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Fully discrete approximation of general nonlinear Sobolev equations

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Abstract
We consider abstract quasilinear evolution equations of Sobolev type in a Hilbert setting. We propose two fully discrete schemes and prove some error estimates under minimal assumptions. Various examples that enter into our abstract framework are considered, for each of them our theoretical results are confirmed by several numerical experiments.

Keywords Fully discrete scheme · Sobolev equations · Error estimates

Mathematics Subject Classification 65N30 · 65M12 · 65M15 · 35G30

1 Introduction

The purpose of our paper is to study different numerical schemes for the abstract quasilinear Sobolev equation

$$\begin{cases} A_1(t,u)u_t + A_2(t,u)u = f(t,u), & \text{in } V',\ 0 < t \le T, \\ u(0) = u_0, & \text{in } V, \end{cases} \tag{1.1}$$

where $A_1(t, u)$ is an isomorphism from a Hilbert space V into its dual V', while $A_2(t, u)$ is a bounded operator from V into V' (plus some assumptions specified below).

The linear or semilinear case, corresponding to the situation when $A_i(t, u)$ do not depend neither on t nor on u, will retain some particular interests.

Such problems are interesting not only because they are generalizations of a standard parabolic problem but also because they arise naturally in a large variety of applications

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(model of fluid flow in fissured porous media [4], two-phase flow in porous media with dynamical capillary pressure [12,20], heat conduction in two-temperature systems [9,42] and shear in second order fluids [11,41]).

Existence results for such problems are proved for semi-linear or non-autonomous equations (i.e., the case when A_1 and A_2 depends only on t) in [7,16,27,38–40] for instance, where the authors exploit the fact that A_1 is invertible. This allows to reduce the problem into a first order evolution equation (see (2.3) below) with a bounded (non-autonomous) operator and existence results easily follow. The same idea is here used to show existence results in the quasilinear situation by using the results from [23].

A large numbers of papers are devoted to the discretization of pseudoparabolic equations. Crank-Nicolson/explicit multistep approximation in time is combined with a finite element method in [2,8,15,28,43], with a Petrov-Galerkin method in [3,14] and with a discontinuous Galerkin method in [17,31,32]. The discretization along characteristics is applied in [34], while a Fourier-Galerkin method is used in [35]. In all these references, the operators A_1 and A_2 are (eventually non linear) second order elliptic operators. Hence in the spirit of [8] our main goal is to perform a general analysis for a fully discrete scheme by combining some error estimates of explicit semi-discrete schemes in time of ordinary differential equations (adapted to Hilbert valued equations) with new error estimates of the corresponding fully discrete schemes based on some "regularity" assumptions (see assumption $\mathbf{H_7}$) and interpolation error estimates. Altogether, if $U_{n,h}$ is the fully discrete approximation of the solution u at time t_n obtained by the Euler scheme or the Runge–Kutta scheme of order 2, we prove the error estimate

$$\|u(t_n) - U_{n,h}\| \leq C((\Delta t)^p + h^{q(s)}), \tag{1.2}$$

for all $n = 1, \ldots, N$, where $p = 1$ (resp. $p = 2$) for the Euler (resp. Runge–Kutta) scheme and $q(s)$ is related to our abstract assumptions (but in practice it depends on the regularity of the initial datum and the chosen finite element space), and C is a positive constant independent of h and Δt. Similarly to [8, see p. 14], the operator $\mathcal{A}(t,u) = A_1(t,u)^{-1}A_2(t,u)$ (involved in (2.3)) satisfies an appropriated Lipschitz property (see Corollary 2.3 below), hence its associated evolution problem is nonstiff. Due to this property, we do not have to impose mesh restrictions, like the CFL one. Note that, contrary to [8], our approach does not require any smallness restriction on the time step and on the meshsize.

We finally illustrate our abstract framework by various examples, like the case when the A_i's are (linear, non-autonomous, quasilinear) second order differential operators in smooth and non-smooth domains. In each case, new analytic results are proved to check that our general assumptions hold and some numerical tests that confirm the orders of convergence are presented.

In the whole paper, the norm of V will be denoted by $\|\cdot\|$ and we will write $a \lesssim b$, for the existence of a generic positive constant C that can depend on the final time T and on the norm of the data but is always independent of a, b, of the time step Δt and the meshsize parameter h such that $a \leq Cb$.

The paper is organized as follows. In Sect. 2, we give the basic assumptions that allow to obtain existence results. Section 3 is devoted to the introduction of the semi-discrete and the fully discrete problems and to the proof of error estimates. Some illustrative examples and numerical tests are presented in Sects. 4 and 5 for semi-linear and quasi-linear equations.

2 Existence results

We associate to each operator $A_i(t,u) : V \to V'$, $t \in [0,T], u \in V, i = 1,2$, a bilinear form $a_i(t;u;\cdot,\cdot)$, via the relation

$$a_i(t;u;v,w) = \langle A_i(t,u)v, w\rangle_{V',V}, \quad \forall v,w \in V. \tag{2.1}$$

In this section, we give some (local) existence results for problem (1.1) under the following assumptions.

H$_1$ (uniform continuity of $a_i(t;u;\cdot,\cdot)$ with respect to t and u) for $i = 1,2$, there exists a constant $M_i > 0$ independent of t and u such that for all $t \in [0,T]$ and $u,v,w \in V$,

$$|a_i(t;u;v,w)| \le M_i\|v\|\|w\|.$$

H$_2$ (uniform coerciveness of $a_1(t;u;\cdot,\cdot)$ with respect to t and u) there exists a constant $\alpha > 0$ independent of t and u such that for all $t \in [0,T]$ and $u,v \in V$,

$$a_1(t;u;v,v) \ge \alpha\|v\|^2. \tag{2.2}$$

The hypothesis **H$_1$** is equivalent to the uniform (in t and u) continuity of $A_i(t,u)$ from V into V', with $\|A_i(t,u)\|_{\mathcal{L}(V,V')} \le M_i$, for any $t \in [0,T]$ and $u \in V$; while the hypothesis **H$_2$** and Lax-Milgram's lemma guarantees that the operator $A_1(t,u), t \in [0,T], u \in V$ is an isomorphism from V into V', with $\|A_1(t,u)^{-1}\|_{\mathcal{L}(V',V)} \le \frac{1}{\alpha}$, for any $t \in [0,T]$ and $u \in V$.

