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More notions and mappings via somewhere dense sets

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Abstract
In this paper, the authors continue studying more properties of somewhere dense sets. They derive some interesting results related to them such as the collection of all somewhere dense subsets of a strongly hyperconnected space (X, τ) forms a filter on X, and a topological space which contains at least two disjoint somewhere dense sets is an ST_1-space. Then they introduce and study a concept of S-limit points of a soft set. Depending on somewhere dense and cs-dense sets, they also define and investigate various maps between topological spaces, namely SD-continuous, SD-irresolute, SD-open, SD-closed and SD-homeomorphism maps.

Keywords Somewhere dense set · S-limit point · SD-continuous map · Topological space

Mathematics Subject Classification 54A05 · 54A20 · 54C10

1 Introduction

Studying different types of generalized open sets is a major area of research in general topology during the last few decades. By generalized open sets, mathematicians generalized many concepts in general topology such as continuity, compactness, connectedness, etc. and studied their properties. Historically, Stone [23] started the study of generalized open sets in 1937. He presented a notion of regular open sets. Then Levine [18] introduced semi open sets in 1963 and Njastad [22] introduced α-open sets in 1965. Mashhour et al. [20] in 1982, and Abd El-Monsef et al. [2], 1983, presented and studied preopen and β-open sets, respectively. In 1996, Andrijević [8] defined and investigated the concept of b-open sets. Recently, the concepts of somewhere dense sets and ST_1-spaces were introduced and studied in detail by Al-shami [6]. The class of somewhere dense sets contains all regular open, α-open, preopen, semi open, β-open and b-open sets with the exception of the empty set. These six types

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of generalized open sets are defined by using interior and closure operators. In a similar way, these kind of generalized open sets were studied on approximation spaces [1] and soft topological spaces [3,4,7,9,10,12,14,16,24]. It is worthy note that one of the significant applications of regular open and semi-open sets in digital topology was presented in [5,13], respectively.

The purpose of this paper is to investigate further properties of somewhere dense and cs-dense sets and to define new maps on topological spaces depending on somewhere dense and cs-dense sets. We point out that the collection of all somewhere dense (resp. cs-dense) subsets of a strongly hyperconnected spaces forms a filter (resp. an ideal) and a topological space which contains at least two disjoint somewhere dense sets is an ST_1-space. Also, we introduce a concept of S-limit points of a soft set and derive its main properties with the help of examples. In the last two sections, we present and study five types of maps, namely SD-continuous, SD-irresolute, SD-open, SD-closed and SD-homeomorphism maps. Their main properties are derived and the relationships among some of them are illustrated. In particular, we investigate under what conditions the restriction of SD-continuous (resp. SD-open, SD-closed) maps is also SD-continuous (resp. SD-open, SD-closed). Moreover, we expound of why we omit the empty and universe sets from the definitions of SD-open and SD-closed maps, respectively.

2 Preliminaries

Hereafter, we use the two ordered pairs (X, τ) and (Y, θ) to indicate topological spaces on which no separation axiom is assumed unless otherwise stated, and we use the notation $S(\tau)$ to stand for the collection of all somewhere dense subsets of (X, τ).

Definition 2.1 A subset E of (X, τ) is said to be:

(i) regular open [23] if $E = int(cl(E))$.
(ii) semi-open [18] if $E \subseteq cl(int(E))$.
(iii) α-open [22] if $E \subseteq int(cl(int(E)))$.
(iv) preopen [20] if $E \subseteq int(cl(E))$.
(v) β-open [2] if $E \subseteq cl(int(cl(E)))$.
(vi) b-open [8] if $E \subseteq int(cl(E)) \bigcup cl(int(E))$.
(vii) somewhere dense [6] if $int(cl(E)) \neq \emptyset$. The complement of a somewhere dense set is said to be cs-dense.

The complement of a ξ-open set is said to be ξ-closed for $\xi \in \{regular, semi, \alpha, pre, \beta, b\}$.

Remark 2.2 We note that the term of preopen sets was used under the name of locally dense by Corson and Michael [11].

Theorem 2.3 [21] *If M is an open subset of (X, τ), then $M \bigcap cl(B) \subseteq cl(M \bigcap B)$ for each $B \subseteq X$.*

Definition 2.4 [21] A topological space (X, τ) with no mutually disjoint non-empty open sets is said to be hyperconnected.

Theorem 2.5 [6] *A subset B of (X, τ) is cs-dense if and only if there is a proper closed subset F of X such that $int(B) \subseteq F$.*

Theorem 2.6 [6] *Every subset of (X, τ) is somewhere dense or cs-dense.*

Theorem 2.7 [6] *The intersection of an open set and a somewhere dense set in a hyperconnected space* (X, τ) *is somewhere dense.*

Theorem 2.8 [6] *Let* $(\prod_{i=1}^{i=s} X_i, T)$ *be a finite product topological space. Then* M_i *is a somewhere dense subset of* (X_i, τ_i) *for each* $i = 1, 2, \ldots, s$ *if and only if* $\prod_{i=1}^{i=s} M_i$ *is a somewhere dense subset of* $(\prod_{i=1}^{i=s} X_i, T)$.

Definition 2.9 [6] (X, τ) is called strongly hyperconnected provided that a subset of X is dense if and only if it is non-empty and open.

Theorem 2.10 [6] *If* M *and* N *are* cs*-dense (resp. somewhere dense) subsets of a strongly hyperconnected space* (X, τ)*, then* $M \bigcup N$ *(resp.* $M \bigcap N$*) is* cs*-dense (resp. somewhere dense).*

Theorem 2.11 [6] *Every superset of a somewhere dense set is somewhere dense.*

Definition 2.12 [6] Let M be a subset of (X, τ). Then:

(i) The S-interior of M ($Sint(M)$, in short) is the union of all somewhere dense sets contained in M.
(ii) The S-closure of M ($Scl(M)$, in short) is the intersection of all cs-dense sets containing M.
(iii) The S-boundary of M ($Sb(M)$, in short) is the set of elements which belong to $(Sint(M) \bigcup Sint(M^c))^c$.

