \quad \textit{uniformly on } \Gamma_\omega.$$

Proof Since I_1 is a compact operator, as in Remark 8, we can establish the result for an operator of rank one, that is

$$I_1 : L_p^+ \to L_p^-, \varphi \to I_1\varphi(x, v) = \theta_1(x, v) \int_{\Gamma_+} \theta_2(x', v')\varphi(x', v')|v'.\nu_{x'}|d\gamma_{x'}dv',$$
$$\forall (x, v) \in \Gamma_-,$$

where $\theta_1 \in L_p^-$ and $\theta_2 \in L_q^+$. So, the operator $I_1 G_\lambda K$ writes in the form

$$(I_1 G_\lambda K\varphi)(x, v) = \theta_1(x, v) \int_{\Gamma_+} \theta_2(x', v')|v'.\nu_{x'}|d\gamma_{x'}dv' \int_0^{\tau(x', v')} e^{-(\lambda+\sigma(v))t} f(v')dt$$
$$\times \int_{\mathbb{R}^N} g(v'')\varphi(x' - tv', v'')dv''.$$

Hence, it can be factorized as

$$I_1 G_\lambda K = A_3 \, A_2(\lambda) \, A_1,$$

where

$$A_1 : \psi \in X_p \to \int_{\mathbb{R}^N} g(v)\psi(x, v)dv \in L_p(\Omega),$$

$$A_2(\lambda) : \varphi \in L_p(\Omega) \to \int_{\Gamma_+} \theta_2(x, v) f(v)|v.\nu_x|d\gamma_x dv \int_0^{\tau(x, v)} e^{-(\sigma(v)+\lambda)t}\varphi(x - tv)dt \in \mathbb{R}$$

and

$$A_3 : \gamma \in \mathbb{R} \to \gamma\theta_1(x, v) \in L_p^-.$$

Since A_1 and A_3 are bounded linear operators independent of the parameter λ, it suffices to establish the lemma for $A_2(\lambda)$. Let $\varepsilon > 0$ and define the operator $A_2^\varepsilon(\lambda)$ by

$$\begin{cases} A_2^\varepsilon(\lambda) \, : \, L^p(\Omega) \longrightarrow \mathbb{R} \\ \varphi \to \int_{\Gamma_+} \theta_2(x,v) f(v) |v.\nu_x| d\gamma_x dv \int_\varepsilon^{\tau(x,v)} e^{-(\sigma(v)+\lambda)t} \varphi(x-tv)dt. \end{cases}$$

Since $A_2^\varepsilon(\lambda) \to A_2(\lambda)$, in the operator norm, as $\varepsilon \to 0$, it suffices to show the lemma for the operator $A_2^\varepsilon(\lambda)$.
Let $(x,v) \in \Gamma_+$. We know from Sect. 2 that $\tau(x,v) = t^-(x,v)$, so using the convexity of Ω, we get the following equivalence

$$0 < t < t^-(x,v) \Longleftrightarrow y = x - tv \in \Omega$$

and therefore the change of variable $y = x - tv$ leads to

$$A_2^\varepsilon(\lambda)\varphi = \int_\Omega \varphi(y) F_{\lambda,\varepsilon}(y) dy,$$

where

$$F_{\lambda,\varepsilon}(y) = \int_\varepsilon^\infty \frac{dt}{t^N} \int_{\partial\Omega^+} \theta_2\left(x, \frac{x-y}{t}\right) f\left(\frac{x-y}{t}\right) |\frac{x-y}{t}.\nu_x| e^{-\left(\lambda+\sigma\left(\frac{x-y}{t}\right)\right)t} d\gamma_x.$$

Now arguing as in the last part of the proof of Lemma 1 we reach the desired result. □

4 Main results

It should be noticed that the streaming operator T_H with conservative boundary conditions does not generate a strongly continuous but it possesses an extension $\widetilde{T}_H$ which generates a strongly continuous semigroup. So, in this work we consider only dissipative boundary conditions $\|H\| < 1$ and multiplying boundary conditions $\|H\| > 1$. A natural hypothesis (from physical view point) is that the boundary operator H is positive in the lattice sense.

4.1 Dissipative boundary conditions

As in the previous section, the collision frequency $\sigma(\cdot)$ is also assumed to be an even on $\mathbb{R}^N$ in this subsection, that is,

$$\sigma(v) = \sigma(-v), \quad \text{for all } v \in \mathbb{R}^N.$$

Now we ready to prove the following result.