As the operator $A_1(t,u), t \in [0,T], u \in V$ is invertible, we can compose the two sides of the first identity of (1.1) by $A_1(t,u)^{-1}$ and obtain the equivalent problem

$$\begin{cases} u_t + \mathcal{A}(t,u)u = g(t,u), & \text{in } V,\ 0 < t \le T,\\ u(0) = u_0, & \text{in } V, \end{cases} \tag{2.3}$$

where $\mathcal{A}(t,u) = A_1(t,u)^{-1}A_2(t,u)$ is a bounded operator from V into itself (uniformly with respect to t and u owing to **H$_1$** and **H$_2$** with $\|\mathcal{A}(t,u)\|_{\mathcal{L}(V,V)} \le \frac{M_2}{\alpha}$, for any $t \in [0,T]$ and $u \in V$) and $g(t,u) = A_1(t,u)^{-1}f(t,u)$. This problem enters into the framework of Kato's theory [23, Theorem 6], hence it suffices to check that the assumptions of this theorem are satisfied to obtain a local existence result. This is made under some additional assumptions on the sesquilinear forms a_i and on f. Before let us make the following definition.

Definition 2.1 Let E and F be two Hilbert spaces. A mapping $f : [0,T] \times E \longrightarrow F$ is called (E,F)-Lipschitz continuous with respect to the second variable uniformly in t, if there exists a positive constant L independent of t such that

$$\|f(t,v) - f(t,w)\|_F \le L\|v-w\|_E, \quad \forall v,w \in E,\ \forall t \in [0,T].$$

If $E = F$, we will say that f is E-Lipschitz continuous with respect to the second variable uniformly in t.

For a fixed open ball W of V, we now introduce the next assumptions:

H$_3$ there exists $\gamma \in (0,1]$ such that for $i = 1$ or 2,

$$\left|a_i(t;y;u,v) - a_i(s;z;u,v)\right| \lesssim \left(|t-s|^\gamma + \|y-z\|\right)\|u\|\|v\|,\ \forall y,z,u,v \in V, t,s \in [0,T]. \tag{2.4}$$

H$_4$ f is (V,V')-Lipschitz continuous with respect to the second variable uniformly in t.

H$_5$ f is bounded from $[0,T] \times V$ into V':

$$\|f(t,v)\|_{V'} \lesssim 1, \quad \forall t \in [0,T],\ \forall v \in V.$$

$\mathbf{H_6}$ For all $v \in W$, the mapping $t \to f(t, v)$ is continuous from $[0, T]$ into V'.

We now give some consequences of these assumptions.

Lemma 2.2 *Under the hypotheses* $\mathbf{H_1}$–$\mathbf{H_3}$, *the function* $(t, v) \to A_1(t, v)^{-1}$ *is Lipschitz continuous on* $[0, T] \times V$ *for the norm of* $\mathcal{L}(V', V)$ *uniformly in* t, v, *namely*

$$\|A_1(t, v)^{-1} - A_1(t_0, v_0)^{-1}\|_{\mathcal{L}(V',V)} \lesssim |t - t_0|^{\gamma} + \|v - v_0\|, \quad \forall t, t_0 \in [0, T], v, v_0 \in V. \tag{2.5}$$

Proof Let $v, v_0 \in V$ and $t, t_0 \in [0, T]$ be arbitrarily fixed. Then by definition we have

$$\|A_1(t, v)^{-1} - A_1(t_0, v_0)^{-1}\|_{\mathcal{L}(V',V)} = \sup_{h \in V', h \neq 0} \frac{\|A_1(t, v)^{-1}h - A_1(t_0, v_0)^{-1}h\|}{\|h\|_{V'}}.$$

Now for $h \in V', h \neq 0$, if we set $\phi_1 = A_1(t, v)^{-1}h$, and $\phi_2 = A_1(t_0, v_0)^{-1}h$, then

$$\begin{cases} a_1(t; v; \phi_1, \psi) = \langle h, \psi \rangle_{V',V}, \quad \forall \psi \in V, \\ a_1(t_0; v_0; \phi_2, \psi) = \langle h, \psi \rangle_{V',V}, \forall \psi \in V, \end{cases}$$

which yields

$$a_1(t; v; \phi_1 - \phi_2, \psi) = a_1(t_0; v_0; \phi_2, \psi) - a_1(t; v; \phi_2, \psi), \quad \forall \psi \in V.$$

Choosing $\psi = \phi_1 - \phi_2$, and using the hypotheses $\mathbf{H_2}$ and $\mathbf{H_3}$, we obtain

$$\begin{aligned} \|\phi_1 - \phi_2\| &\lesssim (|t - t_0|^{\gamma} + \|v - v_0\|)\|\phi_2\| = (|t - t_0|^{\gamma} + \|v - v_0\|)\|A_1(t_0, v_0)^{-1}h\| \\ &\lesssim (|t - t_0|^{\gamma} + \|v - v_0\|)\|h\|_{V'}, \end{aligned}$$

which proves the estimate (2.5). □

Corollary 2.3 *Under the hypotheses* $\mathbf{H_1}$–$\mathbf{H_3}$, *the function* $(t, v) \to \mathcal{A}(t, v)$ *is Lipschitz continuous on* $[0, T] \times V$ *for the norm of* $\mathcal{L}(V)$ *uniformly in* t, v, *namely*

$$\|\mathcal{A}(t, v) - \mathcal{A}(t_0, v_0)\|_{\mathcal{L}(V)} \lesssim |t - t_0|^{\gamma} + \|v - v_0\|, \quad \forall t, t_0 \in [0, T], v, v_0 \in V. \tag{2.6}$$

As a consequence, we have

$$\|\mathcal{A}(t, v)v - \mathcal{A}(t, w)w\| \lesssim (1 + \|w\|)\|v - w\|, \quad \forall t \in [0, T], \quad \forall v, w \in V. \tag{2.7}$$

Proof By definition, for $t, t_0 \in [0, T]$, $u, v, v_0 \in V$ arbitrarily fixed with $u \neq 0$, we have

$$\begin{aligned} \|\mathcal{A}(t, v) - \mathcal{A}(t_0, v_0)\|_{\mathcal{L}(V)} &\leq \|A_1(t, v)^{-1}(A_2(t, v) - A_2(t_0, v_0))\|_{\mathcal{L}(V)} \\ &\quad + \|(A_1(t, v)^{-1} - A_1(t_0, v_0)^{-1})A_2(t_0, v_0)\|_{\mathcal{L}(V)} \\ &\leq \|(A_1(t, v)^{-1}\|_{\mathcal{L}(V',V)}\|A_2(t, v) - A_2(t_0, v_0)\|_{\mathcal{L}(V,V')} \\ &\quad + \|A_1(t, v)^{-1} - A_1(t_0, v_0)^{-1}\|_{\mathcal{L}(V',V)}\|A_2(t_0, v_0)\|_{\mathcal{L}(V,V')}. \end{aligned}$$

