Afrika Matematika (2019) 30:1103–1118
https://doi.org/10.1007/s13370-019-00705-z

Generalized Bernstein–Durrmeyer operators of blending type

Arun Kajla[1] · **Meenu Goyal**[2]

Received: 16 July 2018 / Accepted: 18 July 2019 / Published online: 22 July 2019

Abstract
In this article, we present the Durrmeyer variant of generalized Bernstein operators that preserve the constant functions involving a non-negative parameter ρ. We derive the approximation behaviour of these operators including a global approximation theorem via Ditzian–Totik modulus of continuity and the order of convergence for the Lipschitz type space. Furthermore, we study a Voronovskaja type asymptotic formula, local approximation theorem by means of second order modulus of smoothness and the rate of approximation for absolutely continuous functions having a derivative equivalent to a function of bounded variation. Lastly, we illustrate the convergence of these operators for certain functions using Maple software.

Keywords Positive approximation · Global approximation · Rate of convergence · Modulus of continuity · Steklov mean

Mathematics Subject Classification 41A25 · 26A15

1 Introduction

Bernstein introduced the most famous algebraic polynomials $B_n(f; x)$ in approximation theory in order to give a constructive proof of Weierstrass's theorem which is given by

$$B_n(f; x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x \in [0, 1],$$

✉ Arun Kajla
rachitkajla47@gmail.com

Meenu Goyal
meenu.goyal700@gmail.com

1 Department of Mathematics, Central University of Haryana, Mahendragarh, Haryana 123031, India

2 School of Mathematics, Thapar University, Patiala, India

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and he proved that if $f \in C[0,1]$ then $B_n(f;x)$ converges uniformly to $f(x)$ in $[0,1]$.

The Bernstein operators have been used in many branches of mathematics and computer science. Due to their useful structure, Bernstein polynomials and their generalizations have been intensively studied. Among others we refer the readers to (cf. [4,13,19,26,32,35,36]).

For $f \in C(J)$ with $J = [0,1]$, Chen et al. [15] introduced a vital generalization of the Bernstein operators depending on a non-negative real parameter α $(0 \le \alpha \le 1)$ as

$$T_n^{(\alpha)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) f\left(\frac{k}{n}\right), \quad x \in J \tag{1.1}$$

where $p_{n,k}^{(\alpha)}(x) = \left[\binom{n-2}{k}(1-\alpha)x + \binom{n-2}{k-2}(1-\alpha)(1-x) + \binom{n}{k}\alpha x(1-x)\right] x^{k-1}$ $(1-x)^{n-k-1}$ and $n \ge 2$. They obtained a Voronovskaja type asymptotic formula, the rate of approximation in terms of modulus of smoothness and shape preserving properties for these operators. In the particular case, $\alpha = 1$, these operators reduce to the well-known Bernstein operators. Kajla and Acar [28] introduced the Durrmeyer variant of the operators (1.1) and investigated the rate of approximation of these operators.

Gonska and Păltănea [21] presented genuine Bernstein–Durrmeyer type operators and obtained the simultaneous approximation for these operators. Gupta and Rassias [25] studied approximation behavior of Durrmeyer type of Lupaş operators based on Polya distribution. Goyal et al. [22] derived Baskakov–Szàsz type operators and studied quantitative convergence theorems for these operators. Gupta et al. [23] introduced a hybrid operators based on inverse Polya–Eggenberger distribution and studied the degree of approximation and uniform convergence. Acu and Gupta [8] introduced a summation-integral type operators involving two parameters and studied some direct results e.g. Voronovskaja type asymptotic formula, local approximation and weighted approximation of these operators. Very recently, Kajla and Goyal [31] considered the hybrid operators involving non-negative parameters and investigated their order of approximation. In the literature survey, several researchers have been studied the approximation properties of hybrid operators [cf. [1–3,5–7,9–12,14,20,24,27,29,30,34]].

For $f \in C(J)$, we construct the following Durrmeyer variant of the operators (1.1) depending on a parameter $\rho > 0$ as follows:

$$\mathcal{G}_{n,\rho}^{(\alpha)}(f;x) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x) \int_0^1 \mu_{n,\rho}(t) f(t)dt, \tag{1.2}$$

where $\mu_{n,\rho}(t) = \dfrac{t^{k\rho}(1-t)^{(n-k)\rho}}{B(k\rho+1,(n-k)\rho+1)}$ and $B(k\rho+1,(n-k)\rho+1)$ is the beta function defined by $B(e,f) = \int_0^1 t^{e-1}(1-t)^{f-1}dt = \dfrac{\Gamma(e)\Gamma(f)}{\Gamma(e+f)}$, $e, f > 0$ and $p_{n,k}^{(\alpha)}(x)$ is defined as above. It is seen that the operators $\mathcal{G}_{n,\rho}^{(\alpha)}$ reproduce the constant functions.

The aim of this note is to find the approximation properties for the generalized Bernstein–Durrmeyer operators involving a nonnegative parameter of the operators defined in (1.2). We give a Voronovskaja type theorem, global approximation theorem by means of Ditzian–Totik modulus of smoothness, Lipschitz type space and a local approximation theorem with the help of second order modulus of continuity. Furthermore, we study the rate of approximation for absolutely continuous functions having a derivative equivalent to a function of bounded variation. Lastly, we illustrate the convergence of these operators for certain functions using Maple software.