Theorem 2 *Let K be a regular collision operator and let H be a boundary operator satisfying (13) and (14). If the hypotheses of Theorem 1 hold true, then, $[(\lambda - T_H)^{-1}K]^n$ is compact on X_p $(1 \le p < \infty)$ for $n \ge 4$. Further, for all $r \in [0,1]$, we have*

$$\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda|^r \left\| K(\lambda - T_H)^{-1} K \right\| = 0 \text{ uniformly on } \Gamma_\omega.$$

Proof If $p \in (1, +\infty)$, then, according to Theorem 1(1), the operator $(\lambda - T_H)^{-1}K$ is compact on X_p. If $p = 1$, then, by Theorem 1(2), we infer that $K(\lambda - T_H)^{-1}K$ weakly compact on X_1 and therefore $(\lambda - T_H)^{-1}K(\lambda - T_H)^{-1}K$ is weakly compact on X_1. Next, this together with the fact that X_1 admits the Dunford–Pettis property, implies that $[(\lambda - T_H)^{-1}K(\lambda - T_H)^{-1}K]^2 = [(\lambda - T_H)^{-1}K]^4$ is compact on X_1 which proves the first assertion. The second statement follows immediately from Lemmas 1 and 2. □

The remainder of the subsection is devoted to give a spectral decomposition of the solution to Problem (1). Before going further, we first define the set $\sigma_{as}(A_H)$ (the asymptotic spectrum of the operator A_H) by

$$\sigma_{as}(A_H) := \sigma(A_H) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\lambda^*\}.$$

Lemma 4 *If the hypotheses of Theorem 2 hold true, then*

(1) $\sigma_{as}(A_H)$ *consists of, at most, isolated eigenvalues with finite algebraic multiplicities;*
(2) *if* $\varepsilon > 0$ *then* $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : Re\lambda > -\lambda^* + \varepsilon\}$ *is finite;*
(3) *if* $\varepsilon > 0$ *then* $\|(\lambda - A_H)^{-1}\|$ *is uniformly bounded in* $\{\lambda \in \mathbb{C} : Re\lambda > -\lambda^* + \varepsilon\}$ *for large* $|\mathrm{Im}\lambda|$.

Proof Let λ be such that $\mathrm{Re}\lambda > -\lambda^*$. Since K is a regular collision operator, according to Theorem 1 with $n = 4$, the operator $[(\lambda - T_H)^{-1}K]^4$ is compact on X_p, $1 \leq p \leq \infty$. Next, applying Theorem 2 with $r = 0$, we get

$$\lim_{|\mathrm{Im}\lambda| \to \infty} \left\|[(\lambda - T_H)^{-1}K]^4\right\| = 0 \quad \text{uniformly in } \Gamma_\omega.$$

Now, the result follows from [25, Lemma 1.1]. □

Assume that K is a regular collision operator on X_p and H satisfies the hypotheses (13) and (14). Then, by Lemma 4, the spectrum of A_H in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > -\lambda^* + \varepsilon\}$ consists of, a finite number of isolated eigenvalues with finite algebraic multiplicity which we denote $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with λ_i, $i = 1, 2, \ldots, n$. Then $P := P_1 + \cdots + P_n$ is the spectral projection of the compact set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. According to the spectral decomposition theorem corresponding to the compact set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (see, for example, [12]), we may write

$$V_H(t) = \tilde{V}_H(t) + \sum_{i=1}^{n} e^{\lambda_i t} e^{D_i t} P_i$$

where $\tilde{V}_H(t) := V_H(t)(I - P)$ is the C_0-semigroup on the Banach space $(I - P)X_p$ with generator $\tilde{A}_H := A_H(I - P)$ ($\tilde{A}_H$ is the part of A_H on the closed subspace $(I - P)X_p$).

Now, we are in a position to give the main result in this paper.

Theorem 3 *Let* $p = 1$ *and* K *be a regular collision operator. Assume that the boundary operator satisfies* (13) *and* (14). *Then, there exists* $\varepsilon > 0$ *small enough and* $M > 0$ *such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^{n} e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_1} \leq M e^{(-\lambda^* + \varepsilon)t}, \quad \forall t > 0,$$

where $\psi_0 \in D(A_H)$.

Proof Let λ be a complex number satisfying $\mathrm{Re}\lambda > -\lambda^*$. We know from Theorem 1 that if K is a regular collision operator, then $(\lambda - T_H)^{-1}[K(\lambda - T_H)^{-1}]^4$ is compact on X_1.