Since the assumption $\mathbf{H_3}$ for $i = 2$ is equivalent to

$$\|A_2(t, v) - A_2(t_0, v_0)\|_{\mathcal{L}(V,V')} \lesssim |t - t_0|^{\gamma} + \|v - v_0\|,$$

we conclude that (2.6) holds owing to (2.5) and the assumptions $\mathbf{H_1}$ and $\mathbf{H_2}$. □

Corollary 2.4 *Under the hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_6}$, *the function* $g(t, u) = A_1(t, u)^{-1} f(t, u)$ *satisfies the next properties:*

1. *g is bounded on* $[0, T] \times V$*, i. e.,*

$$\|g(t, v)\| \lesssim 1, \quad \forall u, v \in V, \ \forall t \in [0, T].$$

2. *g is V-Lipschitz continuous with respect to the second variable uniformly in t,*

$$\|g(t, v) - g(t, v_0)\| \lesssim \|v - v_0\|, \quad \forall v, v_0 \in V, \ \forall t \in [0, T]. \tag{2.8}$$

3. *for all* $v \in W$*, the mapping* $t \to g(t, v)$ *is continuous from* $[0, T]$ *into* V*.*

Proof 1. Direct consequence of the hypotheses $\mathbf{H_2}$ and $\mathbf{H_5}$.

2. Let $v, v_0 \in V$ and $t, t_0 \in [0, T]$, then by the assumption $\mathbf{H_2}$, we may write

$$\begin{aligned}\|g(t, v) - g(t_0, v_0)\| &\leq \|A_1(t, v)^{-1}\left(f(t, v) - f(t_0, v_0)\right)\| \\ &\quad + \left\|\left(A_1(t, v)^{-1} - A_1(t_0, v_0)^{-1}\right) f(t_0, v_0)\right\| \\ &\lesssim \|f(t, v) - f(t_0, v_0)\|_{V'} + \|A_1(t, v)^{-1} \\ &\quad - A_1(t_0, v_0)^{-1}\|_{\mathcal{L}(V', V)} \|f(t_0, v_0)\|_{V'}.\end{aligned}$$

Hence by our assumption $\mathbf{H_5}$ and (2.5), we obtain

$$\|g(t, v) - g(t_0, v_0)\| \leq \|f(t, v) - f(t_0, v_0)\|_{V'} + |t - t_0|^{\gamma} + \|v - v_0\|. \tag{2.9}$$

If in particular $t_0 = t$, this estimate and the Lipschitz property of f then yield (2.8).

3. If $v_0 = v \in W$ in the estimate (2.9), we get

$$\|g(t, v) - g(t_0, v)\| \leq \|f(t, v) - f(t_0, v)\|_{V'} + |t - t_0|^{\gamma},$$

and the continuity property on g follows from the assumption $\mathbf{H_6}$. □

We are ready to prove our existence result.

Theorem 2.5 *Fix an open ball W of V and suppose that* $u_0 \in W$*. Under the assumptions* $\mathbf{H_1}$–$\mathbf{H_6}$*, there exists* $T' \in (0, T]$ *such that problem* (2.3) *(or* (1.1)*) admits a unique strong solution u in* $[0, T']$*, i.e., with the regularity* $u \in C([0, T']; W) \cap C^1([0, T']; V)$*.*

Proof We apply Theorem 6 from [23] with the Hilbert space $X = Y = V$ and the fixed open ball W from the statement. The assumptions (A1), (A2) and (A4) of this Theorem are trivially satisfied because the operators $\mathcal{A}(t, v)$ are bounded in V, the assumption (A3) holds owing to Corollary 2.3, while assumption (f1) holds owing to Corollary 2.4. □

Remark 2.6 Note that in the linear or semilinear case and under the assumption $f \in C^1([0, T] \times V; V')$ and $\mathbf{H_1}$–$\mathbf{H_2}$, Theorem 6.1.5 of [33] guarantees the existence of a global solution $u \in C^1([0, T]; V)$ for any initial data in V.

We end up this section with the following comment. If we suppose that there exists another Hilbert space H such that V is continuously embedded into H (denoted by $V \hookrightarrow H$) and such that V is a dense subspace of H, then we can introduce the restriction of $A_i(t, u)$ to H (that, for shortness, is still denoted by $A_i(t, u)$), namely we can define the unbounded operator from H into itself by

$$D(A_i(t, u)) = \{v \in V : \exists g_v \in H \text{ such that } a_i(t; u; v, w) = (g_v, w)_H \text{ for all } w \in V\},$$

and

$$A_i(t, u)v = g_v, \quad \forall v \in D(A_i(t, u)).$$

3 Discretizations of the problem

3.1 Explicit semi-discretization in time

We notice that problem (1.1) can be equivalently written as

$$\begin{cases} u_t = F(t, u), \text{ in } V, \ 0 < t \leq T, \\ u(0) = u_0, \quad \text{ in } V, \end{cases} \tag{3.1}$$

where $F(t, u) = g(t, u) - \mathcal{A}(t, u)u$, which is Hilbert-valued nonlinear ordinary differential equation. Since in our case, $F(\cdot, \cdot)$ is bounded, we can use standard explicit schemes, like the Euler or Runge–Kutta methods as in the case of finite-dimensional ODE. More precisely, we now consider a regular subdivision $(t_i = i\Delta t)_{i=0}^{N}$ of the interval $[0, T']$, where T' is the life time of u, $N \in \mathbb{N}^*$ and $\Delta t = \frac{T'}{N}$ the time step. Given a continuous function $\phi : [0, T] \times V \times [0, \Delta t] \longrightarrow V$, starting from $u(t_0 = 0) = u_0$, we try to estimate the solution u of (3.1) at the points $(t_{n+1}), n = 0, \ldots, N-1$, by estimating step by step the values of $u(t_{n+1})$ using the variation of constants formula

$$u(t_{n+1}) = u(t_n) + \int_{t_n}^{t_{n+1}} F(\tau, u(\tau)) d\tau.$$

Here we restrict ourselves to a one step method that consists to approach the expression $\int_{t_n}^{t_{n+1}} F(\tau, u(\tau)) d\tau$ by $\Delta t \phi(t_n, u(t_n), \Delta t)$, i.e., the approximated solution of problem (3.1) is given by

$$\begin{cases} U_0 = u_0, \\ U_{n+1} = U_n + \Delta t \phi(t_n, U_n, \Delta t), \ n = 0, \ldots, N-1. \end{cases} \tag{3.2}$$

The convergence of this numerical scheme is based on the estimation of the local consistency error.

