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Fuzzy filters of pseudo-BE algebras

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Abstract
The theory of fuzzy filters in pseudo-BE algebras is developed. Various characterizations of fuzzy filters are given. It is proved that the set of all fuzzy filters of a pseudo-BE algebra is a complete lattice. Some characterizations of Noetherian pseudo-BE algebras by fuzzy filters are obtained. Finally, fuzzy commutative filters are defined and studied. Moreover, the homomorphic properties of fuzzy (commutative) filters are provided.

Keywords (Noetherian) pseudo-BE algebra · Filter · Fuzzy (commutative) filter · Homomorphism

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1 Introduction

In 1966, Imai and Iséki [11] introduced BCK algebras which are an algebraic model of BCK-logic. There exist several generalizations of BCK algebras such as BCI algebras [12], BCH algebras [10], BE algebras [13], etc. BE algebras were deeply studied by Rezaei et. al. [1,17–19]. In 2001, Georgescu and Iorgulescu [9] introduced pseudo-BCK algebras as a non-commutative extension of BCK algebras. Next, pseudo-BCI and pseudo-BCH algebras were defined in [6] and [21], respectively. In 2013, Borzooei et al. [2] introduced the notion of pseudo-BE algebra. Ciungu [3] defined commutative pseudo-BE algebras and proved that the class of these algebras coincides with the class of commutative pseudo-BCK algebras.

In 1965, Zadeh [25] introduced fuzzy sets. At present these ideas have been applied to many mathematical branches such as abstract algebra (semigroups, groups, rings, modules, etc.), functional analysis, probability theory, topology and so on. In 1991, Xi [24] applied the concept of fuzzy set to BCK algebras. Fuzzy ideals of BCI and BCH algebras were studied in [16] and [23], respectively. Fuzzy filters of BE algebras were discussed in [7] and

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[20]. Lee [15] established the fuzzyfication of ideals in pseudo-BCI algebras. Fuzzy ideals of pseudo-BCK algebras were investigated in [8]. Walendziak and Wojciechowska–Rysiawa [22] studied fuzzy ideal theory in pseudo-BCH algebras.

In this paper, we develop fuzzy filter theory in pseudo-BE algebras. This theory plays an important role in the investigation of such algebras. We give characterizations of fuzzy filters and provide conditions for a fuzzy set to be a fuzzy filter. We also show that the set of fuzzy filters of a pseudo-BE algebra is a complete lattice. Moreover, we obtain some characterizations of Noetherian pseudo-BE algebras by fuzzy filters. Finally, we introduce the notion of fuzzy commutative filter and investigate some of its properties.

2 Preliminaries

In this section, we review some of the standard facts on pseudo-BE algebras.

Definition 2.1 An algebra $(X; \rightarrow, 1)$ of type $(2, 0)$ is called a *BE algebra* if it satisfies the following equations:

(BE$_1$) $x \rightarrow x = 1$,
(BE$_2$) $x \rightarrow 1 = 1$,
(BE$_3$) $1 \rightarrow x = x$,
(BE$_4$) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Definition 2.2 [2] An algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called a *pseudo-BE algebra* if it satisfies the following axioms:

(pBE$_1$) $x \rightarrow x = x \rightsquigarrow x = 1$,
(pBE$_2$) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$,
(pBE$_3$) $1 \rightarrow x = 1 \rightsquigarrow x = x$,
(pBE$_4$) $x \rightarrow y = 1 \Longleftrightarrow x \rightsquigarrow y = 1$,
(pBE$_5$) $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$.

Definition 2.3 [14] An algebra $(X; \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 0)$ is called *a pseudo-BCK algebra* if it satisfies (pBE$_1$)–(pBE$_4$) and the following axioms:

(pBCK$_1$) $(x \rightarrow y) \rightsquigarrow [(y \rightarrow z) \rightsquigarrow (x \rightarrow z)] = 1$,
(pBCK$_2$) $(x \rightsquigarrow y) \rightarrow [(y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)] = 1$,
(pBCK$_3$) $(x \rightarrow y = 1 \text{ and } y \rightarrow x = 1) \Longrightarrow x = y$.

Remark 2.4 Every pseudo-BCK algebra verifies (pBE$_5$), see [14], and therefore pseudo-BCK algebras are pseudo-BE algebras.

Definition 2.5 A pseudo-BE algebra $(X; \rightarrow, \rightsquigarrow, 1)$ is called *proper* if it is not a pseudo-BCK algebra and $\rightarrow \neq \rightsquigarrow$.

Remark 2.6 If $(X; \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BE algebra, then $(X; \rightsquigarrow, \rightarrow, 1)$ is a pseudo-BE algebra, too. We also note that if the operations $\rightarrow$ and $\rightsquigarrow$ coincide, then $(X; \rightarrow, 1)$ is a BE algebra.

In a pseudo-BE algebra, one can introduce a binary relation $\leq$ by:

$$x \leq y \Longleftrightarrow x \rightarrow y = 1 \Longleftrightarrow x \rightsquigarrow y = 1, \quad \text{for all } x, y \in X.$$

Proposition 2.7 [2] *In a pseudo-BE algebra* $(X; \rightsquigarrow, \to, 1)$, *the following statements hold:*

(i) $x \le y \to x$ *and* $x \le y \rightsquigarrow x$,
(ii) $x \le (x \rightsquigarrow y) \to y$ *and* $x \le (x \to y) \rightsquigarrow y$,
(iii) $1 \le x \Longrightarrow x = 1$.

Example 2.8 Consider the set $X = \{a, b, c, d, e, f, 1\}$ with the operations $\to$ and $\rightsquigarrow$ defined by the following tables:

$\to$	a	b	c	d	e	f	1
a	1	1	1	d	e	f	1
b	c	1	c	d	e	f	1
c	a	b	1	d	e	f	1
d	a	b	c	1	d	1	1
e	a	b	c	1	1	d	1
f	a	b	c	1	d	1	1
1	a	b	c	d	e	f	1

$\rightsquigarrow$	a	b	c	d	e	f	1
a	1	1	1	d	e	f	1
b	a	1	c	d	e	f	1
c	b	b	1	d	e	f	1
d	a	b	c	1	d	1	1
e	a	b	c	1	1	d	1
f	a	b	c	1	d	1	1
1	a	b	c	d	e	f	1

Then $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a pseudo-BE algebra. Since $(e \to d) \rightsquigarrow [(d \to f) \rightsquigarrow (e \to f)] = 1 \rightsquigarrow (1 \rightsquigarrow d) = d \neq 1$, axiom (pBCK$_1$) does not hold. Hence, $\mathfrak{X}$ is not a pseudo-BCK algebra. Therefore, $\mathfrak{X}$ is a proper pseudo-BE algebra.