So, applying Theorem 2 with $r = 1$, we get

$$\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda| \left\| (\lambda - T_H)^{-1}[K(\lambda - T_H)^{-1}]^4 \right\| = 0.$$

Now, according to Proposition 1, we get the compactness of $R_n^H(t)$ for $n \geq 9$. Then by Proposition 2, the C_0-Semigoups $U_H(t)$ and $V_H(t)$ have the same essential type, that is, for all $t \geq 0$ we have

$$r_e(U_H(t)) = r_e(V_H(t)) \leq e^{-\lambda^* t}.$$

This implies that, outside the spectral disc $|\mu| \leq e^{(\varepsilon-\lambda^*)t}$, the spectrum of $V_H(t)$ consists of, at most, a finite number of isolated eigenvalues with finite algebraic multiplicity. On other hand, all points μ' satisfying $\left|\mu'\right| \geq e^{(\varepsilon-\lambda^*)t}$ belong to the resolvent set of $\tilde{V}_H(t)$ and consequently,

$$\left\| \tilde{V}_H(t) \right\| < e^{(\varepsilon-\lambda^*)t}.$$

Hence, $\left\| \tilde{V}_H(t) \right\| = o(e^{(-\lambda^*+\varepsilon)t})$ as $t \to \infty$. This complete the proof. □

In the remainder of this subsection, our aim is to establish a similar result to that of Theorem 3 for $p \in (1, +\infty)$. In fact, we have the following result.

Theorem 4 *Let $p \in (1, \infty)$ and let K be a regular collision operator. Assume that the boundary operator satisfies (13) and (14). Then, there exists $\varepsilon > 0$ small enough and $M > 0$ such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^{n} e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_p} \leq M e^{(-\lambda^*+\varepsilon)t}, \quad \forall t > 0,$$

where $\psi_0 \in D(A_H)$.

The proof of the Theorem 4 is based on the following result.

Proposition 3 *Assume that the hypotheses of Theorem 4 hold true, then the first order remainder term of the Dyson-Phillips expansion, $R_1^H(t)$, is compact on X_p.*

As an immediate consequence of Proposition 3 we have

Corollary 1 *Assume that the hypotheses of Theorem 4 hold true, then*

$$\sigma_{ess}(V_H(t)) = \sigma_{ess}(U_H(t)) \textit{ for any } t > 0.$$

Proof of Proposition 3 Making use Eq. (4), we have

$$R_1^H(t) = \int_0^t U_H(s) K V_H(t-s) ds.$$

Thus, it is obvious that $R_1^H(t)$ depends linearly and continuously, in the norm operator topology, on the collision operator K. On the other hand, according to Remark 8 K has the following form

$$K : X_p \to X_p, \varphi \to K\varphi(x, v) = \alpha(x) \int_{\mathbb{R}^N} f(v) g(v') \varphi(x, v') dv', \ p \in (1, \infty)$$

where f and g are continuous functions with compact supports. Hence, $R_1^H(t)$ maps X_q into itself for all $q \in (1, \infty)$. Taking into account of Krasnoselskii interpolation [14, Theorem 3.10, p. 57], we may restrict ourselves to the Hilbert space X_2. On the other hand, let λ be a complex number such that $\mathrm{Re}\lambda > -\lambda^*$. According to Theorem 1(1), we obtain the compactness of $(\lambda - T_H)^{-1}K$. Furthermore, using Theorem 2 with $r = 0$, we get

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \left\| K(\lambda - T_H)^{-1}K] \right\| = 0 \quad \text{uniformly in } \Gamma_\omega.$$

Finally, the dissipativity of T_H (because H satisfies the condition (14)) together with [28, Theorem 2.2, Corollary 2.1] give the desired result. □

Proof of Theorem 4 It is clear that, according to Corollary 1, we have

$$r_{ess}(V_H(t)) = r_{ess}(U_H(t)), \quad \text{for all } t > 0.$$

Now, arguing as in the last part of the proof of the Theorem 3, we obtain the desired result. □

Remark 4 (a) It should be noticed that, by using the so called resolvent approach, one can give a different proof of Theorems 3 and 4 based on the Riesz-Thorin interpolation theorem. One of the main steps in the proofs consists in showing that the asymptotic spectrum of the operator A_H remains unchanged in all X_p for $p \in [1, \infty)$. Unlike our proofs, this approach uses many complex computations (cf. [20, Section 5]).
(b) It seems that the compactness in the space $L^1(\Omega \times \mathbb{R}^N, dxdv)$ of the first order remainder term of the Dyson-Phillips expansion, $R_1^H(t)$, is an open question.