Proposition 2.13 [6] *Consider a subset* M *of* (X, τ)*. Then:*

(i) $M \subseteq Scl(M)$*; and a set* $M \neq X$ *is* cs*-dense if and only if* $M = Scl(M)$.
(ii) $Sint(M) \subseteq M$*; and a non-empty set* M *is somewhere dense if and only if* $M = Sint(M)$.
(iii) $(Sint(M))^c = Scl(M^c)$ *and* $(Scl(M))^c = Sint(M^c)$.

Definition 2.14 [6] (X, τ) is called an ST_1-space if for any pair of distinct points $a, b \in X$, there are two somewhere dense sets such that one of them contains a but not b and the other contains b but not a.

Theorem 2.15 [6] (X, τ) *is an* ST_1*-space if and only if for each pair of distinct points* $a, b \in X$*, there are two disjoint somewhere dense sets one containing* a *and the other containing* b*.*

Definition 2.16 [15] A non-empty collection $\mathscr{F}$ of subsets of X is called a filter if it meets the next three axioms

(i) $\emptyset \notin \mathscr{F}$.
(ii) Every superset of any member of $\mathscr{F}$ belongs to $\mathscr{F}$.
(iii) The intersection of any two member of $\mathscr{F}$ belongs to $\mathscr{F}$.

Definition 2.17 [17] A non-empty collection $\mathscr{I}$ of subsets of X is called an ideal if it meets the following three axioms

(i) $X \notin \mathscr{I}$.
(ii) Every subset of any member of $\mathscr{I}$ belongs to $\mathscr{I}$.
(iii) The union of any two members of $\mathscr{I}$ belongs to $\mathscr{I}$.

Definition 2.18 [19] Let $f : (X, \tau) \rightarrow (Y, \theta)$ be a map. Then

(i) The graph of f, usually denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.
(ii) The graph of f is called closed if it is a closed subset of the product space $X \times Y$.

3 Further properties of somewhere dense sets

In this section, we continue studying main properties of somewhere dense and cs-dense sets. We investigate the conditions under which the collection of all somewhere dense (resp. cs-dense) sets forms a filter (resp. an ideal). Also, we present the concept of S-limit points and derive main features with the help of examples.

Theorem 3.1 *If (X, τ) is strongly hyperconnected, then $S(\tau)$ forms a filter on X.*

Proof It can be seen the following properties for $S(\tau)$:

(i) From the definition of somewhere dense sets, $\emptyset \notin S(\tau)$.
(ii) It follows from Theorem 2.10 that $A \bigcap B \in S(\tau)$ for each $A, B \in S(\tau)$.
(iii) If $A \in S(\tau)$ and $A \subseteq B$, then from Theorem 2.11, $B \in S(\tau)$.

Hence $S(\tau)$ is a filter on X. □

Theorem 3.2 *The collection of all cs-dense subsets of a strongly hyperconnected space (X, τ) forms an ideal on X.*

Proof Let $\mathscr{I}$ be the collection of all cs-dense subsets of a strongly hyperconnected space (X, τ). Then we have the following properties

(i) Obviously, $X \notin \mathscr{I}$.
(ii) It follows from Theorem 2.10 that $A \bigcup B \in \mathscr{I}$ for each $A, B \in \mathscr{I}$.
(iii) Let $B \in \mathscr{I}$ and $A \subseteq B$. Since the set B is cs-dense, then there is a proper closed subset H of X satisfies that $int(B) \subseteq H$. Since $int(A) \subseteq H$, then from Theorem 2.5, $A \in \mathscr{I}$.

Hence $\mathscr{I}$ is an ideal on X. □

Lemma 3.3 *Let a set X be infinite. Then there is an infinite somewhere dense subset E of (X, τ) such that E^c is infinite.*

Proof Let $\Lambda = \{E_i : i \in I\}$ be the family of all infinite somewhere dense subsets of (X, τ). Since $X \in \Lambda$, then $I \neq \emptyset$. Suppose that every E_i^c is finite. Then every closed set is finite. Take an infinite subset A of X such that A^c is infinite as well. Now, $cl(A) = cl(A^c) = X$. This implies that the sets A and A^c are both somewhere dense and cs-dense. But this contradicts our assumption. Hence the lemma holds. □

Theorem 3.4 *Let X be an infinite set. If $\Lambda = \{E_i : i \in I\}$ is a collection of all infinite somewhere dense subsets of (X, τ) such that E_i^c is infinite for each $i \in I$, then I is infinite.*

Proof Let the given conditions be satisfied. Since $\{E_i^c : i \in I\}$ is the collection of all infinite cs-dense sets, then the collection of all infinite closed sets is contained in $\{E_i^c : i \in I\}$. Let $E_{i_0} \in \Lambda$. Then we have the next two cases

(i) Either $cl(E_{i_0}) = X$. Then $\{E_{i_0} \bigcup \{x_j\} : x_j \in E_{i_0}^c\}$ is a collection of infinite somewhere dense sets. So $\{E_{i_0} \bigcup \{x_j\} : x_j \in E_{i_0}^c\} \subseteq \Lambda$.
(ii) Or there exists $j \in I$ such that $cl(E_{i_0}) = E_j^c$. Then the infinite set E_j^c is both somewhere dense and cs-dense. So $\{E_j^c \bigcup \{x_k\} : x_k \in E_j\}$ is a collection of infinite somewhere dense sets. Thus $\{E_j^c \bigcup \{x_k\} : x_k \in E_j\} \subseteq \Lambda$.

