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A survey on new methods for partial functional differential equations and applications

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Abstract
This work is a survey of many papers dealing with new methods to study partial functional differential equations. We propose a new reduction method of the complexity of partial functional differential equations and its applications. Since, any partial functional differential equation is well-posed in a infinite dimensional space, this presents many difficulties to study the qualitative analysis of the solutions. Here, we propose to reduce the dimension from infinite to finite. We suppose that the undelayed part is not necessarily densely defined and satisfies the Hille–Yosida condition. The delayed part is continuous. We prove the dynamic of solutions are obtained through an ordinary differential equations that is well-posed in a finite dimensional space. The powerty of this results is used to show the existence of almost automorphic solutions for partial functional differential equations. For illustration, we provide an application to the Lotka–Volterra model with diffusion and delay.

Keywords Partial functional differential equations · Variation of constants formula · Reduction of complexity · Almost automorphic solutions

Mathematics Subject Classification 34K08 · 34K14 · 34K40 · 47A10 · 47D06

1 Introduction

This work is a survey of many works that I have been done before on reduction methods for partial functional differential equations with finite and infinite delay. Firstly, we establish variations of constants formulas for partial functional differential equations in finite and infinite delays, this formula will play a crucial role to develop the reduction of complexity from infinite dimension to finite dimension. We propose an application to prove the existence of almost automorphic solutions using classical theorems on ordinary differential equations.

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Here, we are concerned with the following partial functional differential equation with finite delay

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq 0, \\ u_0 = \varphi \in C := C([-r, 0], X), \end{cases} \tag{1.1}$$

where A is a linear operator on a Banach space X not necessarily densely defined and satisfies the Hille–Yosida condition: there exist $M \geq 0$, $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$, and

$$\left| R(\lambda, A)^n \right| \leq \frac{M}{(\lambda - \omega)^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda - A)^{-1}$, C is the space of continuous functions from $[-r, 0]$ to the observable X endowed with the uniform norm topology. L is a bounded linear operator from C into X and f is an almost automorphic function from $\mathbb{R}$ to X, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t + \theta) \text{ for } \theta \in [-r, 0].$$

As an example of Eq. (1.1), we propose the following model arising in many problems in population dynamics and physical systems

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + \displaystyle\int_{-r}^{0} G(\theta)v(t+\theta, x)d\theta + h(t,x) \text{ for } t \geq 0 \text{ and } x \in [0, \pi], \\ u(t,x) = 0 \text{ for } x = 0, \pi \text{ and } t \geq 0, \\ \\ u(\theta, x) = \varphi(\theta, x) \text{ for } \theta \in [-r, 0] \text{ and } x \in [0, \pi], \end{cases}$$

We use the reduction of complexity to prove that the existence of almost automorphic solution of Eq. (1.1) is equivalent to the existence of a bounded solution on $\mathbb{R}^+$. To achieve this goal, we use the variation of constants formula obtained in [1] and we develop new fundamental results about the spectral decomposition of solutions.

Recall that partial functional differential equations are an important area of research in applied mathematics, since many phenomenons in physical and biological systems are modeled using the history of the system. Then a system using delay is well-posed in infinite dimensional spaces and many classical results in differential equations well-posed in finite dimensional spaces cannot be applied. The aim of this chapter is to reduce the complexity of partial functional differential equations. We prove the existence of an ordinary differential equation that is well-posed in finite dimensional spaces and give all the fundamental properties on the qualitative analysis for the whole partial functional differential equations. Recall that the theory of partial functional differential equations was initiated in [2], for more details we refer to the book [3].

Almost automorphic functions are more general than almost periodic functions and they were introduced by Bochner [4], for more details about this topics we refer to the recent book [5] where the author give an important overview about the theory of almost automorphic functions and their applications to differential equations. The existence of almost automorphic solutions for differential equations in infinite dimensional space has been studied by several authors. For example in [6], the author studied the existence of almost automorphic solutions for the following semilinear abstract differential equation

$$\frac{d}{dt}x(t) = \mathcal{C}x(t) + \theta(t) \text{ for } t \geq 0, \tag{1.2}$$

where $\mathcal{C}$ generates an exponentially stable semigroup on a Banach space Y and θ is an almost automorphic function from $\mathbb{R}$ to Y. The author proved that the only bounded mild solution of Eq. (1.2) on $\mathbb{R}$ is almost automorphic.

Recently in [7], the authors studied the existence of almost automorphic solutions for the following partial functional differential equations with infinite delay

$$\begin{cases} \dfrac{dx}{dt}(t) = \mathcal{D}x(t) + \mathcal{L}(t)x_t + \mathcal{K}(t) \text{ for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.3}$$

where $\mathcal{D}$ is the generator of a strongly continuous semigroup of linear operators on a Banach space E which is equivalent by Hille–Yosida's theorem that $\mathcal{D}$ satisfies the Hille–Yosida condition and $\overline{D(\mathcal{D})} = E$. The phase space $\mathcal{B}$ is a linear space of functions mapping $(-\infty, 0]$ into E satisfying some axioms introduced by Hale and Kato [7], for all $t \geq 0$, $\mathcal{L}(t)$ is a bounded linear operator form $\mathcal{B}$ to E and periodic in t. For every $t \geq 0$, the history function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \leq 0.$$

The function $\mathcal{K}$ is an almost automorphic function from $\mathbb{R}$ to E. The authors proved that the existence of a bounded mild solution on $\mathbb{R}^+$ of Eq. (1.3) is equivalent to the existence of an almost automorphic solution.

2 Variation of constants formula for partial functional differential equations

Throughout this chapter, we suppose that
($\mathbf{H_0}$) A satisfies the Hille–Yosida condition.
We consider the following definition and results which are taken from [8].

Definition 2.1 [8] We say that a continuous function u from $[-r, \infty)$ into X is an integral solution of Eq. (1.1), if the following conditions hold

(i) $\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,
(ii) $u(t) = \varphi(0) + A\int_0^t u(s)ds + \int_0^t [L(u_s) + f(s)]\,ds$ for $t \geq 0$,
(iii) $u_0 = \varphi$.

If $\overline{D(A)} = X$, the integral solutions coincide with the known mild solutions. We can see that if u is an integral solution of Eq. (1.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\} \\ A_0 x = Ax \text{ for } x \in D(A_0). \end{cases}$$

Lemma 2.2 [9, Lemma 3.3.12, p. 140] *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

For the existence of the integral solutions, one has the following result.

Theorem 2.3 [8] *Assume that ($\mathbf{H_0}$) holds, then for all $\varphi \in C$ such that $\varphi(0) \in \overline{D(A)}$, Eq. (1.1) has a unique integral solution u on $[-r, +\infty)$. Moreover u is given by*

$$u(t) = T_0(t)\varphi(0) + \lim_{\lambda\to+\infty} \int_0^t T_0(t-s)B_\lambda[L(u_s) + f(s)]ds \textit{ for } t \geq 0,$$

where $B_\lambda = \lambda R(\lambda, A)$ *for* $\lambda > \omega$.

In the sequel of this work, we call integral solutions as solutions
Let C_0 be the phase space of Eq. (1.1):

$$C_0 = \left\{\varphi \in C : \varphi(0) \in \overline{D(A)}\right\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t)\varphi = v_t(\cdot, \varphi),$$

where $v(\cdot, \varphi)$ is the solution of the following linear equation

$$\begin{cases} \dfrac{d}{dt}v(t) = Ay(t) + L(y_t) & \text{for } t \geq 0, \\ v_0 = \varphi \in C, \end{cases}$$

Proposition 2.4 [8] *The family* $(\mathcal{U}(t))_{t\geq 0}$ *is a strongly continuous semigroup of linear operators on* C_0*:*

(i) *for all* $t \geq 0$*,* $\mathcal{U}(t)$ *is a bounded linear operator on* C_0,
(ii) $\mathcal{U}(0) = I$*,*
(iii) $\mathcal{U}(t + s) = \mathcal{U}(t)\mathcal{U}(s$*, for all* $t, s \geq 0$,
(iv) *for all* $\varphi \in C_0$*,* $\mathcal{U}(t)\varphi$ *is a continuous function of* $t \geq 0$ *with values in* C_0*. Moreover,*
(v) $(\mathcal{U}(t))_{t\geq 0}$ *satisfies, for* $t \geq 0$ *and* $\theta \in [-r, 0]$*, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) & \text{if } t + \theta \geq 0 \\ \\ \varphi(t+\theta) & \text{if } t + \theta \leq 0. \end{cases}$$

Theorem 2.5 [1, Theorem 3] *Let* $\mathcal{A}_u$ *be defined on* C_0 *by*

$$\begin{cases} D(\mathcal{A}_u) = \left\{\varphi \in C^1([-r, 0]; X) : \varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi)\right\} \\ \\ \mathcal{A}_u\varphi = \varphi' \text{ for } \varphi \in D(\mathcal{A}_u). \end{cases}$$

Then $\mathcal{A}_u$ *is the infinitesimal generator of the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *on* C_0.

In order to give a variation of constants formula, we need to recall some notations and results which are taken from [1]. Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle = \{X_0 c : c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ c & \text{if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0\rangle$ is equipped with the norm $\|\phi + X_0 c\| = |\phi|_C + |c|$ for $(\phi, c) \in C_0 \times X$, is a Banach space and consider the extension $\widetilde{\mathcal{A}}_{\mathcal{U}}$ of the operator $\mathcal{A}_u$ defined on $C_0 \oplus \langle X_0\rangle$ by

$$\begin{cases} D\left(\widetilde{\mathcal{A}}_{\mathcal{U}}\right) = \left\{\varphi \in C^1([-r, 0]; X) : \varphi(0) \in D(A) \text{ and } \varphi'(0) \in \overline{D(A)}\right\}, \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi \quad = \varphi' + X_0\left(A\varphi(0) + L\varphi - \varphi'(0)\right). \end{cases}$$

Lemma 2.6 [1, Theorem 13 and Lemma 15] *Assume that* $(\mathbf{H_0})$. *Then* $\widetilde{\mathcal{A}}_{\mathcal{U}}$ *satisfies the Hille–Yosida condition on* $C_0 \oplus \langle X_0 \rangle$*: there exists* $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ *such that* $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}}_{\mathcal{U}})$, *and*

$$\left|R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}})^n\right| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } n \in \mathbb{N},\ \lambda > \widetilde{\omega}$$

where $R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}}) = (\lambda - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-1}$. *Moreover, the part of* $\widetilde{\mathcal{A}}_{\mathcal{U}}$ *on* $\overline{D\left(\widetilde{\mathcal{A}}_{\mathcal{U}}\right)} = C_0$ *is exactly the operator* $\mathcal{A}_u$.