4.2 Multiplying compact boundary conditions

The aim of this subsection is to extend the results of the last subsection to the case of multiplying compact boundary conditions. Here we follow the same approach as in the preceding subsection.
We now introduce the following two assumptions required in the sequel. we shall assume that following condition is satisfied:

$$\begin{cases} \text{Assume that } \Omega \text{ is a smooth open subset of } \mathbb{R}^N \text{ for which} \\ \text{there exists } \delta > 0 \text{ such that } \operatorname{ess\,inf}_{(x,v)\in\Gamma_+} \tau(x, v) > \delta. \end{cases} \tag{18}$$

Note that, in general, the sojourn time $\tau(x, v)$ may be arbitrary big as well as arbitrary small. Our assumption means that the boundary operator H vanish on the set $\left\{(x, v) \in \Gamma_+ : \tau(x, v) \leq \delta\right\}$ (the tangential velocities are not taken into account by the boundary operator regardless of its norm). We suppose further that the boundary operator satisfies

$$\|H\| > 1. \tag{19}$$

Remark 5 Note that, for streaming operator T_H in multidimensional geometry with a positive boundary operator satisfying (19), it is established in [22, Theorem 5.2] that, under the condition (18), T_H generates a strongly continuous semigroup on the space $L^1(\Omega \times V, dxd\mu(v))$ where $V \subset \mathbb{R}^N$ is the space of admissible velocities and $\mu(\cdot)$ a positive Radon measure on V. Using the same idea and a renormalized argument it is proved that this result holds also true for L^p-spaces (see [2,24]).

Set

$$\lambda_0 = -\lambda^* + \frac{1}{\delta}\ln(\|H\|).$$

As in Sect. 2, in order to derive the expression of $(\lambda - T_H)^{-1}$, we consider the equation $(\lambda - T_H)\psi = \varphi$, where φ is a given function in X_p, λ is a complex number. The unknown ψ must be sought in $D(T_H)$. For $\mathrm{Re}\lambda > -\lambda^*$, the invertibility of $(\lambda - T_H)$, reduces to the invertibility of the operator $\mathcal{P}_\lambda := I - M_\lambda H$ (cf. [17]). In particular, if λ is such that $\mathrm{Re}\lambda > \lambda_0$, then $\|M_\lambda H\| < 1$. Let Γ_{λ_0} be the set defined by

$$\Gamma_{\lambda_0} := \{\lambda \in \mathbb{C} \text{ such that } \mathrm{Re}\lambda > \lambda_0\}.$$

Clearly, if $\lambda \in \Gamma_{\lambda_0}$, the operator $\mathcal{P}_\lambda$ is invertible and

$$(\mathcal{P}_\lambda)^{-1} = \sum_{n\geq 0} (M_\lambda H)^n. \tag{20}$$

Next, substituting (20) in the Eq. (10), we get

$$(\lambda - T_H)^{-1} = \sum_{n\geq 0} B_\lambda H (M_\lambda H)^n G_\lambda + C_\lambda. \tag{21}$$

Unlike in Sect. 4.1, here the strip $\{\lambda \in \mathbb{C} \text{ such that } -\lambda^* < \mathrm{Re}\lambda \leq \lambda_0\}$ is not necessary contained in $\rho(T_H)$. However, with additional hypotheses (for example, the compactness of H), we can obtain more information about its structure.
As before, we denote by G the set $\{\lambda \in \mathbb{C} \text{ such that } \mathrm{Re}\lambda > -\lambda^*\}$.

Lemma 5 *Let $\lambda \in G$. If H is a compact operator satisfying* (19)*, then there exists a discrete subset S of G such that $G \backslash S \subset \rho(T_H)$.*

Proof Let $\lambda \in G$. It is easy to see that $M_\lambda \to 0$, as $\mathrm{Re}\lambda \to \infty$, in the strong operator topology. Using the compactness of H together with [12, Lemma 3.7, p. 151], we infer that $M_\lambda H \to 0$, as $\mathrm{Re}\lambda \to \infty$, in the operator topology. Applying Gohberg-Smul'yan's theorem [11, Theorem 11.4] we deduce that $(I - M_\lambda H)$ is invertible for all $\lambda \in G \backslash S$ where S is a discrete subset of G. □

Lemma 6 *Let H be a boundary operator satisfying* (19)*. Then the following holds*

$$\lim_{\mathrm{Re}\lambda \to +\infty} \left\|(\lambda - T_H)^{-1}\right\| = 0.$$

