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Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions

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Abstract

In the present paper, we determine necessary and sufficient conditions for $zF(a, b; c; z)$ and $z(2 - F(a, b; c; z))$ where $F(a, b; c; z) = \sum_{n=0}^{\infty}[(a)_n(b)_n/(c)_n(1)_n]z^n$ to be in a certain class of analytic functions with negative coefficients. Furthermore, we consider an integral operator related to hypergeometric functions.

Keywords Analytic functions · Univalent functions · Hypergeometric functions

Mathematics Subject Classification 30C45

1 Introduction and definitions

Let $\mathcal{T}$ denote the class of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Let $T^*(\alpha)$ and $C(\alpha)$ denote the subclasses of $\mathcal{T}$ consisting of starlike and convex functions of order $\alpha (0 \leq \alpha < 1)$, respectively [11].

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A function f of the form (1.1) is in $\mathcal{S}(k,\lambda)$ if it satisfies the condition

$$\left|\frac{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-1}{\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}+1}\right|<k,\ \ (0<k\le 1,\ 0\le\lambda<1, z\in\mathcal{U}) \tag{1.2}$$

and $f\in\mathcal{C}(k,\lambda)$ if and only if $zf'\in\mathcal{S}(k,\lambda)$.

We note that $\mathcal{S}(k,0)=\mathcal{S}(k)$ and $\mathcal{C}(k,0)=\mathcal{C}(k)$, where the classes $\mathcal{S}(k)$ and $\mathcal{C}(k)$ were introduced and studied by Padmanabhan [8] (see also, [4,7]).

Let $F(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n, \tag{1.3}$$

where $c\neq 0,-1,-2,\ldots,$ and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n:=\frac{\Gamma(x+n)}{\Gamma(x)}=\begin{cases}1, & n=0,\\ x(x+1)(x+2)\cdots(x+n-1), & n\in\mathbb{N}:\{1,2,\ldots\}.\end{cases} \tag{1.4}$$

We note that $F(a,b;c;1)$ converges for $\mathrm{Re}(c-a-b)>0$ and is related to the Gamma function by

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{1.5}$$

The Gauss hypergeometric function $F(a,b;c;z)$ is one of the most important special functions in complex analysis. Due to its three defining parameters, its structure is extremely rich and contains almost all special functions as a particular or limiting case. Important orthogonal polynomials such as Chebyshev, Legendre, Gegenbauer, and Jacobi polynomials, can all be expressed with the $F(a,b;c;z)$ function, see [9].

Silverman [12] gave necessary and sufficient conditions for $zF(a,b;c;z)$ to be in the classes $T^*(\alpha)$ and $C(\alpha)$, and also examined a linear operator acting on hypergeometric functions. For more details, see the works done in [1–6,10,13].

In the present paper, we determine necessary and sufficient conditions for $zF(a,b;c;z)$ to be in our new classes $\mathcal{S}(k,\lambda)$ and $\mathcal{C}(k,\lambda)$. Furthermore,we consider an integral operator related to hypergeometric functions.

The proof of Lemma 1.1 below is much akin to that of Theorem 1 in [7], so we choose to omit the details involved.

Lemma 1.1 *(i) A function f of the form (1.1) is in $\mathcal{S}(k,\lambda)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]a_n\le 2k \tag{1.6}$$

where $0<k\le 1$ and $0\le\lambda<1$. The result is sharp.

(ii) A function f of the form (1.1) is in $\mathcal{C}(k,\lambda)$ if and only if it satisfies

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]a_n\le 2k \tag{1.7}$$

where $0<k\le 1$ and $0\le\lambda<1$. The result is sharp.

2 The necessary and sufficient conditions

Unless otherwise mentioned, we shall assume in this paper that $0 < k \leq 1$ and $0 \leq \lambda < 1$.

Theorem 2.1 *(i) If $a, b > -1$, $c > 0$, and $ab < 0$, then $zF(a, b; c; z)$ is in $\mathcal{S}(k, \lambda)$ if and only if*

$$c \geq a + b + 1 - \frac{((1-\lambda) + k(1+\lambda))ab}{2k}. \tag{2.1}$$

(ii) If $a, b > 0$, $c > a + b + 1$, then $F_1(a, b; c; z) = z(2 - F(a, b; c; z))$ is in $\mathcal{S}(k, \lambda)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(1 + \frac{((1-\lambda) + k(1+\lambda))ab}{2k(c-a-b-1)}\right) \leq 2. \tag{2.2}$$

Proof (i) The function $zF(a, b; c; z)$ can be written as

$$\begin{aligned} zF(a, b; c; z) &= z + \frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n \\ &= z - \left|\frac{ab}{c}\right|\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n. \end{aligned} \tag{2.3}$$

According to (1.6) of Lemma 1.1, we must show that

$$\sum_{n=2}^{\infty}[n((1-\lambda) + k(1+\lambda)) + (1-\lambda)(k-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left|\frac{c}{ab}\right| 2k. \tag{2.4}$$

