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Strong convergence theorem for monotone inclusion problem in CAT(0) spaces

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Abstract

In this paper, we introduce Halpern-type proximal point algorithm for approximating a common solution of monotone inclusion problem and fixed point problem. We obtain a strong convergence of the proposed algorithm to a common solution of finite family of monotone inclusion problem and fixed point problem for nonexpansive mappings in complete CAT(0) spaces. Nontrivial application and numerical example were given. Our results complement and extend some recent results in literature.

Keywords Monotone inclusion problem · Fixed point · Nonexpansive mapping · Monotone operators · Resolvent operators

Mathematics Subject Classification 47H09 · 47H10 · 49J20 · 49J40

1 Introduction

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \to X$ such that $c(0) = x$, $c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and (X, d) is said to be uniquely geodesic space if every two points of X are joined by only one geodesic segment. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consist of three points x_1, x_2, x_3 in X (known as

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the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is called a CAT(0) space if all geodesic triangles satisfies the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be its comparison triangle in $\mathbb{R}^2$. Then, Δ is said to satisfy CAT(0) inequality, if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{1.1}$$

If x, y, z are points in CAT(0) space and y_0 is the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{1.2}$$

Inequality (1.2) is known as the (CN) inequality of Bruhat and Titis [5]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the CN inequality (see [4]).

A complete CAT(0) space is called a Hamadard space. It is generally known that a CAT(0) space is a uniquely geodesic space (see for example [9]). Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbb{R}$-trees, Hilbert ball [13], Hyperbolic spaces [28]. See [4,6,14,16] for more equivalent definitions and properties of CAT(0) spaces.

Let $(1 - t)x \oplus ty$ denote the unique point z in the geodesic segment joining x to y for each x, y in a CAT(0) space such that $d(z, x) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$. Set $[x, y] := \{(1 - t)x \oplus ty : t \in [0, 1]\}$, then a subset C of X is said to be convex if $[x, y] \subseteq C$ for all $x, y \in C$.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space (X, d) and let $r(x, \{x_n\}) = \lim_{n \to \infty} \sup d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$ and the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. It is generally known that in a complete CAT(0) space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in a complete CAT(0) space is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n \to \infty} x_n = x$ (see [11]).

Lim [23] introduced the concept of Δ-convergence in general metric spaces. Kirk and Panyanak [21] introduced this concept to CAT(0) space and it coincides with weak convergence in the setting of Banach space. Several other authors have also studied the concept of Δ-convergence in CAT(0) spaces (see, for example [9,12,30] and the references therein). Breg and Nikolaev [3] introduced the concept of quasilineartization for CAT(0) spaces. They denoted a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and called it a vector. Quasilinearization is a map $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right), \quad (a, b, c, d \in X).$$

It can easily be verified that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. The space X is said to satisfy the Cauchy–Schwartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \; \forall a, b, c, d \in X$. It is known that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwartz inequality (see [3, Corollary 3]).

Based on the work of Breg and Nikolaev [3], Kakavandi and Amini [1] introduced the concept of dual space in a complete CAT(0) space X as follows:
Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ define by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \quad (t \in \mathbb{R}, a, b, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy–Schwartz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t| d(a, b)$ $(t \in \mathbb{R}, a, b \in X)$, where $L(\varphi) = \sup \left\{ \frac{\varphi(x) - \varphi(y)}{d(x,y)} : x, y \in X, x \neq y \right\}$ is the Lipschitz semi-norm for any function $\varphi : X \to \mathbb{R}$. A pseudometric D on $\mathbb{R} \times X \times X$ is defined by

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \quad (t, s \in \mathbb{R}, a, b, c, d \in X).$$

In a complete CAT(0) space (X, d), the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subset of the pseudometric space of all real-valued Lipschitz functions $(\text{Lip}(X, \mathbb{R}), L)$. It is shown in [1], that $D((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$. Thus, D induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is defined as

$$[t\overrightarrow{ab}] := \left\{ s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0 \right\}.$$

The set $X^* = \{ [t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X \}$ is a metric space with the metric $D([t\overrightarrow{ab}], [s\overrightarrow{cd}]) := D((t, a, b), (s, c, d))$. The pair (X^*, D) is called the dual space of the metric space (X, d). It is shown in [1] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is H and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R}, a, b \in H$. We also note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle, \quad \left(x^* = [t\overrightarrow{ab}] \in X^*, x, y \in X \right).$$

Let X be a complete CAT(0) space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ with domain $\mathbb{D}(A) := \{ x \in X : Ax \neq \emptyset \}$ is monotone if and only if for all $x, y \in \mathbb{D}(A), x^* \in Ax, y^* \in Ay,$

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0 \quad \text{(see [19]).}$$

A monotone operator A is called a maximal monotone operator if the graph $G(A)$ of A defined by

$$G(A) := \left\{ (x, x^*) \in X \times X^* : x^* \in A(x) \right\},$$

is not properly contained in the graph of any other monotone operator. It is easy to see that a monotone operator is maximal if and only if for each $(x, x^*) \in X \times X^*$, $\langle y^* - x^*, \overrightarrow{xy} \rangle \geq 0, \forall (y, y^*) \in G(A) \implies x^* \in A(x).$
The resolvent of a monotone operator A of order $\lambda > 0$ is the multivalued mapping $J_\lambda^A : X \to 2^X$ defined by (see [19])

$$J_\lambda^A(x) := \left\{ z \in X \mid \left[\frac{1}{\lambda} \overrightarrow{zx} \right] \in Az \right\}.$$

We say that the operator A satisfies the range condition if for every $\lambda > 0$, $\mathbb{D}(J_\lambda^A) = X$ (see [19]). For simplicity, we shall write J_λ for the resolvent of a monotone operator A.

The concept of monotone operator theory is one of the most important aspect in nonlinear and convex analysis due to the role it plays in optimization, variational inequalities, semi group theory, evolution equations, among others. One of the most important problems in monotone operator theory is the problem of finding the solution of the following monotone inclusion problem (MIP).

$$\text{Find } x \in \mathbb{D}(A) \text{ such that } 0 \in Ax, \tag{1.3}$$

where $A : X \rightarrow 2^{X^*}$ is a monotone operator. The solution set of problem (1.3) is denoted by $A^{-1}(0)$, which is known to be closed and convex (see [27, Remark 3.1]).