Since $E_{i_0}^c$ and E_j are infinite in the above both cases, then the index I is infinite. □

Corollary 3.5 *Let X be an infinite set. Then for each $x \in X$, there is an infinite somewhere dense subset E of (X, τ) contains x such that E^c is infinite.*

Theorem 3.6 *(X, τ) is an ST_1-space if and only if it contains at least two disjoint somewhere dense sets.*

Proof *Necessity*: The proof comes immediately from Theorem 2.15.
Sufficiency: Let a, b be two distinct points in X and let E and H be two disjoint somewhere dense subsets of X. Then we have only the following four cases:

(1) $a \in E$ and $b \in H$.
(2) $a, b \in E$ or $a, b \in H$. Say, $a, b \in E$. Since the set H is somewhere dense, then $H \bigcup \{a\}$ and $H \bigcup \{b\}$ are two somewhere dense. Obviously, $H \bigcup \{a\}$ and $H \bigcup \{b\}$ containing a and b, respectively, such that $b \notin H \bigcup \{a\}$ and $a \notin H \bigcup \{b\}$.
(3) $a, b \notin E \bigcup H$. Then $E \bigcup \{a\}$ and $H \bigcup \{b\}$ are two disjoint somewhere dense sets containing a and b, respectively.
(4) E contains only one of the points a, b and H does not contain both points a, b or H contains only one of the points a, b and E does not contain both points a, b. Say, ($a \in E$ and $b \notin H$). Then E and $H \bigcup \{b\}$ are somewhere dense sets containing a and b, respectively.

Since a and b are chosen arbitrary, then (X, τ) is an ST_1-space. □

For the sake of brevity, the proofs of the following two propositions have been omitted.

Proposition 3.7 *Every non-empty proper regular open subset of (X, τ) is both somewhere dense and cs-dense.*

Proposition 3.8 *If (X, τ) is hyperconnected, then the collection of somewhere dense sets coincides with the collection of non-empty β-open sets.*

Definition 3.9 Let E be a subset of (X, τ). Then x is called an S-limit point of E if $[G \backslash \{x\}] \bigcap E \neq \emptyset$ for each somewhere dense set G containing x.
All S-limit points of E is called an S-derived set and denoted by $Sl(E)$.

The following example illustrates that the sets of S-limit points and S-closure points (resp. S-interior points) of a soft set need not be cs-dense (resp. somewhere dense).

Example 3.10 Let $\tau = \{\emptyset, G_n = \{n, n+1, n+2, ...\} : n \in \mathscr{N}\}$ be a topology on the set of real numbers $\mathscr{N}$. Let a set $M = \{2n : n \in \mathscr{N}\}$. Then $Sl(M) = \mathscr{N}$ and $Scl(M) = \mathscr{N}$ which is not cs-dense. Also, $Sint(M) = \emptyset$ which is not somewhere dense.

Proposition 3.11 *Suppose M and N are subsets of (X, τ). Then:*

(i) *If $M \subseteq N$, then $Sl(M) \subseteq Sl(N)$.*
(ii) $Sl(M \bigcap N) \subseteq Sl(M) \bigcap Sl(N)$
(iii) $Sl(M) \bigcup Sl(N) \subseteq Sl(M \bigcup N)$.

We point out in the next example that the inclusion relation in the above proposition can be proper.

Example 3.12 Assume that $(\mathscr{N}, \tau)$ is the same as in Example 3.10. Let $M = \{2n : n \in \mathscr{N}\}$, $N = \{2n+1 : n \in \mathscr{N}\}$, $O = \{1, 3\}$ and $P = \{2\}$. Then

(i) $Sl(M) = Sl(N) = \mathscr{N}$, whereas neither $M \nsubseteq N$ nor $N \nsubseteq M$.
(ii) $Sl(M \bigcap N) = \emptyset$, whereas $Sl(M) \bigcap Sl(N) = \mathscr{N}$.
(iii) $Sl(O) = \{1\}$ and $Sl(P) = \{1\}$. So $Sl(O) \bigcup Sl(P) = \{1\}$, whereas $Sl(O \bigcup P) = \{1, 2\}$.

Proposition 3.13 *If (X, τ) is strongly hyperconnected, then $Sl(M \bigcup N) = Sl(M) \bigcup Sl(N)$ for each subsets M and N of X.*

Proof By observing that $(X, S(\tau) \bigcup \{\emptyset\})$ is a topological space, when (X, τ) is strongly hyperconnected, the proposition holds. □

Theorem 3.14 *(i) A proper subset M of (X, τ) is cs-dense if and only if $Sl(M) \subseteq M$.*
(ii) If $M \bigcup Sl(M) \neq X$, then $M \bigcup Sl(M)$ is cs-dense.
(iii) $Scl(M) = M \bigcup Sl(M)$ for each subset M of (X, τ).

Proof (i) *Necessity*: Assume that M is a cs-dense set and $x \notin M$. Then $x \in M^c \in S(\tau)$. As $M^c \bigcap M = \emptyset$, then $x \notin Sl(M)$. Therefore $Sl(M) \subseteq M$.
Sufficiency: Let $x \in M^c \neq \emptyset$ and $Sl(M) \subseteq M$. Then $x \notin Sl(M)$. Therefore there is a somewhere dense set E_x containing x such that $(E_x \backslash \{x\}) \bigcap M = \emptyset$. As $x \in M^c$, then $E_x \bigcap M = \emptyset$. Now, $E_x \subseteq M^c$. Therefore $M^c = \bigcup\{E_x : x \in M^c\}$. Thus M is cs-dense.