Noting that $(\tau)_n = \tau(\tau+1)_{n-1}$ and then applying (1.5), we have

$$\begin{aligned} &\sum_{n=0}^{\infty}[n((1-\lambda) + k(1+\lambda)) + (1-\lambda)(k-1)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\ &= (1-\lambda) + k(1+\lambda)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + 2k\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\ &= (1-\lambda) + k(1+\lambda)\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + 2k\frac{c}{ab}\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right). \end{aligned}$$

Hence, (2.4) is equivalent to

$$\begin{aligned} &\frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left((1-\lambda) + k(1+\lambda) + 2k\frac{c-a-b-1}{ab}\right) \\ &\leq 2k\left(\frac{c}{|ab|} + \frac{c}{ab}\right) = 0. \end{aligned} \tag{2.5}$$

Thus, (2.5) is valid if and only if $(1-\lambda) + k(1+\lambda) + 2k\frac{c-a-b-1}{ab} \leq 0$ or, $c \geq a + b + 1 - \frac{((1-\lambda)+k(1+\lambda))ab}{2k}$.

(ii) Since

$$F_1(a, b; c; z) = z - \sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

from the condition (1.6), we need only to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq 2k.$$

Now,

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= (1-\lambda)+k(1+\lambda)\sum_{n=1}^{\infty}\frac{n(a)_n(b)_n}{(c)_n(1)_n}+2k\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= \frac{((1-\lambda)+k(1+\lambda))ab}{c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}}+2k\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} \\
&= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{((1-\lambda)+k(1+\lambda))ab}{c-a-b-1}+2k\right)-2k.
\end{aligned}$$

But this last expression is bounded above by $2k$ if and only if (2.2) holds. □

Theorem 2.2 *(i) If $a, b > -1$, $ab < 0$, $c > a+b+2$, then $zF(a,b;c;z)$ is in $\mathcal{C}(k,\lambda)$ if and only if*

$$((1-\lambda)+k(1+\lambda))(a)_2(b)_2+2((1-\lambda)+k(2+\lambda))ab(c-a-b-2)+2k(c-a-b-2)_2 \geq 0. \tag{2.6}$$

(ii) If $a, b > 0$, $c > a+b+2$, then $F_1(a,b;c;z) = z(2-F(a,b;c;z))$ is in $\mathcal{C}(k,\lambda)$ if and only if

$$\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{((1-\lambda)+k(1+\lambda))(a)_2(b)_2}{2k(c-a-b-2)_2}\right. \\
&\quad\left.+\left(\frac{1-\lambda+k(2+\lambda)}{k}\right)\left(\frac{ab}{c-a-b-1}\right)+1\right) \leq 2.
\end{aligned} \tag{2.7}$$

Proof (i) From (2.3) and (1.7) it follows that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \frac{2ck}{|ab|}.$$

Writing

$$\begin{aligned}
&(n+2)[(n+2)((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)] \\
&\quad = ((1-\lambda)+k(1+\lambda))(n+1)^2+((1-\lambda)+k(3+\lambda))(n+1)+2k,
\end{aligned}$$

we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+2)[(n+2)((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} \\
&= ((1-\lambda)+k(1+\lambda))\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} \\
&\quad +((1-\lambda)+k(3-\lambda))\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+2k\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}
\end{aligned}$$

$$= \frac{((1-\lambda)+k(1+\lambda))(a+1)(b+1)}{c+1}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}$$
$$+2[(1-\lambda)+k(2+\lambda)]\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\frac{2kc}{ab}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$=\frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Big(((1-\lambda)+k(1+\lambda))(a+1)(b+1)$$
$$+2[(1-\lambda)+k(2+\lambda)]\,(c-a-b-2)+\frac{2k}{ab}\,(c-a-b-2)_2\Big)-\frac{2kc}{ab}.$$

This last expression is bounded above by $2ck/\,|ab|$ if and only if

$$((1-\lambda)+k(1+\lambda))(a+1)(b+1)+2((1-\lambda)+k(2+\lambda))\,(c-a-b-2)$$
$$+\frac{2k}{ab}\,(c-a-b-2)_2$$
$$\leq 0,$$

which is equivalent to (2.6).

(ii) In view of (1.7), we need only to show that

$$\sum_{n=2}^{\infty}n[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq 2k.$$

Now,

$$\sum_{n=0}^{\infty}(n+2)[(n+2)((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=((1-\lambda)+k(1+\lambda))\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$-(1-\lambda)(1-k)\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \tag{2.8}$$

Writing $n+2=(n+1)+1$, we have

$$\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}},$$
$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$
$$=\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}. \tag{2.9}$$

Substituting (2.9) into the right-hand side of (2.8), we obtain

$$((1-\lambda)+k(1+\lambda))\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n}+2((1-\lambda)+k(2+\lambda))\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}$$
$$+2k\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}. \tag{2.10}$$

Since $(a)_{n+j}=(a)_j(a+k)_n$, we write (2.10) as

$$\frac{((1-\lambda)+k(1+\lambda))(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}$$
$$+2((1-\lambda)+k(2+\lambda))\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}$$
$$+2k\left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right).$$