Theorem 2.7 [1, Theorem 16] *Assume that* $(\mathbf{H_0})$ *holds. Then for all* $\varphi \in C_0$, *the solution* u *of Eq.* (1.1) *is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t)\,\varphi + \lim_{\lambda \to +\infty} \int_0^t \mathcal{U}(t-s)\, \widetilde{B}_\lambda \left(X_0 f(s)\right) ds \text{ for } t \geq 0,$$

where $\widetilde{B}_\lambda = \lambda R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}})$ *for* $\lambda > \widetilde{\omega}$.

In the next, we establish a new variation of constants formula for the following partial functional differential equation with infinite delay

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + Lx_t + f(t) \text{ for } t \geq 0, \\ x_0 = \phi \in \mathcal{B}, \end{cases} \tag{2.1}$$

where $A : D(A) \to X$ is a nondensely defined linear operator on a complex Banach space $(X, |\cdot|)$, $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ into X and satisfying some fundamental Axioms, x_t is an element of $\mathcal{B}$ defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \in (-\infty, 0],$$

L is a bounded linear operator from $\mathcal{B}$ into X, and f is a continuous X-valued function on $\mathbb{R}^+$. We assume that A is a Hille–Yosida operator.

We employ an axiomatic definition of the phase space $\mathcal{B}$ which has been introduced at first by Hale and Kato [10]. In the following, we assume that $\mathcal{B}$ is a normed space of functions mapping $]-\infty, 0]$ into X satisfying the following fundamental axioms:
$(\mathbf{A})$: There exist a positive constant N, a locally bounded functions $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty[$, such that if $x :]-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,
(ii) $t \mapsto x_t$ is continuous with respect to the norm of $\mathcal{B}$ on $[\sigma, a]$,
(iii) $N|x(t)| \leq |x_t| \leq K(t-\sigma)\sup_{\sigma \leq s \leq t}|x(s)| + M(t-\sigma)|x_\sigma|$.

$(\mathbf{B})$: $\mathcal{B}$ is a Banach space.
We assume that
$(\mathbf{D}_1)$: if $(\phi_n)_{n\geq 0}$ is a sequence in $\mathcal{B}$ such that $\phi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then for all $\theta \leq 0$, $(\phi_n(\theta))_{n\geq 0}$ converges to 0 in X.
Let $C(]-\infty, 0], X)$ be the space of continuous functions from $]-\infty, 0]$ into X. We make the following assumptions:
$(\mathbf{D}_2)$: $\mathcal{B} \subset C(]-\infty, 0], X)$,
$(\mathbf{D}_3)$: there exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \lambda_0$ and $x \in X$, we have that $e^{\lambda \cdot}x \in \mathcal{B}$ and

$$K_0 := \sup_{\substack{\operatorname{Re}\lambda > \lambda_0,\, x \in X \\ x \neq 0}} \frac{\left|e^{\lambda \cdot}x\right|}{|x|} < \infty,$$

where $\left(e^{\lambda\cdot}x\right)(\theta) = e^{\lambda\theta}x$ for $\theta \in]-\infty, 0]$ and $x \in$ X.
The following results are taken from [11].

Definition 2.8 [11] A function $u : \mathbb{R} \to X$ is called an integral solution of Eq. (2.1) on $\mathbb{R}^+$ if the following conditions hold

(i) u is continuous on $\mathbb{R}^+$,
(ii) $u_0 = \phi$,
(iii) $\int_0^t u(s)\,ds \in D(A)$ for $t \geq 0$,
(iv) $u(t) = \phi(0) + A\int_0^t u(s)\,ds + \int_0^t Lu_s ds + \int_0^t f(s)\,ds$ for $t \geq 0$.

If the operator A is densely defined, then the integral solution coincides with the mild solution given in [12].

Theorem 2.9 [11, p. 336] *Assume that $\mathcal{B}$ satisfies* (**A**) *and* (**B**). *Then for all $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A)}$, Eq.* (2.1) *has a unique integral solution $u(\cdot, \phi, L, f)$ on $\mathbb{R}^+$ given by*

$$u(t) = \begin{cases} T_0(t)\phi(0) + \lim_{\lambda \to +\infty} \int_0^t T_0(t-s)\lambda R(\lambda, A)[Lu_s + f(s)]\,ds \textit{ for } t \geq 0, \\ \phi(t) \textit{ for } \quad t \leq 0. \end{cases}$$

A continuous function u on $\mathbb{R}$ is said to be an integral solution of Eq. (2.1) on $\mathbb{R}$ if $u_s \in \mathcal{B}$ for $s \in \mathbb{R}$ and

$$u(t) = T_0(t-\sigma)u(\sigma) + \lim_{\lambda \to +\infty} \int_\sigma^t T_0(t-s)\lambda R(\lambda, A)[Lu_s + f(s)]\,ds \text{ for any } t \geq \sigma.$$

Let $\mathcal{B}_A := \left\{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\right\}$ be the phase space corresponding to Eq. (2.1). Define $U(t)$ for $t \geq 0$ by

$$U(t)\phi = u_t(\cdot, \phi, L) \;\text{ for } \phi \in \mathcal{B}_A,$$

where $u(\cdot, \phi, L)$ is the integral solution of Eq. (2.1) with $f = 0$.

Proposition 2.10 [11, Proposition 2] *$(U(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{B}_A$, that's*

(i) *$U(0) = Id$,*
(ii) *$U(t+s) = U(t)U(s)$ for $t, s \geq 0$,*
(iii) *for all $\phi \in \mathcal{B}_A$, $t \mapsto U(t)\phi$ is continuous.*

Moreover $(U(t))_{t\,0}$ satisfies the translation property

$$(U(t)\phi)(\theta) = \begin{cases} U(t+\theta)\phi(0) \textit{ for } t+\theta \geq 0 \\ \\ \phi(t+\theta) \textit{ for } t+\theta \leq 0. \end{cases}$$

In order to establish a new variation of constant formula, we follow the same approach used in [1]. Before we need to recall the following results.

Lemma 2.11 [11, Proposition 5] *Let $\mathcal{B}$ satisfy Axioms* (**A**), (**B**), (**D_1**) *and* (**D_2**). *Then the infinitesimal generator A_U of $(U(t))_{t\geq 0}$ is given by:*

$$\begin{cases} D(A_U) = \left\{ \begin{array}{c} \phi \in C^1(]-\infty, 0], X) \cap \mathcal{B}_A : \phi' \in \mathcal{B}_A, \phi(0) \in D(A) \; and \\ \phi'(0) = A\phi(0) + L(\phi) \end{array} \right\}, \\ A_U\phi \quad = \phi'. \end{cases}$$

By Axiom ($\mathbf{D}_3$), we define for each complex number λ such that $\mathcal{R}e(\lambda) > \lambda_0$, the linear operator $\triangle(\lambda) : D(A) \to X$ by

$$\triangle(\lambda) = \lambda \mathrm{I} - A - L\left(e^{\lambda \cdot}\mathrm{I}\right).$$

Consider the space $\mathfrak{X} := \mathcal{B}_A \oplus \langle X_0 \rangle$, where $\langle X_0 \rangle = \{X_0 x : x \in X\}$ and $X_0 x$ is a function defined by

$$(X_0 x)(\theta) = \begin{cases} 0 \text{ if } \theta \in]-\infty, 0[, \\ x \text{ if } \theta = 0. \end{cases}$$

Then $\mathfrak{X}$ endowed with the norm $\|\phi + X_0 x\| = \|\phi\| + |x|$ is a Banach space.

Theorem 2.12 *Assume that $\mathcal{B}$ satisfies Axioms* (**A**), (**B**), ($\boldsymbol{D}_1$), ($\boldsymbol{D}_2$) *and* ($\boldsymbol{D}_3$). *Then the extension $\widetilde{A_U}$ of the operator A_U defined on $\mathfrak{X}$ by*

$$\begin{cases} D\left(\widetilde{A_U}\right) = \left\{\phi \in \mathcal{B}_A : \phi' \in \mathcal{B}_A, \text{ and } \phi(0) \in D(A)\right\}, \\ \widetilde{A_U}\phi \quad = \phi' + X_0\left(A\phi(0) + L\phi - \phi'(0)\right), \end{cases}$$

is a Hille–Yosida operator on $\mathfrak{X}$.

For the proof we need the following fundamental lemma.