Proof This follows from the fact that T_H is the generator of a C_0-semigroup. □

Lemma 7 *If H is a compact boundary operator satisfying* (19)*, then $(I - M_\lambda H)^{-1}$ exists for $\lambda \in G$ such that $|\mathrm{Im}\lambda|$ is large enough.*

Before giving the proof, we first recall the following lemma established in [13]

Lemma 8 *Let H be a compact boundary operator satisfying (19). If $\lambda \in G$, then*

$$\lim_{|\mathrm{Im}\lambda| \to +\infty} r_\sigma(M_\lambda H) = 0.$$

Proof of Lemma 7 The result is clear if $\lambda \in \Gamma_{\lambda_0}$. Next, let $\lambda \in G \backslash \Gamma_{\lambda_0}$. According to Lemma 8, there exists $M > 0$ such that, for $|\mathrm{Im}\lambda| > M$, we have $r_\sigma(M_\lambda H) < 1$ which conclude the proof. □

Now, we are ready to state one of the main result of this section. Before going further, we first recall that the set Γ_ω is defined by

$$\Gamma_\omega = \Big\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \geq -\lambda^* + \omega\Big\}.$$

where $\omega > 0$ is small enough.

Proposition 4 *Let K be a regular collision operator. If H is a compact boundary operator satisfying* (19)*, then*

$$\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda|^r \left\| K(\lambda - T_H)^{-1} K \right\| = 0 \ \textit{uniformly on} \quad \Gamma_\omega,$$

where $r \in [0, 1]$.

Proof According to Remark 8, it suffices to establish the result for a one rank collision operator which we define by

$$K : X_p \to X_p, \varphi \to K\varphi(x, v) = \int_{\mathbb{R}^N} f(v) g(v')\varphi(x, v')dv',$$

where $f(\cdot) \in L_p(\mathbb{R}^N)$ et $g(\cdot) \in L_q(\mathbb{R}^N)$.
Since, $K(\lambda - T_H)^{-1}K = KB_\lambda H(I - M_\lambda H)^{-1}G_\lambda K + KC_\lambda K$ (use Eq. (21)), it follows from Lemma 1, that $\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda|^r \, \|KC_\lambda K\| = 0$ uniformly on Γ_ω. Hence, it suffices to establish the result for the operator $KB_\lambda H(I - M_\lambda H)^{-1}G_\lambda K$. To do so, we shall proceed in two steps.
(i) We will first prove that the family of operators $\{H(I - M_\lambda H)^{-1}G_\lambda K, -\lambda^* + \omega \le \mathrm{Re}\lambda \le \lambda_0\}$ is collectively compact. Let B be the unit ball of X_p and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\cup_\lambda H(I - M_\lambda H)^{-1}G_\lambda KB, \ \lambda \in \{\lambda, \ -\lambda^* + \omega \le \mathrm{Re}\lambda \le \lambda_0\}$. Then there exists $(q_n)_{n\in\mathbb{N}}$ in B such that $x_n = H(I - M_\lambda H)^{-1}G_\lambda Kq_n \ n = 1, 2, \ldots$ It is clear that the sequence $(y_n = (I - M_\lambda H)^{-1}G_\lambda Kq_n)_{n\in\mathbb{N}}$ is bounded in L_p^+. So, it follows from the compactness of H that $(x_n = Hy_n)_{n\in\mathbb{N}}$ has converging subsequence in $\overline{\cup_\lambda H(I - M_\lambda H)^{-1}G_\lambda KB}$.
(ii) Now we shall establish that $\{|\mathrm{Im}\lambda|^r \, KB_\lambda, -\lambda^* + \omega \le \ \mathrm{Re}\lambda \le \lambda_0\}$ converges strongly to zero as $|\mathrm{Im}\lambda|$ goes to infinity. For $\varphi \in L_p^-$ and $\lambda = \eta + i\,\mathrm{Im}\lambda$, we have

$$(KB_\lambda\varphi)(x, v) = f(v) \int_{\mathbb{R}^N} g(v')e^{-i\mathrm{Im}\lambda t^-(x,v')} B_\eta\varphi(x, v')dv'.$$

So, it can be decomposed as $KB_\lambda := A_2(\mathrm{Im}\lambda)A_1(\eta)$, where

$$A_1(\eta) : L_p^+ \to X_p, \psi \to (A_1(\eta)\psi)(x, v) = B_\eta\psi(x, v)$$

and

$$A_2(\mathrm{Im}\lambda) : X_p \to X_p, \varphi \to (A_2(\mathrm{Im}\lambda)\varphi)(x, v) = f(v) \int_{\mathbb{R}^N} g(v')e^{-i\mathrm{Im}\lambda t^-(x,v')}\varphi(x, v')dv'.$$