Definition 2.9 [5] *A pseudo-BE algebra with the condition* (A) or *a pseudo-BE(A) algebra* for short, is a pseudo-BE algebra $(X; \to, \rightsquigarrow, 1)$ satisfying

$$\text{(A)} \quad x \le y \Longrightarrow (y \to z \le x \to z \text{ and } y \rightsquigarrow z \le x \rightsquigarrow z).$$

Example 2.10 [5] Let $X = \{a, b, c, d, 1\}$. Define the operations $\to$ and $\rightsquigarrow$ on X as follows:

$\to$	a	b	c	d	1
a	1	c	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	c	c	1	1
1	a	b	c	d	1

$\rightsquigarrow$	a	b	c	d	1
a	1	b	c	1	1
b	d	1	1	d	1
c	d	1	1	d	1
d	1	b	c	1	1
1	a	b	c	d	1

It is easy to check that $\mathfrak{X} = (X; \to, \rightsquigarrow, 1)$ is a pseudo-BE(A) algebra. Since $b \to c = 1$ and $c \to b = 1$, axiom (pBCK$_3$) is not satisfied. Hence, $\mathfrak{X}$ is not a pseudo-BCK algebra.

Definition 2.11 [3] A pseudo-BE algebra $(X; \to, \rightsquigarrow, 1)$ is said to be *commutative* if it verifies the equations:

$$(x \to y) \rightsquigarrow y = (y \to x) \rightsquigarrow x \quad \text{and} \quad (x \rightsquigarrow y) \to y = (y \rightsquigarrow x) \to x.$$

Proposition 2.12 [3] *Any commutative pseudo-BE algebra is a pseudo-BCK algebra, therefore commutative pseudo-BE algebras coincide with commutative pseudo-BCK algebras.*

Definition 2.13 [2] Let $\mathfrak{X}$ be a pseudo-BE algebra. A subset F of X is called a *filter* of $\mathfrak{X}$ if for all $x, y \in X$:

(F1) $1 \in F$,
(F2) if $x \to y \in F$ and $x \in F$, then $y \in F$.

Proposition 2.14 [2] *Let $\mathfrak{X}$ be a pseudo-BE algebra and F be a subset of X satisfying (F1). Then F is a filter of $\mathfrak{X}$ if and only if for all $x, y \in X$,*

$(F2')$ *if* $x \rightsquigarrow y \in F$ *and* $x \in F$, *then* $y \in F$.

We will denote by $\mathsf{F}(\mathfrak{X})$ the set of all filters of $\mathfrak{X}$. Obviously, $\{1\}, X \in \mathsf{F}(\mathfrak{X})$.

Example 2.15 Let $\mathfrak{X}$ be the pseudo-BE algebra from Example 2.8. We have $\mathsf{F}(\mathfrak{X}) = \{\{1\}, \{1, b\}, \{1, c\}, X_1, X_2, \{b\} \cup X_2, \{c\} \cup X_2, X_1 \cup X_2\}$, where $X_1 = \{a, b, c, 1\}$ and $X_2 = \{d, e, f, 1\}$.

Proposition 2.16 *Let* $\mathfrak{X}$ *be a pseudo-BE algebra and let* $F \in \mathsf{F}(\mathfrak{X})$. *For any* $x, y \in X$, *if* $x \in F$ *and* $x \leq y$, *then* $y \in F$.

Proof Straightforward. □

3 Fuzzy filters

We now review some fuzzy logic concepts. First, for $\Gamma \subseteq [0, 1]$ we define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a fuzzy set in X is a function $\bar{\mu} : X \longrightarrow [0, 1]$.

For any fuzzy sets $\overline{\mu}$ and $\overline{\nu}$ in X, we define

$$\bar{\mu} \leq \bar{\nu} \Longleftrightarrow \bar{\mu}(x) \leq \bar{\nu}(x) \quad \text{for all } x \in X.$$

It is easy to check that this relation is an order relation in the set of fuzzy sets in X.

Let X and Y be any two sets, $\overline{\mu}$ be any fuzzy set in X, and $f : X \to Y$ be any function. Write $f^{\leftarrow}(y) = \{x \in A : f(x) = y\}$ for $y \in Y$. The fuzzy set $\overline{\nu}$ in Y defined by

$$\overline{\nu}(y) = \begin{cases} \bigvee\{\overline{\mu}(x) : x \in f^{\leftarrow}(y)\} & \text{if } f^{\leftarrow}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in Y$, is called the *image of* $\overline{\mu}$ *under* f and is denoted by $f(\overline{\mu})$.

Let X and Y be any two sets, $f : X \to Y$ be any function, and $\overline{\nu}$ be any fuzzy set in $f(X)$. The fuzzy set $\overline{\mu}$ in X defined by

$$\overline{\mu}(x) = \overline{\nu}(f(x)) \quad \text{for all } x \in X$$

is called the *preimage of* $\overline{\nu}$ *under* f and is denoted by $f^{\leftarrow}(\overline{\nu})$.

Next we define fuzzy filters of pseudo-BE algebras. From now on, $\mathfrak{X}$ is a pseudo-BE algebra, unless it is stated.

Definition 3.1 A fuzzy set $\bar{\mu}$ in X is called a *fuzzy filter* of $\mathfrak{X}$ if it satisfies the following conditions:

(FF1) $\bar{\mu}(1) \geq \bar{\mu}(x)$,
(FF2) $\bar{\mu}(y) \geq \bar{\mu}(x) \wedge \bar{\mu}(x \to y)$.

Let $\mathsf{FF}(\mathfrak{X})$ denote the set of all fuzzy filters of a pseudo-BE algebra $\mathfrak{X}$.

Example 3.2 Let $\mathfrak{X}$ be the pseudo-BE algebra given in Example 2.10. Define a fuzzy set $\bar{\mu} : X \longrightarrow [0, 1]$ by $\bar{\mu}(a) = \bar{\mu}(d) = 0.5$, $\bar{\mu}(b) = \bar{\mu}(\mathrm{c}) = 0.6$ and $\bar{\mu}(1) = 0.7$. It is easily seen that $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$.