Lemma 2.13 *There exist $\omega_1 > \lambda_0$ and $M_1 \in \mathbb{R}$ such that for $\lambda > \omega_1$ we have*

(i) $\triangle(\lambda)$ *is invertible and* $\left|\triangle(\lambda)^{-1}\right| \leq \frac{M_0}{\lambda - \omega_1}$.
(ii) $D\left(\widetilde{A_U}\right) = D(A_U) \oplus \left\langle e^{\lambda \cdot}\right\rangle$, *where*

$$\left\langle e^{\lambda \cdot}\right\rangle = \left\{e^{\lambda \cdot} x : x \in D(A)\right\}.$$

(iii) $\lambda \in \rho\left(\widetilde{A_U}\right)$, *and for $n \in \mathbb{N}^*$, $(\phi, x) \in \mathcal{B}_A \times X$, one has*

$$R\left(\lambda, \widetilde{A_U}\right)^n (\phi + X_0 x) = R(\lambda, A_U)^n \phi + R(\lambda, A_U)^{n-1}\left(e^{\lambda \cdot} \triangle(\lambda)^{-1} x\right).$$

Proof of the lemma *a*) For $\lambda > \overline{\omega} := \max\{0, \omega_0, \lambda_0\}$, one has

$$\triangle(\lambda) = \lambda \mathrm{I} - A - L\left(e^{\lambda \cdot}\mathrm{I}\right) = (\lambda \mathrm{I} - A)\left(\mathrm{I} - R(\lambda, A) L\left(e^{\lambda \cdot}\mathrm{I}\right)\right),$$

and

$$\left|R(\lambda, A) L\left(e^{\lambda \cdot} x\right)\right| \leq \frac{M_0 |L|}{\lambda - \omega_0}\left|e^{\lambda \cdot} x\right| \leq \frac{M_0 K_0 |L|}{\lambda - \omega_0}|x| \text{ for } x \in X.$$

Consequently

$$\left|R(\lambda, A) L\left(e^{\lambda \cdot}\mathrm{I}\right)\right| \leq \frac{\overline{M}}{\lambda - \omega_0} < 1 \quad \text{for all } \lambda > \omega_1 := \overline{\omega} + \overline{M},$$

where $\overline{M} := M_0 K_0 |L|$. We conclude that the operator $\left(\mathrm{I} - R(\lambda, A) L\left(e^{\lambda \cdot}\mathrm{I}\right)\right)$ is invertible, and

$$\left|\left(\mathrm{I} - R(\lambda, A) L\left(e^{\lambda \cdot}\mathrm{I}\right)\right)^{-1}\right| \leq \frac{1}{1 - \left|R(\lambda, A) L\left(e^{\lambda \cdot}\mathrm{I}\right)\right|} \leq \frac{\lambda - \omega_0}{\lambda - \omega_0 - \overline{M}}.$$

Consequently, $\triangle(\lambda)$ is invertible for $\lambda > \omega_1$ and

$$|\triangle(\lambda)^{-1}| \leq \frac{M_0}{\lambda - \bar{\omega}}.$$

b) Let $\lambda > \omega_1$ and $\left(e^{\lambda\cdot}x\right) \in D(A_U) \cap \langle e^{\lambda\cdot}\rangle$. Then $\lambda x = Ax + L\left(e^{\lambda\cdot}x\right)$, that is

$$\Delta(\lambda)x = 0.$$

Since $\triangle(\lambda)$ is invertible for $\lambda > \omega_1$, we conclude that $D(A_U) \cap \langle e^{\lambda\cdot}\rangle = \{0\}$. On the other hand, let $\tilde{\psi} \in D\left(\widetilde{A_U}\right)$ and ψ given by

$$\psi = \tilde{\psi} + e^{\lambda\cdot} \triangle(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right).$$

Then

$$\begin{aligned} A\psi(0) + L\psi &= A\tilde{\psi}(0) + L\tilde{\psi} + A\Delta(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right) \\ &\quad + L\left(e^{\lambda\cdot}\Delta(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right)\right) \\ &= A\tilde{\psi}(0) + L\tilde{\psi} - \Delta(\lambda)\Delta(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right) \\ &\quad + \lambda\Delta(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right) \\ &= \tilde{\psi}'(0) + \lambda\Delta(\lambda)^{-1}\left(A\tilde{\psi}(0) + L\tilde{\psi} - \tilde{\psi}'(0)\right) \\ &= \psi'(0). \end{aligned}$$

Hence $\psi \in D(A_U)$, which implies that $D\left(\widetilde{A_U}\right) = D(A_U) \oplus \langle e^{\lambda\cdot}\rangle$.
c) Let $\lambda > \omega_1$ and $\tilde{\psi} \in \mathfrak{X}$. Then $\tilde{\psi} = \psi + X_0 x$ for some $\psi \in \mathcal{B}_A$ and $x \in X$. We seek for $\tilde{\phi} = \phi + e^{\lambda\cdot}a \in D\left(\widetilde{A_U}\right)$ such that $\left(\lambda I - \widetilde{A_U}\right)\tilde{\phi} = \tilde{\psi}$, where $\phi \in D(A_U)$ and $a \in D(A)$. We have $\left(\lambda I - \widetilde{A_U}\right)\left(\phi + e^{\lambda\cdot}a\right) = \psi + X_0 x$, which is equivalent to find $(a, \phi) \in D(A) \times D(A_U)$ such that

$$\begin{cases} (\lambda I - A_U)\phi = \psi, \\ \triangle(\lambda)a \qquad = x. \end{cases}$$

For ω_1 large enough, it follows that, $\left(\lambda I - \widetilde{A_U}\right)^{-1}$ exists for $\lambda > \omega_1$, and

$$\left(\lambda I - \widetilde{A_U}\right)^{-1}(\psi + X_0 x) = (\lambda I - A_U)^{-1}\psi + e^{\lambda\cdot}\triangle(\lambda)^{-1}x.$$

Consequently, for $n \in \mathbb{N}^*$, we have

$$R\left(\lambda, \widetilde{A_U}\right)^n(\psi + X_0 x) = R(\lambda, A_U)^n\psi + R(\lambda, A_U)^{n-1}\left(e^{\lambda\cdot}\triangle(\lambda)^{-1}x\right).$$

□

Proof of Theorem 2.12 Since A_U is the generator of the semigroup $(U(t)_{t\geq 0})$ on $\mathcal{B}_A$, by Hille and Yosida's Theorem [13] there exists a positive constant $\tilde{M}$ such that

$$\sup_{n\in\mathbb{N},\ \lambda>\omega_1}\left|(\lambda - \omega_1)^n R(\lambda, A_U)^n\right| \leq \tilde{M}.$$

By Lemma 2.13, there exist ω_1 and $M_1 > 0$ such that

$$\sup_{n\in\mathbb{N},\ \lambda>\omega_1}\left|(\lambda - \omega_1)^n R\left(\lambda, \widetilde{A_U}\right)^n\right| \leq M_1.$$

□

Lemma 2.14 *The part of* $\widetilde{A_U}$ *in* $\overline{D\left(\widetilde{A_U}\right)}$ *is the operator* A_U.

Proof From Lemma 2.11, the operator A_u generates a strongly continuous semigroup on $\mathcal{B}_A$, by Hille and Yosida's Theorem $\overline{D(A_U)} = \mathcal{B}_A$. Since, $D(A_U) \subset D\left(\widetilde{A_U}\right) \subset \mathcal{B}_A$, then

$$\overline{D(A_U)} = \overline{D\left(\widetilde{A_U}\right)} = \mathcal{B}_A.$$

Let C be the part of $\widetilde{A_U}$ in $\overline{D\left(\widetilde{A_U}\right)}$, which is defined by

$$\begin{cases} D(C) = \left\{\phi \in D\left(\widetilde{A_U}\right) : \widetilde{A_U}\phi \in \mathcal{B}_A\right\}, \\ C\phi \quad = \widetilde{A_U}\phi. \end{cases}$$

Then $D(A_U) \subseteq D(C)$ and $A_U\phi = C\phi$ for all $\phi \in D(A_U)$.
Conversely, let $\phi \in D(C)$. Then

$$\begin{cases} \phi \in C^1(]-\infty, 0], X) \cap \mathcal{B}_A,\ \phi' \in \mathcal{B}_A, \phi(0) \in D(A) \\ \phi' + X_0\left(A\phi(0) + L\phi - \phi'(0)\right) \in \mathcal{B}_A. \end{cases}$$

By assumption ($\mathbf{D}_2$), it follows that

$$\begin{cases} \phi \in D\left(\widetilde{A_U}\right) \text{ and } \phi'(0) = A\phi(0) + L\phi \\ C\phi = \phi'. \end{cases}$$

From which we conclude that $C = A_U$. □

Consider the following evolution equation

$$\begin{cases} \dfrac{d}{dt}\xi(t) = \widetilde{A_U}\xi(t) + X_0 f(t) \text{ for } t \geq 0 \\ \xi(0) \quad = \widetilde{\phi} \in \mathfrak{X}. \end{cases} \tag{2.2}$$

Definition 2.15 A continuous function $\xi : [0, +\infty[\to \mathcal{B}_A$ is called an integral solution of Eq. (2.2) if

(i) $\displaystyle\int_0^t \xi(s)\,ds \in D\left(\widetilde{A_U}\right)$ for $t \geq 0$,

(ii) $\displaystyle\xi(t) = \widetilde{\phi} + \widetilde{A_U}\int_0^t \xi(s)\,ds + \int_0^t X_0 f(s)\,ds$ for $t \geq 0$.

Theorem 2.16 *Assume that* ($\boldsymbol{D}_1$), ($\boldsymbol{D}_2$) *and* ($\boldsymbol{D}_3$) *hold. If u is an integral solution of Eq.* (2.1)*, then the function given by $\xi(t) = u_t,\ t \geq 0$, is an integral solution of Eq.* (2.2) *for $\widetilde{\phi} = \phi$. Conversely, if ξ is an integral solution of Eq.* (2.2) *with $\widetilde{\phi} = \phi$, then the function u defined by*

$$u(t) = \begin{cases} \xi(t)(0) & if\ t \geq 0 \\ \phi(t) & if\ t \leq 0 \end{cases}$$

is an integral solution of Eq. (2.1).

Proof Let $\phi \in \mathcal{B}_A$ and u be the integral solution of Eq. (2.1). Define $\xi : [0, \infty) \to \mathcal{B}_A$ by

$$\xi(t) = u_t \text{ for } t \geq 0.$$

To compute the integral in $\mathcal{B}$ in term of the integral in X, we need the following lemma.

