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# Fekete–Szegö functional problems for some subclasses of bi-univalent functions defined by Frasin differential operator

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**Abstract**
In this paper, we introduce a new subclass $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$ of bi-univalent functions defined by a new differential operator of analytic functions involving binomial series due to Frasin (Bol Soc Paran Mat (in press), 2019) in the open unit disk. We obtain coefficient bounds for the Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ of the function $f \in \mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$. Furthermore, we solve the Fekete–Szegö functional problem for functions in $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$. The results presented in this paper improve or generalize the earlier results of Peng and Han (Acta Math Sci 34(1):228–240, 2014) and Tang et al. (J Math Inequal 10(4):1063–1092, 2016).

**Keywords** Analytic functions · Univalent functions · Bi-univalent functions · Taylor–Maclaurin series · Binomial series · Coefficient inequalities · Fekete–Szegö problems

**Mathematics Subject Classification** 30C45

## 1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right)$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be in $\Sigma$, the class of bi-univalent functions in $\mathbb{U}$, if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Lewin [9] showed that $|a_2| < 1.51$ for every function $f \in \Sigma$ given by (1.1). Posteriorly, Brannan and Clunie [3] improved Lewin's result and conjectured that $|a_2| \leq \sqrt{2}$ for every function $f \in \Sigma$ given by (1.1). Later, Netanyahu [11] showed that $\max\limits_{f\in\Sigma}|a_2| = \frac{4}{3}$. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \quad (n \in \mathbb{N} = \{1, 2, \ldots\}; n \geq 4)$$

is still an open problem (see, for details, [15]). Since then, many researchers (see [2,5–7,14,16,18,19]) investigated several interesting subclasses of the class $\Sigma$ and found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In fact, its worth to mention that by making use of the Faber polynomial coefficient expansions Jahangiri and Hamidi [8] have obtained estimates for the general coefficients $|a_n|$ for bi-univalent functions subject to certain gap series.

Let $\mathcal{P}$ denote the class of function of $p$ analytic in $\mathbb{U}$ such that $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$, where

$$p(z) = 1 + p_1z + p_2z^2 + \ldots \quad (z \in \mathbb{U}).$$

Now we recall the unified subordination due to Ma and Minda [10]:

Let $\varphi$ be an analytic function with positive real part in the unit disk $\mathbb{U}$ such that

$$\varphi(0) = 1, \varphi'(0) > 0$$

and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis and has a series expansion of the form:

$$\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \quad (B_1 > 0).$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $\mathbb{U}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1z + b_2z^2 + b_3z^3 + \cdots \text{ and } v(w) = c_1w + c_2w^2 + c_3w^3 + \cdots . \tag{1.2}$$

We observe that

$$|b_1| \leq 1,\ |b_2| \leq 1 - |b_1|^2,\ |c_1| \leq 1 \text{ and } |c_2| \leq 1 - |c_1|^2 . \tag{1.3}$$

Further we have

$$\varphi(u(z)) = 1 + B_1b_1z + (B_1b_2 + B_2b_1^2)z^2 + \cdots \quad (|z| < 1) \tag{1.4}$$

and

$$\varphi(v(w)) = 1 + B_1c_1w + (B_1c_2 + B_2c_1^2)w^2 + \cdots \quad (|w| < 1). \tag{1.5}$$

Making use of the binomial series

$$(1-\gamma)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \gamma^j \quad (m \in \mathbb{N} = \{1, 2, \ldots\}, \quad j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

recently for $f \in \mathcal{A}$, Frasin [4] defined the differential operator $D^{\zeta}_{m,\lambda} f(z)$ as follows:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1_{m,\gamma} f(z) &= (1-\gamma)^m f(z) + (1-(1-\gamma)^m) z f'(z) = D_{m,\lambda} f(z),\ \gamma > 0; m \in \mathbb{N}, \\ D^{\zeta}_{m,\gamma} f(z) &= D_{m,\gamma}(D^{\zeta-1} f(z)) \\ &= z + \sum_{n=2}^{\infty} \left[1 + (n-1) C^m_j(\gamma)\right]^{\zeta} a_n z^n; \zeta \in \mathbb{N}_0, \end{aligned} \tag{1.6}$$

where

$$C^m_j(\gamma) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \gamma^j.$$

Using the relation (1.6), it is easily verified that

$$C^m_j(\gamma) z (D^{\zeta}_{m,\gamma} f(z))' = D^{\zeta+1}_{m,\gamma} f(z) - (1 - C^m_j(\gamma)) D^{\zeta}_{m,\gamma} f(z). \tag{1.7}$$