Proposition 3.3 *Every fuzzy filter* $\bar{\mu}$ *of* $\mathfrak{X}$ *satisfies the following assertions:*

(i) *if* $x \leq y$, *then* $\bar{\mu}(x) \leq \bar{\mu}(y)$,

(ii) *if* $x \leq y$, *then* $\bar{\mu}(x) \leq \bar{\mu}(z \rightarrow y)$ *and* $\bar{\mu}(x) \leq \bar{\mu}(z \rightsquigarrow y)$,
(iii) $\bar{\mu}(y) \leq \bar{\mu}(x \rightarrow y)$ *and* $\bar{\mu}(y) \leq \bar{\mu}(x \rightsquigarrow y)$.

Proof (i) Assume that $x \leq y$. Then $x \rightarrow y = 1$. Applying (FF2) and (FF1), we have

$$\bar{\mu}(y) \geq \bar{\mu}(x) \wedge \bar{\mu}(x \rightarrow y)) = \bar{\mu}(x) \wedge \bar{\mu}(1) = \bar{\mu}(x).$$

(ii) Let $x \leq y$. From (pBE$_5$) and (pBE$_2$) we get $x \rightsquigarrow (z \rightarrow y) = z \rightarrow (x \rightsquigarrow y) = z \rightarrow 1 = 1$, and so $x \leq z \rightarrow y$. Using (i), we obtain $\bar{\mu}(x) \leq \bar{\mu}(z \rightarrow y)$. The second inequality is obtained by a similar argument.
(iii) By Proposition 2.7(i), $y \leq x \rightarrow y$ and $y \leq x \rightsquigarrow y$. Applying (i), we have (iii).

□

Proposition 3.4 *A fuzzy set* $\bar{\mu}$ *in* X *is a fuzzy filter of* $\mathfrak{X}$ *if and only if* $\bar{\mu}$ *verifies* (FF1) *and for all* $x, y \in X$,

(FF3) $\bar{\mu}(y) \geq \bar{\mu}(x) \wedge \bar{\mu}(x \rightsquigarrow y)$.

Proof It suffices to prove that if (FF2) is satisfied, then (FF3) is also satisfied. The proof of the converse of this implication is similar. From Proposition 2.7(ii) we see that $x \leq (x \rightsquigarrow y) \rightarrow y$. By Proposition 3.3(i), $\bar{\mu}(x) \leq \bar{\mu}((x \rightsquigarrow y) \rightarrow y)$. Hence

$$\bar{\mu}(x) \wedge \bar{\mu}(x \rightsquigarrow y) \leq \bar{\mu}((x \rightsquigarrow y) \rightarrow y) \wedge \bar{\mu}(x \rightsquigarrow y)) \leq \bar{\mu}(y).$$

□

Proposition 3.5 *A fuzzy set* $\bar{\mu}$ *in* X *is a fuzzy filter of* $\mathfrak{X}$ *if and only if* $\bar{\mu}$ *verifies* (FF1) *and for all* $x, y, z \in X$,

(*FF4*) *if* $x \leq y \rightarrow z$, *then* $\bar{\mu}(z) \geq \bar{\mu}(x) \wedge \bar{\mu}(y)$.

Proof Assume that $\bar{\mu}$ is a fuzzy filter and $x \leq y \rightarrow z$. Using (FF2) and Proposition 3.3(i), we have:

$$\bar{\mu}(z) \geq \bar{\mu}(y) \wedge \bar{\mu}(y \rightarrow z) \geq \bar{\mu}(y) \wedge \bar{\mu}(x).$$

Conversely, let $\bar{\mu}$ satisfy (FF4). Since $y \rightarrow z \leq y \rightarrow z$, we obtain:

$$\bar{\mu}(z) \geq \bar{\mu}(y \rightarrow z) \wedge \bar{\mu}(y).$$

Thus, $\bar{\mu}$ satisfies (FF2), and hence $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$. □

Proposition 3.6 *A fuzzy set* $\bar{\mu}$ *in* X *is a fuzzy filter of* $\mathfrak{X}$ *if and only if* $\bar{\mu}$ *verifies* (FF1) *and for all* $x, y, z \in X$,

(*FF5*) *if* $x \leq y \rightsquigarrow z$, *then* $\bar{\mu}(z) \geq \bar{\mu}(x) \wedge \bar{\mu}(y)$,

Proof Similar to the proof of Proposition 3.5. □

Theorem 3.7 *A fuzzy set* $\bar{\mu}$ *in* X *is a fuzzy filter of* $\mathfrak{X}$ *if and only if its nonempty level subset* $U(\bar{\mu}, \alpha) = \{x \in X : \bar{\mu}(x) \geq \alpha\}$ *is a filter of* $\mathfrak{X}$ *for all* $\alpha \in [0, 1]$.

Proof Let $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$ and let $\alpha \in [0, 1]$. Assume $U(\bar{\mu}, \alpha) \neq \emptyset$. Then there exists $x_0 \in X$ such that $\bar{\mu}(x_0) \geq \alpha$. Since $\bar{\mu}(1) \geq \bar{\mu}(x)$, we have $1 \in U(\bar{\mu}, \alpha)$. Let $x, x \rightarrow y \in U(\bar{\mu}, \alpha)$. Hence $\bar{\mu}(x) \geq \alpha$ and $\bar{\mu}(x \rightarrow y) \geq \alpha$. It follows from (FF2) that

$$\bar{\mu}(y) \geq \bar{\mu}(x) \wedge \bar{\mu}(x \rightarrow y) \geq \alpha.$$

Therefore, $y \in U(\bar{\mu}, \alpha)$.