2 Auxiliary results

Lemma 1 *Let* $e_i(x) = x^i, i = \overline{0,4}$. *For the generalized Bernstein–Durrmeyer operators* $\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)$, *we have*

(i)

$$\mathcal{G}_{n,\rho}^{(\alpha)}(e_0;x) = 1;$$

(ii)

$$\mathcal{G}_{n,\rho}^{(\alpha)}(e_1;x) = \frac{n\rho x + 1}{n\rho + 2};$$

(iii)

$$\mathcal{G}_{n,\rho}^{(\alpha)}(e_2;x) = \frac{x^2\rho^2\left(n^2 + 2(\alpha-1) - n\right)}{(n\rho+3)(n\rho+2)} + \frac{x\rho\left(n\rho^2 + 3n\rho - 2(\alpha-1)\rho^2\right)}{(n\rho+3)(n\rho+2)} + \frac{2}{(n\rho+3)(n\rho+2)};$$

(iv)

$$\begin{aligned}\mathcal{G}_{n,\rho}^{(\alpha)}(e_3;x) = {} & \frac{x^3\rho^3\left(n^3 + 6n\alpha - 3n^2 - 4n - 12(\alpha-1)\right)}{(n\rho+4)(n\rho+3)(n\rho+2)} \\ & + \frac{3x^2\rho^2\left(6n^2 + 3n\rho + 3n^2\rho - 6n\alpha\rho - 6n + 6(\alpha-1)(2+3\rho)\right)}{(n\rho+4)(n\rho+3)(n\rho+2)} \\ & + \frac{x\rho\left(n\rho^2 + 6n\rho + 11n - 6(\alpha-1)\rho(2+\rho)\right)}{(n\rho+4)(n\rho+3)(n\rho+2)} \\ & + \frac{6}{(n\rho+4)(n\rho+3)(n\rho+2)};\end{aligned}$$

(v)

$$\begin{aligned}\mathcal{G}_{n,\rho}^{(\alpha)}(e_4;x) = {} & \frac{x^4\rho^4\left(n^4 - 6n^3 + 72(\alpha-1) - 6n(10\alpha-9) + n^2(12\alpha-1)\right)}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)} \\ & + \frac{x^3\rho^3}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}\Big[10n^3 - 30n^2 + 10n(6\alpha-4) \\ & - 7n^2\rho + 6n^3\rho + 6n(6\alpha-5)\rho + 6n(10\alpha-9)\rho + n^2(12\alpha-1)\rho \\ & - 24(\alpha-1)(6\rho+5)\Big] + \frac{x^2\rho^2}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)} \\ & \times\Big[35n(n-1) - 10n\rho + 30n^2\rho - 10n(6\alpha-4)\rho - n\rho^2 + 7n^2\rho^2 \\ & - 6n(6\alpha-5)\rho^2 + 2(\alpha-1)(43\rho^2 + 90\rho + 35)\Big] \\ & + \frac{x\rho\left(35n\rho + 50n + 10n\rho^2 + n\rho^3 - 2(\alpha-1)\rho(7\rho^2 + 30\rho + 35)\right)}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)} \\ & + \frac{24}{(n\rho+5)(n\rho+4)(n\rho+3)(n\rho+2)}.\end{aligned}$$

Lemma 2 *For $m = 1, 2$, the m^{th} order central moments of $\mathcal{G}_{n,\rho}^{(\alpha)}$ defined as $\tau_{n,\rho,m}^{(\alpha)}(x) = \mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^m; x)$ we get*

(i)

$$\tau_{n,\rho,1}^{(\alpha)}(x) = \frac{1-2x}{(n\rho+2)};$$

(ii)

$$\tau_{n,\rho,2}^{(\alpha)}(x) = \frac{x(1-x)\,(\rho(n+(n-2\alpha+2)\rho)-6)}{(n\rho+2)(n\rho+3)} + \frac{2}{(n\rho+2)(n\rho+3)}.$$

Remark 1 For every $x \in J$, we have

$$\lim_{n\to\infty} n\,\tau_{n,\rho,1}^{(\alpha)}(x) = \frac{1-2x}{\rho},$$
$$\lim_{n\to\infty} n\,\tau_{n,\rho,2}^{(\alpha)}(x) = \frac{x(1-x)(1+\rho)}{\rho},$$
$$\lim_{n\to\infty} n^2\,\tau_{n,\rho,4}^{(\alpha)}(x) = \frac{3x^2(1-x)^2(1+\rho)^2}{\rho^2}.$$

Lemma 3 *For $n \in \mathbb{N}$, we obtain*

$$\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2; x) \le \frac{\mathcal{X}_\rho^{(\alpha)}\;x(1-x)}{(1+n\rho)},$$

where $\mathcal{X}_\rho^{(\alpha)}$ is a positive constant depending on α and ρ.

3 Direct estimates

Theorem 1 *Let $f \in C(J)$. Then $\lim_{n\to\infty} \mathcal{G}_{n,\rho}^{(\alpha)}(f; x) = f(x)$, uniformly on J.*

Proof In view of Lemma 1, $\mathcal{G}_{n,\rho}^{(\alpha)}(1; x) = 1$, $\mathcal{G}_{n,\rho}^{(\alpha)}(e_1; x) \to x$, $\mathcal{G}_{n,\rho}^{(\alpha)}(e_2; x) \to x^2$ as $n \to \infty$, uniformly in J. Applying Bohman-Korovkin criterion, it follows that $\mathcal{G}_{n,\rho}^{(\alpha)}(f; x) \to f(x)$ as $n \to \infty$, uniformly on J. □

3.1 Voronovskaja type theorem

In this section we prove Voronvoskaja type theorem for the operators $\mathcal{G}_{n,\rho}^{(\alpha)}$.

Theorem 2 *Let $f \in C(J)$. If f'' exists at a point $x \in J$, then we have*

$$\lim_{n\to\infty} n\left[\mathcal{G}_{n,\rho}^{(\alpha)}(f; x) - f(x)\right] = \frac{1-2x}{\rho} f'(x) + \frac{(1+\rho)x(1-x)}{2\rho} f''(x).$$

Proof By Taylor's expansion of f, we get

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2, \tag{3.1}$$

where $\lim_{t\to x} \varpi(t,x)=0$. By applying the linearity of the operator $\mathcal{G}_{n,\rho}^{(\alpha)}$, we obtain

$$\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)-f(x)=\mathcal{G}_{n,\rho}^{(\alpha)}((t-x);x)f'(x)+\frac{1}{2}\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2;x)f''(x)$$
$$+\mathcal{G}_{n,\rho}^{(\alpha)}(\varpi(t,x)(t-x)^2;x).$$