Lemma 2.17 [11] *Assume that* ($\mathbf{D}_1$) *holds, and $F : [0, a] \to \mathcal{B}$ is continuous, then*

$$\left(\int_0^a F(s)\,ds\right)(\theta) = \int_0^a F(s)(\theta)\,ds \ \textit{ for all } \theta \leq 0.$$

By Lemma 2.17, we have

$$\begin{aligned}\frac{d}{d\theta}\left(\int_0^t u_s ds\right)(\theta) &= \frac{d}{d\theta}\left(\int_0^t u(s+\theta)\,ds\right)\\ &= \frac{d}{d\theta}\left(\int_\theta^{t+\theta} u(s)\,ds\right)\\ &= u_t(\theta) - \phi(\theta).\end{aligned}$$

Then

$$\widetilde{A_U}\left(\int_0^t \xi(s)\,ds\right) = u_t - \phi + X_0\left(A\int_0^t u(s)\,ds + L\left(\int_0^t u_s ds\right) - u(t) - \phi(0)\right).$$

Since u is an integral solution of Eq. (2.1), it follows that

$$u(t) = \phi(0) + A\int_0^t u(s)\,ds + L\left(\int_0^t u_s ds\right) + \int_0^t f(s)\,ds,$$

which implies that

$$\xi(t) = \phi + \widetilde{A_U}\int_0^t \xi(s)\,ds + X_0\int_0^t f(s)\,ds \quad \text{for } t \geq 0.$$

Consequently ξ is an integral solution of Eq. (2.2). Conversely, let ξ be an integral solution of Eq. (2.2) for $\widetilde{\phi} = \phi$. Then ξ satisfies the following translation property

$$\xi(t)(\theta) = \begin{cases} \xi(t+\theta)(0) & \text{if } t+\theta \geq 0,\\ \phi(t+\theta) & \text{if } t+\theta \leq 0,\end{cases}$$

In fact, for $t+\theta \geq 0$,

$$\xi(t)(\theta) = (U(t)\phi)(\theta) + \lim_{\lambda\to+\infty}\int_0^t \left(U(t-s)\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(\theta)\,ds.$$

Then

$$\begin{aligned}\xi(t)(\theta) = (U(t+\theta)\phi)(0) &+ \lim_{\lambda\to+\infty}\int_0^{t+\theta}\left(U(t+\theta-s)\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(0)\,ds\\ &+ \lim_{\lambda\to+\infty}\int_{t+\theta}^t \left(U(t-s)\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(\theta)\,ds.\end{aligned}$$

Since

$$\begin{aligned}&\lim_{\lambda\to+\infty}\int_{t+\theta}^t \left(U(t-s)\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(\theta)\,ds\\ &= \lim_{\lambda\to+\infty}\int_{t+\theta}^t \left(\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(t-s+\theta)\,ds\\ &= \lim_{\lambda\to+\infty}\int_{t+\theta}^t e^{\lambda(t-s+\theta)}\lambda\,\triangle(\lambda)^{-1} f(s)\,ds = 0.\end{aligned}$$

which gives that

$$\begin{aligned}\xi(t)(\theta) &= (U(t+\theta)\phi)(0) + \lim_{\lambda\to+\infty}\int_0^{t+\theta}\left(U(t+\theta-s)\lambda R\left(\lambda, \widetilde{A_U}\right)X_0 f(s)\right)(0)\,ds\\ &= \xi(t+\theta)(0).\end{aligned}$$

If we consider the function

$$u(t) = \begin{cases} \xi(t)(0) & \text{if } t > 0, \\ \phi(t) & \text{if } t \le 0. \end{cases}$$

Then $\xi(t) = u_t$ for all t 0 and

$$u_t = \phi + \widetilde{A_U}\left(\int_0^t u_s ds\right) + \int_0^t X_0 f(s) ds \text{ for } t \ge 0.$$

Which implies that u is an integral solution of Eq. (2.1). □

Theorem 2.18 *Assume that* $(\boldsymbol{D}_1)$, $(\boldsymbol{D}_2)$ *and* $(\boldsymbol{D}_3)$ *hold. Then the integral solution* x *of Eq.* (2.1) *is given by the following variation of constants formula*

$$x_t = U(t)\phi + \lim_{n\to+\infty}\int_0^t U(t-s)\widetilde{B_n}(X_0 f(s)) ds \ \ for\ t \ge 0, \tag{2.3}$$

where $\widetilde{B_n} = n\left(n - \widetilde{A_u}\right)^{-1}$.

Proof This theorem is a consequence from Theorem 2.16 and the following lemma.

Lemma 2.19 [14] *Let* C *be a Hille–Yosida operator on a Banach space* Y *and* $\alpha : \mathbb{R}^+ \to Y$ *be a continuous function. Consider the following problem*

$$\begin{cases} \frac{d}{dt}x(t) = Cx(t) + \alpha(t) \ \ for \ \ t \ge 0, \\ x(0) \ \ = x_0 \in Y. \end{cases}$$

If $x_0 \in \overline{D(C)}$, *then there exists a unique continuous function* x *such that*

(i) $\int_0^t x(s) ds \in D(C)$ *for* $t \ge 0$

(ii) $x(t) = x_0 + C\int_0^t x(s) ds + \int_0^t \alpha(s) ds$ *for* $t \ge 0$.

Moreover, x *is given by*

$$x(t) = S_0(t)x_0 + \lim_{\lambda\to+\infty}\int_0^t S_0(t-s) C_\lambda \alpha(s) ds \ for\ t \ge 0,$$

where $C_\lambda := \lambda(\lambda I - C)^{-1}$ *and* $(S_0(t))_{t\ge 0}$ *is the semigroup generated by the part of* C *in* $\overline{D(C)}$.

3 Reduction of complexity for partial functional differential equations with finite delay

In the following, we assume that:
$(\mathbf{H_1})$ The operator $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

Theorem 3.1 *Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H_1})$ *hold, then* $\mathcal{U}(t)$ *is compact for* $t > r$.

As a consequence from the compactness property and [15, Theorem 5.3.7, p. 333], we have the following spectral decomposition result.

Corollary 3.2 [1] *C_0 is decomposed as follows:*

$$C_0 = S \oplus V,$$

where S is $\mathcal{U}$-invariant and there are positive constants α and N such that

$$|\mathcal{U}(t)\varphi|_C \le Ne^{-\alpha t}|\varphi|_C \text{ for each } t \ge 0 \text{ and } \varphi \in S. \tag{3.1}$$

V is a finite dimensional space and the restriction of $\mathcal{U}$ to V becomes a group.

In the sequel, $\mathcal{U}^s(t)$ and $\mathcal{U}^v(t)$ denote the restriction of $\mathcal{U}(t)$ respectively on S and V which correspond to the above decomposition.

Let $d = \dim V$ with a basis vectors $\Phi = \{\phi_1, \ldots, \phi_d\}$. Then, there exist d-elements $\{\psi_1, \ldots, \psi_d\}$ in C_0^* such that

$$\begin{cases} \langle \psi_i, \phi_j \rangle = \delta_{ij}, \\ \langle \psi_i, \phi \rangle = 0 \text{ for all } \phi \in S \text{ and } i \in \{1, \ldots, d\}, \end{cases} \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between C_0^* and C_0 and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Let $\Psi = col\{\psi_1, \ldots, \psi_d\}$, $\langle \Psi, \Phi \rangle$ is a $d \times d$-matrix, where the (i, j)-component is $\langle \psi_i, \phi_j \rangle$. Denote by Π^s and Π^v the projections respectively on S and V. For each $\varphi \in C_0$, we have

$$\Pi^v\varphi = \Phi \langle \Psi, \varphi \rangle.$$

In fact, for $\varphi \in C_0$, we have $\varphi = \Pi^s\varphi + \Pi^v\varphi$ with $\Pi^v\varphi = \sum_d^{i=1} \alpha_i\phi_i$ and $\alpha_i \in \mathbb{R}$. By (3.2), we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle.$$

Hence

$$\begin{aligned} \Pi^v\varphi &= \sum_{i=1}^{d} \langle \psi_i, \varphi \rangle \phi_i \\ &= \Phi \langle \Psi, \varphi \rangle. \end{aligned}$$

Since $(\mathcal{U}^v(t))_{t\ge 0}$ is a group on V, then there exists a $d \times d$-matrix G such that

$$\mathcal{U}^v(t)\Phi = \Phi e^{tG} \text{ for } t \in \mathbb{R}.$$

Moreover, $\sigma(G) = \{\lambda \in \sigma(\mathcal{A}_u) : \mathrm{Re}(\lambda) \ge 0\}$.
For $n, n_0 \in N$ such that $n \ge n_0 \ge \widetilde{\omega}$ and $i \in \{1, \ldots, d\}$, we define the linear mapping $x_{i,n}^*$ by

$$x_{i,n}^*(a) = \langle \psi_i, \widetilde{B}_n X_0 a \rangle \text{ for } a \in X.$$

Since $|\widetilde{B}_n| \le \frac{n}{n-\widetilde{\omega}}\widetilde{M}$, for any $n \ge n_0$, then $x_{i,n}^*$ is a bounded linear operator from X to $\mathbb{R}$ with

$$|x_{i,n}^*| \le \frac{n}{n - n_0}\widetilde{M}|\psi_i| \text{ for any } n \ge n_0.$$

Define the d-column vector $x_n^* = col\left(x_{1,n}^*, \ldots, x_{d,n}^*\right)$, then

$$\langle x_n^*, a \rangle = \langle \Psi, \widetilde{B}_n X_0 a \rangle \text{ for } a \in X,$$

with

$$\left\langle x_n^*, a\right\rangle_i = \left\langle \psi_i, \widetilde{B}_n X_0 a\right\rangle \text{ for } i = 1, \ldots, d \text{ and } a \in X.$$

Consequently,

$$\sup_{n \geq n_0} \left|x_n^*\right| < \infty,$$

which implies that $\left(x_n^*\right)_{n\geq n_0}$ is a bounded sequence in $\mathcal{L}(X, \mathbb{R}^d)$. We have the following important result of this work.

Theorem 3.3 *There exists $x^* \in \mathcal{L}(X, \mathbb{R}^d)$, such that $\left(x_n^*\right)_{n\geq n_0}$ converges weakly to x^* in the sense that*

$$\left\langle x_n^*, x\right\rangle \to \left\langle x^*, x\right\rangle \text{ as } n \to \infty \text{ for all } x \in X.$$

For the proof, we need the following fundamental Theorem in functional analysis.