Conversely, suppose that for each $\alpha \in [0, 1]$, $U(\bar{\mu}, \alpha) = \emptyset$ or $U(\bar{\mu}, \alpha)$ is a filter of $\mathfrak{X}$. If (FF1) does not hold, then there exists $x_0 \in X$ such that $\bar{\mu}(1) < \bar{\mu}(x_0) := \beta$. Then $U(\bar{\mu}, \beta) \neq \emptyset$ and by assumption, $U(\bar{\mu}, \beta)$ is a filter of $\mathfrak{X}$. Hence $1 \in U(\bar{\mu}, \beta)$ and consequently, $\bar{\mu}(1) \geq \beta$. This is a contradiction, and so (FF1) holds. Now, assume that (FF2) is not satisfied. Then there are $x_0, y_0 \in X$ such that $\bar{\mu}(y_0) < \bar{\mu}(x_0) \wedge \bar{\mu}(x_0 \to y_0)$. Taking

$$\beta = \frac{1}{2}(\bar{\mu}(y_0) + (\bar{\mu}(x_0) \wedge \bar{\mu}(x_0 \to y_0)),$$

we get $\bar{\mu}(y_0) < \beta < \bar{\mu}(x_0) \wedge \bar{\mu}(x_0 \to y_0) \leq \bar{\mu}(x_0 \to y_0)$ and $\beta < \bar{\mu}(x_0)$. Therefore, $x_0, x_0 \to y_0 \in U(\bar{\mu}, \beta)$ but $y_0 \notin U(\bar{\mu}, \beta)$. This is impossible, and so $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$. □

Corollary 3.8 *If $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$, then the set $X_a := \{x \in X : \bar{\mu}(x) \geq \bar{\mu}(a)\}$ is a filter of $\mathfrak{X}$ for all $a \in X$.*

Corollary 3.9 *If $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$, then the set $X_{\bar{\mu}} := \{x \in X : \bar{\mu}(x) = \bar{\mu}(1)\}$ is a filter of $\mathfrak{X}$.*

The following example shows that the converse of Corollary 3.9 is not true in general.

Example 3.10 Let $\mathfrak{X}$ be a pseudo-BE algebra. Define a fuzzy set $\bar{\mu}$ in X by:

$$\bar{\mu}(x) = \begin{cases} 0.3 & \text{if } x = 1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then $X_{\bar{\mu}} = \{1\}$ and it is a filter of $\mathfrak{X}$ but $\bar{\mu}$ is not a fuzzy filter, since $\bar{\mu}$ does not satisfy (FF1).

Let $\overline{\mu}_i \in \mathsf{FF}(\mathfrak{X})$ for $i \in I$. The meet $\bigwedge_{i \in I} \overline{\mu}_i$ of fuzzy filters $\overline{\mu}_i$ is defined as follows:

$$\left(\bigwedge_{i \in I} \overline{\mu}_i\right)(x) = \bigwedge\{\overline{\mu}_i(x) : i \in I\}.$$

Proposition 3.11 *Let $\bar{\mu}_i \in FF(\mathfrak{X})$ for $i \in I$. Then $\bigwedge_{i \in I} \bar{\mu}_i \in FF(\mathfrak{X})$.*

Proof Let $\bar{\mu} := \bigwedge_{i \in I} \bar{\mu}_i$. Then, by (FF1),

$$\bar{\mu}(1) = \bigwedge\{\bar{\mu}_i(1) : i \in I\} \geq \bigwedge\{\bar{\mu}_i(x) : i \in I\} = \bar{\mu}(x)$$

for all $x \in X$. Let $x, y \in X$. Since $\bar{\mu}_i \in \mathsf{FF}(\mathfrak{X})$, we have $\bar{\mu}_i(x) \geq \bar{\mu}_i(y \to x) \wedge \bar{\mu}_i(y)$. Hence, by (FF2),

$$\begin{aligned} \bigwedge\{\bar{\mu}_i(x) : i \in I\} &\geq \bigwedge\{\bar{\mu}_i(y \to x) \wedge \bar{\mu}_i(y) : i \in I\} \\ &= \bigwedge\{\bar{\mu}_i(y \to x) : i \in I\} \wedge \bigwedge\{\bar{\mu}_i(y) : i \in I\}. \end{aligned}$$

Consequently, $\bar{\mu}(x) \geq \bar{\mu}(y \to x) \wedge \bar{\mu}(y)$, and therefore $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$. □

Let $\bar{\nu}$ be a fuzzy set in X. A fuzzy filter $\bar{\mu}$ of $\mathfrak{X}$ is said to be *generated by* $\bar{\nu}$ if $\bar{\nu} \leq \bar{\mu}$ and for any fuzzy filter $\bar{\rho}$ of $\mathfrak{X}$, $\bar{\nu} \leq \bar{\rho}$ implies $\bar{\mu} \leq \bar{\rho}$. The fuzzy filter generated by $\bar{\nu}$ will be denoted by $[\bar{\nu})$. The fuzzy filter $[\bar{\nu})$ we can define equivalently as follows:

$$[\bar{\nu}) = \bigwedge\{\bar{\rho} : \bar{\rho} \in \mathsf{FF}(\mathfrak{X}) \quad \text{and} \quad \bar{\nu} \leq \bar{\rho}\}.$$

Let $\bar{\mu}, \bar{\nu}$ be two fuzzy filters in $\mathfrak{X}$. Denote the join of $\bar{\mu}$ and $\bar{\nu}$ by $\bar{\mu} \vee \bar{\nu}$, that is, $\bar{\mu} \vee \bar{\nu} = [\bar{\rho})$, where $\bar{\rho}$ is the fuzzy set in X defined by $\bar{\rho}(x) = \bar{\mu}(x) \vee \bar{\nu}(x)$.

Theorem 3.12 *Let $\mathfrak{X}$ be a pseudo-BE algebra. Then $(FF(\mathfrak{X}); \wedge, \vee)$ is a complete lattice.*

Proof The proof is straightforward. □

Definition 3.13 A pseudo-BE algebra $\mathfrak{X}$ is said to satisfy the *ascending chain condition* if for every ascending sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$ of filters of $\mathfrak{X}$, there exists a natural number k such that $F_n = F_k$ for all $n \geq k$. If $\mathfrak{X}$ satisfies the ascending chain condition, we say $\mathfrak{X}$ is a *Noetherian pseudo-BE algebra.*

Theorem 3.14 *Let $\mathfrak{X}$ be a pseudo-BE algebra. The following statements are equivalent:*

(i) *$\mathfrak{X}$ is Noetherian,*
(ii) *For each fuzzy filter $\overline{\mu}$ of $\mathfrak{X}$, $Im(\overline{\mu})$ is a well-ordered set.*
(iii) *For each fuzzy filter $\overline{\mu}$ of $\mathfrak{X}$, if $Im(\overline{\mu}) \subseteq \{t_1, t_2, \ldots\} \cup \{0\}$, where (t_n) is a strictly decreasing sequence in $(0, 1]$, then there exists $k \in \mathbb{N}$ such that $Im(\overline{\mu}) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.*