(ii) Let $x \notin (M \bigcup Sl(M))$. Then $x \notin M$ and $x \notin Sl(M)$. Therefore there is a somewhere dense set E containing x such that

$$E \bigcap M = \emptyset \tag{3.1}$$

On the other hand, for each $x \in E$, we have $x \notin M$. Therefore $(E \backslash \{x\}) \bigcap M = \emptyset$. Thus $x \notin Sl(M)$ and this implies that

$$E \bigcap Sl(M) = \emptyset \tag{3.2}$$

From (3.1) and (3.2), we obtain $E \bigcap (M \bigcup Sl(M)) = \emptyset$ and this implies that $x \notin Sl(M \bigcup Sl(M))$. Hence $Sl(M \bigcup Sl(M)) \subseteq (M \bigcup Sl(M))$. By (i), we get $M \bigcup Sl(M)$ is cs-dense.

(iii) To prove this result, we consider the following two cases

(1) Either $Scl(M) = X$. Suppose that there is $x \in X$ such that $x \notin (M \bigcup Sl(M))$. Then $x \notin M$ and $x \notin Sl(M)$. So there is a somewhere dense set E containing x such that $E \bigcap M = \emptyset$. This implies that $M \subseteq E^c \neq X$. But this contradicts that $Scl(M) = X$. Thus $Scl(M) = M \bigcup Sl(M)$.
(2) Or $Scl(M) \neq X$. Then $Scl(M)$ is cs-dense. By (ii), we have $Sl(M) \subseteq Scl(M)$. So $M \bigcup Sl(M) \subseteq Scl(M)$. Also, $Scl(M)$ is the smallest cs-dense set containing M and by (ii) $M \bigcup Sl(M)$ is a cs-dense set containing M. Then $Scl(M) \subseteq M \bigcup Sl(M)$. So $Scl(M) = M \bigcup Sl(M)$.

From (1) and (2), we conclude that $Scl(M) = M \bigcup Sl(M)$ for each subset M of (X, τ). □

4 Somewhere dense continuous maps

The notions of somewhere dense continuous and somewhere dense irresolute maps are introduced and their characterizations are investigated. The equivalent conditions for them are given. Some results which associate somewhere dense continuous maps with restricted and graph maps are investigated.

Definition 4.1 A map $g : (X, \tau) \to (Y, \theta)$ is said to be somewhere dense continuous (briefly, SD-continuous) at $a \in X$ if for any open set U containing $g(a)$, there is a somewhere dense set E containing a such that $g(E) \subseteq U$.

Definition 4.2 A map $g : (X, \tau) \to (Y, \theta)$ is said to be SD-continuous if it is SD-continuous for each $a \in X$.

Theorem 4.3 *A map $g : (X, \tau) \to (Y, \theta)$ is SD-continuous if and only if the inverse image of each open set is empty or somewhere dense.*

Proof To prove the "if" part, let U be an open set in Y. If U is empty, then the proof is trivial. So consider U is non-empty. If $g^{-1}(U) = \emptyset$, then this part holds. If $g^{-1}(U) \neq \emptyset$, then for each $a \in g^{-1}(U)$, we have a somewhere dense subset E_a of X containing a such that $g(E_a) \subseteq U$. Thus $a \in E_a \subseteq g^{-1}(U)$ and $\cup\{E_a : a \in g^{-1}(U)\} = g^{-1}(U)$. Hence $g^{-1}(U)$ is somewhere dense.
To prove the "only if" part, assume that $a \in X$ and U is an open set including $g(a)$. Then $g^{-1}(U)$ is a somewhere dense set containing a and satisfies that $g(g^{-1}(U)) \subseteq U$. So g is SD-continuous at a. Since a is chosen arbitrary, then g is SD-continuous. □

Corollary 4.4 *A surjective map $g : (X, \tau) \to (Y, \theta)$ is SD-continuous if and only if the inverse image of each non-empty open set is somewhere dense.*

Proposition 4.5 *Every j-continuous map is SD-continuous for each $j \in \{\beta, b, semi, pre, \alpha\}$.*

Proof Consider that $g : (X, \tau) \to (Y, \theta)$ is a j-continuous map and let E be a non-empty open subset of Y. Then $g^{-1}(E)$ is a j-open subset of X for each $j \in \{\beta, b, semi, pre, \alpha\}$. Since any somewhere dense set contains all non-empty j-open set ($j \in \{\beta, b, semi, pre, \alpha\}$), then it follows from Theorem 4.3 that $g^{-1}(E)$ is empty or somewhere dense. Hence g is SD-continuous. □

The next example shows that the above theorem is not conversely.

Example 4.6 Consider $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}\}$ and $\theta = \{\emptyset, Y, \{y, z\}\}$ are topologies on $X = \{1, 2, 3\}$ and $Y = \{x, y, z\}$, respectively. Take a map g of (X, τ) into (Y, θ) which is defined as: $g(1) = y$, $g(2) = x$ and $g(3) = z$. Then g is SD-continuous. Since $g^{-1}(\{y, z\}) = \{1, 3\}$ and $\{1, 3\}$ is not j-open, then a map g is not j-continuous for each $j \in \{\beta, b, semi, pre, \alpha\}$.

Lemma 4.7 *If (A, τ_A) is an open subspace of (X, τ) and $E \subseteq X$, then $A \bigcap cl(E) = cl_A(A \bigcap E)$.*

Proof Consider $x \notin A \bigcap cl(E)$. Then we have two possible cases

- (i) Either $x \notin A$. So $x \notin cl_A(A \bigcap E)$.
- (ii) Or $x \in A$ and $x \notin cl(E)$. This means there is an open subset G of (X, τ) containing x such that $G \bigcap E = \emptyset$. Now, $x \in G \bigcap A$ and A is open. This implies that $(G \bigcap A) \bigcap (E \bigcap A) = \emptyset$. Thus $x \notin cl_A(A \bigcap E)$. □

Thus $cl_A(A \bigcap E) \subseteq A \bigcap cl(E)$.