Many mathematical problems such as optimization problems, equilibrium problems, variational inequality problems, saddle point problems, among others, can be modelled as a MIP (1.3). Thus, MIP is of central importance in nonlinear and convex analysis. The most popular method for finding solutions of MIP, is the proximal point algorithm introduced in Hilbert space by Martinet [24] and Rockafellar [29], as follows:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = J_{\lambda_n}^A x_n, \ n \geq 0, \end{cases} \tag{1.4}$$

where $J_{\lambda_n}^A = (I + \lambda_n A)^{-1}$ is the resolvent of the operator A and $\{\lambda_n\}$ is a sequence of positive real numbers. Rockafellar [29] proved that the sequence $\{x_n\}$ generated by Algorithm (1.4) is weakly convergent to a solution of MIP (1.3), provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. Güler [15] gave an example to show that the sequence $\{x_n\}$ generated by the proximal point algorithm (1.4) may not converge strongly even if the maximal monotone operator is the subdifferential of a convex, proper and lower semicontinuous function.

Another important algorithm for approximating solutions of MIP is the Mann-type proximal point algorithm which was first introduced in Hilbert space by Kamimura and Takahashi [18]:

$$\begin{cases} x_0 \in H, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n}^A x_n, \ n \geq 0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\} \subset [0, 1]$. Under some suitable conditions, Kamimura and Takahashi [18] proved weak convergence of (1.5) in Hilbert spaces. In order to obtain strong convergence result, Kamimura and Takahashi [18] proposed the following Halpern-type proximal point algorithm for approximating a solution of MIP (1.3):

$$\begin{cases} u, x_0 \in H, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{\lambda_n}^A x_n. \end{cases} \tag{1.6}$$

Other authors have also studied the MIP (1.3) in the setting of real Banach spaces (see, for example [26] and the references therein).

In [2], Bačák proved the Δ-convergence of proximal point algorithm in CAT(0) spaces when the operator A is the subdifferential of a convex, proper and lower semicontinuous function. In 2016, Khatibzadeh and Ranjbar [19] introduced and studied the following proximal point algorithm in CAT(0) spaces:

$$\begin{cases} x_0 \in X, \\ \left[\frac{1}{\lambda_n} \overrightarrow{x_n x_{n-1}} \right] \in Ax_n. \end{cases} \tag{1.7}$$

They obtained a strong and Δ-convergence result of the proximal point algorithm (1.7) to a solution of MIP (1.3). Very recently, Ranjbar and Khatibzadeh [27] proposed the following

Mann-type proximal point algorithm in a complete CAT(0) space for finding a solution of (1.3) and obtained a Δ-convergence result.

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J_{\lambda_n} x_n, \end{cases} \tag{1.8}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. In the same paper, Ranjbar and Khatibzadeh [27] proposed the following Halpern-type proximal point algorithm in order to obtain a strong convergence result:

$$\begin{cases} u, x_0 \in X, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_{\lambda_n} x_n, \end{cases} \tag{1.9}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$.

In this paper, we propose and study the modified Halpern-type algorithm for approximating a common solution of finite family of MIP and fixed point problem for nonexpansive mapping in CAT(0) spaces. We establish the strong convergence of the proposed algorithm. Our results extend and complement the results of Bačák [2], Khatibzadeh and Ranjbar [19] and Ranjbar and Khatibzadeh [27].

2 Preliminary

We state some known and useful results which will be needed in the proof of our main theorem. Throughout this paper, we shall denote the strong and Δ-convergence by "$\longrightarrow$" and "$\rightharpoonup$" respectively.

Lemma 2.1 *Let X be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0, 1]$:*

  (i) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$, *(see [9]),*
 (ii) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$, *(see [9]),*
(iii) $d(tx \oplus (1 - t)y, sx \oplus (1 - s)y) \leq |t - s|d(x, y)$, *(see [7]),*
(iv) $d(tx \oplus (1 - t)y, tx \oplus (1 - t)z) \leq (1 - t)d(y, z)$, *(see [20]).*

Lemma 2.2 [8] *Let X be a CAT(0) space and $a, b, c \in X$. Then for each $\lambda \in [0, 1]$,*

$$d^2(\lambda a \oplus (1 - \lambda)b, c) \leq \lambda^2 d^2(a, c) + (1 - \lambda)^2 d^2(b, c) + 2\lambda(1 - \lambda)\langle \overrightarrow{ac}, \overrightarrow{bc} \rangle.$$

Lemma 2.3 [9,22] *Every bounded sequence in a complete CAT(0) space has a Δ-convergence subsequence.*

Lemma 2.4 [17] *Let X be a complete CAT(0) space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\lim_{n \to \infty} \sup \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0 \ \forall y \in X$.*

Lemma 2.5 [10] *Let X be a complete CAT(0) space and $T : X \to X$ be a nonexpansive mapping, then the conditions that $\{x_n\}$ Δ-converges to x and $d(x_n, Tx_n) \to 0$, implies $x = Tx$.*

Lemma 2.6 [31]. *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,

(iii) $\gamma_n \geq 0(n \geq 0)$, $\Sigma_{n=0}^{\infty}\gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [25]. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1} \forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ when the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad and \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

Definition 2.8 Let X be a complete CAT(0) space and C be a nonempty closed and convex subset of X. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if $Tx = x$. The set of fixed points of T is denoted by F(T).

The mapping T is said to be

(i) firmly nonexpansive if (see [19])

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \quad \forall x, y \in C,$$

(ii) nonexpansive if

$$d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in C.$$

From Cauchy–Schwartz inequality, it is clear that the class of nonexpasive mappings is more general than the class of firmly nonexpansive mappings.

Theorem 2.9 [19] *Let X be a CAT(0) space and J_λ be the resolvent of the operator A of order λ. We have*

(i) *For any $\lambda > 0$, $\mathbb{R}(J_\lambda) \subset \mathbb{D}(A)$, $F(J_\lambda) = A^{-1}(0)$.*

(ii) *If A is monotone then J_λ is a single-valued and firmly nonexpansive mapping.*

We make the following remark which is a consequence of Theorem 2.9.

Remark 2.10 If X is a CAT(0) space and J_λ is the resolvent of a monotone operator A of order λ, then

$$d^2(u, J_\lambda x) + d^2(J_\lambda x, x) \leq d^2(u, x)$$

for all $u \in A^{-1}(0)$, $x \in \mathbb{D}(J_\lambda)$ and $\lambda > 0$.