Definition 3.1 The local consistancy error E_l relative to the exact solution u of (3.1) is defined by

$$E_l(t_{n+1}) = u(t_{n+1}) - u(t_n) - \Delta t \phi(t_n, u(t_n), \Delta t), \quad \forall n = 0, \ldots, N-1. \tag{3.3}$$

The next Theorem is a direct generalization of a well-known result for ODE in the form $u_t = F(t, u)$, where F has values in $\mathbb{R}^k$, $k \in \mathbb{N}^*$ (see for example Theorem 3.5 of [19]).

Theorem 3.2 *Let the assumptions of Theorem 2.5 be satisfied (or Remark 2.6 in the linear or semilinear case). Let u be the exact solution of problem (3.1). Suppose that ϕ is locally Lipschitz continuous with respect to the second variable uniformly in t, i.e., there exists a positive constant L independent of t and Δt such that*

$$\|\phi(t, u, \Delta t) - \phi(t, v, \Delta t)\| \leq L\|u - v\|, \quad \forall t \in [0, T], u \in W, v \in V, \tag{3.4}$$

and suppose that there exists $p \in \mathbb{N}$ such that the local errors satisfy

$$\|E_l(t_{n+1})\| \lesssim (\Delta t)^{p+1}, \quad \forall n = 0, 1, \ldots, N-1.$$

Then the global errors $e_n = u(t_n) - U_n$ satisfy

$$\|e_n\| \lesssim (\Delta t)^p, \quad \forall n = 0, 1, \ldots, N. \tag{3.5}$$

Proof We have from (3.3),

$$u(t_{n+1}) = u(t_n) + \Delta t \phi(t_n, u(t_n), \Delta t) + E_l(t_{n+1}), \ n = 0, 1, \ldots, N.$$

Then, from (3.2), the Lipschitz property on ϕ and the fact that $u \in C([0, T'], W)$, we obtain

$$\begin{aligned}\|e_{n+1}\| &\leq \|e_n\| + \Delta t\|\phi(t_n, u(t_n), \Delta t) - \phi(t_n, U_n, \Delta t)\| + \|E_l(t_{n+1})\| \\ &\leq (1 + L\Delta t)\|e_n\| + C(\Delta t)^{p+1}.\end{aligned} \tag{3.6}$$

By induction on n, we deduce that

$$\|e_n\| \leq (1 + L\Delta t)^n \|e_0\| + C(\Delta t)^{p+1} \sum_{k=0}^{n-1} (1 + L\Delta t)^{n-k-1}, \quad \forall\, 1 \leq n \leq N.$$

Since $e_0 = 0$, we find that (3.5) holds. □

Lemma 3.3 *Assume that $u_0 \in W$. Under the assumptions $\mathbf{H_1}$–$\mathbf{H_5}$, the function F is locally Lipschitz with respect to the second variable uniformly in t, i.e., for all $t \in [0, T]$,*

$$\|F(t, u) - F(t, v)\| \lesssim \|u - v\|, \quad \forall u \in W,\ \forall v \in V.$$

Proof By definition of F, it is easy to see that

$$\|F(t, u) - F(t, v)\| \leq \|g(t, u) - g(t, v)\| + \|\mathcal{A}(t, v)v - \mathcal{A}(t, u)u\|.$$

The result follows thanks to Corollary 2.2, (2.7) and the fact that $u \in W$. □

Remark 3.4 In the linear or semilinear case, the assumptions $\mathbf{H_1}$, $\mathbf{H_2}$ and $\mathbf{H_4}$ guarantee that F is V-Lipschitz continuous with respect to the second variable uniformly in t.

We now concentrate on two particular schemes.

3.1.1 Explicit Euler scheme

This scheme corresponds to the choice $\phi(t, u, \Delta t) = F(t, u)$, and then takes the form

$$\begin{cases} U_0 = u_0. \\ U_{n+1} = U_n + \Delta t F(t_n, U_n),\ n = 0,\ \dots\ , N - 1. \end{cases} \tag{3.7}$$

In this case, we have the next error estimate.

Proposition 3.5 *Under the assumptions of Theorem 2.5, assume that the solution u of (1.1) has the extra regularity $C^2([0, T']; V)$. If U_n is the approximated solution given by the explicit Euler scheme (3.7), then the local errors E_l satisfy the following estimate*

$$\|E_l(t_{n+1})\| \lesssim (\Delta t)^2, \quad \forall\, 0 \leq n \leq N - 1. \tag{3.8}$$

Furthermore the global errors satisfy

$$\|u(t_n) - U_n\| \lesssim \Delta t, \quad \forall\, 0 \leq n \leq N. \tag{3.9}$$

Proof By a Taylor development with integral remainder at order 1, we have

$$u(t_{n+1}) = u(t_n) + \Delta t u'(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)u''(\tau)d\tau.$$

Consequently, one has

$$E_l(t_{n+1}) = \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)u''(\tau)d\tau$$

and we conclude that (3.8) holds by our assumption.

For the second assertion, we simply notice that Lemma 3.3 guarantees that $\phi(t, u, \Delta t) = F(t, u)$ satisfies (3.4) and we conclude by Theorem 3.2. □

Remark 3.6 In the linear or semilinear case, it is not difficult to check that the assumption $f \in C^1([0, T'] \times V; V')$ implies the extra regularity $u \in C^2([0, T']; V)$. In the general situation, we further need that the mapping $(t, v) \to A_i(t, v)$, $i = 1, 2$, is Fréchet differentiable on $[0, T'] \times V$, with

$$\left\| \frac{\partial A_i}{\partial t}(t, v) \right\|_{\mathcal{L}(\mathbb{R}; \mathcal{L}(V; V'))} + \left\| \frac{\partial A_i}{\partial v}(t, v) \right\|_{\mathcal{L}(V; \mathcal{L}(V; V'))} \lesssim 1, \quad \forall t \in [0, T], v \in V.$$

3.1.2 Heun's scheme (or Runge–Kutta of order 2)

This scheme corresponds to the choice

$$\phi(t, u, \Delta t) = \frac{1}{2}\left(F(t, u) + F(t + \Delta t, u + \Delta t F(t, u))\right),$$

and then may be written as

$$\begin{cases} U_0 = u_0 \\ U^*_{n+1} = U_n + \Delta t F(t_n, U_n), \\ U_{n+1} = U_n + \frac{\Delta t}{2}[F(t_n, U_n) + F(t_{n+1}, U^*_{n+1})], n = 0, \ldots, N-1. \end{cases} \tag{3.10}$$