Note that $A_1(\eta)$ independent of $\mathrm{Im}\lambda$ and $t^-(x, v') \in (\delta, \mathsf{d})$ (d stands for the diameter of Ω). Now, arguing as in the last part of the proof of the Lemma 1, we get

$$\lim_{|\mathrm{Im}\lambda|\to\infty} |\mathrm{Im}\lambda|^r \, \|A_2(\mathrm{Im}\lambda)\varphi\|_{X_p} = 0.$$

Now, according to (i), (ii) and [12, Lemma 3.7, p.151], we get the desired result. □

Let $\overline{\lambda}$ be an element of S with the greater real part (S is the subset obtained in Lemma 5).

Lemma 9 *Let K be a regular collision operator and let H be compact boundary operator satisfying* (19)*. Then*

(1) $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \mathrm{Re}\overline{\lambda}\}$ *consists of, at most, isolated eigenvalues with finite algebraic multiplicities,*
(2) *if* $\varepsilon > 0$ *then* $\sigma(A_H) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \mathrm{Re}\overline{\lambda} + \varepsilon\}$ *is finite,*

(3) *if* $\varepsilon > 0$ *then* $\left\|(\lambda - A_H)^{-1}\right\|$ *is uniformly bounded in* $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \mathrm{Re}\overline{\lambda} + \varepsilon\}$ *for large* $|\mathrm{Im}\lambda|$.

Proof The proof is similar to that of Lemma 4 and so it is omitted. □

Assume that K is a regular collision operator on X_p and H is an arbitrary compact boundary operator satisfying (19). Then, by Lemma 9, the spectrum of A_H in the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > \mathrm{Re}\overline{\lambda} + \varepsilon\}$ consists of, a finite number of isolated eigenvalues with finite algebraic multiplicity which we denote $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let P_i and D_i denote, respectively, the spectral projection and the nilpotent operator associated with λ_i, $i = 1, 2, \ldots, n$. Then $P := P_1 + \cdots + P_n$ is the spectral projection of the compact set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. According to the spectral decomposition theorem corresponding to the compact set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ (see, for example, [12]), we may write

$$V_H(t) = \tilde{V}_H(t) + \sum_{i=1}^{n} e^{\lambda_i t} e^{D_i t} P_i$$

where $\tilde{V}_H(t) := V_H(t)(I - P)$ is the C_0-semigroup on the Banach space $(I - P)X_p$ with generator $\tilde{A}_H := A_H(I - P)$ ($\tilde{A}_H$ is the part of A_H on the closed subspace $(I - P)X_p$). We are in a position to state the main result of this subsection.

Theorem 5 *Assume that* $\lambda^* = \inf_{v\in V} \sigma(v) > 0$ *and* $\delta \geq \dfrac{\ln(\|H\|)}{\lambda^*}$. *If the hypotheses of Lemma 9 hold true, then, there exists* $\varepsilon > 0$ *small enough and* $M > 0$ *such that*

$$\left\| V_H(t)\psi_0 - \sum_{i=1}^{n} e^{(\lambda_i + D_i)t} P_i \psi_0 \right\|_{X_p} \leq M e^{(\varepsilon+\lambda_0)t} \quad \forall t > 0$$

where $\psi_0 \in D(A_H)$ *and* $p \in [1, \infty)$.

Proof According to Proposition 4 and Theorem 1, $(\lambda - T_H)^{-1}K$ is power compact. Further, for all $r \in [0, 1]$, we have $|\mathrm{Im}\lambda|^r \left\|K(\lambda - T_H)^{-1}K\right\| \to 0$ as $|\mathrm{Im}\lambda| \to \infty$. Now the rest of the proof may modeled in a similar way to those of Theorems 3 and 4. □

Remark 6 We close this section by noticing that it is not difficult to prove that the first order remainder term of the Dyson-Phillips, $R_1^H(t)$, is compact on X_p for $p \in (1, +\infty)$. The proof may be derived in the same way as that of Proposition 3. However, the weak compactness of $R_1^H(t)$ on X_1, even for dissipative boundary conditions, is still an open problem.

5 Appendix

In this appendix, we shall give some definitions and properties concerning a large class of collision operations.
Note that in neutron transport theory, in general, the collision operator K has the form

$$K : \varphi \in X_p \longrightarrow K\varphi(x, v) = \int_{\mathbb{R}^N} \kappa(x, v, v')\varphi(x, v')dv', \tag{22}$$

where $\kappa(\cdot, \cdot, \cdot)$ is a non-negative measurable function. It is a partial integral operator.