On the otherhand, let $x \notin cl_A(A \bigcap E)$. Then the subspace (A, τ_A) containing open set H satisfies that $x \in H$ and $H \bigcap (A \bigcap E) = \emptyset$. By hypotheses, A is an open subset of X, So H is also an open subset of (X, τ). Now, $H \bigcap (A \bigcap E) = (H \bigcap A) \bigcap E = \emptyset$. then $x \notin A \bigcap cl(E)$. Thus $A \bigcap cl(E) \subseteq cl_A(A \bigcap E)$. Hence the proof is complete. □

Theorem 4.8 *If $f : (X, \tau) \to (Y, \theta)$ is SD-continuous and A is an open dense subset of (X, τ), then the restriction map $f_A : (A, \tau_A) \to (Y, \theta)$ is SD-continuous.*

Proof Let G be an open subset of Y. Then $f^{-1}(G)$ is an empty set or a somewhere dense set. If the set $f^{-1}(G)$ is empty, then the theorem holds. If the set $f^{-1}(G)$ is somewhere dense, then there is a non-empty open set U such that $U \subseteq cl(f^{-1}(G))$. Now, $U \bigcap A \subseteq cl(f^{-1}(G)) \bigcap A = cl_A(f^{-1}(G) \bigcap A)$. Since A is open dense, then $U \bigcap A$ is a non-empty open set in (A, τ_A). Therefore $f^{-1}(G) \bigcap A$ is a somewhere dense subset of (A, τ_A). Thus f_A is SD-continuous. □

Corollary 4.9 *If A is an open subset of a hyperconnected space (X, τ) and $f : (X, \tau) \to (Y, \theta)$ is SD-continuous, then $f_A : (A, \tau_A) \to (Y, \theta)$ is SD-continuous.*

The equivalent conditions for an SD-continuous map are presented in the next theorem.

Theorem 4.10 *Consider $g : (X, \tau) \to (Y, \theta)$ is a map. Then we have the next five equivalent statements:*

(i) *g is SD-continuous;*
(ii) *The inverse image of each closed subset of (Y, θ) is X or cs-dense;*
(iii) *$Scl(g^{-1}(K)) \subseteq g^{-1}(cl(K))$ for each $K \subseteq Y$;*
(iv) *$g(Scl(H)) \subseteq cl(g(H))$ for each $H \subseteq X$;*
(v) *$g^{-1}(int(K)) \subseteq Sint(g^{-1}(K))$ for each $K \subseteq Y$.*

Proof (i) $\Rightarrow$ (ii): Suppose that F is a closed subset of (Y, θ). Then F^c is open. Therefore $g^{-1}(F^c) = X - g^{-1}(F)$ is empty or somewhere dense. So $g^{-1}(F)$ is X or cs-dense.

(ii) $\Rightarrow$ (iii): Let $K \subseteq Y$. Then $g^{-1}(cl(K))$ is X or cs-dense. So we have two cases

(1) Either $g^{-1}(cl(K)) = X$. Then $Scl(g^{-1}(K)) \subseteq X = g^{-1}(cl(K))$.
(2) Or $g^{-1}(cl(K))$ is cs-dense. Then $Scl(g^{-1}(K)) \subseteq Scl(g^{-1}(cl(K))) = g^{-1}(cl(K))$.

(iii) $\Rightarrow$ (iv): Let $H \subseteq X$. Then $Scl(H) \subseteq Scl(g^{-1}(g(H)))$. By (iii), $Scl(g^{-1}(g(H))) \subseteq g^{-1}(cl(g(H)))$. Therefore $g(Scl(H) \subseteq g(g^{-1}(cl(g(H)))) \subseteq cl(g(H))$.

(iv) $\Rightarrow$ (v): Let K be any set of Y and $H = g^{-1}(Y - K)$. By (iv), $g(Scl(g^{-1}(K^c)) \subset cl(g(g^{-1}(K^c))) \subset cL(K^c)$. Hence $Scl((g^{-1}(K))^c) \subset g^{-1}((int(K))^c$ and hence $g^{-1}(int(K)) \subset Sint(g^{-1}(K))$.

(v) $\Rightarrow$ (i): Suppose that K is an open subset of Y. By (v), we obtain that $g^{-1}(K) = g^{-1}(int(K)) \subseteq Sint(g^{-1}(K))$. Since $Sint(g^{-1}(K)) \subseteq g^{-1}(K)$, then $g^{-1}(K) = Sint(g^{-1}(K))$. Therefore $g^{-1}(K)$ is empty or somewhere dense. Thus g is SD-continuous. □

Theorem 4.11 *Let f be a map of a hyperconnected space (X, τ) into (Y, θ) and let $g : (X, \tau) \to (X \times Y, T)$ be the graph map of f, where T is the product topology on $X \times Y$. Then f is SD-continuous if and only if g is SD-continuous.*

Proof *Necessity*: Let $a \in X$ and $g(a) \in W \in T$. Then there exist $G \in \tau$ and $H \in \theta$ such that $g(a) = (a, f(a)) \in G \times H \subseteq W$. Now, $a \in G$ and $f(a) \in H$. Since f is SD-continuous, then there is a somewhere dense subset E of X containing a such that $f(E) \subseteq H$. By Theorem 2.7, $G \bigcap E$ is a somewhere dense set containing a. Therefore $g(G \bigcap E) = (G \bigcap E, f(G \bigcap E)) \subseteq (G, f(E)) \subseteq G \times H \subseteq W$. Thus g is SD-continuous at a. Since a is an arbitrary point, then g is SD-continuous.