Proposition 3.7 *Under the assumptions of Theorem 2.5, assume that the solution u of (1.1) has the extra regularity $C^3([0, T']; V)$. If U_n is the approximated solution given by the Runge–Kutta scheme, then*

$$\|E_l(t_{n+1})\| \lesssim (\Delta t)^3, \quad \forall\, 0 \le n \le N-1.$$

As a consequence, we have

$$\|u(t_n) - U_n\| \lesssim (\Delta t)^2, \quad \forall\, 0 \le n \le N.$$

Proof The first assumption follows by a Taylor development with integral remainder at the order 2. Thanks to Lemma 3.3, we easily check that

$$\|\phi(t, u, \Delta t) - \phi(t, v, \Delta t)\| \lesssim (1 + \Delta t)\|u - v\|, \quad \forall t \in [0, T], u \in W,\ v \in V,$$

which implies that ϕ satisfies (3.4). The second assertion then follows from Theorem 3.2. □

Remark 3.8 In the linear or semilinear case, the assumption $f \in C^2([0, T'] \times V; V')$ implies the extra regularity $u \in C^3([0, T']; V)$. In the general situation, we further need that the mapping $(t, v) \to A_i(t, v)$, $i = 1, 2$, is twicely Fréchet differentiable on $[0, T'] \times V$, with

$$\left\| \frac{\partial^2 A_i}{\partial t^2}(t, v) \right\|_{\mathcal{L}(\mathbb{R}, \mathbb{R}; \mathcal{L}(V; V'))} + \left\| \frac{\partial^2 A_i}{\partial v^2}(t, v) \right\|_{\mathcal{L}(V, V; \mathcal{L}(V; V'))} + \left\| \frac{\partial^2 A_i}{\partial t \partial v}(t, v) \right\|_{\mathcal{L}(\mathbb{R}, V; \mathcal{L}(V; V'))} \lesssim 1, \quad \forall t \in [0, T], v \in V.$$

3.2 Fully discrete scheme

For a positive parameter h (that plays the rule of a mesh size), we suppose given a finite dimensional subspace V_h of V and build a fully discrete approximation of problem (1.1). For that purpose, let us introduce some useful notations. For an arbitrary element u_h in V_h, we consider the approximation $A_{i,h}(t, u_h)$ of $A_i(t, u_h)$, $i = 1, 2$, defined by

$$\langle A_{i,h}(t, u_h)v_h, w_h\rangle_{V_h',V_h} = a_i(t; u_h; v_h, w_h), \quad \forall v_h, w_h \in V_h. \tag{3.11}$$

For further uses, for any $t \in [0, T]$ and any $u \in V$, we define the orthogonal projection $P_h(t, u)$ associated with the bilinear form $a_1(t; u; \cdot, \cdot)$, i.e., for any $v \in V$, $P_h(t, u)v \in V_h$ is the unique solution of

$$a_1(t; u; P_h(t, u)v, w_h) = a_1(t; u; v, w_h), \quad \forall w_h \in V_h.$$

Similarly we introduce the orthogonal projection Q_h in V on V_h associated with the inner product $(\cdot, \cdot)_V$.

We first consider the discrete (in space) version of (1.1), namely we look for $u_h \in C^1([0, T], V_h)$ solution of

$$\begin{cases} A_{1,h}(t, u_h)u_{h,t} + A_{2,h}(t, u_h)u_h = f_h(t, u_h), & \text{in } V_h,\ 0 < t \le T, \\ u_h(0) = P_h(t_0, u_0)u_0, & \text{in } V_h, \end{cases} \tag{3.12}$$

where $f_h(t, u_h) = I_h f(t, u_h)$ and $I_h : V' \to V_h'$ is the linear and continuous operator defined by

$$\langle f_h(t, u_h), v_h\rangle_{V_h',V_h} = \langle f(t, u_h), v_h\rangle_{V',V}, \quad \forall v_h \in V_h.$$

As in the continuous case, the operator $A_{1,h}(t, u_h)$, $t \in [0, T]$, $u_h \in V_h$ being invertible, this problem is then equivalent to

$$\begin{cases} u_{h,t} + \mathcal{A}_h(t, u_h)u_h = g_h(t, u_h), & \text{in } V_h,\ 0 < t \le T, \\ u_h(0) = P_h(t_0, u_0)u_0, & \text{in } V_h, \end{cases} \tag{3.13}$$

where $\mathcal{A}_h(t, u_h) = A_{1,h}(t, u_h)^{-1}A_{2,h}(t, u_h)$ and $g_h(t, u_h) = A_{1,h}(t, u_h)^{-1} f_h(t, u_h)$. The next Lemma shows that the operator $\mathcal{A}_h(t, u_h)$ is bounded (uniformly with respect to h) from V_h into itself.

Lemma 3.9 *Under the hypotheses* $\mathbf{H_1}$ *and* $\mathbf{H_2}$, *for any* $t \in [0, T]$ *and* $v_h \in V_h$, *one has*

$$\|\mathcal{A}_h(t, v_h)\|_{\mathcal{L}(V_h)} \le \frac{M_2}{\alpha}. \tag{3.14}$$

Proof Let us fix $t \in [0, T]$ and $v_h \in V_h$. Then by definition we have

$$\|\mathcal{A}_h(t, v_h)\|_{\mathcal{L}(V_h)} = \sup_{w_h \in V_h, w_h \neq 0} \frac{\|\mathcal{A}_h(t, v_h)w_h\|}{\|w_h\|}.$$

For a fixed $w_h \in V_h$, $w_h \neq 0$, we set $v_h' = A_{1,h}(t, v_h)^{-1}A_{2,h}(t, v_h)w_h$, then owing to (3.11), $v_h' \in V_h$ is the unique solution of

$$a_1(t; v_h; v_h', \varphi_h) = a_2(t; v_h; w_h, \varphi_h), \quad \forall \varphi_h \in V_h.$$