Indeed for any $u \in A^{-1}(0)$, $x \in \mathbb{D}(J_\lambda)$ and $\lambda > 0$, we obtain from Theorem 2.9 (i) and (ii), and by the definition of firmly nonexpansive mapping that

$$
\begin{aligned}
d^2(J_\lambda x, u) &\leq \langle \overrightarrow{J_\lambda x\, u}, \overrightarrow{xu} \rangle \\
&= \frac{1}{2}\left(d^2(J_\lambda x, u) + d^2(u, x) - d^2(J_\lambda x, x)\right),
\end{aligned}
$$

which implies

$$d^2(u, J_\lambda x) + d^2(J_\lambda x, x) \leq d^2(u, x).$$

3 Main results

Theorem 3.1 *Let X be a complete CAT(0) space and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping and for each $i = 1, 2, \ldots N$, let J_λ^i be the resolvent of monotone operators A_i of order $\lambda > 0$. Then*

$$F\left(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1\right)$$
$$= F(T) \cap F\left(J_\lambda^N\right) \cap F\left(J_\lambda^{N-1}\right) \cap \cdots \cap F\left(J_\lambda^2\right) \cap F\left(J_\lambda^1\right).$$

Proof Clearly, $F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1) \subseteq F(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1)$. We now show that $F(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) \subseteq F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$.

Let $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$ and $\Phi_\lambda^0 = I$, then for any $x \in F(T \circ \Phi_\lambda^N)$ and $y \in F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$, we have that

$$\begin{aligned}
d^2(x, y) &= d^2\left(T\Phi_\lambda^N x, T\Phi_\lambda^N y\right) \\
&\leq d^2\left(\Phi_\lambda^N x, \Phi_\lambda^N y\right) \\
&= d^2\left(\Phi_\lambda^N x, y\right).
\end{aligned} \tag{3.1}$$

From Remark 2.10 and (3.1), we have

$$d^2\left(J_\lambda^N\left(\Phi_\lambda^{N-1}x\right), \Phi_\lambda^{N-1}x\right) \leq d^2\left(\Phi_\lambda^{N-1}x, y\right) - d^2\left(J_\lambda^N\left(\Phi_\lambda^{N-1}x\right), y\right)$$
$$\vdots$$
$$\leq d^2(x, y) - d^2\left(\Phi_\lambda^N x, y\right)$$
$$\leq d^2\left(\Phi_\lambda^N x, y\right) - d^2\left(\Phi_\lambda^N x, y\right),$$

which implies

$$\Phi_\lambda^N x = \Phi_\lambda^{N-1} x, \tag{3.2}$$

where the last inequality follows from (3.1).

Also, from Remark 2.10 and (3.1), we have

$$d^2\left(\Phi_\lambda^{N-1}x, \Phi_\lambda^{N-2}x\right) \leq d^2\left(\Phi_\lambda^{N-2}x, y\right) - d^2\left(\Phi_\lambda^{N-1}x, y\right)$$
$$\vdots$$
$$\leq d^2(x, y) - d^2\left(\Phi^{N-1}x, y\right)$$
$$\leq d^2\left(\Phi_\lambda^N x, y\right) - d^2\left(\Phi_\lambda^N x, y\right),$$

which implies

$$\Phi_\lambda^{N-1} x = \Phi_\lambda^{N-2} x. \tag{3.3}$$

Continuing in this manner, we obtain that

$$\Phi_\lambda^N x = \Phi_\lambda^{N-1} x = \Phi_\lambda^{N-2} x = \Phi_\lambda^{N-3} x = \cdots = \Phi_\lambda^2 x = \Phi_\lambda^1 x = \Phi_\lambda^0 x = x. \tag{3.4}$$

From (3.4), we obtain

$$x = J_\lambda^1 x. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$x = \Phi_\lambda^2 x = J_\lambda^2 \left(J_\lambda^1 x \right) = J_\lambda^2 x. \tag{3.6}$$

Continuing in this manner, we obtain

$$x = J_\lambda^1 x = J_\lambda^2 x = \cdots = J_\lambda^{N-1} x = J_\lambda^N x. \tag{3.7}$$

Finally, from (3.4), we get

$$x = T \left(\Phi_\lambda^N x \right) = Tx. \tag{3.8}$$

Thus, we have from (3.7) and (3.8) that $F(T \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) \subseteq F(T) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$, which completes the proof. $\quad\square$

Theorem 3.2 *Let X be a complete CAT(0) space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfies the range condition and T be a nonexpansive mapping on X. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^N A_i^{-1}(0) \right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 (x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{3.9}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$, satisfying the following conditions

C1: $\lim_{n \to \infty} \alpha_n = 0$,
C2: $\sum_{n=1}^\infty \alpha_n = \infty$,
C3: $\sum_{n=1}^\infty |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to an element of Γ.

Proof Let $p \in \Gamma$, $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$, where $\Phi_\lambda^0 = I$. Then from (3.9), we have

$$\begin{aligned} d(x_{n+1}, p) &= d \left(\alpha_n u \oplus (1 - \alpha_n) T y_n, p \right) \\ &\leq \alpha_n d(u, p) + (1 - \alpha_n) d(T y_n, p) \\ &\leq \alpha_n d(u, p) + (1 - \alpha_n) d \left(\Phi_\lambda^N x_n, p \right) \\ &\leq \alpha_n d(u, p) + (1 - \alpha_n) d \left(\Phi_\lambda^{N-1} x_n, p \right) \\ &\vdots \\ &\leq \alpha_n d(u, p) + (1 - \alpha_n) d(x_n, p) \\ &\leq \max\{d(u, p), d(x_n, p)\}, \end{aligned}$$

which implies by mathematical induction that

$$d(x_n, p) \leq \max\{d(u, p), d(x_1, p)\}, \quad \forall n \geq 1. \tag{3.10}$$

Therefore, $\{d(x_n, p)\}$ is bounded. Consequently, $\{x_n\}$, $\{y_n\}$ and $\{T y_n\}$ are all bounded. We now divide our proof into two cases.

Case 1: Suppose that $\{d(x_n, p)\}$ is monotone sequence, then we assume that $\{d(x_n, p)\}$ is monotone decreasing. Then $\lim_{n\to\infty} d(x_n, p)$ exists. Consequently,

$$\lim_{n\to\infty} \left[d(x_{n+1}, p) - d(x_n, p) \right] = 0. \tag{3.11}$$

From Lemma 2.1 and Remark 2.10, we have

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2\left(\alpha_n u \oplus (1 - \alpha_n) T y_n, p \right) \\
&\leq \alpha_n d^2(u, p) + (1 - \alpha_n) d^2(T y_n, p) - \alpha_n(1 - \alpha_n) d^2(u, T y_n) \\
&\leq \alpha_n d^2(u, p) + (1 - \alpha_n) d^2(T y_n, p) \\
&\leq \alpha_n d^2(u, p) + (1 - \alpha_n) d^2(y_n, p) \\
&\leq \alpha_n d^2(u, p) + d^2(y_n, p) \\
&\leq \alpha_n d^2(u, p) + d^2\left(\Phi_\lambda^{N-1} x_n, p \right) - d^2\left(\Phi_\lambda^{N-1} x_n, y_n \right),
\end{aligned} \tag{3.12}$$

which implies

$$\begin{aligned}
d^2(y_n, \Phi_\lambda^{N-1} x_n) &\leq \alpha_n d^2(u, p) + d^2\left(\Phi_\lambda^{N-1} x_n, p \right) - d^2(x_{n+1}, p) \\
&\leq \alpha_n d^2(u, p) + d^2(x_n, p) - d^2(x_{n+1}, p) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