Sufficiency: Let a be any point of X and $f(a) \in V \in \theta$. Then $(a, f(a)) \in X \times V \in T$. Since g is SD-continuous, then there is a somewhere dense subset H of X containing a such that $g(H) \subseteq X \times V$. Now, $g(H) = (H, f(H))$ implies that $f(H) \subseteq V$. Thus f is SD-continuous. □

Theorem 4.12 *Let $f : (X, \tau) \to (Y, \theta)$ be an SD-continuous map and (Y, θ) be a T_2-space. Then the graph of f is a cs-dense subset of $X \times Y$.*

Proof Let $(x, y) \in (G(f))^c$. Then $y \neq f(x)$. Since (Y, θ) is a T_2-space, then there exist two disjoint open subsets H and W of Y containing y and $f(x)$, respectively. By hypothesis, f is SD-continuous. Then there is a somewhere dense subset U containing x such that $f(U) \subseteq W$. From Theorem 2.8, $U \times H$ is a somewhere dense subset of X. Since $f(U) \bigcap H = \emptyset$, then $(U \times H) \bigcap G(f) = \emptyset$. Therefore $(H \times U) \subseteq (G(f))^c$. Since the point (x, y) is arbitrarily chosen, then $(G(f))^c$ is a somewhere dense subset of $X \times Y$. This completes the proof. □

Theorem 4.13 *Let $p : (X, \tau) \to (Y, \theta)$ be a surjective i-continuous map for each $i \in \{\beta, b, semi, pre, \alpha\}$, and let $q : (Y, \theta) \to (Z, \mu)$ be a j-continuous map for each $j \in \{semi, \alpha\}$. Then the map $q \circ p$ is SD-continuous.*

Proof It is sufficient to prove the theorem in case of $i = \beta$ and $j = semi$, and the other follow easily.
Let G be an open subset of Z. Then $q^{-1}(G)$ is a $semi$-open subset of Y.

(1) If $q^{-1}(G) = \emptyset$, then $p^{-1}(q^{-1}(G)) = \emptyset$.
(2) If $q^{-1}(G) \neq \emptyset$, then there is a non-empty open subset H of Y satisfies that $H \subseteq q^{-1}(G)$. Now, $p^{-1}(H) \subseteq p^{-1}(q^{-1}(G))$. Since p is surjective β-continuous, then $p^{-1}(H)$ is a non-empty β-open subset of X. Therefore $p^{-1}(H)$ is somewhere dense and this implies that $p^{-1}(q^{-1}(G))$ is somewhere dense.

Thus $q \circ p$ is SD-continuous. □

Theorem 4.14 *If $p : (X, \tau) \to (Y, \theta)$ is a surjective continuous map and $q : (Y, \theta) \to (Z, \mu)$ is an SD-continuous map, then $q \circ p$ is SD-continuous.*

Proof Let U be a non-empty open subset of Z. Then $q^{-1}(U)$ is empty or a somewhere dense subset of Y.

(1) If $q^{-1}(U) = \emptyset$, then $p^{-1}(q^{-1}(U)) = \emptyset$.
(2) If $q^{-1}(U) \neq \emptyset$, then there is a non-empty open subset H of Y such that $H \subseteq cl(q^{-1}(U))$. Now, $p^{-1}(H) \subseteq p^{-1}(cl(q^{-1}(U)))$. Since p is continuous surjective, then $p^{-1}(H)$ is a non-empty open subset of X. Also, p^{-1} is continuous implies that $p^{-1}(cl(q^{-1}(U))) \subseteq cl(p^{-1}(q^{-1}(U)))$. Therefore $p^{-1}(q^{-1}(U))$ is somewhere dense.

Thus $q \circ p$ is SD-continuous. □

Definition 4.15 A map $g : (X, \tau) \to (Y, \theta)$ is said to be SD-irresolute provided that the inverse image of each somewhere dense subset of Y is empty or a somewhere dense subset of X.

Proposition 4.16 *Every SD-irresolute map is SD-continuous.*

Proof The proof is obvious. □

The above theorem is not conversely as it is illustrated in the next example.

Example 4.17 Let $\tau = \{\emptyset, X, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ and $\theta = \{\emptyset, Y, \{x, y\}, \{w, z\}\}$ be topologies on $X = \{1, 2, 3, 4\}$ and $Y = \{x, y, w, z\}$, respectively. Take a map $g : (X, \tau) \to (Y, \theta)$ which is defined as: $g(1) = x$, $g(2) = z$, $g(3) = y$ and $g(4) = w$. Then the map g is SD-continuous. On the other hand, $\{y\}$ is somewhere dense and $g^{-1}(\{y\}) = \{3\}$ is not somewhere dense. So the map g is not SD-irresolute.

Theorem 4.18 *For a map $g : (X, \tau) \to (Y, \theta)$, the following statements are equivalent:*

(i) *g is SD-irresolute;*
(ii) *The inverse image of each cs-dense subset of (Y, θ) is cs-dense;*
(iii) *$Scl(g^{-1}(A)) \subseteq g^{-1}(Scl(A))$ for each $A \subseteq Y$;*
(iv) *$g(Scl(H)) \subseteq Scl(g(H))$ for each $H \subseteq X$;*
(v) *$g^{-1}(Sint(A)) \subseteq Sint(g^{-1}(A))$ for each $A \subseteq Y$.*

Proof The proof is similar to that of Theorem 4.10. □

Theorem 4.19 *A map $g : (X, \tau) \to (Y, \theta)$ is SD-irresolute if one of the next conditions holds.*

(i) *$cl(g^{-1}(K)) \subseteq g^{-1}(Scl(K))$ for each $K \subseteq Y$.*
(ii) *$g(cl(H)) \subseteq Scl(g(H))$ for each $H \subseteq X$.*
(iii) *$g^{-1}(Sint(K)) \subseteq int(g^{-1}(K))$ for each $K \subseteq Y$.*

Proof (i) It is clear that $Scl(K) \subseteq cl(K)$ for each $K \subseteq Y$. If condition (i) holds, then $Scl(g^{-1}(K)) \subseteq cl(g^{-1}(K)) \subseteq g^{-1}(Scl(K))$. By (iii) of Theorem 4.18, g is SD-irresolute.