Proof (i) $\Rightarrow$ (ii): Assume that $\mathfrak{X}$ is Noetherian and $\overline{\mu}$ is a fuzzy filter of $\mathfrak{X}$ such that $\mathrm{Im}(\overline{\mu})$ is not a well-ordered subset of $[0, 1]$. Then, there exists a strictly decreasing sequence $(\mu(x_n))$, where $x_n \in X$. Let $t_n = \mu(x_n)$ and $U_n = U(\overline{\mu}; t_n)$. By Theorem 3.7, U_n is a filter of $\mathfrak{X}$ for every $n \in \mathbb{N}$. So $U_1 \subset U_2 \subset \ldots$ is a strictly ascending sequence of filters of $\mathfrak{X}$. This contradicts the assumption that is Noetherian. Then $\mathrm{Im}(\overline{\mu})$ is a well-ordered set for each fuzzy filter $\overline{\mu}$ of $\mathfrak{X}$.

(ii) $\Rightarrow$ (iii): Assume that (ii) is true. Let $\overline{\mu}$ be a fuzzy filter of $\mathfrak{X}$ such that $\mathrm{Im}(\overline{\mu}) \subseteq \{t_1, t_2, \ldots\} \cup \{0\}$. Since $\mathrm{Im}(\overline{\mu})$ is a well-ordered subset of $[0, 1]$ and (t_n) is a strictly decreasing sequence in $(0, 1]$, there is $k \in \mathbb{N}$ such that $\mathrm{Im}(\overline{\mu}) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.

(iii) $\Rightarrow$ (i): Suppose that $\mathfrak{X}$ is not Noetherian. Then there exists a strictly ascending sequence $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ of filters of $\mathfrak{X}$. Let $\overline{\mu}$ be a fuzzy set in X such that

$$\overline{\mu}(x) = \begin{cases} \frac{1}{n} & \text{if } x \in F_n - F_{n-1} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{if } x \notin F_n \text{ for each } n \in \mathbb{N}, \end{cases}$$

where $F_0 = \emptyset$. Let $F = \bigcup_{n\in\mathbb{N}} F_n$. It is easy to see that F is a filter of $\mathfrak{X}$. Obviously, $\overline{\mu}(1) = 1 \geq \overline{\mu}(x)$ for all $x \in X$, that is, (FF1) holds. Now we show that $\overline{\mu}$ satisfies (FF2). Let $x, y \in X$. We consider two cases.

Case 1: $x \notin F$.
Then $y \to x \notin F$ or $y \notin F$. Therefore, $\overline{\mu}(y \to x) \wedge \overline{\mu}(y) = 0 = \overline{\mu}(x)$.

Case 2: $x \in F_n - F_{n-1}$ for some $n \in \mathbb{N}$.
Then $y \to x \notin F_{n-1}$ or $y \notin F_{n-1}$. Hence $\overline{\mu}(y \to x) \leq \frac{1}{n}$ or $\overline{\mu}(y) \leq \frac{1}{n}$. So $\overline{\mu}(y \to x) \wedge \overline{\mu}(y) \leq \frac{1}{n} = \overline{\mu}(x)$. Thus (FF2) is also satisfied, and consequently $\overline{\mu}$ is a fuzzy filter of $\mathfrak{X}$. We have $\mathrm{Im}(\overline{\mu}) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. Obviously, $\mathrm{Im}(\overline{\mu}) \nsubseteq \left\{1, \frac{1}{2}, \ldots, \frac{1}{k}\right\} \cup \{0\}$ for every $k \in \mathbb{N}$, which is a contradiction. Therefore $\mathfrak{X}$ is Noetherian, and the proof is complete. □

Corollary 3.15 *If for every fuzzy filter $\overline{\mu}$ of $\mathfrak{X}$, $Im(\overline{\mu})$ is a finite set, then $\mathfrak{X}$ is Noetherian.*

4 Fuzzy commutative filters

Ciungu [4] defined commutative deductive systems and showed that a pseudo-BCK algebra $\mathfrak{X}$ is commutative if and only if all the deductive systems of $\mathfrak{X}$ are commutative. In this section, we introduce the notion of a fuzzy commutative filter of a pseudo-BE algebra and study some of its properties.

Definition 4.1 [4] We say that a filter F of a pseudo-BE algebra $\mathfrak{X}$ is *commutative* if for all $x, y \in X$:

(CF1) $y \to x \in F \Longrightarrow [(x \to y) \rightsquigarrow y] \to x \in F$,
(CF2) $y \rightsquigarrow x \in F \Longrightarrow [(x \rightsquigarrow y) \to y] \rightsquigarrow x \in F$.

Let $\mathsf{CF}(\mathfrak{X})$ denote the set of all commutative filters of $\mathfrak{X}$.

Example 4.2 Let $\mathfrak{X}$ be the pseudo-BE algebra given in Example 2.8. It is easy to see that $\{a, b, c, 1\} \in \mathsf{CF}(\mathfrak{X})$, while the filter $\{1\}$ is not commutative, since $a \to c = 1$ but $[(c \to a) \rightsquigarrow a] \to c = c \notin \{1\}$.

Definition 4.3 A fuzzy filter $\bar{\mu}$ of $\mathfrak{X}$ is called a *fuzzy commutative filter* if for all $x, y \in X$,

(FCF1) $\bar{\mu}((x \to y) \rightsquigarrow y) \to x) \geq \bar{\mu}(y \to x)$,
(FCF2) $\bar{\mu}((x \rightsquigarrow y) \to y) \rightsquigarrow x) \geq \bar{\mu}(y \rightsquigarrow x)$.

If $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$ satisfies (FCF1) and (FCF2), then we also say that $\bar{\mu}$ is *commutative.* Let $\mathsf{FCF}(\mathfrak{X})$ denote the set of all fuzzy commutative filters of a pseudo-BE algebra $\mathfrak{X}$.

Proposition 4.4 *A fuzzy set $\bar{\mu}$ in X is a fuzzy commutative filter of $\mathfrak{X}$ if and only if it satisfies the following conditions:*

(i) $\bar{\mu}(1) \geq \bar{\mu}(x)$,
(ii) $\bar{\mu}(((x \to y) \rightsquigarrow y) \to x) \geq \bar{\mu}(z \to (y \to x)) \wedge \bar{\mu}(z)$,
(iii) $\bar{\mu}(((x \rightsquigarrow y) \to y) \rightsquigarrow x) \geq \bar{\mu}(z \rightsquigarrow (y \rightsquigarrow x)) \wedge \bar{\mu}(z)$.