Now, applying Cauchy–Schwarz property, we can get

$$n\mathcal{G}_{n,\rho}^{(\alpha)}(\varpi(t,x)(t-x)^2;x)\le\sqrt{\mathcal{G}_{n,\rho}^{(\alpha)}(\varpi^2(t,x);x)}\sqrt{n^2\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^4;x)}.$$

From Theorem 1, we have $\lim_{n\to\infty}\mathcal{G}_{n,\rho}^{(\alpha)}(\varpi^2(t,x);x)=\varpi^2(x,x)=0$, since $\varpi(t,x)\to 0$ as $t\to x$, and Remark 1 for every $x\in J$, we may write

$$\lim_{n\to\infty} n^2\mathcal{G}_{n,\rho}^{(\alpha)}\left((t-x)^4;x\right)=\frac{3x^2(1-x)^2(1+\rho)^2}{\rho^2}. \tag{3.2}$$

Hence,

$$n\mathcal{G}_{n,\rho}^{(\alpha)}(\varpi(t,x)(t-x)^2;x)=0.$$

Applying Remark 1, we get

$$\lim_{n\to\infty} n\mathcal{G}_{n,\rho}^{(\alpha)}(t-x;x)=\frac{1-2x}{\rho},$$
$$\lim_{n\to\infty} n\mathcal{G}_{n,\rho}^{(\alpha)}\left((t-x)^2;x\right)=\frac{(1+\rho)x(1-x)}{\rho}. \tag{3.3}$$

Collecting the results from above the theorem is completed. □

3.2 Local approximation

We begin by recalling the following K-functional :

$$K_2(f,\delta)=\inf\{||f-g||+\delta||g''||:g\in W^2\}\ (\delta>0),$$

where $W^2=\{g:g''\in C(J)\}$ and $||.||$ is the uniform norm on $C(J)$. By [16], ∃ a positive constant $M>0$ such that

$$K_2(f,\delta)\le M\omega_2(f,\sqrt{\delta}), \tag{3.4}$$

where the modulus of smoothness of second order for $f\in C(J)$ is defined as

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\le\sqrt{\delta}}\ \sup_{x,x+2h\in J}|f(x+2h)-2f(x+h)+f(x)|.$$

The modulus of continuity for $f\in C(J)$ is defined by

$$\omega(f,\delta)=\sup_{0<h\le\delta}\ \sup_{x,x+h\in J}|f(x+h)-f(x)|.$$

The Steklov mean is defined as

$$f_h(x)=\frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2f(x+u+v)-f(x+2(u+v))]\,du\,dv. \tag{3.5}$$

The Steklov mean satisfies the following inequality:

(a) $\|f_h - f\|_{C(J)} \le \omega_2(f,h)$.

(b) $f_h', f_h'' \in C(J)$ and $\|f_h'\|_{C(J)} \le \frac{5}{h}\omega(f,h), \quad \|f_h''\|_{C(J)} \le \frac{9}{h^2}\omega_2(f,h)$,

Theorem 3 *Let $f \in C(J)$. Then for each $x \in J$, we have*

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x)\right| \le 5\omega\left(f, \sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)}\right).$$

Proof For $x \in J$, and applying the Steklov mean f_h that is given by (3.5), we can write

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x)\right| \le \mathcal{G}_{n,\rho}^{(\alpha)}(|f - f_h|;x) + |\mathcal{G}_{n,\rho}^{(\alpha)}(f_h - f_h(x);x)| + |f_h(x) - f(x)|. \tag{3.6}$$

From (1.2), for each $f \in C(J)$ we obtain

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)\right| \le ||f||. \tag{3.7}$$

By assumption (a) of the Steklov mean and (3.7), we get

$$\mathcal{G}_{n,\rho}^{(\alpha)}(|f - f_h|;x) \le \|\mathcal{G}_{n,\rho}^{(\alpha)}(f - f_h)\| \le \|f - f_h\| \le \omega_2(f,h).$$

Applying Taylor's expansion and Cauchy–Schwarz inequality, we have

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f_h - f_h(x);x)\right| \le \|f_h'\|\sqrt{\mathcal{G}_{n,\rho}^{(\alpha)}\left((t-x)^2;x\right)} + \frac{1}{2}\|f_h''\|\mathcal{G}_{n,\rho}^{(\alpha)}\left((t-x)^2;x\right).$$

By Lemma 2 and property (b) of the Steklov mean, we get

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f_h - f_h(x);x)\right| \le \frac{5}{h}\omega(f,h)\sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)} + \frac{9}{2h^2}\omega_2(f,h)\tau_{n,\rho,2}^{(\alpha)}(x).$$

Finally, choosing $h = \sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)}$, we obtain the desired result. □

3.3 Global approximation

Now, we recall the definitions of the Ditzian–Totik first order modulus of continuity and the K-functional [17]. Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C(J)$. The first order modulus of smoothness is defined by

$$\omega_\phi(f,t) = \sup_{0<h\le t}\left\{\left|f\left(x + \frac{h\phi(x)}{2}\right) - f\left(x - \frac{h\phi(x)}{2}\right)\right|, x \pm \frac{h\phi(x)}{2} \in J\right\},$$

and the corresponding K-functional is given by

$$\overline{K}_\phi(f,t) = \inf_{g\in W_\phi}\{||f-g|| + t||\phi g'|| + t^2||g'||\} \ (t > 0),$$

where $W_\phi = \{g : g \in AC_{loc}, ||\phi g'|| < \infty, ||g'|| \le \infty\}$ and $||.||$ is the uniform norm on $C(J)$. It is well known that (Theorem 3.1.2, [17]) $\overline{K}_\phi(f,t) \sim \omega_\phi(f,t)$ which means that there exists a constant $M > 0$ such that

$$M^{-1}\omega_\phi(f,t) \le \overline{K}_\phi(f,t) \le M\omega_\phi(f,t). \tag{3.8}$$

Now, we establish the order of approximation with the aid of the Ditzian–Totik modulus of the first and second order.

Theorem 4 *Let f be in $C(J)$ and $\phi(x)=\sqrt{x(1-x)}$, then for each $x\in[0,1)$, we get*

$$|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)-f(x)|\le C\omega_{\phi}\left(f,\sqrt{\frac{\mathcal{X}_{\rho}^{(\alpha)}}{(1+n\rho)}}\right),$$

where $\mathcal{X}_{\rho}^{(\alpha)}$ is defined in Lemma 3 and $C>0$ is a constant.