That is

$$\lim_{n\to\infty} d^2\left(y_n, \Phi_\lambda^{N-1} x_n \right) = 0. \tag{3.13}$$

From (3.12), we have

$$\begin{aligned}
d^2(x_{n+1}, p) &\leq \alpha_n d^2(u, p) + d^2(y_n, p) \\
&\leq \alpha_n d^2(u, p) + d^2\left(\Phi_\lambda^{N-1} x_n, p \right) \\
&\leq \alpha_n d^2(u, p) + d^2\left(\Phi_\lambda^{N-2} x_n, p \right) - d^2\left(\Phi_\lambda^{N-2} x_n, \Phi_\lambda^{N-1} x_n \right),
\end{aligned}$$

which implies

$$\begin{aligned}
d^2\left(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n \right) &\leq \alpha_n d^2(u, p) + d^2\left(\Phi_\lambda^{N-2} x_n, p \right) - d^2(x_{n+1}, p) \\
&\leq \alpha_n d^2(u, p) + d^2(x_n, p) - d^2(x_{n+1}, p) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

That is

$$\lim_{n\to\infty} d^2\left(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n \right) = 0. \tag{3.14}$$

Continuing in the same manner, we have that

$$\begin{aligned}
\lim_{n\to\infty} d^2\left(\Phi_\lambda^{N-2} x_n, \Phi_\lambda^{N-3} x_n \right) &= \cdots = \lim_{n\to\infty} d^2\left(\Phi_\lambda^2 x_n, \Phi_\lambda^1 x_n \right) \\
&= \lim_{n\to\infty} d^2\left(\Phi_\lambda^1 x_n, \Phi_\lambda^0 x_n \right) = 0.
\end{aligned} \tag{3.15}$$

Thus,

$$\begin{aligned}
d(y_n, x_n) &\leq d^2\left(y_n, \Phi_\lambda^{N-1} x_n \right) + d^2\left(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n \right) \\
&\quad + \cdots + d^2\left(\Phi_\lambda^2 x_n, \Phi_\lambda^1 x_n \right) + d^2\left(\Phi_\lambda^1 x_n, x_n \right),
\end{aligned}$$

which implies from (3.13), (3.14) and (3.15) that

$$\lim_{n \to \infty} d(y_n, x_n) = \lim_{n \to \infty} d(\Phi_\lambda^N x_n, x_n) = 0. \tag{3.16}$$

From (3.9) and Lemma (2.1), we have

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1 - \alpha_n) T y_n, \alpha_{n-1} u \oplus (1 - \alpha_{n-1}) T y_{n-1}) \\
&\leq d(\alpha_n u \oplus (1 - \alpha_n) T y_n, \alpha_n u \oplus (1 - \alpha_n) T y_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1 - \alpha_n) T y_{n-1}, \alpha_{n-1} u \oplus (1 - \alpha_{n-1}) T y_{n-1}) \\
&\leq (1 - \alpha_n) d(T y_n, T y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T y_{n-1}) \\
&\leq (1 - \alpha_n) d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T y_{n-1}) \\
&\leq (1 - \alpha_n) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T y_{n-1}). \tag{3.17}
\end{aligned}$$

By applying condition C2 and C3 in (3.17), we have from Lemma 2.6 that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{3.18}$$

From (3.9), (3.16), (3.18), Lemma 2.1 (i) and condition C1, we obtain

$$\begin{aligned}
d(x_n, T x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T y_n) + d(T y_n, T x_n) \\
&\leq d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1 - \alpha_n) T y_n, T y_n) + d(y_n, x_n) \\
&\leq d(x_n, x_{n+1}) + \alpha_n d(u, T y_n) + d(y_n, x_n) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

That is

$$\lim_{n \to \infty} d(x_n, T x_n) = 0. \tag{3.19}$$

From (3.16) and (3.19), we have

$$\begin{aligned}
d(x_n, T y_n) &\leq d(x_n, T x_n) + d(T x_n, T y_n) \\
&\leq d(x_n, T x_n) + d(x_n, y_n) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

That is

$$\lim_{n \to \infty} d(x_n, T y_n) = \lim_{n \to \infty} d\left(x_n, T \Phi_\lambda^N x_n\right) = 0. \tag{3.20}$$

Since $\{x_n\}$ is bounded and X is a complete CAT(0) space, then from Lemma 2.3, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k \to \infty} x_{n_k} = z$. Also, since $T \circ \Phi_\lambda^N$ is the composition of nonexpansive mappings, it implies that $T \circ \Phi_\lambda^N$ is nonexpansive. Thus, it follows from (3.20) and Lemma 2.5 that $z \in F(T \circ \Phi_\lambda^N)$. Hence, by Theorem 3.1, we obtain that $z \in \Gamma$.

Furthermore, from Lemma 2.4, we obtain

$$\limsup_{k \to \infty} \langle \overrightarrow{uz}, \overrightarrow{x_{n_k} z} \rangle \leq 0. \tag{3.21}$$

By using the quasilinearization properties, we obtain

$$\begin{aligned}
\langle \overrightarrow{uz}, \overrightarrow{T y_{n_k} z} \rangle &= \langle \overrightarrow{uz}, \overrightarrow{T y_{n_k} x_{n_k}} \rangle + \langle \overrightarrow{uz}, \overrightarrow{x_{n_k} z} \rangle \\
&\leq d(u, z) d(T y_{n_k}, x_{n_k}) + \langle \overrightarrow{uz}, \overrightarrow{x_{n_k} z} \rangle,
\end{aligned}$$

which implies from (3.20) and (3.21) that

$$\limsup_{k \to \infty} \langle \overrightarrow{uz}, \overrightarrow{T y_{n_k} z} \rangle \leq 0. \tag{3.22}$$

By condition C1 and inequality (3.22), we get

$$\limsup_{k\to\infty} \left(\alpha_n d^2(u, z) + 2(1 - \alpha_n)\langle \overrightarrow{uz}, \overrightarrow{Ty_{n_k}z} \rangle \right) \leq 0. \tag{3.23}$$