(ii) It is clear that $Scl(H) \subseteq cl(H)$ for each $H \subseteq X$. If condition (ii) holds, then $g(Scl(H)) \subseteq g(cl(H)) \subseteq Scl(g(H))$. By (iv) of Theorem 4.18, g is SD-irresolute.

(iii) It is clear that $int(K) \subseteq Sint(K)$ for each $K \subseteq Y$. If condition (iii) holds, then $g^{-1}(Sint(K)) \subseteq int(g^{-1}(K)) \subseteq Sint(g^{-1}(K))$. It follows from (v) of Theorem 4.18 that g is SD-irresolute. □

In the following, we present an example to show that the above theorem is not conversely.

Example 4.20 Consider τ is the indiscrete topology on $X = \{a, b\}$ and $\theta = \{\emptyset, Y, \{z\}\}$ is a topology on $Y = \{x, y, z\}$. If a map $p : (X, \tau) \to (Y, \theta)$ is given by the following $p(a) = x$ and $p(b) = y$. Then p is SD-irresolute, whereas the three conditions which mentioned in the above theorem are not satisfied as pointed out in the following:

(i) Let $K = \{y\}$. Then $cl(p^{-1}(K)) = X$ and $p^{-1}(Scl(K)) = \{b\}$.
Therefore $cl(p^{-1}(K)) \nsubseteq p^{-1}(Scl(K))$.
(ii) Let $H = \{a\}$. Then $p(cl(H)) = \{x, y\}$ and $p(H) = \{x\}$.
Therefore $p(cl(H)) \nsubseteq Scl(p(H))$.
(iii) Let $K = \{y, z\}$. Then $p^{-1}(Sint(K)) = \{b\}$ and $int(p^{-1}(K)) = \emptyset$.
Therefore $p^{-1}(Sint(K)) \nsubseteq int(p^{-1}(K))$.

Proposition 4.21 *Consider a map $g : (X, \tau) \to (Y, \theta)$ is injective SD-irresolute. If (Y, θ) is an ST_1-space, then (X, τ) is an ST_1-space.*

Proof Let $a \neq b$ in X and let g be injective. Then $g(a) \neq g(b)$. Therefore there are two disjoint somewhere dense sets G and H containing $g(a)$ and $g(b)$, respectively. Since $g^{-1}(G)$ and $g^{-1}(H)$ are disjoint somewhere dense sets containing a and b, respectively, then (X, τ) is an ST_1-space. □

5 *SD*-closed (*SD*-open, *SD*-homeomorphism) maps

We devote this section to introducing SD-closed (SD-open, SD-homeomorphism) maps and to seeking main properties. In particular, we give the sufficient conditions of restricted maps of SD-closed (resp. SD-open) maps to be SD-closed (resp. SD-open).

Definition 5.1 A map g of (X, τ) into (Y, θ) is said to be SD-closed (resp. SD-open) provided that the image of each proper closed (resp. non-empty open) subset of X is a cs-dense (resp. a somewhere dense) subset of Y.

We omit the empty set from the definition of an SD-open map because the image of the empty set is empty and the empty set is not somewhere dense. Also, we omit the universe set from the definition of an SD-closed map because the image of the universe set under a surjective map is the universe set in the codomain and the universe set is not cs-dense.

Theorem 5.2 *A map $g : (X, \tau) \rightarrow (Y, \theta)$ is SD-closed if and only if $Scl(g(K)) \subseteq g(cl(K))$ for each subset K of X.*

Proof *Necessity*: Assume that g is an SD-closed map. For any subset K of X, we have two cases:

(1) Either $cl(K) = X$. Then $g(cl(K)) = g(X)$. So $Scl(g(K)) \subseteq g(X) = g(cl(K))$.
(2) Or $cl(K) \neq X$. Then $g(cl(K))$ is a cs-dense set. Since $g(K) \subseteq g(cl(K))$, then $Scl(g(K)) \subseteq g(cl(K))$.

Sufficiency: Assume that K is a proper closed subset of X. By hypothesis, $g(K) \subseteq Scl(g(K)) \subseteq g(cl(K)) = g(K)$. Therefore $g(K) = Scl(g(K))$. Since $g(K) \neq Y$, then $g(K)$ is cs-dense. So g is SD-closed. □

Theorem 5.3 *A map $g : (X, \tau) \rightarrow (Y, \theta)$ is SD-open if and only if $g(int(K)) \subseteq Sint(g(K))$ for every subset K of X.*

Proof *Necessity*: Assume that g is an SD-open map and let K be a subset of X. Then we have two cases:

(1) Either $int(K) = \emptyset$. Then the theorem holds.
(2) Or $int(K) \neq \emptyset$. Then $g(int(K))$ is a somewhere dense set. Since $g(int(K)) \subseteq g(K)$, then $g(int(K)) \subseteq Sint(g(K))$.