By specializing the parameters we observe that, for $m = 1$, $D^{\zeta}_{1,\lambda}$ defined by Al-Oboudi [1] and for $m = \gamma = 1$, $D^{\zeta}_{1,1}$ defined by Sălăgean [13].

The main object of this paper is to introduce the following new subclasses of bi-univalent functions involving Frasin differential operator $D^{\zeta}_{m,\lambda}$ [4] and to obtain bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Further, we discuss Fekete–Szegö functional problems for functions in these new classes.

## 2 The function class $\mathcal{B}^{\zeta}_{\Sigma}(m, \gamma, \lambda; \varphi)$

**Definition 2.1** A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{B}^{\zeta}_{\Sigma}(m, \gamma, \lambda; \varphi)$ if and only if

$$(1-\lambda)\frac{D^{\zeta}_{m,\gamma} f(z)}{z} + \lambda (D^{\zeta}_{m,\gamma} f(z))' \prec \varphi(z)$$

and

$$(1-\lambda)\frac{D^{\zeta}_{m,\gamma} g(w)}{w} + \lambda (D^{\zeta}_{m,\gamma} g(w))' \prec \varphi(w)$$

where $0 \le \lambda \le 1$, $z, w\ in\ \mathbb{U}$ and $g(w) = f^{-1}(w)$.

For $\lambda = 1$, the class $\mathcal{B}^{\zeta}_{\Sigma}(m, \gamma, \lambda; \varphi)$ reduces to the following class.

**Definition 2.2** A function $f(z) \in \Sigma$ is said to be in the class $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma; \varphi)$ if and only if

$$(D_{m,\gamma}^{\zeta} f(z))' \prec \varphi(z)$$

and

$$(D_{m,\gamma}^{\zeta} g(w))' \prec \varphi(w)$$

where $z, w \; in \; \mathbb{U}$ and $g(w) = f^{-1}(w)$.

***Remark 2.3*** We note that $\mathcal{B}_{\Sigma}^{0}(m, \gamma, \lambda; \varphi) = \mathcal{B}_{\Sigma}(\lambda; \varphi)$ and $\mathcal{B}_{\Sigma}^{0}(m, \gamma; \varphi) = \mathcal{H}_{\Sigma}(\varphi)$, where the classes $\mathcal{B}_{\Sigma}(\lambda; \varphi)$ and $\mathcal{H}_{\Sigma}(\varphi)$ were introduced and studied by Peng and Han [12].

In the following theorem we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$.

**Theorem 2.4** *If $f(z)$ given by (1.1) is in the class $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\chi + B_1(1+\lambda)^2\left(1 + C_j^m(\gamma)\right)^{2\zeta}}} \tag{2.1}$$

*and*

$$|a_3| \leq \begin{cases} \dfrac{B_1}{(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta}} & \text{if } B_1 < \dfrac{(1+\lambda)^2\left(1+C_j^m(\gamma)\right)^{2\zeta}}{(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta}} \\ \dfrac{\chi B_1 + (1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} B_1^3}{(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta}\left(\chi + (1+\lambda)^2\left(1+C_j^m(\gamma)\right)^{2\zeta} B_1\right)} & \text{if } B_1 \geq \dfrac{(1+\lambda)^2\left(1+C_j^m(\gamma)\right)^{2\zeta}}{(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta}} \end{cases}, \tag{2.2}$$

*where*

$$\chi = \left| B_1^2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} - B_2(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta} \right|.$$