Proof By using the relation $g(t)=g(x)+\int_x^t g'(u)du$, we can write

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(g;x)-g(x)\right|=\left|\mathcal{G}_{n,\rho}^{(\alpha)}\left(\int_x^t g'(u)du;x\right)\right|. \tag{3.9}$$

For any $x,t\in(0,1)$, we get

$$\left|\int_x^t g'(u)du\right|\le||\phi g'||\left|\int_x^t\frac{1}{\phi(u)}du\right|. \tag{3.10}$$

Therefore,

$$\begin{aligned}\left|\int_x^t\frac{1}{\phi(u)}du\right|&=\left|\int_x^t\frac{1}{\sqrt{u(1-u)}}du\right|\le\left|\int_x^t\left(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\right)du\right|\\&\le 2\left(\,|\sqrt{t}-\sqrt{x}\,|+|\sqrt{1-t}-\sqrt{1-x}\,|\,\right)\\&=2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right)\\&<2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right)\le\frac{2\sqrt{2}\,|t-x|}{\phi(x)}.\end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11) and applying Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\mathcal{G}_{n,\rho}^{(\alpha)}(g;x)-g(x)|&<2\sqrt{2}||\phi g'||\phi^{-1}(x)\mathcal{G}_{n,\rho}^{(\alpha)}(|t-x|;x)\\&\le 2\sqrt{2}||\phi g'||\phi^{-1}(x)\left(\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2;x)\right)^{1/2}.\end{aligned}$$

From Lemma 3, we get

$$|\mathcal{G}_{n,\rho}^{(\alpha)}(g;x)-g(x)|<C\sqrt{\frac{\mathcal{X}_{\rho}^{(\alpha)}}{(1+n\rho)}}||\phi g'||. \tag{3.12}$$

Applying Lemma 1 and (3.12), we get

$$\begin{aligned}|\,\mathcal{G}_{n,\rho}^{(\alpha)}(f)-f\,|&\le|\,\mathcal{G}_{n,\rho}^{(\alpha)}(f-g;x)\,|+|f-g|+|\,\mathcal{G}_{n,\rho}^{(\alpha)}(g;x)-g(x)\,|\\&\le C\left(||f-g||+\sqrt{\frac{\mathcal{X}_{\rho}^{(\alpha)}}{(1+n\rho)}}||\phi g'||\right).\end{aligned} \tag{3.13}$$

Taking infimum on the right hand side of (3.13) over all $g\in W_{\phi}$, we may write

$$|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)-f(x)|\le C\overline{K}_{\phi}\left(f;\sqrt{\frac{\mathcal{X}_{\rho}^{(\alpha)}}{(1+n\rho)}}\right). \tag{3.14}$$

Using $\overline{K_{\phi}}(f,t)\sim\omega_{\phi}(f,t)$, we immediately arrive to the required relation. □

[33] Let us consider the Lipschitz-type space with two parameters $\kappa_1 \geq 0, \kappa_2 > 0$, we have

$$Lip_M^{(\kappa_1,\kappa_2)}(\sigma) := \left\{ f \in C(J) : |f(t) - f(x)| \leq M \frac{|t-x|^\sigma}{(t + \kappa_1 x^2 + \kappa_2 x)^{\frac{\sigma}{2}}}; t \in J, x \in (0,1] \right\},$$

where $0 < \sigma \leq 1$.

Theorem 5 *Let $f \in Lip_M^{(\kappa_1,\kappa_2)}(\sigma)$. Then for all $x \in (0,1]$, we have*

$$\left| \mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x) \right| \leq M \left(\frac{\tau_{n,\rho,2}^{(\alpha)}(x)}{\kappa_1 x^2 + \kappa_2 x} \right)^{\sigma/2}.$$

Proof Let us prove the theorem for the case $0 < \sigma \leq 1$, using Holder's property with $p = \frac{2}{\sigma}, q = \frac{2}{2-\sigma}$.

$$\begin{aligned}
\left| \mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x) \right| &\leq \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 |f(t) - f(x)| \, \mu_{n,\rho}(t) dt \\
&\leq \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \left(\int_0^1 |f(t) - f(x)|^{\frac{2}{\sigma}} \mu_{n,\rho}(t) dt \right)^{\frac{\sigma}{2}} \\
&\leq \left\{ \sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 |f(t) - f(x)|^{\frac{2}{\sigma}} \mu_{n,\rho}(t) dt \right\}^{\frac{\sigma}{2}} \\
&\quad \times \left(\sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 \mu_{n,\rho}(t) dt \right)^{\frac{2-\sigma}{2}} \\
&= \left(\sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 |f(t) - f(x)|^{\frac{2}{\sigma}} \mu_{n,\rho}(t) dt \right)^{\frac{\sigma}{2}} \\
&\leq M \left(\sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 \frac{(t-x)^2}{(t + \kappa_1 x^2 + \kappa_2 x)} \mu_{n,\rho}(t) dt \right)^{\frac{\sigma}{2}} \\
&\leq \frac{M}{(\kappa_1 x^2 + \kappa_2 x)^{\frac{\sigma}{2}}} \left(\sum_{k=0}^n p_{n,k}^{(\alpha)}(x) \int_0^1 (t-x)^2 \mu_{n,\rho}(t) dt \right)^{\frac{\sigma}{2}} \\
&= \frac{M}{(\kappa_1 x^2 + \kappa_2 x)^{\frac{\sigma}{2}}} \mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2; x)^{\frac{\sigma}{2}} \\
&= \frac{M}{(\kappa_1 x^2 + \kappa_2 x)^{\frac{\sigma}{2}}} (\tau_{n,\rho,2}^{(\alpha)}(x))^{\frac{\sigma}{2}}.
\end{aligned}$$

□

Theorem 6 *For $f \in C^1(J)$ and $x \in J$, we have*

$$\left| \mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x) \right| \leq \left| \frac{1-2x}{(n\rho + 2)} \right| |f'(x)| + 2\sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)} \, \omega \left(f', \sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)} \right). \quad (3.15)$$