Taking $\varphi_h = v_h'$ and using the hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$, we find that

$$\alpha\|v_h'\|^2 \le a_2(t; v_h; w_h, v_h') \le M_2\|w_h\|\|v_h'\|,$$

which implies (3.14). □

Note that problem (3.13) can be equivalently written as an ODE in V_h:

$$\begin{cases} u_{h,t} = F_h(t, u_h), & \text{in } V_h,\ 0 < t \leq T, \\ u_h(0) = P_h(t_0, u_0)u_0, & \text{in } V_h, \end{cases} \tag{3.15}$$

where $F_h(t, u_h) = g_h(t, u_h) - \mathcal{A}_h(t, u_h)u_h$. Therefore its approximation by an explicit Euler scheme or by a Runge–Kutta scheme of order 2, takes the respective forms

$$\begin{cases} U_{0,h} = P_h(t_0, u_0)u_0 \\ U_{n+1,h} = U_{n,h} + \Delta t F_h(t_n, U_{n,h}),\ n = 0, \ldots, N-1. \end{cases} \tag{3.16}$$

or

$$\begin{cases} U_{0,h} = P_h(t_0, u_0)u_0 \\ U^*_{n+1,h} = U_{n,h} + \Delta t F_h(t_n, U_{n,h}), \\ U_{n+1,h} = U_{n,h} + \frac{\Delta t}{2}[F_h(t_n, U_{n,h}) + F_h(t_{n+1}, U^*_{n+1,h})],\ n = 0, \ldots, N-1. \end{cases} \tag{3.17}$$

Remark 3.10 For any $u_h \in V_h$, the hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$ yield

$$g_h(t, u_h) = P_h(t, u_h)g(t, u_h),\ \forall t \in [0, T]. \tag{3.18}$$

Indeed as $A_{1,h}(t, u_h)g_h(t, u_h) = I_h f(t, u_h)$, we obtain

$$a_1(t; u_h; g_h(t, u_h), v_h) = \langle f(t, u_h), v_h\rangle_{V',V}, \quad \forall v_h \in V_h.$$

On one hand, the definition of $P_h(t, u_h)$ implies that

$$a_1(t; u_h; P_h(t, u_h)g(t, u_h), v_h) = a_1(t; u_h; g(t, u_h), v_h) = a_1(t; u_h; A_1^{-1}(t, u_h)f(t, u_h), v_h),$$

on the other hand, by the definition of the bilinear form $a_1(t; u_h; \cdot, \cdot)$, we find that

$$a_1(t; u_h; g_h(t, u_h), v_h) = a_1(t; u_h; P_h(t, u_h)g(t, u_h), v_h), \quad \forall v_h \in V_h.$$

This proves (3.18) by Lax–Milgram's lemma.

Our error analysis of the fully discrete scheme is based on the following assumptions.

$\mathbf{H_7}$ There exist two Hilbert spaces D_s and $\tilde{D}_{s-1}$ ($s \geq 1$ being a parameter that could take different occurrence or not) such that $D_s \hookrightarrow V$ and such that $A_1(t, u)$ is an isomorphism from D_s into $\tilde{D}_{s-1}$, while $A_2(t, u)$ is only supposed to be bounded from D_s into $\tilde{D}_{s-1}$, with

$$\|A_1(t, u)^{-1}\|_{\mathcal{L}(\tilde{D}_{s-1}, D_s)} + \|A_2(t, u)\|_{\mathcal{L}(D_s, \tilde{D}_{s-1})} \lesssim 1,$$

for all $t \in [0, T]$ and $u \in V$.

$\mathbf{H_8}$ For each parameter s from assumption $\mathbf{H_7}$, there exists a positive real number $q(s)$ such that

$$\|\varphi - Q_h\varphi\| \lesssim h^{q(s)}\|\varphi\|_{D_s}, \quad \forall \varphi \in D_s. \tag{3.19}$$

In practice, under the assumption that $V \hookrightarrow H$, D_s corresponds to the domain of powers of $A_1(t, u)$ or a subdomain of it (hence $\tilde{D}_{s-1} = A_1(t, u)D_s$), while the estimate (3.19) follows from an interpolation error estimate. In particular for $s = 1$, we can chose $D_1 = D(A_1(t, u))$ and $\tilde{D}_0 = H$, if $D(A_1(t, u))$ is independent of t and u, and if we can check the above assumptions. We refer to Sects. 4 and 5 for some concrete illustrations.

Now, if we denote by

$$\|\cdot\|_{t,v} = \sqrt{a_1(t; v; \cdot, \cdot)},$$

the norm on V associated with the bilinear form $a_1(t; v; \cdot, \cdot)$, by the continuity and the uniform coercivity of $a_1(t; v; \cdot, \cdot)$ (hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$), we notice that

$$\sqrt{\alpha}\|u\| \leq \|u\|_{t,v} \leq \sqrt{M_1}\|u\|, \quad \forall t \in [0, T], \forall u, v \in V. \tag{3.20}$$

Proposition 3.11 *Let the hypotheses* $\mathbf{H_1}$–$\mathbf{H_8}$ *be satisfied. If we suppose that* $f \in C([0, T] \times V; \tilde{D}_{s-1})$*, then for all* $t \in [0, T]$ *and* $(v, v_h) \in D_s \times V_h$*, we have*

$$\|F_h(t, v_h) - F(t, v)\| \lesssim (1 + \|v\|)\|v - v_h\| + h^{q(s)}(\|v\|_{D_s} + \|f(t, v)\|_{\tilde{D}_{s-1}}). \tag{3.21}$$

Proof Fix $t \in [0, T]$ and $(v, v_h) \in D_s \times V_h$. By the triangular inequality, we have

$$\|F_h(t, v_h) - F(t, v)\| \leq \|g_h(t, v_h) - g(t, v)\| + \|\mathcal{A}(t, v)v - \mathcal{A}_h(t, v_h)v_h\|.$$

We start by the estimating the first term of this right-hand side. The identity (3.18) yields

$$\begin{aligned}\|g_h(t, v_h) - g(t, v)\| &\leq \|g_h(t, v_h) - P_h(t, v_h)g(t, v)\| + \|(P_h(t, v_h) - I)g(t, v)\| \\ &= \|P_h(t, v_h)(g(t, v_h) - g(t, v))\| + \|(P_h(t, v_h) - I)g(t, v)\|.\end{aligned}$$

On one hand using (3.20), we obtain

$$\begin{aligned}\|P_h(t, v_h)(g(t, v_h) - g(t, v))\| &\leq \frac{1}{\sqrt{\alpha}}\|P_h(t, v_h)(g(t, v_h) - g(t, v))\|_{t,v_h} \\ &\leq \frac{1}{\sqrt{\alpha}}\|g(t, v_h) - g(t, v)\|_{t,v_h} \\ &\leq \sqrt{\frac{M_1}{\alpha}}\|g(t, v_h) - g(t, v)\| \\ &\lesssim \sqrt{\frac{M_1}{\alpha}}\|v - v_h\|,\end{aligned}$$

where the last estimate is due to Corollary 2.4, 2. On the other hand, using again (3.20) we obtain

$$\begin{aligned}\|(P_h(t, v_h) - I)g(t, v)\| &\leq \frac{1}{\sqrt{\alpha}}\|(P_h(t, v_h) - I)g(t, v)\|_{t,v_h} \\ &\leq \frac{1}{\sqrt{\alpha}}\|(Q_h - I)g(t, v)\|_{t,v_h} \\ &\leq \sqrt{\frac{M_1}{\alpha}}\|(Q_h - I)g(t, v)\|.\end{aligned}$$