***Proof*** Let $f(z) \in \mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$. Then there are analytic functions $u$ and $v$, with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, given by (1.2) and satisfying the following conditions:

$$(1-\lambda)\frac{D_{m,\gamma}^{\zeta} f(z)}{z} + \lambda(D_{m,\gamma}^{\zeta} f(z))' = \varphi(u(z)) \tag{2.3}$$

and

$$(1-\lambda)\frac{D_{m,\gamma}^{\zeta} g(w)}{w} + \lambda(D_{m,\gamma}^{\zeta} g(w))' = \varphi(v(w)), \tag{2.4}$$

where $g(w) = f^{-1}(w)$. Since

$$\begin{aligned} &(1-\lambda)\frac{D_{m,\gamma}^{\zeta} f(z)}{z} + \lambda(D_{m,\gamma}^{\zeta} f(z))' \\ &= 1 + (1+\lambda)(1+C_j^m(\gamma))^{\zeta} a_2 z + (1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} a_3 z^2 + \cdots \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &(1-\lambda)\frac{D_{m,\gamma}^{\zeta} g(w)}{w} + \lambda(D_{m,\gamma}^{\zeta} g(w))' \\ &= 1 - (1+\lambda)(1+C_j^m(\gamma))^{\zeta} a_2 w + (1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}(2a_2^2 - a_3)w^2 + \cdots, \end{aligned} \tag{2.6}$$

it follows from (1.4), (1.5), (2.5) and (2.6) that

$$(1+\lambda)\left(1+C_j^m(\gamma)\right)^{\zeta} a_2 = B_1 b_1, \tag{2.7}$$

$$(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta} a_3 = B_1 b_2 + B_2 b_1^2, \tag{2.8}$$

$$-(1+\lambda)\left(1+C_j^m(\gamma)\right)^{\zeta} a_2 = B_1 c_1, \tag{2.9}$$

and

$$(1+2\lambda)\left(1+2C_j^m(\gamma)\right)^{\zeta} (2a_2^2 - a_3) = B_1 c_2 + B_2 c_1^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$c_1 = -b_1 \tag{2.11}$$

and

$$2\left[\left(1+C_j^m(\gamma)\right)^{\zeta}(1+\lambda)\right]^2 a_2^2 = B_1^2(b_1^2+c_1^2). \tag{2.12}$$

By adding (2.8) to (2.10), we have

$$2(1+2C_j^m(\gamma))^{\zeta}(1+2\lambda)a_2^2 = B_1(b_2+c_2) + B_2(b_1^2+c_1^2). \tag{2.13}$$

Therefore, from equalities (2.12) and (2.13) we find that

$$[2(1+2C_j^m(\gamma))^{\zeta}(1+2\lambda)B_1^2 - 2B_2\left((1+C_j^m(\gamma))^{\zeta}(1+\lambda)\right)^2]a_2^2 = B_1^3(b_2+c_2). \tag{2.14}$$

Then, in view of (2.7), (2.11) and (1.3), we obtain

$$\left|[2(1+2C_j^m(\gamma))^{\zeta}(1+2\lambda)B_1^2 - 2B_2\left((1+C_j^m(\gamma))^{\zeta}(1+\lambda)\right)^2]\right| |a_2|^2$$
$$\le B_1^3(|b_2|+|c_2|) \le 2B_1^3(1-|b_1|^2) = 2B_1^3 - 2B_1\left((1+C_j^m(\gamma))^{\zeta}(1+\lambda)\right)^2 |a_2|^2 .$$

Thus, we get

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\chi + B_1(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}}},$$

where

$$\chi = \left|B_1^2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} - B_2(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}\right|.$$

Next, in order to find the bound on $|a_3|$, subtracting (2.10) from (2.8) and using (2.11), we get

$$2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} a_3 = 2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} a_2^2 + B_1(b_2-c_2). \tag{2.15}$$

Then in view of (1.3) and (2.11), we have

$$2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}\,|a_3| \le 2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}\,|a_2|^2 + B_1(|b_2|+|c_2|)$$
$$\le 2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}\,|a_2|^2 + 2B_1(1-|b_1|^2).$$