Proof Let $f \in C^1(J)$. For any $t, x \in J$, we have

$$f(t) - f(x) = f'(x)(t - x) + \int_x^t \big(f'(u) - f'(x)\big)\, du.$$

Using $\mathcal{G}_{n,\rho}^{(\alpha)}(\cdot; x)$ on both sides of the above relation, we may write

$$\mathcal{G}_{n,\rho}^{(\alpha)}(f(t) - f(x); q_n, x) = f'(x)\mathcal{G}_{n,\rho}^{(\alpha)}(t - x; x) + \mathcal{G}_{n,\rho}^{(\alpha)}\left(\int_x^t \big(f'(u) - f'(x)\big)\, du; x\right)$$

Using the well-known inequality of modulus of continuity $|f(t)-f(x)| \le \omega(f, \delta)\left(\frac{|t-x|}{\delta} + 1\right)$, $\delta > 0$, we obtain

$$\left|\int_x^t \big(f'(u) - f'(x)\big)\, du\right| \le \omega(f', \delta)\left(\frac{(t - x)^2}{\delta} + |t - x|\right),$$

it follows that

$$\begin{aligned}\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f; x) - f(x)\right| &\le |f'(x)| \quad |\mathcal{G}_{n,\rho}^{(\alpha)}(t - x; x)| \\ &\quad +\omega(f', \delta)\left\{\frac{1}{\delta}\mathcal{G}_{n,\rho}^{(\alpha)}((t - x)^2; x) + \mathcal{G}_{n,\rho}^{(\alpha)}(|t - x|; x)\right\}.\end{aligned}$$

From Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f; x) - f(x)\right| &\le |f'(x)| \quad |\mathcal{G}_{n,\rho}^{(\alpha)}(t - x; x)| \\ &\quad +\omega(f', \delta)\left\{\frac{1}{\delta}\sqrt{\mathcal{G}_{n,\rho}^{(\alpha)}((t - x)^2; x)} + 1\right\}\sqrt{\mathcal{G}_{n,\rho}^{(\alpha)}((t - x)^2; x)}.\end{aligned}$$

Now, choosing $\delta = \sqrt{\tau_{n,\rho,2}^{(\alpha)}(x)}$, the required result follows. □

3.4 Rate of convergence

Let $DBV_{(J)}$ be the class of all absolutely continuous functions f defined on J, having on J a derivative f' equivalent to a function of bounded variation on J. We observed that the functions $f \in DBV_{(J)}$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0)$$

where $g \in BV_{(J)}$, i.e., g is a function of bounded variation on J.

The operators $\mathcal{G}_{n,\rho}^{(\alpha)}(f; x)$ also admit the integral representation

$$\mathcal{G}_{n,\rho}^{(\alpha)}(f; x) = \int_0^1 \mathcal{U}_{n,\rho}^{(\alpha)}(x, t) f(t)dt, \tag{3.16}$$

where the kernel $\mathcal{U}_{n,\rho}^{(\alpha)}(x, t)$ is given by

$$\mathcal{U}_{n,\rho}^{(\alpha)}(x, t) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(x)\mu_{n,\rho}(t).$$

Lemma 4 *For a fixed $x \in (0, 1)$ and sufficiently large n, we have*

(i) $\gamma_{n,\rho}^{(\alpha)}(x,y)=\int_0^y \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt\le \frac{\mathcal{X}_\rho^{(\alpha)}}{(1+n\rho)}\frac{x(1-x)}{(x-y)^2},\ 0\le y<x,$

(ii) $1-\gamma_{n,\rho}^{(\alpha)}(x,z)=\int_z^1 \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt\le \frac{\mathcal{X}_\rho^{(\alpha)}}{(1+n\rho)}\frac{x(1-x)}{(z-x)^2},\ x<z<1,$

where $\mathcal{X}_\rho^{(\alpha)}$ is defined in Lemma 3.

Proof (i) From Lemma 3, we get

$$\gamma_{n,\rho}^{(\alpha)}(x,y)=\int_0^y \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt\le\int_0^y\left(\frac{x-t}{x-y}\right)^2\mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt$$
$$=\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2;x)(x-y)^{-2}\le\frac{\mathcal{X}_\rho^{(\alpha)}}{(1+n\rho)}\frac{x(1-x)}{(x-y)^2}.$$

The proof of (ii) is similar hence the details are missing. □

Theorem 7 *Let $f\in DBV(J)$. Then for every $x\in(0,1)$ and sufficiently large n, we have*

$$\left|\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)-f(x)\right|\le\frac{(1-2x)}{(n\rho+2)}\frac{|f'(x+)+f'(x-)|}{2}$$
$$+\sqrt{\frac{\mathcal{X}_\rho^{(\alpha)}x(1-x)}{(1+n\rho)}}\frac{|f'(x+)-f'(x-)|}{2}$$
$$+\frac{\mathcal{X}_\rho^{(\alpha)}(1-x)}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x)$$
$$+\frac{\mathcal{X}_\rho^{(\alpha)}x}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f'_x)+\frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f'_x),$$

where $\bigvee_c^d(f'_x)$ denotes the total variation of f'_x on $[c,d]$ and f'_x is defined by

$$f'_x(t)=\begin{cases}f'(t)-f'(x-), & 0\le t<x\\ 0, & t=x\\ f'(t)-f'(x+) & x<t<1.\end{cases}\tag{3.17}$$

Proof Since $\mathcal{G}_{n,\rho}^{(\alpha)}(1;x)=1$, by using (3.16), for every $x\in(0,1)$ we may write

$$\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)-f(x)=\int_0^1\mathcal{U}_{n,\rho}^{(\alpha)}(x,t)(f(t)-f(x))dt$$
$$=\int_0^1\mathcal{U}_{n,\rho}^{(\alpha)}(x,t)\left(\int_x^t f'(u)du\right)dt.\tag{3.18}$$