Owing to the hypothesis (3.19), we find

$$\|(P_h(t, v_h) - I)g(t, v)\| \lesssim h^{q(s)}\|g(t, v)\|_{D_s} \lesssim h^{q(s)}\|f(t, v)\|_{\tilde{D}_{s-1}}.$$

For the second term, we may write

$$\begin{aligned}\|\mathcal{A}(t, v)v - \mathcal{A}_h(t, v_h)v_h\| \leq{}& \|\mathcal{A}(t, v)v - \mathcal{A}(t, v_h)v\| + \|(I - P_h(t, v_h))\mathcal{A}(t, v_h)v\| \\ &+\|P_h(t, v_h)\mathcal{A}(t, v_h)v - \mathcal{A}_h(t, v_h)v_h\|.\end{aligned} \tag{3.22}$$

Corollary 2.3 directly furnishes

$$\|\mathcal{A}(t, v)v - \mathcal{A}(t, v_h)v\| \lesssim \|v - v_h\|\|v\|.$$

Further owing to (3.20), we have

$$\|(I - P_h(t, v_h))\mathcal{A}(t, v_h)v\| \leq \sqrt{\frac{M_1}{\alpha}}\|(I - Q_h)\mathcal{A}(t, v_h)v\|.$$

As $\mathcal{A}(t, v_h)v \in D_s$ due to the assumption $v \in D_s$, we deduce from the estimate (3.19) that

$$\|(I - P_h(t, v_h))\mathcal{A}(t, v_h)v\| \lesssim h^{q(s)}\|\mathcal{A}(t, v_h)v\|_{D_s} \lesssim h^{q(s)}\|v\|_{D_s}.$$

It then remains to estimate the last term of the right-hand side of (3.22). For that purpose, setting $w = \mathcal{A}(t, v_h)v$ and $w_h = \mathcal{A}_h(t, v_h)v_h$, by the definition of a_1 and a_2, we have

$$a_1(t; v_h; w, \psi) = a_2(t; v_h; v, \psi), \quad \forall \psi \in V,$$
$$a_1(t; v_h; w_h, \psi_h) = a_2(t; v_h; v_h, \psi_h), \quad \forall \psi_h \in V_h,$$

and by the definition of $P_h(t, v_h)$, we also have

$$a_1(t; v_h; P_h(t, v_h)w, \psi_h) = a_1(t; v_h; w, \psi_h), \quad \forall \psi_h \in V_h.$$

Hence

$$a_1(t; v_h; P_h(t, v_h)w - w_h, \psi_h) = a_2(t; v_h; v - v_h, \psi_h), \quad \forall \psi_h \in V_h. \tag{3.23}$$

Choosing in (3.23) $\psi_h = P_h(t, v_h)w - w_h$, we obtain owing to the hypotheses $\mathbf{H_1}$ and $\mathbf{H_2}$

$$\|P_h(t, v_h)w - w_h\| \lesssim \|v - v_h\|.$$

Consequently

$$\|P_h(t, v_h)\mathcal{A}(t, v)v - \mathcal{A}_h(t, v_h)v_h\| \lesssim \|v - v_h\|.$$

Altogether we have shown that (3.21) is valid. □

Lemma 3.12 *Suppose that $u_0 \in D_s$, that $f \in C([0, T] \times V; \tilde{D}_{s-1})$ is $(V, \tilde{D}_{s-1})$-Lipschitz continuous with respect to the second variable uniformly in t, and that $\mathbf{H_7}$ hold. Then the sequence $(U_n)_{n=1}^N$ defined by (3.10) is (uniformly) bounded in D_s, i.e.,*

$$\|U_n\|_{D_s} \lesssim 1, \quad \forall n = 1, \ldots, N.$$

Proof By construction (see (3.10)), we have

$$\|U_{n+1}\|_{D_s} \leq \|U_n\|_{D_s} + \frac{\Delta t}{2}\Big(\|F(t_n, U_n)\|_{D_s} + \|F\big(t_{n+1}, U_n + \Delta t F(t_n, U_n)\big)\|_{D_s}\Big). \tag{3.24}$$

We then estimate each term of this right-hand side separately by using the assumptions that $A_1(t, v)^{-1}$ is (uniformly) bounded from $\tilde{D}_{s-1}$ into D_s, that $\mathcal{A}(t, v)$ is (uniformly) bounded in D_s and taking into account the Lipschitz hypothesis on f.

For the first term, by the triangle inequality, for any $u \in D_s$ we have

$$\|F(t_n, u)\|_{D_s} \leq \|F(t_n, u) - F(t_n, 0)\|_{D_s} + \|F(t_n, 0)\|_{D_s}. \tag{3.25}$$

Since $F(t_n, 0) = A_1(t_n, 0)^{-1}f(t_n, 0)$, we directly get

$$\|F(t_n, 0)\|_{D_s} \lesssim \|f(t_n, 0)\|_{\tilde{D}_{s-1}}, \quad \forall\, n = 0, 1, \ldots, N. \tag{3.26}$$

Furthermore for any $u \in D_s$, and for any $t \in [0, T]$, we can write

$$\begin{aligned}\|F(t, u) - F(t, 0)\|_{D_s} &\leq \|\mathcal{A}(t, u)u\|_{D_s} + \|g(t, u) - g(t, 0)\|_{D_s} \\ &\leq \|\mathcal{A}(t, u)u\|_{D_s} + \|A_1(t, u)^{-1}f(t, u) - A_1(t, 0)^{-1}f(t, 0)\|_{D_s}\end{aligned}$$

$$\lesssim \|u\|_{D_s} + \|A_1(t,u)^{-1}(f(t,u) - f(t,0))\|_{D_s} \\ +\|A_1(t,u)^{-1} f(t,0) - A_1(t,0)^{-1} f(t,0)\|_{D_s} \\ \lesssim \|u\|_{D_s} + \|f(t,u) - f(t,0)\|_{\tilde{D}_{s-1}} + \|f(t,0)\|_{\tilde{D}_{s-1}}$$

By the Lipschitz property on f, we then conclude that

$$\|F(t,u) - F(t,0)\|_{D_s} \lesssim \|u\|_{D_s} + \|f(t,0)\|_{\tilde{D}_{s-1}}. \tag{3.27}$$