Theorem 3.4 [16, p. 776] *(*Banach-Alaoglu-Bourbaki*) Let Y be any separable Banach space and $\left(z_n^*\right)_{n\in\mathbb{N}}$ any bounded sequence in Y^*. Then there exists a subsequence $\left(z_{n_k}^*\right)_{k\in\mathbb{N}}$ of $\left(z_n^*\right)_{n\in\mathbb{N}}$ which converges weakly in Y^*in the sense that there exists $z^* \in Y^*$ such that*

$$\left\langle z_{n_k}^*, x\right\rangle \to \left\langle z^*, x\right\rangle \text{ as } n \to \infty \text{ for all } x \in Y.$$

Proof Let Z_0 be any closed separable subspace of X. Since $\left(x_n^*\right)_{n\geq n_0}$ is a bounded sequence, then by Theorem 3.4 we get that the sequence $\left(x_n^*\right)_{n\geq n_0}$ has a subsequence $\left(x_{n_k}^*\right)_{k\in\mathbb{N}}$ which converges weakly to some $x_{Z_0}^*$ in Z_0. We claim that all the sequence $\left(x_n^*\right)_{n\geq n_0}$ converges weakly to $x_{Z_0}^*$ in Z_0. In fact, we proceed by contradiction and suppose that there exists a subsequence $\left(x_{n_p}^*\right)_{p\in\mathbb{N}}$ of $\left(x_n^*\right)_{n\geq n_0}$ which converges we****akly to some $\widetilde{x}_{Z_0}^*$ with $\widetilde{x}_{Z_0}^* \neq x_{Z_0}^*$. Let $u_t(\cdot, \sigma, \varphi, f)$ denote the solution of Eq. (1.1). Then

$$\Pi^v u_t(\cdot, \sigma, 0, f) = \lim_{n\to+\infty} \int_\sigma^t \mathcal{U}^v\,(t-\xi)\,\Pi^v\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi,$$

and

$$\Pi^v\left(\widetilde{B}_n X_0 f(\xi)\right) = \Phi\left\langle \Psi, \widetilde{B}_n X_0 f(\xi)\right\rangle = \Phi\left\langle x_n^*, f(\xi)\right\rangle.$$

It follows that

$$\begin{aligned}\Pi^v u_t(\cdot, \sigma, 0, f) &= \lim_{n\to+\infty} \Phi \int_\sigma^t e^{(t-\xi)G}\left\langle \Psi, \widetilde{B}_n X_0 f(\xi)\right\rangle d\xi,\\ &= \lim_{n\to+\infty} \Phi \int_\sigma^t e^{(t-\xi)G}\left\langle x_n^*, f(\xi)\right\rangle d\xi.\end{aligned}$$

For any $a \in Z_0$, set $f(\cdot) = a$, then

$$\lim_{k\to+\infty} \int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_k}^*, a\right\rangle d\xi = \lim_{p\to+\infty} \int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_p}^*, a\right\rangle d\xi \text{ for } a \in Z_0,$$

which implies that

$$\int_\sigma^t e^{(t-\xi)G}\left\langle x_{Z_0}^*, a\right\rangle d\xi = \int_\sigma^t e^{(t-\xi)G}\left\langle \widetilde{x}_{Z_0}^*, a\right\rangle d\xi \text{ for } a \in Z_0,$$

consequently $x^*_{Z_0} \equiv \widetilde{x}^*_{Z_0}$, which gives a contradiction. We conclude that the whole sequence $\left(x^*_n\right)_{n\geq n_0}$ converges weakly to $x^*_{Z_0}$ in Z_0. Let Z_1 be another closed separable subspace of X, by using the same argument as above, we get that $\left(x^*_n\right)_{n\geq n_0}$ converges weakly to $x^*_{Z_1}$ in Z_1. Since $Z_0 \cap Z_1$ is a closed separable subspace of X, we get that $x^*_{Z_1} \equiv x^*_{Z_0}$ in $Z_0 \cap Z_1$. For any $x \in X$, we define x^* by

$$\left\langle x^*, x\right\rangle = \left\langle x^*_Z, x\right\rangle,$$

where Z is any closed separable subspace of X such that $x \in Z$. Then x^* is well defined on X and x^* is a bounded linear from X to $\mathbb{R}^d$ such that

$$\left|x^*\right| \leq \sup_{n\geq n_0} \left|x^*_n\right| < \infty,$$

and $\left(x^*_n\right)_{n\geq n_0}$ converges weakly to x^* in X.

As a consequence, we conclude that

Corollary 3.5 *For any continuous function $h : \mathbb{R} \to X$, we have*

$$\lim_{n\to+\infty} \int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\widetilde{B}_n X_0 h(\xi)\right)d\xi = \Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*, h(\xi)\right\rangle d\xi \textit{ for all } t, \sigma \in \mathbb{R}.$$

Theorem 3.6 *Assume that* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *hold. Let u be a solution of Eq.* (1.1) *on* $\mathbb{R}$. *Then* $z\left(t\right) = \left\langle\Psi, u_t\right\rangle$ *is a solution of the ordinary differential equation*

$$\frac{d}{dt}z\left(t\right) = Gz\left(t\right) + \left\langle x^*, f\left(t\right)\right\rangle \textit{ for } t \in \mathbb{R}. \tag{3.3}$$

Conversely, if f is a bounded function on $\mathbb{R}$ *and z is a solution of Eq.* (3.3) *on* $\mathbb{R}$, *then the function u given by*

$$u(t) = \left[\Phi z\left(t\right) + \lim_{n\to+\infty}\int_{-\infty}^t \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n X_0 f\left(\xi\right)\right)d\xi\right](0) \textit{ for } t \in \mathbb{R},$$

is a solution of Eq. (1.1) *on* $\mathbb{R}$.

Let u be a solution of Eq. (1.1) on $\mathbb{R}$. Then

$$u_t = \Pi^s u_t + \Pi^v u_t \text{ for all } t \in \mathbb{R},$$

and

$$\Pi^v u_t = \mathcal{U}^v\left(t-\sigma\right)\Pi^v u_\sigma + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\widetilde{B}_n X_0 f\left(\xi\right)\right)d\xi \text{ for } t, \sigma \in \mathbb{R}.$$

Since $\Pi^v u_t = \Phi\left\langle\Psi, u_t\right\rangle$ and by Corollary 3.5, we get that

$$\begin{aligned}\Phi\left\langle\Psi, u_t\right\rangle &= \mathcal{U}^v\left(t-\sigma\right)\Phi\left\langle\Psi, u_\sigma\right\rangle + \Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi \text{ for } t, \sigma \in \mathbb{R},\\ &= \Phi e^{(t-\sigma)G}\left\langle\Psi, u_\sigma\right\rangle + \Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi \text{ for } t, \sigma \in \mathbb{R}.\end{aligned}$$

Let $z\left(t\right) = \left\langle\Psi, u_t\right\rangle$. Then

$$z(t) = e^{(t-\sigma)G}z(\sigma) + \int_\sigma^t e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi \text{ for } t, \sigma \in \mathbb{R}.$$

Consequently, z is a solution of the ordinary differential Eq. (3.3) on $\mathbb{R}$. Conversely, assume that f is bounded on $\mathbb{R}$, then $\int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi$ is well defined on $\mathbb{R}$. Let z be a solution of (3.3) on $\mathbb{R}$ and v be defined by

$$v(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \text{ for } t \in \mathbb{R}.$$

Since

$$z(t) = e^{(t-\sigma)G} z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G} \left\langle x^*, f(\xi)\right\rangle d\xi \text{ for } t, \sigma \in \mathbb{R},$$

Using Corollary 3.5, the function v_1 given by

$$v_1(t) = \Phi z(t) \quad \text{for } t \in \mathbb{R},$$

satisfies

$$v_1(t) = \mathcal{U}^v(t-\sigma)v_1(\sigma) + \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}^v(t-\xi)\Pi^v\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \quad \text{for } t, \sigma \in \mathbb{R}.$$

Moreover, the function v_2 given by

$$v_2(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \text{ for } t \in \mathbb{R},$$

satisfies

$$v_2(t) = \mathcal{U}^s(t-\sigma)v_2(\sigma) + \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}^s(t-\xi)\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \text{ for all } t \geq \sigma.$$

Then, for all $t \geq \sigma$ with $t, \sigma \in \mathbb{R}$, one has

$$\begin{aligned}
\mathcal{U}(t-\sigma)v(\sigma) &= \mathcal{U}^v(t-\sigma)v_1(\sigma) + \mathcal{U}^s(t-\sigma)v_2(\sigma), \\
&= v_1(t) - \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}^v(t-\xi)\Pi^v\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi + v_2(t) \\
&\quad - \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}^s(t-\xi)\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi, \\
&= v(t) - \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}(t-\xi)\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi.
\end{aligned}$$

Therefore

$$v(t) = \mathcal{U}(t-\sigma)v(\sigma) + \lim_{n\to+\infty} \int_{\sigma}^{t} \mathcal{U}(t-\xi)\left(\widetilde{B}_n X_0 f(\xi)\right) \xi \text{ for } t \geq \sigma.$$

By Theorem 2.7, we obtain that the function u defined by $u(t) = v(t)(0)$ is a solution of Eq. (1.1) on $\mathbb{R}$.

4 Reduction of complexity of partial functional differential equations in fading memory spaces

Let C_{00} be the space of X-valued continuous function on $]-\infty, 0]$ with compact support.