For any $f\in DBV(J)$, by (3.17) we can write

$$f'(u)=f'_x(u)+\frac{1}{2}(f'(x+)+f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)$$
$$+\delta_x(u)\left[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right],\tag{3.19}$$

where

$$\delta_x(u) = \begin{cases} 1\ , & u = x \\ 0\ , & u \neq x. \end{cases}$$

Obviously,

$$\int_0^1 \left(\int_x^t \left(f'(u) - \frac{1}{2}(f'(x+) + f'(x-)) \right) \delta_x(u) du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt = 0.$$

By (3.16) and simple calculations we find

$$\begin{aligned} &\int_0^1 \left(\int_x^t \frac{1}{2}(f'(x+) + f'(x-)) du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt \\ &= \frac{1}{2}(f'(x+) + f'(x-)) \int_0^1 (t-x)\mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt \\ &= \frac{1}{2}(f'(x+) + f'(x-)) \mathcal{G}_{n,\rho}^{(\alpha)}((t-x);x) \end{aligned}$$

and

$$\begin{aligned} &\left| \int_0^1 \mathcal{U}_{n,\rho}^{(\alpha)}(x,t) \left(\int_x^t \frac{1}{2}(f'(x+) - f'(x-)) sgn(u-x) du \right) dt \right| \\ &\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \int_0^1 |t-x| \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt \\ &\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \mathcal{G}_{n,\rho}^{(\alpha)}(|t-x|;x) \\ &\leq \frac{1}{2} \mid f'(x+) - f'(x-) \mid \left(\mathcal{G}_{n,\rho}^{(\alpha)}((t-x)^2;x) \right)^{1/2}. \end{aligned}$$

By Lemmas 2 and 3, using (3.18)–(3.19) we find

$$\begin{aligned} |\mathcal{G}_{n,\rho}^{(\alpha)}(f;x) - f(x)| \leq \frac{1}{2}|f'(x+) - f'(x-)| &\sqrt{\frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)}} \\ &+ \left| \int_0^x \left(\int_x^t f_x'(u)du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt \right. \\ &\left. + \int_x^1 \left(\int_x^t f_x'(u)du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt \right|. \end{aligned} \tag{3.20}$$

Let

$$\begin{aligned} \mathcal{S}_{n,\rho}^{(\alpha)}(f_x',x) &= \int_0^x \left(\int_x^t f_x'(u)du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt, \\ \mathcal{T}_{n,\rho}^{(\alpha)}(f_x',x) &= \int_x^1 \left(\int_x^t f_x'(u)du \right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt. \end{aligned}$$

To complete the proof, it is sufficient to determine the terms $\mathcal{S}_{n,\rho}^{(\alpha)}(f_x',x)$ and $\mathcal{T}_{n,\rho}^{(\alpha)}(f_x',x)$. Since $\int_c^d d_t\gamma_{n,\rho}^{(\alpha)}(x,t) \leq 1$ for all $[c,d] \subseteq J$, applying the integration by parts and applying

Lemma 4 with $y = x - (x/\sqrt{n})$, we have

$$\begin{aligned}
|\mathcal{S}_{n,\rho}^{(\alpha)}(f_x', x)| &= \left| \int_0^x \left(\int_x^t f_x'(u)du \right) d_t \gamma_{n,\rho}^{(\alpha)}(x,t) \right| \\
&= \left| \int_0^x \gamma_{n,\rho}^{(\alpha)}(x,t) f_x'(t)dt \right| \\
&\leq \left(\int_0^y + \int_y^x \right) |f_x'(t)| \, |\gamma_{n,\rho}^{(\alpha)}(x,t)|dt \\
&\leq \frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_0^y \bigvee_t^x (f_x')(x-t)^{-2}dt + \int_y^x \bigvee_t^x (f_x')dt \\
&\leq \frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_0^{x-(x/\sqrt{n})} \bigvee_t^x (f_x')(x-t)^{-2}dt + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f_x').
\end{aligned}$$

By the substitution of $u = x/(x-t)$, we have

$$\begin{aligned}
\frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f_x')dt &= \frac{\mathcal{X}_\rho^{(\alpha)} (1-x)}{(1+n\rho)} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^x (f_x')du \\
&\leq \frac{\mathcal{X}_\rho^{(\alpha)} (1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/u)}^x (f_x')du \\
&\leq \frac{\mathcal{X}_\rho^{(\alpha)} (1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f_x').
\end{aligned}$$

Thus,

$$\left| \mathcal{S}_{n,\rho}^{(\alpha)}(f_x', x) \right| \leq \frac{\mathcal{X}_\rho^{(\alpha)} (1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f_x') + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f_x'). \tag{3.21}$$

Using the integration by parts and Lemma 4 with $z = x + ((1-x)/\sqrt{n})$, we can write

$$
\begin{aligned}
\left|\mathcal{T}_{n,\rho}^{(\alpha)}(f_x', x)\right| &= \left|\int_x^1 \left(\int_x^t f_x'(u)du\right) \mathcal{U}_{n,\rho}^{(\alpha)}(x,t)dt\right| \\
&= \left|\int_x^z \left(\int_x^t f_x'(u)du\right) d_t(1-\gamma_{n,\rho}^{(\alpha)}(x,t)) \right.\\
&\quad \left. + \int_z^1 \left(\int_x^t f_x'(u)du\right) d_t(1-\gamma_{n,\rho}^{(\alpha)}(x,t))\right| \\
&= \left|\left[\int_x^t f_x'(u)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))du\right]_x^z - \int_x^z f_x'(t)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))dt \right.\\
&\quad \left. + \int_z^1 \left(\int_x^t f_x'(u)du\right) d_t(1-\gamma_{n,\rho}^{(\alpha)}(x,t))\right| \\
&= \left|\int_x^z f_x'(u)du(1-\gamma_{n,\rho}^{(\alpha)}(x,z)) - \int_x^z f_x'(t)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))dt \right.\\
&\quad + \left[\int_x^t f_x'(u)du(1-\gamma_{n,\rho}^{(\alpha)}(x,t))\right]_z^1 \\
&\quad \left. - \int_z^1 f_x'(t)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))dt\right| \\
&= \left|\int_x^z f_x'(t)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))dt + \int_z^1 f_x'(t)(1-\gamma_{n,\rho}^{(\alpha)}(x,t))dt\right| \\
&\le \frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_z^1 \bigvee_x^t (f_x')(t-x)^{-2}dt + \int_x^z \bigvee_x^t (f_x')dt \\
&= \frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x^t (f_x')(t-x)^{-2}dt \\
&\quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x').
\end{aligned}
$$