Next, we show that $\{x_n\}$ converges strongly to z. From (3.9) and Lemma 2.2, we obtain

$$d^2(x_{n+1}, z) \leq \alpha_n^2 d^2(u, z) + (1 - \alpha_n)^2 d^2(Ty_n, z) + 2\alpha_n(1 - \alpha_n)\langle \overrightarrow{uz}, \overrightarrow{Ty_n z} \rangle,$$

which implies

$$d^2(x_{n+1}, z) \leq (1 - \alpha_n)d^2(x_n, z) + \alpha_n \left(\alpha_n d^2(u, z) + 2(1 - \alpha_n)\langle \overrightarrow{uz}, \overrightarrow{Ty_n z} \rangle \right). \tag{3.24}$$

It follows from (3.23) and Lemma 2.6 that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d(x_n, p)\}$ is not monotone decreasing sequence. Then, there exists a subsequence $\{d(x_{n_i}, p)\}$ of $\{d(x_n, p)\}$ such that $d(x_{n_i}, p) < d(x_{n_i+1}, p)$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.7, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$

$$d(x_{m_k}, p) \leq d(x_{m_k+1}, p) \quad \text{and} \quad d(x_k, p) \leq d(x_{m_k+1}, p) \quad \forall k \in \mathbb{N}.$$

Thus, we have

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \left(d(x_{m_k+1}, p) - d(x_{m_k}, p) \right) \\
&\leq \limsup_{n\to\infty} \left(d(x_{n+1}, p) - d(x_n, p) \right) \\
&\leq \limsup_{n\to\infty} \left(\alpha_n d(u, p) + (1 - \alpha_n)d(x_n, p) - d(x_n, p) \right) \\
&\leq \limsup_{n\to\infty} \alpha_n \left(d(u, p) - d(x_n, p) \right) = 0,
\end{aligned}$$

which implies

$$\lim_{k\to\infty} \left(d(x_{m_k+1}, p) - d(x_{m_k}, p) \right) = 0. \tag{3.25}$$

Using C2 and C3 in (3.17), we obtain from Lemma 2.6 that

$$\lim_{k\to\infty} d(x_{m_k+1}, x_{m_k}) = 0.$$

Following the same line of argument as in Case 1, we obtain

$$d(x_{m_k}, Tx_{m_k}) \to 0 \quad \text{as} \quad k \to \infty \quad \text{and} \quad d(x_{m_k}, Ty_{m_k}) \to 0 \quad \text{as} \quad k \to \infty.$$

Also, as in Case 1, we obtain

$$\lim_{k\to\infty} \left(\alpha_{m_k} d^2(u, z) + 2(1 - \alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{Ty_{m_k}z} \rangle \right) \leq 0. \tag{3.26}$$

Also from (3.24), we have

$$d^2(x_{m_k+1}, z) \leq (1 - \alpha_{m_k})d^2(x_{m_k}, z) + \alpha_{m_k} \left(\alpha_{m_k} d^2(u, z) + 2(1 - \alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{Ty_{m_k} z} \rangle \right).$$

Since $d(x_{m_k}, z) \leq d(x_{m_k+1}, z)$, we have

$$d^2(x_{m_k}, z) \leq \left(\alpha_{m_k} d^2(u, z) + 2(1 - \alpha_{m_k})\langle \overrightarrow{uz}, \overrightarrow{Ty_{m_k} z} \rangle \right),$$

which implies from (3.26) that

$$\lim_{k\to\infty} d(x_{m_k}, z) = 0. \tag{3.27}$$

Since $d(x_k, z) \leq d(x_{m_k+1}, z)$, we obtain from (3.27) and (3.25) that $\lim_{n \to \infty} d(x_k, z) = 0$. Thus, from Case 1 and Case 2, we conclude that $\{x_n\}$ converges to $z \in \Gamma$.   □

By setting $N = 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.3 *Let X be a complete CAT(0) space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping that satisfies the range condition and T be a nonexpansive mapping on X. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{3.28}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$, satisfying the following conditions

C1: $\lim_{n \to \infty} \alpha_n = 0$,
C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,
C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to an element of Γ.

By setting $T = I$ (I is the identity mapping on X) in Theorem 3.2, we obtain the following result.

Corollary 3.4 *Let X be a complete CAT(0) space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition. Suppose that $\cap_{i=1}^{N} A_i^{-1}(0) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 (x_n), \ n \geq 1, \tag{3.29}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$, satisfying the following conditions

C1: $\lim_{n \to \infty} \alpha_n = 0$,
C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,
C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to an element of $\cap_{i=1}^{N} A_i^{-1}(0)$.

4 Application and numerical example

4.1 Application to minimization problem

Let X be a complete CAT(0) space and X^* be its dual space. Let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous and convex function with domain $\mathbb{D}(f) := \{x \in X : f(x) < +\infty\}$. Then the subdifferential of f is a set-valued function $\partial f : X \to 2^{X^*}$ defined by

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz} \rangle, \ (z \in X)\}, & \text{if } x \in \mathbb{D}(f), \\ \emptyset, & \text{otherwise.} \end{cases}$$

It has been shown in [1] that:

(i) ∂f is a monotone operator,
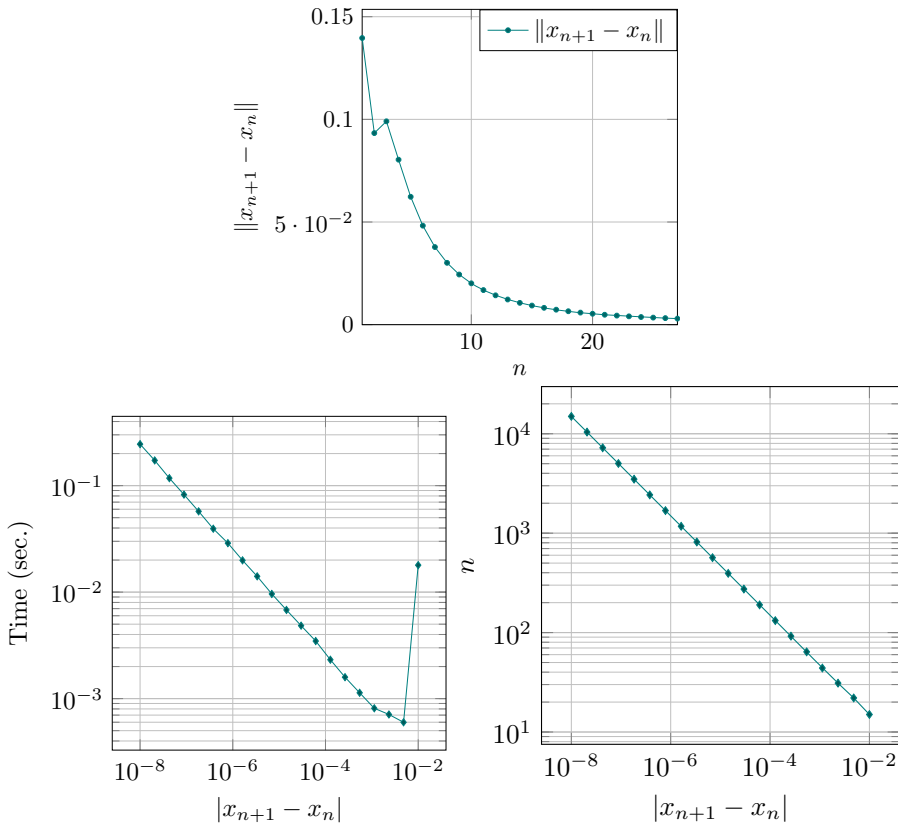
Fig. 1 Case I(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