(C) : If a uniformly bounded sequence $(\varphi_n)_{n\in\mathbb{N}}$ in C_{00} converges to a function φ compactly on $]-\infty,0]$, then φ is in $\mathcal{B}$ and $|\varphi_n-\varphi|\to 0$ as $n\to\infty$.
Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 := \{\phi\in\mathcal{B} : \phi(0)=0\}$$

by

$$(S_0(t)\phi)(\theta)=\begin{cases}\phi(t+\theta) & \text{if } t+\theta\leq 0,\\ 0 & \text{if } t+\theta\geq 0.\end{cases}$$

Definition 4.1 Assume that the space $\mathcal{B}$ satisfies Axioms **(B)** and **(C)**. $\mathcal{B}$ is said to be a fading memory space if for all $\phi\in\mathcal{B}_0$,

$$S_0(t)\phi \underset{t\to\infty}{\longrightarrow} 0 \text{ in } \mathcal{B}_0.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space if

$$|S_0(t)| \underset{t\to\infty}{\longrightarrow} 0, \text{ with respect to the operator norm.}$$

Lemma 4.2 [17, pp 190] *The following statements hold:*

(i) *If $\mathcal{B}$ is a fading memory space, then the functions $K(\cdot)$ and $M(\cdot)$ in axiom* **(A)** *can be chosen to be constants.*
(ii) *If $\mathcal{B}$ is a uniform fading memory space, then we can choose the function $K(\cdot)$ constant and the function $M(\cdot)$ such that $M(t)\to 0$ as $t\to\infty$.*

Proposition 4.3 [17] *If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty,0],X)$ of bounded continuous X-valued functions on $]-\infty,0]$ endowed with the uniform norm topology is continuously embedding in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies $(\boldsymbol{D}_3)$, for $\lambda_0>0$.*

In this section, we assume that
$(\mathbf{H}_2)$ $\mathcal{B}$ is a uniform fading memory space.
Let V be a bounded subset of a Banach space Y, the Kuratowski measure of noncompactness $\alpha(V)$ of V is given by

$$\alpha(v)=\inf\left\{\begin{array}{c} d>0 \text{ such that there exists a finite number of sets } V_1,\ldots,V_n \text{ with}\\ diam(V_i)\leq d \text{ such that } V\subseteq \overset{n}{\underset{i=1}{\cup}} V_i\end{array}\right\},$$

and for a bounded linear operator F on Y, we define $|F|_\alpha$ by

$$|F|_\alpha=\inf\{k>0:\ \alpha(F(V))\leq k\alpha(V),\ \text{for all bounded set } V \text{ of } Y\}.$$

For a strongly continuous semigroup $(S(t))_{t\geq 0}$, we define the essential growth bound $\omega_{ess}(S)$ by

$$\omega_{ess}(S)=\lim_{t\to\infty}\frac{1}{t}\log|S(t)|_\alpha.$$

Theorem 4.4 [18] *Assume that $\mathcal{B}$ satisfies Axioms* **(A)**, **(B)**, $(\mathbf{D}_1)$ *and assumptions* $(\mathbf{H}_0)$, $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *hold. Then*

$$\omega_{ess}(U)<0.$$

From [15, Corollary IV.2.11], it follows that

$$\sigma_u(A_U) := \{\lambda \in \sigma(A_U) : \operatorname{Re}(\lambda) \geq 0\}$$

is a finite subset and $\mathcal{B}_A$ is decomposed as follows:

$$\mathcal{B}_A = \mathcal{S} \oplus \mathcal{V},$$

where $\mathcal{S}$, $\mathcal{V}$ are two closed subspaces of $\mathcal{B}_A$ which are invariant by $(U(t))_{t\geq 0}$. Let $U^{\mathcal{S}}(t)$ be the restriction of $U(t)$ on $\mathcal{S}$, then there exist positive constants N and μ such that

$$\left|U^{\mathcal{S}}(t)\phi\right| \leq Ne^{-\mu t}|\phi| \text{ for all } \phi \in \mathcal{S},$$

$\mathcal{V}$ is a finite dimensional space and the restriction $U^{\mathcal{V}}(t)$ of $U(t)$ on $\mathcal{V}$ becomes a group. Let $\Pi^{\mathcal{S}}$ and $\Pi^{\mathcal{V}}$ denote the projections on $\mathcal{S}$ and $\mathcal{V}$ respectively. Let $d = \dim \mathcal{V}$ and take a basis $\{\phi_1, \ldots, \phi_d\}$ in $\mathcal{V}$. Then there exist d-elements $\{\psi_1, \ldots, \psi_d\}$ in the dual space $\mathcal{B}_A^*$ of $\mathcal{B}_A$, such that $\langle \psi_i, \phi_j \rangle = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\psi_i = 0$ on $\mathcal{S}$, where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between the dual space and the original space. Denote by $\Phi := (\phi_1, \ldots, \phi_d)$ and Ψ is the transpose of $(\psi_1, \ldots, \psi_d)$, in particular one has

$$\Psi\Phi = \mathbb{I}_{\mathbb{R}^d},$$

where $\mathbb{I}_{\mathbb{R}^d}$ is the identity $d \times d$ matrix. For each $\phi \in \mathcal{B}_A$, $\Pi^{\mathcal{V}}\phi$ is computed by:

$$\Pi^{\mathcal{V}}\phi = \Phi\langle \Psi, \phi \rangle = \sum_{i=1}^{d} \langle \psi_i, \phi \rangle \phi_i.$$

Let $\zeta(t) := (\zeta_1(t), \ldots, \zeta_d(t))$ be the component of $\Pi^{\mathcal{V}}x_t$ in the basis vector Φ, then

$$\Pi^{\mathcal{V}}x_t = \Phi\zeta(t), \text{ and } \zeta(t) = \langle \Psi, x_t \rangle.$$

Since $\left(U^{\mathcal{V}}(t)\right)_{t\geq 0}$ is a group on a finite dimensional space $\mathcal{V}$, then there exists a $d \times d$ matrix G such that

$$U^{\mathcal{V}}(t)\phi = \Phi e^{Gt}\langle \Psi, \phi \rangle \text{ for all } t \in \mathbb{R} \text{ and } \phi \in \mathcal{V},$$

which means that

$$U^{\mathcal{V}}(t)\Phi = \Phi e^{Gt} \text{ for all } t \in \mathbb{R}.$$

For $n > \omega_1$ and $i \in \{1, \ldots, d\}$, we define the functional x_n^{*i} by

$$\left\langle x_n^{*i}, x \right\rangle = \left\langle \psi_i, \widetilde{B_n}(X_0 x) \right\rangle \text{ for all } x \in X.$$

Then x_n^{*i} is a bounded linear operator on X with $\left|x_n^{*i}\right| \leq K_0 M_1 |\psi_i|$. Define the d-column vector x_n^* as an element of $\mathcal{L}(X, \mathbb{R}^d)$ (the space of bounded linear operator from X into $\mathbb{R}^d$) given by the transpose of $\left(x_n^{*1}, \ldots, x_n^{*d}\right)$. Then, for all $n \geq 1$, $x \in X$

$$\left\langle x_n^*, x \right\rangle = \left\langle \Psi, \widetilde{B_n}(X_0 x) \right\rangle \text{ and } \sup_{n\geq\omega_1} \left|x_n^*\right| \leq K_0 M_1 \sup_{i=1,\ldots,d} |\psi_i| < \infty.$$

Theorem 4.5 *The sequence $\left(x_n^*\right)_{n\geq 0}$ converges weakly in $\mathcal{L}(X, \mathbb{R}^d)$, in the sense that*

$$\left\langle x_n^*, x\right\rangle \underset{n\to\infty}{\longrightarrow} \left\langle x^*, x\right\rangle \text{ for all } x \in X.$$

Let Y_0 be any separable closed subspace of X. By Theorem 3.4 , the restriction $\left(x_n^{Y_0^*}\right)_{n\geq 0}$ of $\left(x_n^*\right)_{n\geq 0}$ in Y_0 has a subsequence $\left(x_{n_k}^{Y_0^*}\right)_{k\geq 0}$ such that

$$\lim_{k\to\infty}\left\langle x_{n_k}^{Y_0^*}, y\right\rangle = \left\langle x^{Y_0^*}, y\right\rangle \text{ for all } y \in Y_0,$$

where $x^{Y_0^*} \in Y_0^*$. We claim that the whole sequence $\left(x_n^{Y^*}\right)_{n\geq 0}$ converges weakly in Y_0^* to $x^{Y_0^*}$. We proceed by contradiction and assume that there exists a subsequence $\left(x_{m_k}^{Y^*}\right)_{k\geq 0}$ of $\left(x_n^{Y^*}\right)_{n\geq 0}$ such that $x_{m_k}^{Y_0^*} \underset{k\to\infty}{\longrightarrow} x_1^{Y_0^*}$ weakly in Y_0, with $x^{Y_0^*} \neq x_1^{Y_0^*}$. To conclude we need the following lemma.

Lemma 4.6 *For any continuous function $h : \mathbb{R}^+ \to X$ one has:*

$$\lim_{n\to\infty}\int_0^t U^{\mathcal{V}}(t-s)\,\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 h(s))\right)ds = \Phi\lim_{n\to\infty}\int_0^t e^{(t-s)G}\left\langle x_n^*, h\,(s)\right\rangle ds.$$

Proof of the Lemma In fact, we have

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t U^{\mathcal{V}}(t-s)\,\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 h(s))\right)ds\\
&= \lim_{n\to\infty}\int_0^t \left(U^{\mathcal{V}}(t-s)\,\Phi\right)\left\langle \Psi, \widetilde{B_n}\,(X_0 h(s))\right\rangle ds,\\
&= \lim_{n\to\infty}\int_0^t \Phi e^{(t-s)G}\left\langle x_n^*, h\,(s)\right\rangle ds,\\
&= \Phi\lim_{n\to\infty}\int_0^t e^{(t-s)G}\left\langle x_n^*, h\,(s)\right\rangle ds.
\end{aligned}$$

Let $h(\cdot) = y$ for any $y \in Y_0$. Then

$$\int_0^t e^{(t-s)G}\left\langle x^{Y_0^*}, y\right\rangle ds = \int_0^t e^{(t-s)G}\left\langle x_1^{Y_0^*}, y\right\rangle ds \text{ for any } y \in Y_0.$$

This is true if and only if $\langle x^{Y_0^*}, y\rangle = \langle x_1^{Y_0^*}, y\rangle$, for all $y \in Y_0$, which gives a contradiction. Consequently the whole sequence $(x_n^{Y_0^*})_{n\geq 0}$ converges weakly in $\mathcal{L}(Y_0, \mathbb{R}^d)$ to $x^{Y_0^*}$.
Let Y_1 be another separable closed space of X. Then the restriction $(x_n^{Y_1^*})_{n\geq 0}$ of $(x_n^*)_{n\geq 0}$ in Y_1 converges weakly to some $x^{Y_1^*} \in Y_1^*$, and we get that $x^{Y_0^*} = x^{Y_1^*}$ in $Y_0 \cap Y_1$. Since $(x_n^*)_{n\,0}$ converges weakly in $Y_0 \cap Y_1$, and by the uniqueness of the limit we obtain that $x^{Y_0^*} = x^{Y_1^*}$ in $Y_0 \cap Y_1$. Let x^* be the operator defined by

$$\left\langle x^*, x\right\rangle = \left\langle x^{Y^*}, x\right\rangle,$$

for any separable closed space Y of X such that $x \in Y$. Then x^* is well defined and belongs to $\mathcal{L}(X, \mathbb{R}^d)$. Moreover

$$\left\langle x_n^*, x\right\rangle \underset{n\to\infty}{\longrightarrow} \left\langle x^*, x\right\rangle \text{ for all } x \in X.$$

□

Consequently, we get the following.