From (2.7), we immediately have

$$\begin{aligned} & B_1(1+2\lambda)(1+2C_j^m(\gamma))^\zeta\,|a_3| \\ & \leq \left|B_1(1+2\lambda)(1+2C_j^m(\gamma))^\zeta-(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}\right||a_2|^2+B_1^2. \end{aligned}$$

Now the assertion (2.2) follows from (2.1). This evidently completes the proof of Theorem 2.4. □

By taking $\lambda=1$ in Theorem 2.4, we have

**Corollary 2.5** *If $f(z)$ given by (1.1) is in the class $\mathcal{B}_\Sigma^\zeta(m,\gamma;\varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\tau+4B_1(1+C_j^m(\gamma))^{2\zeta}}} \tag{2.16}$$

*and*

$$|a_3| \leq \begin{cases} \frac{B_1}{3(1+2C_j^m(\gamma))^\zeta} & \text{if } B_1<\frac{4(1+C_j^m(\gamma))^{2\zeta}}{3(1+2C_j^m(\gamma))^\zeta} \\ \frac{\tau B_1+3(1+2C_j^m(\gamma))^\zeta B_1^3}{3(1+2C_j^m(\gamma))^\zeta\left(\tau+4(1+C_j^m(\gamma))^{2\zeta}B_1\right)} & \text{if } B_1\geq\frac{4(1+C_j^m(\gamma))^{2\zeta}}{3(1+2C_j^m(\gamma))^\zeta} \end{cases}, \tag{2.17}$$

*where*

$$\tau=\left|3B_1^2(1+2C_j^m(\gamma))^\zeta-4B_2(1+C_j^m(\gamma))^{2\zeta}\right|.$$

Putting $\zeta=0$ in Theorem 2.4, we have

**Corollary 2.6** [12] *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda;\varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2(1+2\lambda)-B_2(1+\lambda)^2\right|+B_1(1+\lambda)^2}} \tag{2.18}$$

*and*

$$|a_3| \leq \begin{cases} \frac{B_1}{(1+2\lambda)} & \text{if } B_1<\frac{(1+\lambda)^2}{(1+2\lambda)} \\ \frac{\left|B_1^2(1+2\lambda)-B_2(1+\lambda)^2\right|B_1+(1+2\lambda)B_1^3}{(1+2\lambda)\left(\left|B_1^2(1+2\lambda)-B_2(1+\lambda)^2\right|+(1+\lambda)^2\right)} & \text{if } B_1\geq\frac{(1+\lambda)^2}{(1+2\lambda)} \end{cases}. \tag{2.19}$$

Putting $\zeta=0$ in Corollary 2.5, we have

**Corollary 2.7** [12] *If $f(z)$ given by (1.1) is in the class $\mathcal{H}_\Sigma(\varphi)$. Then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{\left|3B_1^2-4B_2\right|+4B_1}} \tag{2.20}$$

*and*

$$|a_3| \leq \begin{cases} \frac{B_1}{3} & \text{if } B_1<\frac{4}{3} \\ \frac{\left|3B_1^2-4B_2\right|B_1+3B_1^3}{3\left(\left|3B_1^2-4B_2\right|+4B_1\right)} & \text{if } B_1\geq\frac{4}{3} \end{cases}. \tag{2.21}$$

***Remark 2.8*** If

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots \ (0 < \alpha \leq 1) \tag{2.22}$$

in Corollary 2.6, then we have Theorem 2.2 in [7].

If

$$\varphi(z) = \frac{1 + (1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \cdots \quad (0 < \alpha \leq 1), \tag{2.23}$$

then we have Theorem 3.2 in [7].

Also, if $\zeta = 0$ and $\lambda = 1$, we have Theorem 2.1 in [12].

## 3 Fekete–Szegö inequalities for the function class $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$

Now, we are ready to find the sharp bounds of Fekete–Szegö functional $a_3 - \delta a_2^2$ defined for $f \in \mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$ given by (1.1).