Proof Assume that $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$. By (FF1), condition (i) holds. Since $\bar{\mu}$ is commutative, using (FCF1) and (FF2), we have

$$\begin{aligned}\bar{\mu}(((x \to y) \rightsquigarrow y) \to x) &\geq \bar{\mu}(y \to x)\\ &\geq \bar{\mu}(z \to (y \to x)) \wedge \bar{\mu}(z).\end{aligned}$$

Similarly, $\bar{\mu}(((x \rightsquigarrow y) \to y) \rightsquigarrow x) \geq \bar{\mu}(z \rightsquigarrow (y \rightsquigarrow x)) \wedge \bar{\mu}(z)$, and so (ii) and (iii) hold.

Conversely, let $x, z \in X$ and put $y := 1$ in conditions (ii) and (iii). Applying (pBE_2) and (pBE_3), we obtain

$$\bar{\mu}(x) = \bar{\mu}(((x \to 1) \rightsquigarrow 1) \to x) \geq \bar{\mu}(z \to (1 \to x)) \wedge \bar{\mu}(z) = \bar{\mu}(z \to x) \wedge \bar{\mu}(z).$$

Hence $\bar{\mu}$ satisfies (FF2). From this and (i) we see that $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$.
To prove that $\bar{\mu}$ is commutative, set $z := 1$ in conditions (ii) and (iii). By (pBE_2) and (i), we get

$$\begin{aligned}\bar{\mu}(((x \to y) \rightsquigarrow y) \to x) &\geq \bar{\mu}(1 \to (y \to x)) \wedge \bar{\mu}(1)\\ &= \bar{\mu}(y \to x) \wedge \bar{\mu}(1)\\ &= \bar{\mu}(y \to x)\end{aligned}$$

and

$$\begin{aligned}\bar{\mu}(((x \rightsquigarrow y) \to y) \rightsquigarrow x) &\geq \bar{\mu}(1 \rightsquigarrow (y \rightsquigarrow x)) \wedge \bar{\mu}(1)\\ &= \bar{\mu}(y \rightsquigarrow x) \wedge \bar{\mu}(1)\\ &= \bar{\mu}(y \rightsquigarrow x).\end{aligned}$$

Therefore, $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$. $\square$

Theorem 4.5 *A fuzzy set $\bar{\mu}$ in X is a fuzzy commutative filter of $\mathfrak{X}$ if and only if its nonempty level subset $U(\bar{\mu},\alpha)$ is a commutative filter of $\mathfrak{X}$ for all $\alpha \in [0,1]$.*

Proof Let $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$ and $\alpha \in [0,1]$. Assume that $U(\bar{\mu},\alpha) \neq \emptyset$. From Theorem 3.7 we deduce that $U(\bar{\mu},\alpha)$ is a filter of $\mathfrak{X}$. Let $y \to x \in U(\bar{\mu},\alpha)$. Hence $\bar{\mu}(y \to x) \geq \alpha$. Since $\bar{\mu}(((x \to y) \rightsquigarrow y) \to x) \geq \bar{\mu}(y \to x) \geq \alpha$, it follows that $\bar{\mu}(((x \to y) \rightsquigarrow y) \to x) \geq \alpha$, and so $((x \to y) \rightsquigarrow y) \to x \in U(\bar{\mu},\alpha)$. Similarly, $((x \rightsquigarrow y) \to y) \rightsquigarrow x \in U(\bar{\mu},\alpha)$. Therefore, $U(\bar{\mu},\alpha) \in \mathsf{CF}(\mathfrak{X})$.

Conversely, suppose that for each $\alpha \in [0,1]$, $U(\bar{\mu},\alpha) = \emptyset$ or $U(\bar{\mu},\alpha) \in \mathsf{CF}(\mathfrak{X})$. By Theorem 3.7, $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$. Now observe that $\bar{\mu}$ satisfies (FCF1). On the contrary, assume that there are $x_0, y_0 \in X$ such that $\bar{\mu}(y_0 \to x_0) > \bar{\mu}(((x_0 \to y_0) \rightsquigarrow y_0) \to x_0)$. Set $\bar{\mu}(y_0 \to x_0) := s$ for some $s \in [0,1]$. Hence $y_0 \to x_0 \in U(\bar{\mu},s)$. Since $U(\bar{\mu},s)$ is a commutative filter, we get $((x_0 \to y_0) \rightsquigarrow y_0) \to x_0 \in U(\bar{\mu},s)$. Thus $\bar{\mu}(((x_0 \to y_0) \rightsquigarrow y_0) \to x_0) \geq \bar{\mu}(y_0 \to x_0)$. This is impossible, and so $\bar{\mu}$ satisfies (FCF1). By a similar argument, $\bar{\mu}$ also satisfies (FCF2). Consequently, $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$. □

Corollary 4.6 *A nonempty subset $F \subseteq X$ is a commutative filter of $\mathfrak{X}$ if and only if χ_F is a fuzzy commutative filter of $\mathfrak{X}$.*

Proof The proof is straightforward. □

Example 4.7 Let $\mathfrak{X}$ be the pseudo-BE algebra from Example 2.8. Since $F := \{a,b,c,1\} \in \mathsf{CF}(\mathfrak{X})$, by Corollary 4.6, χ_F is a fuzzy commutative filter of $\mathfrak{X}$.