Corollary 4.7 *For any continuous function* $h : [0, a] \to X$*:*

$$\lim_{n\to\infty}\int_0^t U^{\mathcal{V}}(t-s)\,\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 h(s))\right)ds = \Phi\int_0^t e^{(t-s)G}\left\langle x^*, h\,(s)\right\rangle ds \text{ for all } t\in[0,a].$$

Theorem 4.8 *Assume that* (**A**), (**B**), ($\mathbf{D}_1$), ($\mathbf{D}_2$), ($\mathbf{H}_0$), ($\mathbf{H}_1$) *and* ($\mathbf{H}_2$) *hold. Let u be an integral solution of Eq.* (2.1) *on* $\mathbb{R}$*. Then* $\zeta(t) = \langle\Psi, u_t\rangle$, $t\in\mathbb{R}$ *is a solution of the following ordinary differential equation*

$$\dot{\zeta}\,(t) = G\zeta\,(t) + \left\langle x^*, f\,(t)\right\rangle \text{ for } t\in\mathbb{R}. \tag{4.1}$$

Conversely, if f is bounded and ζ is a solution of Eq. (4.1)*, then the function*

$$\left(\Phi\zeta\,(t) + \lim_{n\to+\infty}\int_{-\infty}^t U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)ds\right)(0) \tag{4.2}$$

is an integral solution of Eq. (2.1) *on* $\mathbb{R}$*.*

Proof Using the variation of constants formula (2.3), we obtain that for $t\geq\sigma$

$$\begin{aligned}\langle\Psi, x_t\rangle &= \langle\Psi, U\,(t-\sigma)\,x_\sigma\rangle + \left\langle\Psi, \lim_{n\to+\infty}\int_\sigma^t U\,(t-s)\left(\widetilde{B_n}\,(X_0 f(s))\right)ds\right\rangle,\\ &= e^{(t-\sigma)G}\langle\Psi, x_\sigma\rangle + \lim_{n\to\infty}\int_\sigma^t e^{(t-s)G}\left\langle\Psi, \left(\widetilde{B_n}\,(X_0 f(s))\right)\right\rangle ds,\\ &= e^{(t-\sigma)G}\langle\Psi, x_\sigma\rangle + \lim_{n\to\infty}\int_\sigma^t e^{(t-s)G}\left\langle x_n^*, f\,(s)\right\rangle ds,\\ &= e^{(t-\sigma)G}\langle\Psi, x_\sigma\rangle + \int_\sigma^t e^{(t-s)G}\left\langle x^*, f\,(s)\right\rangle ds,\end{aligned}$$

which means that $\zeta\,(t) = \langle\Psi, x_t\rangle$, $t\in\mathbb{R}$ is a solution of the ordinary differential Eq. (4.1). Conversely, if we assume that f is bounded on $\mathbb{R}$, then formula (4.2) is well defined, since the restriction of the solution semigroup on $\mathcal{S}$ is exponentially stable. Let y be defined by:

$$y\,(t) := \lim_{\lambda\to+\infty}\int_{-\infty}^t U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)ds \text{ for } t\in\mathbb{R}.$$

Then for $t\geq\sigma$,

$$\begin{aligned}&U^{\mathcal{S}}(t-\sigma)\,y\,(\sigma) + \lim_{n\to+\infty}\int_\sigma^t U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)ds\\ &\quad= \lim_{n\to+\infty}\left(\int_{-\infty}^\sigma U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)ds\right.\\ &\qquad\left.+\int_\sigma^t U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)ds\right)\\ &\quad= y\,(t)\,.\end{aligned} \tag{4.3}$$

Moreover the solution ζ of Eq. (4.1) is given by

$$\zeta\,(t) = e^{(t-\sigma)G}\zeta\,(\sigma) + \int_\sigma^t e^{(t-s)G}\left\langle x^*, f\,(s)\right\rangle ds \text{ for } t\geq\sigma.$$

Corollary 4.7, gives that

$$\Phi\zeta\,(t) = \Phi e^{(t-\sigma)G}\zeta\,(\sigma) + \lim_{n\to\infty}\int_{\sigma}^{t} U^{\mathcal{V}}(t-s)\,\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 f(s))\right) ds \text{ for } t \geq \sigma,$$

and

$$\Phi\zeta\,(t) = U^{\mathcal{V}}(t-\sigma)\,\Phi\zeta\,(\sigma) + \lim_{n\to\infty} n\int_{\sigma}^{t} U^{\mathcal{V}}(t-s)\,\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 f(s))\right) ds \text{ for } t \geq \sigma. \quad (4.4)$$

Set $\xi\,(t) = \Phi\zeta\,(t) + y\,(t)$ on $\mathbb{R}$, by (4.3) and (4.4), we obtain that

$$\begin{aligned} \xi\,(t) &= U\,(t-\sigma)\,(\Phi\zeta\,(\sigma) + y\,(\sigma)) \\ &\quad + \lim_{n\to\infty} n\int_{\sigma}^{t} U\,(t-s)\left[\Pi^{\mathcal{V}} + \Pi^{\mathcal{S}}\right]\left(\widetilde{B_n}\,(X_0 f(s))\right) ds \text{ for } t \geq \sigma. \\ &= U\,(t-\sigma)\,\xi\,(\sigma) + \lim_{n\to\infty}\int_{\sigma}^{t} U\,(t-s)\left(\widetilde{B_n}\,(X_0 f(s))\right) ds \text{ for } t \geq \sigma. \end{aligned}$$

From Theorem 2.16, we conclude that the function

$$\left(\Phi\zeta\,(t) + \lim_{n\to+\infty}\int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\,\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right) ds\right)(0)$$

is an integral solution of Eq. (2.1).

5 Application: almost automorphic solutions for Eq. (1.1)

We recall some properties about almost automorphic functions. Let $\mathcal{BC}(\mathbb{R}, X)$ be the space of all bounded continuous functions from $\mathbb{R}$ to X, provided with the uniform norm topology. Let $h \in \mathcal{BC}(\mathbb{R}, X)$ and $\tau \in \mathbb{R}$, we define the function h_τ by

$$h_\tau(s) = h(\tau + s) \text{ for all } s \in \mathbb{R}.$$

Definition 5.1 [19, Definition 1.1.1, p. 1] A bounded continuous function $h : \mathbb{R} \to X$ is said to be almost periodic if

$$\{h_\tau : \tau \in \mathbb{R}\} \text{ is relatively compact in } \mathcal{BC}(\mathbb{R}, X).$$

Definition 5.2 (Bochner [5, Theorem 5.8, p. 86]) A continuous function $h : \mathbb{R} \to X$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_n$ there exists a subsequence $(s_n)_n$ such that

$$\lim_{n\to\infty} h(t + s_n) = k(t) \text{ exists for all } t \text{ in } \mathbb{R}$$

and

$$\lim_{n\to\infty} k(t - s_n) = h(t) \text{ for all } t \text{ in } \mathbb{R}.$$

Remark. If the convergence in the both limits is uniform, then h is almost periodic. The concept of almost automorphy is much larger than almost periodicity. By the pointwise convergence, the function k is just measurable and not necessarily continuous.

Definition 5.3 (Bochner [5, Theorem 5.8, p. 86]) A continuous function $h : \mathbb{R} \to X$ is said to be compact almost automorphic if for every sequence of real numbers $(s'_n)_n$, there exists a subsequence $(s_n)_n$ such that

$$\lim_{m\to\infty}\lim_{n\to\infty} h(t + s_n - s_m) = h(t) \text{ exists uniformly on any compact set in } \mathbb{R}.$$

Theorem 5.4 [5] *If we equip $AA(X)$, the space of almost automorphic X-valued functions with the sup norm, then $AA(X)$ turns out to be a Banach space.*

Consider the following ordinary differential equation

$$\frac{d}{dt}x(t) = Gx(t) + e(t) \text{ for } t \in \mathbb{R} \tag{5.1}$$

where G is a constant $n \times n$-matrix and $e : \mathbb{R} \to \mathbb{R}^n$ is a continuous function.

Theorem 5.5 [5, Theorem 5.8, p. 86] *Assume that e is an almost automorphic function. Then the following are equivalent:*

i) *existence of a bounded solution on $\mathbb{R}^+$ of Eq. (5.1),*
ii) *existence of an almost automorphic solution of Eq. (5.1).*

Moreover every bounded solution of Eq. (5.1) on the whole line is almost automorphic.