Since K operates only on the velocity variable v', x may be viewed merely as a parameter in Ω. So, it may be regarded as an operator valued mapping from Ω into $\mathcal{L}(L^p(\mathbb{R}^N, dv))$, that is,

$$\Omega \ni x \longrightarrow K(x) \in \mathcal{L}(L^p(\mathbb{R}^N, dv)),$$

where

$$L^p(\mathbb{R}^N) \ni \varphi \longrightarrow K(x)\varphi = \int_{\mathbb{R}^N} \kappa(x, v, v')\varphi(v')dv'.$$

The function $K(\cdot)$ is assumed to be strongly measurable in the following sense

$$\Omega \ni x \to K(x)\psi \in L^p(\mathbb{R}^N) \quad \text{is measurable for any} \quad \psi \in L^p(\mathbb{R}^N)$$

and bounded, i.e.

$$\operatorname*{ess\,sup}_{x \in \Omega} \|K(x)\|_{\mathcal{L}(L^p(\mathbb{R}^N))} < +\infty.$$

We now recall the definition of collectively compact (resp. collectively weakly compact) operators on Banach spaces.

Definition 2 Let X and Y be two normed spaces and denote by $\mathbf{B}$ the closed unit ball of X.

(1) A set $\mathcal{C}$ of $\mathcal{L}(X, Y)$ is said to be collectively compact if, and only if, the set $\mathcal{C}(B) = \{C(x),\ C \in \mathcal{C},\ x \in \mathbf{B}\}$ is relatively compact in Y [1].
(2) A set $\mathcal{W}$ of $\mathcal{L}(Y)$ is said to be collectively weakly compact if, and only if, the set $\mathcal{W}(B) = \{W(x),\ W \in \mathcal{W},\ x \in \mathbf{B}\}$ is relatively weakly compact in Y.

□

Now we are ready to state the definition of the class of regular collisions operators [27] (see also [26]) which will play cornerstone in this work.

Definition 3 Let $p \in (1, +\infty)$. A collision operator K is said to be regular on X_p if:

(1) $\{K(x) : x \in \Omega\}$ is a set of collectively compact operators on $L^p(\mathbb{R}^N, dv)$, i.e.

$$\left\{K(x)\varphi, ;\ x \in \Omega,\ \|\varphi\| \le 1\right\} \text{ is relatively compact in } L^p(\mathbb{R}^N,\ dv).$$

(2) For $\varphi' \in L^q(\mathbb{R}^N,\ dv)$,

$$\left\{K'(x)\varphi', ;\ x \in \Omega,\ \left\|\varphi'\right\| \le 1\right\} \text{ is relatively compact in } L^q(\mathbb{R}^N,\ dv).$$

Here $K'(x)$ denotes the dual operator of $K(x)$ and $q = \dfrac{p}{p-1}$.

□

Regular collision operators is a wide class of operators. In fact, it is the closure (in the operator norm) of the set of operators in the form (22) whose kernels are in given by (23).

Lemma 10 [27, Proposition 4.1] *The class of regular collision operators is the closure in the operator norm of the class of collision operator with kernels of the form*

$$\kappa(x, v, v') = \sum_{i \in I} \alpha_i(x) f_i(v) g_i(v') \tag{23}$$

where I is finite, $\alpha(\cdot) \in L^\infty(\Omega; dx)$, $f_i(\cdot) \in L_p(\mathbb{R}^N; dv)$ and $g_i(\cdot) \in L_q(\mathbb{R}^N; dv)$ with $q = \frac{p}{p-1}$.

For $p = 1$, there is a more appropriate definition of regular collision operators [23].

Definition 4 We say that K is a regular collision operator on X_1 if, for almost all $x \in \Omega$, the operator

$$\phi \in L^1(\mathbb{R}^N, dv) \longrightarrow \int_{\mathbb{R}^N} \kappa(x, v, v')\phi(v')dv' \in L^1(\mathbb{R}^N, dv)$$

is weakly compact on $L^1(\mathbb{R}^N, dv)$ and the family of such operators on $L^1(\mathbb{R}^N, dv)$ indexed by $x \in \Omega$ is collectively weakly compact. □

Remark 7 Obviously the Dunford–Pettis criterion of weak compactness [8, Theorem 15, p. 76] shows that if K is a regular collision operator, then $|K|$ is also a regular collision operator where $|K|$ is defined by

$$\varphi \to (|K|\varphi)(x, v) = \int_{\mathbb{R}^N} |k(x, v, v')|\varphi(x, v')dv'.$$

□

We now recall recall the following useful property of non-negative regular collision operators on X_1.