(ii) ∂f satisfies the range condition. That is, $\mathbb{D}(J_\lambda^{\partial f}) = X$ for all $\lambda > 0$,

(iii) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$.

Now, consider the following minimization problem (MP): find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{4.1}$$

It follows from (iii) above that the above MP (4.1) can be formulated as follows: find $x \in X$ such that

$$0 \in \partial f(x).$$

Thus, in view of (i) and (ii) above, and by setting $A = \partial f$ in Theorem 3.2, we obtain the following result (Figs. 1, 2, 3).

Theorem 4.1 *Let X be a complete CAT(0) space and X^* be its dual space. Let f_i : $X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper, lower semicontinuous and convex function and let T be a nonexpansive mapping on X. Suppose that $\Gamma^* := F(T) \cap \left(\cap_{i=1}^N \partial f_i^{-1}(0) \right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*
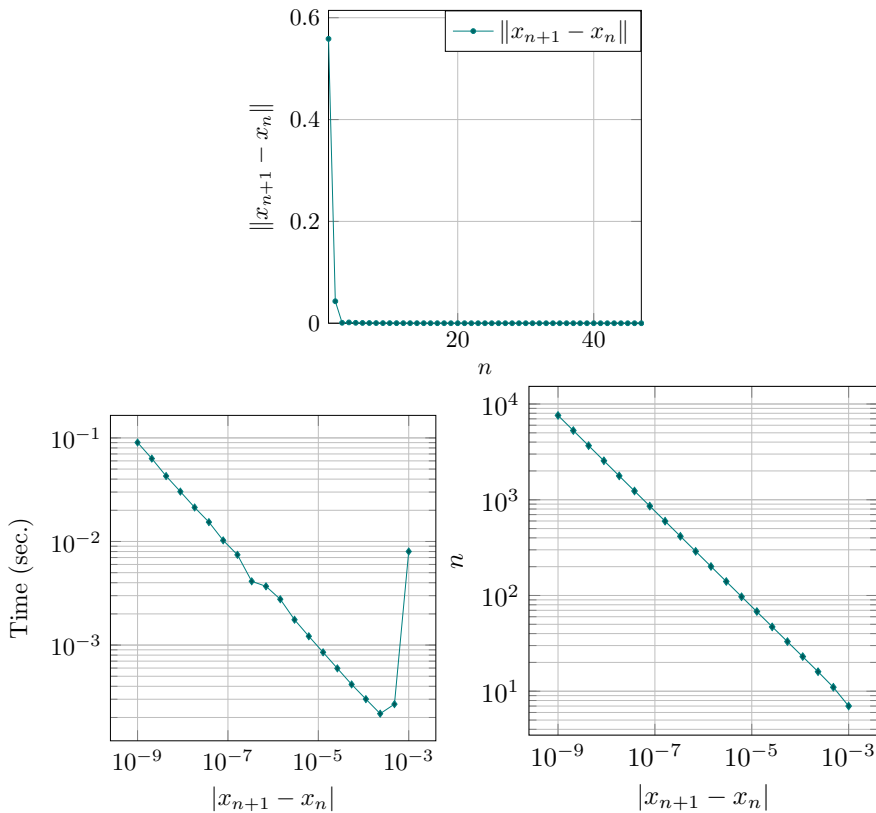
Fig. 2 Case I(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

$$\begin{cases} y_n = J_\lambda^{\partial f_N} \circ J_\lambda^{\partial f_{N-1}} \circ \cdots \circ J_\lambda^{\partial f_2} \circ J_\lambda^{\partial f_1}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \quad n \geq 0, \end{cases} \tag{4.2}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$ satisfying the following conditions

C1: $\lim_{n \to \infty} \alpha_n = 0,$
C2: $\sum_{n=1}^{\infty} \alpha_n = \infty,$
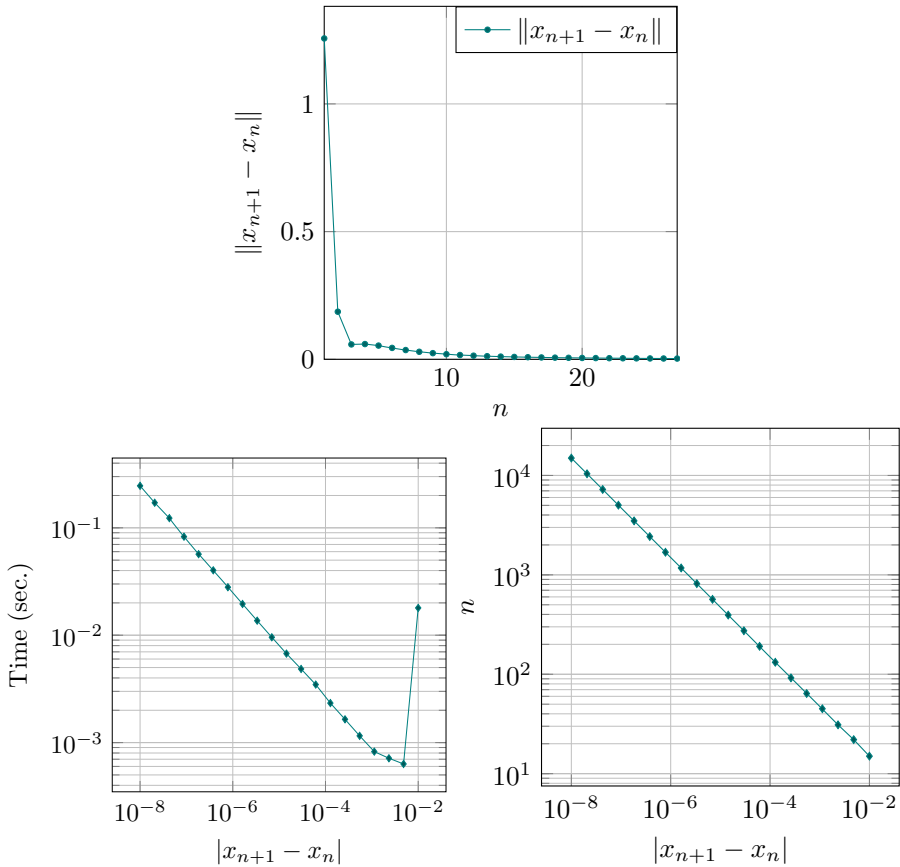C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty.$

Then $\{x_n\}$ converges strongly to an element of Γ^*.