By simplification, we see that the last expression is bounded above by $(1-\alpha)$ if and only if (3.6) holds. □

3 An integral operator

In the next theorems, we obtain similar-type in connections with a particular integral operator $G(a,b;c;z)$ acting on $F(a,b;c;z)$ as follows:

$$G(a,b;c;z)=\int_0^z F(a,b;c;t)dt. \tag{3.1}$$

Theorem 3.1 *Let $a,b>-1$, $ab<0$, and $c>\max\{0,a+b\}$. Then, $G(a,b;c;z)$ defined by (3.1) is in $\mathcal{S}(k,\lambda)$ if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{((1-\lambda)+k(1+\lambda))}{ab}-\frac{(1-\lambda)(1-k)(c-a-b)}{(a-1)_2(b-1)_2}\right)$$
$$+\frac{(1-\lambda)(1-k)(c-1)_2}{(a-1)_2(b-1)_2}\le 0. \tag{3.2}$$

Proof Since

$$G(a,b;c;z)=z-\frac{|ab|}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}z^n,$$

by (1.6), we need to show that

$$\sum_{n=2}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_n}\le\frac{2kc}{|ab|}.$$

Now,

$$\begin{aligned}
&\sum_{n=0}^{\infty}[n((1-\lambda)+k(1+\lambda))-(1-\lambda)(1-k)]\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+2}}\\
&\quad=((1-\lambda)+k(1+\lambda))\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}-(1-\lambda)(1-k)\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}\\
&\quad=\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1-\lambda)+k(1+\lambda)}{ab}-\frac{(1-\lambda)(1-k)(c-a-b)}{(a-1)_2(b-1)_2}\right)\\
&\qquad+\frac{(1-\lambda)(1-k)(c-1)_2}{(a-1)_2(b-1)_2}-\frac{2kc}{ab}\le\frac{2kc}{|ab|},
\end{aligned}$$

which is equivalent to (3.2). □

Now, we observe that $G(a,b;c;z)\in\mathcal{C}(k,\lambda)$ if and only if $zF(a,b;c;z)\in\mathcal{S}(k,\lambda)$. Thus, any result of functions belonging to the class $\mathcal{S}(k,\lambda)$ about zF leads to that of functions belonging to the class $\mathcal{C}(k,\lambda)$. Hence, we obtain the following analogues to Theorem 2.1.

Theorem 3.2 *Let $a,b>-1$, $ab<0$, and $c>a+b+2$. Then, $G(a,b;c;z)$ defined by (3.1) is in $\mathcal{C}(k,\lambda)$ if and only if*

$$c\ge a+b+1-\frac{((1-\lambda)+k(1+\lambda))ab}{2k}.$$

Letting $\lambda=0$ in Theorems 2.1, 2.2, 3.1 and 3.2, we obtain the following corollaries.

Corollary 3.3 *(i) If $a,b>-1$, $c>0$, and $ab<0$, then $zF(a,b;c;z)$ is in $\mathcal{S}(k)$ if and only if*

$$c\ge a+b+1-\frac{(1+k)ab}{2k}. \tag{3.3}$$

(ii) If $a,b>0$, $c>a+b+1$, then $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ is in $\mathcal{S}(k)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(1+\frac{(1+k)ab}{2k(c-a-b-1)}\right)\le 2. \tag{3.4}$$

Corollary 3.4 *(i) If $a,b>-1$, $ab<0$, $c>a+b+2$, then $zF(a,b;c;z)$ is in $\mathcal{C}(k)$ if and only if*

$$(1+k)(a)_2(b)_2+2(1+2k)ab(c-a-b-2)+2k(c-a-b-2)_2\ge 0. \tag{3.5}$$

(ii) If $a,b>0$, $c>a+b+2$, then $F_1(a,b;c;z)=z(2-F(a,b;c;z))$ is in $\mathcal{C}(k,\lambda)$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1+k)(a)_2(b)_2}{2k(c-a-b-2)_2}+\left(\frac{1+2k}{k}\right)\left(\frac{ab}{c-a-b-1}\right)+1\right)\le 2. \tag{3.6}$$

Corollary 3.5 *Let $a,b>-1$, $ab<0$, and $c>\max\{0,a+b\}$. Then, $G(a,b;c;z)$ defined by (3.1) is in $\mathcal{S}(k)$ if and only if*

$$\frac{\Gamma(c+1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left(\frac{(1+k)}{ab}-\frac{(1-k)(c-a-b)}{(a-1)_2(b-1)_2}\right)+\frac{(1-k)(c-1)_2}{(a-1)_2(b-1)_2}\le 0. \tag{3.7}$$

Let $a, b > -1$, $ab < 0$, and $c > a + b + 2$. Then, $G(a, b; c; z)$ defined by (3.1) is in $\mathcal{C}(k)$ if and only if

$$c \geq a + b + 1 - \frac{(1 + k)ab}{2k}.$$

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