Lemma 11 [23] *Let K be a regular non-negative collision operator. Then there exists a sequence $(K_n)_n$ of $\mathcal{L}(X_1)$ such that*

(1) $0 \leq K_n \leq K$ *for any $n \in \mathbb{N}$;*
(2) *for any $n \in \mathbb{N}$, K_n is dominated by a rank-one operator in $\mathcal{L}(L^1(V; dv))$;*
(3) $\lim_{n\to\infty} \|K - K_n\| = 0$.

The Item (2) of the last lemma means that each operator K_n is dominated by an operator on X_1 which acts as follows

$$\varphi \in X_1 \to f_n(v) \int_{\mathbb{R}^N} \varphi(x, v')dv'$$

Remark 8 Note that, if K is a regular collision operator, then, according to Lemmas 10 and 11, it suffices to suppose that K has the following form

$$K : X_p \to X_p, \varphi \to K\varphi(x, v) = \alpha(x) \int_{\mathbb{R}^N} f(v)g(v')\varphi(x, v')dv', \tag{24}$$

where $\alpha(\cdot) \in L^\infty(\Omega, dx)$, $f(\cdot) \in L^p(\mathbb{R}^N, dv)$ and $g(\cdot) \in L^q(\mathbb{R}^N, dv)$. By approximating f and g by continuous functions with compact support, we may suppose, that f and g are continuous with compact supports. Since $\alpha(\cdot) \in L^\infty(\Omega, dx)$, without loss of generality, we assume that the function $\alpha(\cdot)$ is constant equal to one. □

We conclude this Appendix by the proof of Theorem 1.

Proof of Theorem 1 The first item of the theorem was established in [17]. So it suffices to prove the second assertion.
Recall that

$$K(\lambda - T_H)^{-1}K = KB_\lambda H(I - M_\lambda)^{-1}G_\lambda K + KC_\lambda K.$$

Note however that, C_λ is nothing else but the resolvent of the streaming operator with vacuum boundary condition T_0. So, we know from [25] that, if K is a regular collision operator on

X_1, then the operator $KC_\lambda K$ is weakly compact operator on X_1. Thus, in order to prove the weakly compactness of $K(\lambda - T_H)^{-1}K$, it suffices to show that the operator KB_λ is weakly compact. Since K is a regular collision operator, according to Lemma 11, it suffices to establish the result for a collision operator of the form

$$\varphi \in X_1 \to f(v) \int_{\mathbb{R}^N} \varphi(x, v')dv'$$

where $f(\cdot) \in L^1(\mathbb{R}^N; dv)$. For $\varphi \in L^{1,-}$, one can write

$$KB_\lambda\varphi(x, v) := f(v) \int_{\mathbb{R}^N} B_\lambda\varphi(x, v')dv',$$

where $B_\lambda\varphi(x, v) = \varphi(x - t^-(x, v)v, v)e^{-t^-(x,v)(\lambda+\sigma(v))}$ and $\|B_\lambda\| \le (\mathrm{Re}\lambda + \lambda^*)^{-1}$. This yields that

$$\begin{aligned} \|KB_\lambda\varphi(x, v)\|_{X_1} &\le \int_{\mathbb{R}^N} |f(v)|dv \int_{\Omega\times\mathbb{R}^N} |B_\lambda\varphi(x, v')|dxdv' \\ &\le \|f(\cdot)\|_{L^1(\mathbb{R}^N)} \|B_\lambda\varphi(\cdot, \cdot)\|_{X_1} \\ &\le (\mathrm{Re}\lambda + \lambda^*)^{-1} \|f(\cdot)\|_{L^1(\mathbb{R}^N)} \|\varphi\|_{L^{1,-}} . \end{aligned}$$

So, we conclude that

$$\|KB_\lambda\| \le (\mathrm{Re}\lambda + \lambda^*)^{-1} \|f(\cdot)\|_{L^1(\mathbb{R}^N)} . \tag{25}$$

The estimate (25) shows that KB_λ depends continuously (for the uniform topology) on $f(\cdot) \in L^1(\mathbb{R}^N)$. So, by approximating $f(\cdot)$ (in the L^1-norm) by bounded functions, KB_λ is a limit (for operator topology) of integral operators with bounded kernel. Hence, KB_λ is weakly compact on X_1 (cf., [9, Corollary 11, p. 294]). □

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