Proposition 4.8 (Extension property) *Let $\mathfrak{X}$ be a pseudo-BE(A) algebra and let $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$. If $\overline{\nu}$ is a fuzzy filter of $\mathfrak{X}$ such that $\bar{\mu} \leq \overline{\nu}$ and $\bar{\mu}(1) = \overline{\nu}(1)$, then $\overline{\nu} \in \mathsf{FCF}(\mathfrak{X})$.*

Proof Assume that $x, y \in X$ and set $u := y \to x$. By Proposition 2.7 (ii), $y \to (u \rightsquigarrow x) = 1$. Since $\bar{\mu}$ is a commutative fuzzy filter and $\bar{\mu} \leq \overline{\nu}$, we have:

$$\begin{aligned}\overline{\nu}(1) = \bar{\mu}(1) = \bar{\mu}(y \to (u \rightsquigarrow x)) &\leq \bar{\mu}((((u \rightsquigarrow x) \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)) \\ &\leq \overline{\nu}((((u \rightsquigarrow x) \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)).\end{aligned}$$

Therefore

$$\overline{\nu}((((u \rightsquigarrow x) \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)) = \overline{\nu}(1). \tag{4.1}$$

By Proposition 2.7(i), $x \leq u \rightsquigarrow x$. Applying condition (A), we conclude that $(u \rightsquigarrow x) \to y \leq x \to y$, hence that $(x \to y) \rightsquigarrow y \leq ((u \rightsquigarrow x) \to y) \rightsquigarrow y$, finally that $(((u \rightsquigarrow x) \to y) \rightsquigarrow y) \to (u \rightsquigarrow x) \leq ((x \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)$. From (4.1) and Proposition 3.3 (i) we deduce that

$$\overline{\nu}(1) \leq \overline{\nu}(((x \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)),$$

and so $\overline{\nu}(((x \to y) \rightsquigarrow y) \to (u \rightsquigarrow x)) = \overline{\nu}(1)$. Therefore, using ($\mathrm{pBE}_5$), we see that

$$\overline{\nu}(u \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x))) = \overline{\nu}(1). \tag{4.2}$$

By (FF3), (4.2) and (FF1),

$$\begin{aligned}\overline{\nu}(((x \to y) \rightsquigarrow y) \to x) &\geq \overline{\nu}(u) \wedge \overline{\nu}(u \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x))) \\ &= \overline{\nu}(u) \wedge \overline{\nu}(1) \\ &= \overline{\nu}(u) \\ &= \overline{\nu}(y \to x).\end{aligned}$$

Thus $\overline{\nu}$ satisfies (FCF1). Similarly, $\overline{\nu}$ also satisfies (FCF2). Consequently, $\overline{\nu} \in \mathsf{FCF}(\mathfrak{X})$. □

Theorem 4.9 *If $\mathfrak{X}$ is a commutative pseudo-BE algebra, then* $\mathsf{FF}(\mathfrak{X}) = \mathsf{FCF}(\mathfrak{X})$.

Proof Since every fuzzy commutative filter is a fuzzy filter, it is sufficient to prove that $\mathsf{FF}(\mathfrak{X}) \subseteq \mathsf{FCF}(\mathfrak{X})$. Assume that $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$. By Proposition 2.7(ii) and commutativity, we have:

$$y \rightarrow x \leq ((y \rightarrow x) \rightsquigarrow x) \rightarrow x = ((x \rightarrow y) \rightsquigarrow y) \rightarrow x.$$

Hence $\bar{\mu}(y \rightarrow x) \leq \bar{\mu}(((x \rightarrow y) \rightsquigarrow y) \rightarrow x)$. Similarly, we get $\bar{\mu}(y \rightsquigarrow x) \leq \bar{\mu}(((x \rightsquigarrow y) \rightsquigarrow y) \rightsquigarrow x)$. Thus $\bar{\mu} \in \mathsf{FCF}(\mathfrak{X})$, therefore $\mathsf{FF}(\mathfrak{X}) \subseteq \mathsf{FCF}(\mathfrak{X})$, and finally $\mathsf{FF}(\mathfrak{X}) = \mathsf{FCF}(\mathfrak{X})$. $\square$

The following two theorems give the homomorphic properties of fuzzy (commutative) filters.

Theorem 4.10 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BF algebras and let $f : X \rightarrow Y$ be a homomorphism. If $\overline{\nu}$ is a fuzzy (commutative) filter of $\mathfrak{Y}$, then $f^{\leftarrow}(\overline{\nu})$ is a fuzzy (commutative) filter of $\mathfrak{X}$.*

Proof Let $x \in X$. Since $f(x) \in Y$ and $\overline{\nu} \in \mathsf{FF}(\mathfrak{Y})$, we have $\overline{\nu}(1) \geq \overline{\nu}(f(x)) = (f^{\leftarrow}(\overline{\nu}))(x)$, but $\overline{\nu}(1) = \overline{\nu}(f(1)) = (f^{\leftarrow}(\overline{\nu}))(1)$. Thus we get $(f^{\leftarrow}(\overline{\nu}))(1) \geq (f^{\leftarrow}(\overline{\nu}))(x)$ for any $x \in X$, that is, $f^{\leftarrow}(\overline{\nu})$ satisfies (FF1). Let now $x, y \in X$. Since $\overline{\nu}$ is a filter of $\mathfrak{Y}$, we obtain

$$\overline{\nu}(f(x)) \geq \overline{\nu}(f(y) \rightarrow f(x)) \wedge \overline{\nu}(f(y)) = \overline{\nu}(f(y \rightarrow x)) \wedge \overline{\nu}(f(y))$$

and hence, $f^{\leftarrow}(\overline{\nu})(x) \geq f^{\leftarrow}(\overline{\nu})(y \rightarrow x) \wedge f^{\leftarrow}(\overline{\nu})(y)$. Consequently, $f^{\leftarrow}(\overline{\nu}) \in \mathsf{FF}(\mathfrak{X})$. We now suppose that $\overline{\nu}$ satisfies (FCF1) and let $x, y \in X$. Then

$$\overline{\nu}(f(y) \rightarrow f(x)) \leq \overline{\nu}(((f(x) \rightarrow f(y)) \rightsquigarrow f(y)) \rightarrow f(x)).$$

Therefore,

$$\overline{\nu}(f(y \rightarrow x)) \leq \overline{\nu}(f(((x \rightarrow y) \rightsquigarrow y) \rightarrow x)).$$

Hence, $f^{\leftarrow}(\overline{\nu})(y \rightarrow x) \leq f^{\leftarrow}(\overline{\nu})(((x \rightarrow y) \rightsquigarrow y) \rightarrow x))$, so $f^{\leftarrow}(\overline{\nu})$ satisfies (FCF1). Similarly, if $\overline{\nu}$ satisfies (FCF2), then $f^{\leftarrow}(\overline{\nu})$ satisfies (FCF2). Thus $f^{\leftarrow}(\overline{\nu}) \in \mathsf{FCF}(\mathfrak{X})$. $\square$

Lemma 4.11 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BE algebras and let $f : X \rightarrow Y$ be a homomorphism. If $\bar{\mu} \in \mathsf{FF}(\mathfrak{X})$ and $\bar{\mu}$ is constant on* $\ker f = f^{\leftarrow}(1)$, *then* $f^{\leftarrow}(f(\bar{\mu})) = \bar{\mu}$.