In the following, we assume that:
$(\mathbf{H_3})$ f is an almost automorphic function.
Consider now the following equation in the whole line $\mathbb{R}$

$$\frac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}. \tag{5.2}$$

Theorem 5.6 *Assume that $(\mathbf{H_0})$, $(\mathbf{H_1})$ and $(\mathbf{H_3})$ hold. If there exists $\varphi \in C$ such that Eq. (1.1) has a bounded solution on $\mathbb{R}^+$. Then Eq. (5.2) has an almost automorphic integral solution.*

Proof Let u be a bounded solution of Eq. (1.1) on $\mathbb{R}^+$. Then by Theorem 3.6, the function $z(t) = \langle \Psi, u_t\rangle$ for $t \geq 0$, is a solution of the ordinary differential Eq. (3.3) and z is bounded on $\mathbb{R}^+$. Moreover, the function

$$\varrho(t) = \left\langle x^*, f(t)\right\rangle \text{ for } t \in \mathbb{R},$$

is almost automorphic from $\mathbb{R}$ to $\mathbb{R}^d$. By Theorem 5.5, we get that the reduced system (3.3) has an almost automorphic solution $\widetilde{z}$. Consequently $\Phi\widetilde{z}(\cdot)$ is an almost automorphic function on $\mathbb{R}$. By Theorem 3.6, the function $u(t) = v(t)(0)$, where

$$v(t) = \Phi\widetilde{z}(t) + \lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \text{ for } t \in \mathbb{R},$$

is a solution of Eq. (5.2) on $\mathbb{R}$. We claim that v is almost automorphic. In fact, consider the function y by

$$y(t) = \lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n X_0 f(\xi)\right) d\xi \text{ for } t \in \mathbb{R},$$

Since f is almost automorphic, then for any sequence of real numbers $(\alpha'_p)_{p\geq 0}$ there exists a subsequence $(\alpha_p)_{p\geq 0}$ of $(\alpha'_p)_{p\geq 0}$ such that

$$\lim_{p\to\infty} f(t+\alpha_p) = h(t) \text{ for all } t \in \mathbb{R}$$

and

$$\lim_{p\to\infty} h(t-\alpha_p) = f(t) \text{ for all } t \in \mathbb{R}.$$

Now

$$y(t+\alpha_p) = \lim_{n\to+\infty} \int_{-\infty}^{t+\alpha_p} \mathcal{U}^s\left(t+\alpha_p-\xi\right)\Pi^s\left(\widetilde{B}_n X_0 f\left(\xi\right)\right)d\xi \text{ for } t \in \mathbb{R},$$

which gives that

$$y(t+\alpha_p) = \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n X_0 f\left(\xi+\alpha_p\right)\right)d\xi \text{ for } t \in \mathbb{R},$$

By the Lebesgue's dominated convergence theorem, we get that

$$y(t+\alpha_p) \to w(t) \text{ as } p \to \infty,$$

where w is given by

$$w(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n X_0 h\left(\xi\right)\right)d\xi \text{ for } t \in \mathbb{R}.$$

Using the the same argument as above, we prove that

$$w(t-\alpha_p) \to \lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n X_0 f\left(\xi\right)\right)d\xi \text{ as } p \to \infty,$$

which implies that y is almost automorphic. Consequently, v is an almost automorphic integral solution of Eq. (5.2).

6 Lotka–Volterra model

In order to apply the previous results, we consider the model of Lotka–Volterra with diffusion which is taken from [2,3]

$$\begin{cases} \dfrac{\partial}{\partial t}v(t,x) = \dfrac{\partial^2}{\partial x^2}v(t,x) + \displaystyle\int_{-r}^{0} G(\theta)v(t+\theta,x)d\theta + h\left(t,x\right) \text{ for } t \geq 0 \text{ and } x \in [0,\pi], \\ u(t,x) = 0 \text{ for } x = 0, \pi \text{ and } t \geq 0, \\ u(\theta,x) = \varphi_0(\theta,x) \text{ for } \theta \in [-r,0] \text{ and } x \in [0,\pi], \end{cases} \tag{6.1}$$

where $G : [-r,0] \to \mathbb{R}$, $\varphi_0 : [-r,0]\times[0,\pi] \to \mathbb{R}$ and $h : \mathbb{R}\times[0,\pi] \to \mathbb{R}$ are continuous functions.

Let $X = C\left([0,\pi];\mathbb{R}\right)$ be the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. Define the operator $A : D(A) \subset X \to X$ by

$$\begin{cases} D(A) = \left\{y \in C^2\left([0,\pi];\mathbb{R}\right) : y(0) = y(\pi) = 0\right\}, \\ Ay = y''. \end{cases}$$

Lemma 6.1 [20, Proposition 14.6 , p. 319–320]

$$(0,+\infty) \subset \rho(A) \textit{ and } \left|(\lambda - A)^{-1}\right| \leq \frac{1}{\lambda} \textit{ for } \lambda > 0.$$

Moreover,

$$\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\}.$$

This Lemma implies that condition ($\mathbf{H}_0$) is satisfied.
We introduce $L : C := C([-r, 0], X) \to X$ by

$$L(\phi)(x) = \int_{-r}^{0} G(\theta)\phi(\theta)(x)d\theta \text{ for } x \in [0, \pi] \text{ and } \phi \in C.$$

$f : \mathbb{R} \longrightarrow X$ is defined by

$$f(t)(x) = h(t, x) \text{ for } t \in \mathbb{R} \text{ and } x \in [0, \pi].$$

The initial function $\varphi \in C$ is given by

$$\varphi(\theta)(x) = \varphi_0(\theta, x) \text{ for } (\theta, x) \in [-r, 0]\text{x}[0, \pi]$$

L is a bounded linear operator from C to X and by form continuity of h, we get that f is a continuous function from $\mathbb{R}$ to X. Equation (6.1) takes the following abstract form

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \geq 0, \\ u_0 = \varphi \in C. \end{cases} \tag{6.2}$$

Let A_0 be the part of A in $\overline{D(A)}$. Then, A_0 is given by

$$\begin{cases} D(A_0) = \left\{ y \in C^2([0, \pi]; \mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0 \right\}, \\ A_0 y = Ay \text{ for } y \in D(A_0). \end{cases}$$

It is well known from [15, Example 1.4.34 , p. 123], that A_0 generates a strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$ and

$$|T_0(t)| \leq e^{-t} \text{ for } t \geq 0,$$

Let $\varphi_0 \in C([-r, 0] \times [0, \pi]; \mathbb{R})$ be such that

$$\varphi_0(0, 0) = \varphi_0(0, \pi) = 0.$$

Then by Theorem 2.3, we deduce that Eq. (6.2) has a unique integral solution on $[-r, +\infty)$.
In order to study the existence of an almost automorphic solution of the following Equation

$$\frac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \text{ for } t \in \mathbb{R}. \tag{6.3}$$

We suppose that
($\mathbf{H_4}$) h is almost automorphic in t uniformly for $x \in [0, \pi]$, which means that there exists a measurable function $g : \mathbb{R} \times [0, \pi] \to \mathbb{R}$ such that

$$\lim_{n\to\infty} h(t + s_n, x) = g(t, x) \text{ exists for all } t \text{ in } \mathbb{R} \text{ uniformly in } x \in [0, \pi]$$

and

$$\lim_{n\to\infty} g(t - s_n, x) = h(t, x) \text{ for all } t \text{ in } \mathbb{R} \text{ uniformly in } x \in [0, \pi]$$

Moreover, we suppose that:

($\mathbf{H_5}$) there exists a constant $\beta \in (0, 1)$ such that

$$\int_{-r}^{0} |G(\theta)|\, d\theta \leq (1 - \beta)\,.$$

Proposition 6.2 *Assume that* ($\mathbf{H_4}$) *and* ($\mathbf{H_5}$) *hold. Then there exists* $\varphi \in C$ *such that Eq.* (6.2) *has a bounded solution on* $\mathbb{R}^+$*. Consequently Eq.* (6.3) *has an almost automorphic solution.*

Proof The first goal is to prove that Eq. (6.2) has a bounded solution on $\mathbb{R}^+$. Let $\rho = (1 + \dfrac{|f|_\infty}{\beta})$, where $|f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|$. Consider $\varphi \in C_0$ such that $|\varphi|_C < \rho$. We claim that

$$|u\,(t)| \leq \rho \text{ for all } t \geq 0. \tag{6.4}$$

We proceed by contradiction. Let t_0 be the first time such that (6.4) is not true. Then

$$t_0 = \inf\{t > 0 : |u\,(t)| > \rho\}\,.$$

By continuity of u, one has

$$|u\,(t_0)| = \rho,$$

and there exists a positive constant $\varepsilon > 0$ such that

$$|u\,(t)| > \rho \text{ for } t \in (t_0, t_0 + \varepsilon).$$

We have,

$$u\,(t_0) = T_0(t_0)\varphi(0) + \lim_{\lambda\to+\infty} \int_0^{t_0} T_0(t_0 - s) B_\lambda [L(u_s) + f(s)] ds$$

which implies that

$$|u\,(t_0)| \leq e^{-t_0}\rho + \int_0^{t_0} e^{-(t_0-s)} \left[\int_{-r}^{0} |G(\theta)|\, |u(s+\theta)|\, d\theta + |f|_\infty\right] ds.$$

Since $|u\,(t)| \leq \rho$ for $t \leq t_0$. Then

$$|u\,(t)| \leq \rho \text{ for } t \in [-r, t_0]\,.$$

Therefore

$$|u\,(t_0)| \leq e^{-t_0}\rho + \left(1 - e^{-t_0}\right)\left[\int_{-r}^{0} |G(\theta)|\, d\theta\ \rho + |f|_\infty\right].$$

Condition ($\mathbf{H_5}$) implies that

$$|u\,(t_0)| \leq e^{-t_0}\rho + \left(1 - e^{-t_0}\right)\left[(1-\beta)\,\rho + |f|_\infty\right],$$

and

$$|u\,(t_0)| \leq e^{-t_0}\rho + \left(1 - e^{-t_0}\right)\rho + \left(1 - e^{-t_0}\right)\left[-\beta\rho + |f|_\infty\right].$$

Consequently, we obtain that

$$|u\,(t_0)| \leq \rho - \left(1 - e^{-t_0}\right)\beta < \rho,$$

by continuity of u, there exists a positive ε_0 such that

$$|u(t)| < \rho \text{ for } t \in (t_0, t_0 + \varepsilon_0),$$

which gives a contradiction and we deduce that Eq. (6.2) has a bounded integral solution u on $\mathbb{R}^+$, and by Theorem 5.6, we get that Eq. (6.3) has an almost automorphic solution.

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