Sufficiency: Assume that K is a non-empty open subset of X. Then $g(int(K)) = g(K) \subseteq Sint(g(K))$. Therefore $g(K) = Sint(g(K))$. Thus g is SD-open. □

Proposition 5.4 *A bijective map g of (X, τ) into (Y, θ) is SD-open if and only if it is SD-closed.*

Proof To prove the 'if' part, let G be a proper closed subset of (X, τ). Since g is SD-open, then $g(G^c)$ is somewhere dense and since g is bijective, then $g(G^c) = (g(G))^c$. So $g(G)$ is cs-dense. Hence g is SD-closed.
To prove the 'only if' part, we follow similar lines. □

Proposition 5.5 *Let $g : (X, \tau) \rightarrow (Y, \theta)$ be an SD-closed map and A be a closed subset of X. Then $g_A : (A, \tau_A) \rightarrow (Y, \theta)$ is SD-closed.*

Proof Suppose that H is a closed subset of (A, τ_A). Then there exists a closed subset F of (X, τ) such that $H = F \bigcap A$. Since A is a closed subset of (X, τ), then H is also a closed subset of (X, τ). Therefore $g_A(H) = g(H)$ which is cs-dense. Thus g_A is an SD-closed map. □

Proposition 5.6 *Let $g : (X, \tau) \rightarrow (Y, \theta)$ be an SD-open map and A be an open subset of X. Then $g_A : (A, \tau_A) \rightarrow (Y, \theta)$ is SD-open.*

Proof The proof is similar to that of Proposition 5.5. □

The proofs of the next two propositions are easy and so will be omitted.

Proposition 5.7 *Every injective j-closed map is SD-closed for each $j \in \{\beta, b, semi, pre, \alpha\}$.*

Proposition 5.8 *Every j-open map is SD-open, for each $j \in \{\beta, b, semi, pre, \alpha\}$.*

Proposition 5.9 *The next four statements hold for the two maps $p : (X, \tau) \to (Y, \theta)$ and $q : (Y, \theta) \to (Z, \sigma)$.*

(i) *If p is i-open for $i = \{\alpha, semi\}$ and q is j-open for $j = \{\beta, b, semi, pre, \alpha\}$, then $q \circ p$ is SD-open.*
(ii) *If $q \circ p$ is SD-open and p is continuous surjective, then q is SD-open.*
(iii) *If $q \circ p$ is open and q is SD-continuous injective, then p is SD-open.*
(iv) *If $q \circ p$ is SD-open and q is SD-irresolute injective map, then p is SD-open.*

Proof (i) We only prove in case of $i = semi$ and $j = \beta$, and the other follow easily. Let G be a non-empty open subset of X. Then $p(G)$ is a non-empty semi open subset of Y. Therefore there is a non-empty open subset H of Y such that $H \subseteq p(G)$. Now, $q(H) \subseteq q(p(G))$. Since q is β-open, then $q(H)$ is a non-empty β-open subset of Z. Therefore $q(H)$ is somewhere dense and this implies that $q(p(G))$ is somewhere dense. Thus $q \circ p$ is somewhere dense.
(ii) Suppose that $G \neq \emptyset$ is an open subset of Y. Then $p^{-1}(G) \neq \emptyset$ is an open subset of X. Therefore $(q \circ p)(p^{-1}(G))$ is a somewhere dense subset of Z. Since p is surjective, then $(q \circ p)(p^{-1}(G)) = q(p(p^{-1}(G))) = q(G)$. Thus q is an SD-open map.
(iii) Let $G \neq \emptyset$ be an open subset of X. Then $(q \circ p)(G) \neq \emptyset$ is an open subset of Z. Therefore $q^{-1}(q \circ p(G))$ is somewhere dense. Since q is injective, then $q^{-1}(q \circ p(G)) = (q^{-1}q)(p(G)) = p(G)$. Thus p is an SD-open map.
(iv) The proof is similar to that of (iii). □

The proof of the next proposition is similar with that of the above proposition.

Proposition 5.10 *The following four statements hold for the maps $f : (X, \tau) \to (Y, \theta)$ and $g : (Y, \theta) \to (Z, \sigma)$.*

(i) *If f is injective i-closed for $i = \{\alpha, semi\}$ and g is injective j-closed for $j = \{\beta, b, semi, pre, \alpha\}$, then $g \circ f$ is SD-open.*
(ii) *If $g \circ f$ is SD-open and f is surjective continuous, then g is SD-open.*
(iii) *If $g \circ f$ is closed and g is injective SD-continuous, then f is SD-closed.*
(iv) *If $g \circ f$ is SD-closed and g is SD-irresolute injective map, then f is SD-closed.*

Definition 5.11 A bijective map g in which is SD-continuous and SD-open is called an SD-homeomorphism.

Theorem 5.12 *For a bijective map $g : (X, \tau) \to (Y, \theta)$, the following properties are equivalent:*

(i) *g is an SD-homeomorphism;*
(ii) *g and g^{-1} is SD-continuous;*
(iii) *g is SD-closed and SD-continuous.*

Proof Straightforward. □

Theorem 5.13 *A bijective map $p : (X, \tau) \to (Y, \theta)$ is an SD-homeomorphism if and only if one of the following conditions holds.*

(i) $p(Scl(E)) \subseteq cl(p(E))$ and $Scl(p(E)) \subseteq p(cl(E))$, for each $E \subseteq X$.
(ii) $p(int(E)) \subseteq Sint(p(E))$ and $p^{-1}(int(A)) \subseteq Sint(p^{-1}(A))$, for each $E \subseteq X$ and $A \subseteq Y$.

Proof (i) The proof is obtained from Theorems 4.10 and 5.2.
(ii) The proof is obtained from Theorems 4.10 and 5.3. □

6 Conclusion

This article is divided into three main sections: Sects. 3, 4 and 5. We devote Sect. 3 to investigating further properties of somewhere dense sets and to studying a concept of S-limit points. In Sect. 4, we introduce two new maps depending on somewhere dense sets, namely SD-continuous and SD-irresolute maps. We characterize them and discuss main properties. In the last section, we define the notions of SD-closed (SD-open, SD-homeomorphism) maps and derive main properties. In particular, we give the sufficient conditions of restriction SD-closed (resp. SD-open) maps to be SD-closed (resp. SD-open). In the upcoming papers, we plan to apply the concepts initiated in this work to the field of ordered spaces, information systems and digital topologies.

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