4.2 Numerical example

We give a numerical example in $(\mathbb{R}^2, ||.||_2)$ (where $\mathbb{R}^2$ is the Euclidean plane) to support our main result. Let $N = 2$ in Theorem 3.2, then for $i = 1$, we define $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A_1(x) = (x_1 + x_2, \ x_2 - x_1).$$

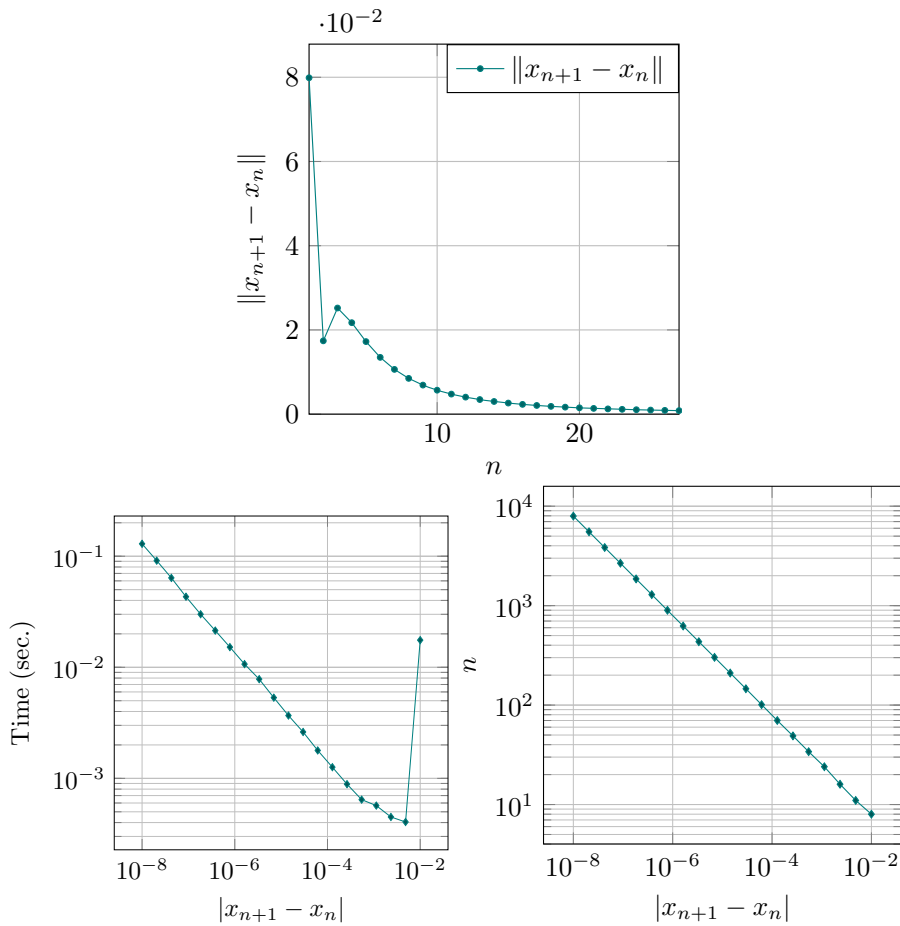Then A_1 is a monotone operator (Figs. 4, 5, 6).

Fig. 3 Case II(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

Recall that $[t\overrightarrow{ab}] \equiv t(b - a)$, for all $t \in \mathbb{R}$ and $a, b \in \mathbb{R}^2$ (see [1]). Using this, we have for each $x \in \mathbb{R}^2$ that

$$
\begin{aligned}
J_\lambda^1(x) = z &\Longleftrightarrow \frac{1}{\lambda}(x - z) = A_1 z \\
&\Longleftrightarrow x = (I + \lambda A_1)z \\
&\Longleftrightarrow z = (I + \lambda A_1)^{-1}x.
\end{aligned}
$$

Hence, we compute the resolvent of A_1 as follows:

$$
\begin{aligned}
J_\lambda^1(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda & \lambda \\ -\lambda & \lambda \end{bmatrix} \right)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} 1 + \lambda & \lambda \\ -\lambda & 1 + \lambda \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \frac{1}{1 + 2\lambda + 2\lambda^2} \begin{bmatrix} 1 + \lambda & -\lambda \\ \lambda & 1 + \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

Fig. 4 Case II(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

$$= \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1 + 2\lambda + 2\lambda^2}, \ \frac{\lambda x_1 + (1+\lambda)x_2}{1 + 2\lambda + 2\lambda^2} \right).$$
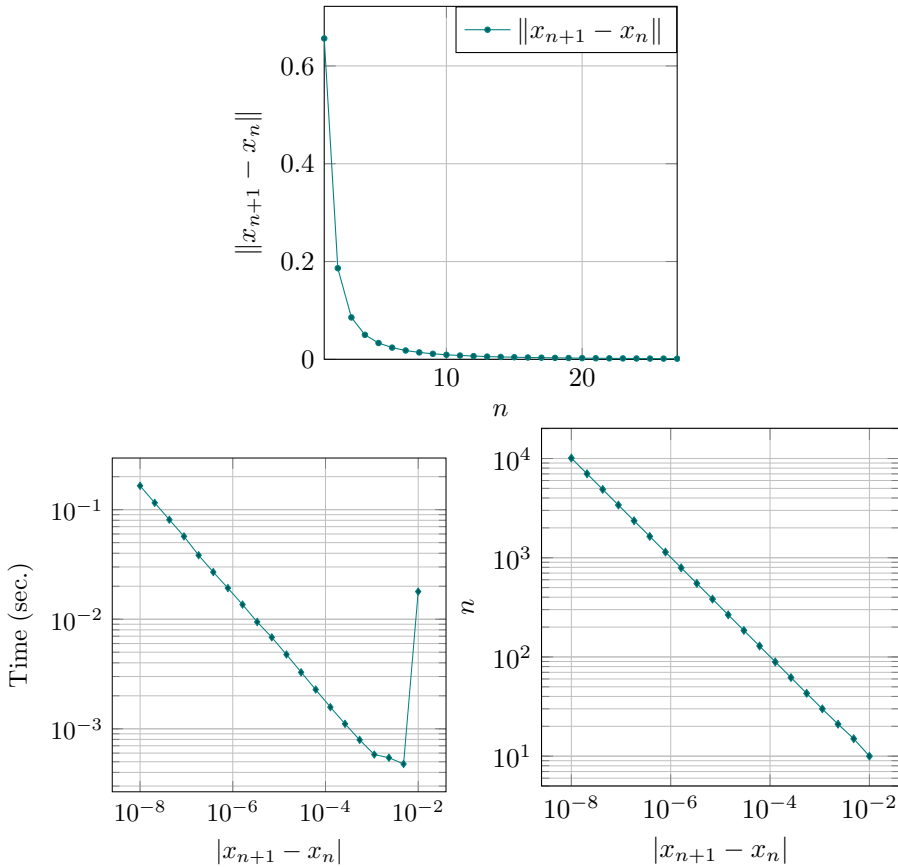
Thus,