By the substitution of $v = (1-x)/(t-x)$, we have

$$
\begin{aligned}
\left|\mathcal{T}_{n,\rho}^{(\alpha)}(f_x', x)\right| &\le \frac{\mathcal{X}_\rho^{(\alpha)} x(1-x)}{(1+n\rho)} \int_1^{\sqrt{n}} \bigvee_x^{x+((1-x)/v)} (f_x')(1-x)^{-1}dv \\
&\quad + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x') \\
&\le \frac{\mathcal{X}_\rho^{(\alpha)} x}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x+((1-x)/v)} (f_x')dv + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})} (f_x') \\
&= \frac{\mathcal{X}_\rho^{(\alpha)} x}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+((1-x)/k)} (f_x') + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x))/\sqrt{n}} (f_x'). \qquad (3.22)
\end{aligned}
$$

Combining (3.20)–(3.22), we get the desired relation. □

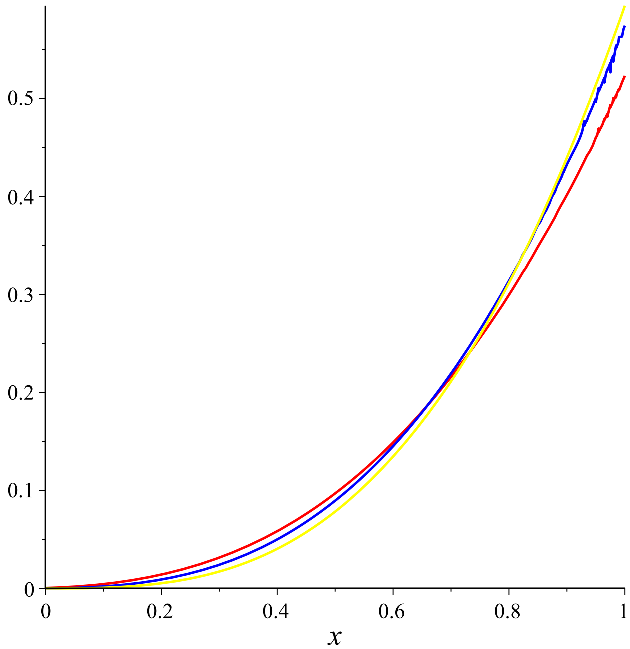

Fig. 1 The convergence of $\mathcal{G}_{20,4}^{(0.3)}(f;x)$ and $D_{20}(f;x)$ to $f(x)$

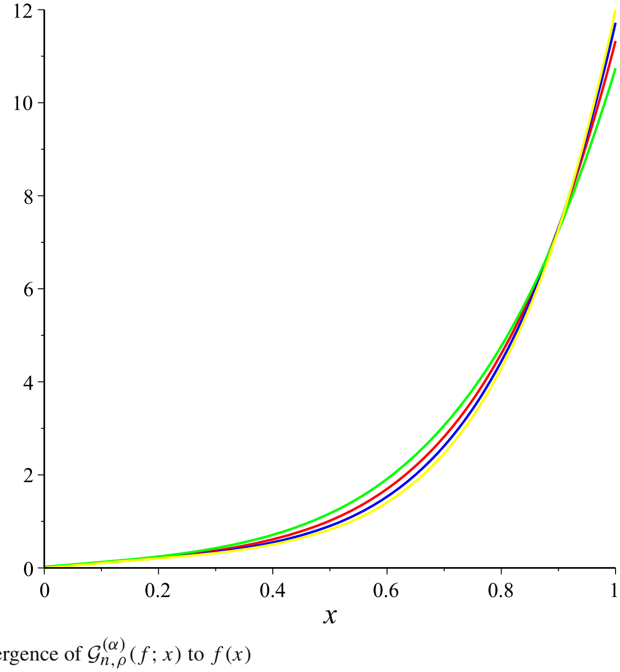

Fig. 2 The convergence of $\mathcal{G}_{n,\rho}^{(\alpha)}(f;x)$ to $f(x)$

4 Numerical examples

Example 1 In Fig. 1, for $n = 20, \alpha = 0.3, \rho = 4$, the comparison of convergence of $\mathcal{G}_{20,4}^{(0.3)}(f; x)$ (blue) and the Bernstein–Durrmeyer $D_n(f; x)$ [18] (red) operators to $f(x) = x^2 \sin(2x/\pi)$(yellow) is illustrated. It is observed that the $\mathcal{G}_{20,4}^{(0.3)}(f; x)$ operators gives a better approximation to $f(x)$ than Bernstein–Durrmeyer $D_n(f; x)$ for $n = 20, \alpha = 0.3, \rho = 4$.

Example 2 For $n \in \{10, 20, 50\}$, $\alpha = 0.2$ and $\rho = 4$, the convergence of the operators $\mathcal{G}_{10,4}^{(0.2)}(f; x)$ (green), $\mathcal{G}_{20,4}^{(0.2)}(f; x)$ (red) and $\mathcal{G}_{50,4}^{(0.2)}(f; x)$ (blue) to $f(x) = x^7 + 10x^5 + x$ (yellow) is illustrated in Fig. 2. We observed that for the values of n increasing, the graph of $\mathcal{G}_{n,\rho}^{(\alpha)}(f; x)$ goes to the graph of the function $f(x)$.