Proof Let $x \in X$ and $f(x) = y$. Hence

$$(f^{\leftarrow}(f(\bar{\mu})))(x) = (f(\bar{\mu}))(f(x)) = (f(\bar{\mu}))(y) = \bigvee\{\bar{\mu}(a) : a \in f^{\leftarrow}(y)\}.$$

For all $a \in f^{\leftarrow}(y)$, we have $f(x) = f(a)$. Hence $f(x \rightarrow a) = 1$, i.e., $x \rightarrow a \in \ker f$. Thus $\bar{\mu}(x \rightarrow a) = \bar{\mu}(1)$. Therefore, $\bar{\mu}(a) \geq \bar{\mu}(x \rightarrow a) \wedge \bar{\mu}(x) = \bar{\mu}(1) \wedge \bar{\mu}(x) = \bar{\mu}(x)$. Similarly, $\bar{\mu}(x) \geq \bar{\mu}(a)$. Then $\bar{\mu}(x) = \bar{\mu}(a)$. Consequently,

$$(f^{\leftarrow}(f(\bar{\mu})))(x) = (f(\bar{\mu}))(y) = \bigvee\{\bar{\mu}(a) : a \in f^{\leftarrow}(y)\} = \bar{\mu}(x),$$

that is, $f^{\leftarrow}(f(\bar{\mu})) = \bar{\mu}$. $\square$

Theorem 4.12 *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be pseudo-BE algebras and $f : X \rightarrow Y$ be a surjective homomorphism. Let $\bar{\mu}$ be a fuzzy filter of $\mathfrak{X}$ such that $X_{\bar{\mu}} \supseteq$* $\ker f$. *Then $f(\bar{\mu})$ is a fuzzy filter of $\mathfrak{Y}$. Moreover, if $\bar{\mu}$ is commutative, then $f(\bar{\mu})$ is also commutative.*

Proof Since $\bar{\mu}$ is a fuzzy filter of $\mathfrak{X}$ and $1 \in f^{\leftarrow}(1)$, we have

$$(f(\bar{\mu}))(1) = \bigvee \{\bar{\mu}(a) : a \in f^{\leftarrow}(1)\} = \bar{\mu}(1) \geq \bar{\mu}(x)$$

for any $x \in X$. Hence

$$(f(\bar{\mu}))(1) \geq \bigvee \{\bar{\mu}(x) : x \in f^{\leftarrow}(y)\} = (f(\bar{\mu}))(y)$$

for any $y \in Y$. Thus $f(\bar{\mu})$ satisfies (FF1). Observe that $f(\bar{\mu})$ also satisfies (FF2). On the contrary, suppose that

$$f(\bar{\mu})(x_0') < f(\bar{\mu})(y_0' \to x_0') \wedge f(\bar{\mu})(y_0')$$

for some $x_0', y_0' \in Y$. Since f is surjective, there are $x_0, y_0 \in A$ such that $f(x_0) = x_0'$ and $f(y_0) = y_0'$. Hence

$$f(\bar{\mu})(f(x_0)) < f(\bar{\mu})(f(y_0 \to x_0)) \wedge f(\bar{\mu})(f(y_0)).$$

Therefore

$$f^{\leftarrow}(f(\bar{\mu}))(x_0) < f^{\leftarrow}(f(\bar{\mu}))(y_0 \to x_0)) \wedge f^{\leftarrow}(f(\bar{\mu}))(y_0)).$$

Since $X_{\bar{\mu}} \supseteq \ker f$, we conclude that $\bar{\mu}$ is constant on $\ker f$. Hence, by Lemma 4.11, we get

$$\bar{\mu}(x_0) < \bar{\mu}(y_0 \to x_0) \wedge \bar{\mu}(y_0),$$

which is a contradiction with a fact that $\bar{\mu}$ is a filter. Therefore, $f(\bar{\mu})$ satisfies (FF2). Thus, $f(\bar{\mu}) \in \mathsf{FF}(\mathfrak{Y})$.

Assume now that $\bar{\mu}$ is commutative. To prove that $f(\bar{\mu})$ satisfies (FCF1), suppose on the contrary that

$$f(\bar{\mu})(y' \to x') > f(\bar{\mu})(((x' \to y') \rightsquigarrow y') \to x')$$

for some $x', y' \in Y$. Since f is surjective, there are $x, y \in X$ such that $f(x) = x'$ and $f(y) = y'$. We have

$$f(\bar{\mu})(f(y \to x) > f(\bar{\mu})(f(((x \to y) \rightsquigarrow y) \to x)).$$

Hence

$$f^{\leftarrow}(f(\bar{\mu}))(y \to x) > f^{\leftarrow}(f(\bar{\mu}))(((x \to y) \rightsquigarrow y) \to x).$$

From Lemma 4.11 we conclude that $\bar{\mu}(y \to x) > \bar{\mu}(((x \to y) \rightsquigarrow y) \to x)$, a contradiction. Consequently, $f(\bar{\mu})$ satisfies (FCF1). Similarly, $f(\bar{\mu})$ satisfies (FCF2). Thus $f(\bar{\mu}) \in \mathsf{FCF}(\mathfrak{Y})$. □

5 Conclusion and future research

This paper begins by considering the notion of fuzzy filter in pseudo-BE algebras (these algebras are a non-commutative extension of BE algebras and a generalization of pseudo-BCK algebras). For the general development of pseudo-BE algebras fuzzy filter theory plays an important role (see, for example, Theorems 3.14 and 4.5).

Various characterizations of fuzzy filters were given and conditions for a fuzzy set to be a fuzzy filter were provided. It was proved that the set of all fuzzy filters of a pseudo-BE algebra is a complete lattice. Next, some characterizations of Noetherian pseudo-BE algebras by fuzzy filters were obtained. Moreover, we have introduced the notion of fuzzy commutative filter of pseudo-BE algebras and derived its basic properties. Finally, we have given the relationships between fuzzy filters and fuzzy commutative filters of a pseudo-BE algebra and also provided the homomorphic properties of fuzzy (commutative) filters.

The next step in studying fuzzy filters in pseudo-BE algebras may be introducing and investigating the notions of fuzzy maximal filter and fuzzy prime filter. We shall also study fuzzy congruence relations on pseudo-BE algebras.

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