$$J_\lambda^1(x) = \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1 + 2\lambda + 2\lambda^2}, \ \frac{\lambda x_1 + (1+\lambda)x_2}{1 + 2\lambda + 2\lambda^2} \right).$$

Now, for $i = 2$, let $A_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A_2(x) = (x_2, \ -x_1).$$

So that by the same argument as in above, we obtain

$$J_\lambda^2(x) = \left(\frac{x_1 - \lambda x_2}{1 + \lambda^2}, \ \frac{x_2 + \lambda x_1}{1 + \lambda^2} \right).$$

Fig. 5 Case III(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

Thus for $i = 1, 2$, we obtain

$$J_\lambda^2(J_\lambda^1 x) = \left(\frac{(1 + \lambda - \lambda^2)x_1 - (2\lambda + \lambda^2)x_2}{(1 + \lambda^2)(1 + 2\lambda + 2\lambda^2)}, \frac{(2\lambda + \lambda^2)x_1 + (1 + \lambda - \lambda^2)x_2}{(1 + \lambda^2)(1 + 2\lambda + 2\lambda^2)} \right).$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Then T is a nonexpansive mapping. Take $\alpha_n = \frac{1}{n+1}$, then $\{\alpha_n\}$ satisfies the conditions in Theorem 3.2.

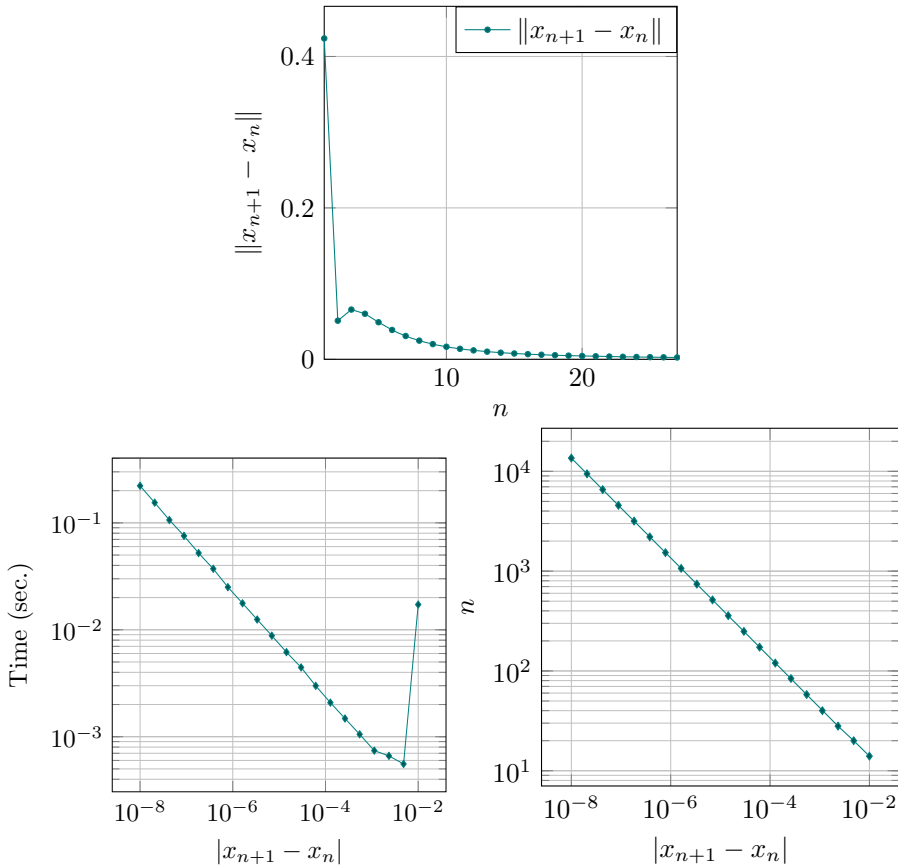Hence, for $u, x_1 \in \mathbb{R}^2$, our Algorithm (3.9) becomes:

$$\begin{cases} y_n = J_\lambda^2(J_\lambda^1 x_n), \\ x_{n+1} = \frac{u}{n+1} + \frac{n}{2(n+1)} y_n, \quad n \geq 1. \end{cases} \tag{4.3}$$

Case I

(a) Take $x_1 = (0.5, 0.25)^T$, $u = (1, 0.5)^T$ and $\lambda = 0.001$.
(b) Take $x_1 = (0.5, 0.25)^T$, $u = (1, 0.5)^T$ and $\lambda = 0.000002$.

Case II

(a) Take $x_1 = (1, 0.5)^T$, $u = (-1, 0.5)^T$ and $\lambda = 0.0002$.

Fig. 6 Case III(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right)

(b) Take $x_1 = (0.1, \ 0.03)^T$, $u = (0.3, \ 0.1)^T$ and $\lambda = 0.0002$.

Case III

(a) Take $x_1 = (-1, \ -0.5)^T$, $u = (-0.5, \ 0.1)^T$ and $\lambda = 0.00004$.
(b) Take $x_1 = (0.3, \ 0.06)^T$, $u = (0.2, \ 0.9)^T$ and $\lambda = 0.000009$.

Mathlab version R2014a is used to obtain the graphs of errors against number of iterations, execution time against accuracy and number of iterations against accuracy.

Compliance with ethical standards

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