References

1. Abel, U., Gupta, V., Ivan, M.: Asymptotic approximation of functions and their derivatives by generalized Baskakov–Szász–Durrmeyer operators. Anal. Theory Appl. **21**(1), 15–26 (2005)
2. Acar, T., Aral, A., Raşa, I.: Modified Bernstein–Durrmeyer operators. Gen. Math. **22**(1), 27–41 (2014)
3. Acar, T.: Asymptotic formulas for generalized Szász–Mirakyan operators. Appl. Math. Comput. **263**, 223–239 (2015)
4. Acar, T., Aral, A.: On pointwise convergence of q-Bernstein operators and their q-derivatives. Numer. Funct. Anal. Optim. **36**(3), 287–304 (2015)
5. Acar, T., Ulusoy, G.: Approximation properties of generalized Szász–Durrmeyer operators. Period. Math. Hungar. **72**(1), 64–75 (2016)
6. Acar, T., Gupta, V., Aral, A.: Rate of convergence for generalized Szász operators. Bull. Math. Sci. **1**(1), 99–113 (2011)
7. Acu, A.M., Hodiş, S., Raşa, I.: A survey on estimates for the differences of positive linear operators. Constr. Math. Anal. **1**(2), 113–127 (2018)
8. Acu, A.M., Gupta, V.: Direct results for certain summation-integral type Baskakov–Szász operators. Results. Math. https://doi.org/10.1007/s00025-016-0603-2
9. Agrawal, P.N., Goyal, M., Kajla, A.: $q-$Bernstein-Schurer-Kantorovich type operators. Boll. Unione Mat. Ital. **8**, 169–180 (2015)
10. Agrawal, P.N., Gupta, V., Sathish Kumar, A., Kajla, A.: Generalized Baskakov–Szász type operators. Appl. Math. Comput. **236**, 311–324 (2014)
11. Bărbosu, D.: On the remainder term of some bivariate approximation formulas based on linear and positive operators. Constr. Math. Anal. **1**(2), 73–87 (2018)
12. Bărbosu, D., Muraru, C.V.: Approximating B-continuous functions using GBS operators of Bernstein–Schurer–Stancu type based on q-integers. Appl. Math. Comput. **259**, 80–87 (2015)
13. Cárdenas-Morales, D., Garrancho, P., Raşa, I.: Asymptotic formulae via a Korovkin type result. Abstr. Appl. Anal. Art ID 217464, pp. 12 (2012)
14. Costarelli, D., Vinti, G.: A Quantitative estimate for the sampling Kantorovich series in terms of the modulus of continuity in orlicz spaces. Constr. Math. Anal. **2**(1), 8–14 (2019)
15. Chen, X., Tan, J., Liu, Z., Xie, J.: Approximation of functions by a new family of generalized Bernstein operators. J. Math. Anal. Appl. **450**, 244–261 (2017)
16. Devore, R.A., Lorentz, G.G.: Constructive Approximation, Grundlehren der Mathematischen Wissenschaften, vol. 303. Springer, Berlin (1993)
17. Ditzian, Z., Totik, V.: Moduli of Smoothness. Springer, New York (1987)
18. Durrmeyer, J.L.: Une formula d'inversion, de la transformee de Laplace: Application a la theorie des Moments. These de 3e Cycle, Faculte des Sciences de l'universite de Paris, Paris (1967)
19. Gadjiev, A.D., Ghorbanalizaeh, A.M.: Approximation properties of a new type Bernstein–Stancu polynomials of one and two variables. Appl. Math. Comput. **216**, 890–901 (2010)
20. Gal, S.G., Trifa, S.: Quantitative estimates for L^p-approximation by Bernstein–Kantorovich–Choquet polynomials with respect to distorted lebesgue measures. Constr. Math. Anal. **2**(1), 15–21 (2019)
21. Gonska, H., Păltănea, R.: Simultaneous approximation by a class of Bernstein–Durrmeyer operators preserving linear functions. Czech. Math. J. **60**(135), 783–799 (2010)
22. Goyal, M., Gupta, V., Agrawal, P.N.: Quantitative convergence results for a family of hybrid operators. Appl. Math. Comput. **271**, 893–904 (2015)

23. Gupta, V., Acu, A.M., Sofonea, D.F.: Approximation of Baskakov type Pòlya–Durrmeyer operators. Appl. Math. Comput. **294**, 318–331 (2017)
24. Gupta, V., Agarwal, R.P.: Convergence Estimates in Approximation Theory. Springer, Berlin (2014)
25. Gupta, V., Rassias, T.M.: Lupas-Durrmeyer operators based on Polya distribution. Banach J. Math. Anal. **8**(2), 146–155 (2014)
26. Heilmann, M., Raşa, I.: On the decomposition of Bernstein operators. Numer. Funct. Anal. Optim. **36**(1), 72–85 (2015)
27. Kajla, A., Acu, A.M., Agrawal, P.N.: Baskakov–Szász type operators based on inverse Pólya–Eggenberger distribution. Ann. Funct. Anal. **8**, 106–123 (2017)
28. Kajla, A., Acar, T.: Blending type approximation by generalized Bernstein–Durrmeyer type operators. Miskolc Math. Notes **19**, 319–336 (2018)
29. Kajla, A., Acar, T.: A new modification of Durrmeyer type mixed hybrid operators. Carpathian J. Math. **34**, 47–56 (2018)
30. Kajla, A., Agrawal, P.N.: Szász–Durrmeyer type operators based on Charlier polynomials. Appl. Math. Comput. **268**, 1001–1014 (2015)
31. Kajla, A., Goyal, M.: Blending type approximation by Bernstein–Durrmeyer type operators. Matematicki Vesnik **70**(1), 40–54 (2018)
32. Mursaleen, M., Ansari, K.J., Khan, A.: On $(p, q)-$analogue of Bernstein operators. Appl. Math. Comput. **266**, 874–882 (2015)
33. Özarslan, M.A., Aktuğlu, H.: Local approximation for certain King type operators. Filomat **27**, 173–181 (2013)
34. Păltănea, R.: Optimal estimates with moduli of continuity. Result. Math. **32**, 318–331 (1997)
35. Taşdelen, F., Başcanbaz-Tunca, G., Erençin, A.: On a new type Bernstein–Stancu operators. Fasci. Math. **48**, 119–128 (2012)
36. Yang, M., Yu, D., Zhou, P.: On the approximation by operators of Bernstein–Stancu types. Appl. Math. Comput. **246**, 79–87 (2014)