**Theorem 3.1** *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma, \lambda; \varphi)$. Then*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \frac{B_1}{(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}} & \text{for } 0 \leq |h(\delta)| < \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}} \\ 2B_1\,|h(\delta)| & \text{for } |h(\delta)| \geq \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}} \end{cases}, \tag{3.1}$$

*where*

$$h(\delta) = \frac{B_1^2(1-\delta)}{2B_1^2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} - 2B_2(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}}.$$

***Proof*** From (2.14) and (2.15), we get

$$a_2^2 = \frac{B_1^3(b_2+c_2)}{2\left[B_1^2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} - B_2(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}\right]} \tag{3.2}$$

and

$$a_3 = \frac{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} a_2^2 + B_1(b_2-c_2)}{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}}. \tag{3.3}$$

From the Eqs. (3.2) and (3.3), it follows that

$$\begin{aligned} a_3 - \delta a_2^2 = B_1 \Bigg[ &\left(h(\delta) + \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}}\right) b_2 \\ &+ \left(h(\delta) - \frac{1}{2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta}}\right) c_2 \Bigg], \end{aligned}$$

where

$$h(\delta) = \frac{B_1^2(1-\delta)}{2B_1^2(1+2\lambda)(1+2C_j^m(\gamma))^{\zeta} - 2B_2(1+\lambda)^2(1+C_j^m(\gamma))^{2\zeta}}.$$

Since all $B_i$ are real and $B_1 > 0$, which implies the assertion (3.1). This completes the proof of Theorem 3.1. □

By taking $\lambda = 1$ in Theorem 3.1, we have

**Corollary 3.2** *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{\zeta}(m, \gamma; \varphi)$. Then*

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} \frac{B_1}{3(1+2C_j^m(\gamma))^\zeta} & \text{for } 0 \le |h(\delta)| < \frac{1}{6(1+2C_j^m(\gamma))^\zeta} \\ 2B_1\,|h(\delta)| & \text{for } |h(\delta)| \ge \frac{1}{6(1+2C_j^m(\gamma))^\zeta} \end{cases}, \qquad (3.4)$$

*where*

$$h(\delta) = \frac{B_1^2(1-\delta)}{6B_1^2(1+2C_j^m(\gamma))^\zeta - 8B_2(1+C_j^m(\gamma))^{2\zeta}}.$$

***Remark 3.3*** Putting $\zeta = 0$ in Corollary 3.2, we get Corollary 4 in [17].

Putting $\zeta = 0$ in Theorem 3.1, we have

**Corollary 3.4** *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\lambda; \varphi)$. Then*

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} \frac{B_1}{1+2\lambda} & \text{for } 0 \le |h(\delta)| < \frac{1}{2(1+2\lambda)} \\ 2B_1\,|h(\delta)| & \text{for } |h(\delta)| \ge \frac{1}{2(1+2\lambda)} \end{cases},$$

*where*

$$h(\delta) = \frac{B_1^2(1-\delta)}{2[B_1^2(1+2\lambda) - B_2(1+\lambda)^2]}.$$

***Remark 3.5*** If $\varphi(z)$ is given by (2.22) then by Corollary 3.4, we have

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} \frac{2\alpha}{1+2\lambda} & \text{for } 0 \le |h(\delta)| < \frac{1}{2(1+2\lambda)} \\ 4\alpha\,|h(\delta)| & \text{for } |h(\delta)| \ge \frac{1}{2(1+2\lambda)} \end{cases},$$

where

$$h(\delta) = \frac{1-\delta}{1+2\lambda-\lambda^2}.$$

Also, if $\varphi(z)$ is given by (2.23) then we have

$$\left|a_3 - \delta a_2^2\right| \le \begin{cases} \frac{2(1-\alpha)}{(1+2\lambda)} & \text{for } 0 \le |h(\delta)| < \frac{1}{2(1+2\lambda)} \\ 4(1-\alpha)\,|h(\delta)| & \text{for } |h(\delta)| \ge \frac{1}{2(1+2\lambda)} \end{cases},$$

where $h(\delta) = \frac{(1-\alpha)(1-\delta)}{2(1-\alpha)(1+2\lambda)-(1+\lambda)^2}$.

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