These two estimates in (3.25) leads to

$$\|F(t_n,u)\|_{D_s} \lesssim \|u\|_{D_s} + \|f(t_n,0)\|_{\tilde{D}_{s-1}}. \tag{3.28}$$

To estimate the second term of the right-hand side of (3.24), writting

$$F\big(t_{n+1}, U_n + \Delta t F(t_n,U_n)\big) = F\big(t_{n+1}, U_n + \Delta t F(t_n,U_n)\big) - F\big(t_{n+1},0\big) + F\big(t_{n+1},0\big),$$

and using (3.26) and (3.27), we obtain

$$\|F\big(t_{n+1}, U_n + \Delta t F(t_n,U_n)\big)\|_{D_s} \lesssim \|U_n + \Delta t F(t_n,U_n)\|_{D_s} + \|f(t_{n+1},0)\|_{\tilde{D}_{s-1}} \\ \lesssim \|U_n\|_{D_s} + \Delta t\|F(t_n,U_n)\|_{D_s} + \|f(t_{n+1},0)\|_{\tilde{D}_{s-1}}.$$

Therefore (3.28) allows to conclude that

$$\|F\big(t_{n+1}, U_n+\Delta t F(t_n,U_n)\big)\|_{D_s} \lesssim (1+\Delta t)\|U_n\|_{D_s} + \Delta t\|f(t_n,0)\|_{\tilde{D}_{s-1}} + \|f(t_{n+1},0)\|_{\tilde{D}_{s-1}}.$$

Using this estimate and (3.28) in (3.24) leads to

$$\|U_{n+1}\|_{D_s} \lesssim (1+\Delta t)\|U_n\|_{D_s} + \Delta t\|f(t_n,0)\|_{\tilde{D}_{s-1}} + \Delta t\|f(t_{n+1},0)\|_{\tilde{D}_{s-1}}.$$

By iteration, we find

$$\|U_n\|_{D_s} \lesssim (1+\Delta t)^n\|u_0\|_{D_s} + \Delta t\sum_{k=0}^{n-1}(1+\Delta t)^{n-1-k}\big(\|f(t_k,0)\|_{\tilde{D}_{s-1}} + \|f(t_{k+1},0)\|_{\tilde{D}_{s-1}}\big).$$

As for all $0 \le k \le N$

$$(1+\Delta t)^k \le (1+\Delta t)^N \lesssim e^T,$$

the result follows since f is continuous from $[0,T] \times V$ into $\tilde{D}_{s-1}$. □

Lemma 3.13 *Let the hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_8}$ *and the hypotheses of Lemma 3.12 hold. Let* U_n *(resp.* $U_{n,h}$*) be the approximated solution given by* (3.10) *(resp.* (3.17)*). Then the error* $e_{n,h} = U_n - U_{n,h}$ *is bounded as follows*

$$\|e_{n,h}\| \lesssim h^{q(s)}. \tag{3.29}$$

Proof By the triangular inequality, we can write

$$\|e_{n+1,h}\| \le \|e_{n,h}\| + \frac{\Delta t}{2}\|F_h(t_n,U_{n,h}) - F(t_n,U_n)\| \\ + \frac{\Delta t}{2}\|F_h\big(t_{n+1}, U_{n,h} + \Delta t F_h(t_n,U_{n,h})\big) - F\big(t_{n+1}, U_n + \Delta t F(t_n,U_n)\big)\|.$$

As $U_n \in D_s$, for all $1 \le n \le N$, Proposition 3.11 and Lemma 3.12 lead to

$$\|F_h(t_n,U_{n,h}) - F(t_n,U_n)\| \lesssim \|e_{n,h}\| + h^{q(s)}\big(\|U_n\|_{D_s} + \|f(t_n,U_n)\|_{\tilde{D}_{s-1}}\big).$$

Using again this proposition and the previous estimate, we find

$$\|F_h\big(t_{n+1}, U_{n,h} + \Delta t F_h(t_n, U_{n,h})\big) - F\big(t_{n+1}, U_n + \Delta t F(t_n, U_n)\big)\| \lesssim (1+\Delta t)\|e_{n,h}\| \\ +h^{q(s)}\big(\|U_n\|_{D_s} + \|f(t_n, U_n)\|_{\tilde{D}_{s-1}} + \|f\big(t_{n+1}, U_n + \Delta t F(t_n, U_n)\big)\|_{\tilde{D}_{s-1}}\big).$$

Consequently

$$\|e_{n+1,h}\| \lesssim (1+\Delta t)\|e_{n,h}\| + h^{q(s)}\Delta t\big(\|U_n\|_{D_s} + \|f(t_n, U_n)\|_{\tilde{D}_{s-1}} \\ +\|f(t_{n+1}, U_n + \Delta t F(t_n, U_n))\|_{\tilde{D}_{s-1}}\big).$$

By iteration we deduce that

$$\|e_{n,h}\| \lesssim (1+\Delta t)^n\|e_{0,h}\| + h^{q(s)}\Delta t\sum_{k=0}^{n-1}(1+\Delta t)^{n-k-1}\big(\|U_k\|_{D_s} \\ +\|f(t_k, U_k)\|_{\tilde{D}_{s-1}} + \|f(t_{k+1}, U_k + \Delta t F(t_k, U_k))\|_{\tilde{D}_{s-1}}\big). \tag{3.30}$$

As $u_0 \in D_s$, our hypothesis (3.19) guarantees that

$$\|e_{0,h}\| = \|u_0 - P_h(t_0, u_0)u_0\| \lesssim h^{q(s)}\|u_0\|_{D_s}. \tag{3.31}$$

By the Lipschitz continuity of f and since D_s is continously embedded into V, we obtain

$$\|f(t_k, U_k)\|_{\tilde{D}_{s-1}} \lesssim \|U_k\|_{D_s} + \|f(t_k, 0)\|_{\tilde{D}_{s-1}}. \tag{3.32}$$

Similarly one has

$$\|f(t_{k+1}, U_k + \Delta t F(t_k, U_k)\|_{\tilde{D}_{s-1}} \lesssim \|U_k\|_{D_s} + \Delta t\|F(t_k, U_k)\|_{D_s} + \|f(t_{k+1}, 0)\|_{\tilde{D}_{s-1}}$$

With the help of the estimate (3.28), we obtain

$$\|f(t_{k+1}, U_k + \Delta t F(t_k, U_k)\|_{\tilde{D}_{s-1}} \lesssim (1+\Delta t)\|U_k\|_{D_s} + \|f(t_{k+1}, 0)\|_{\tilde{D}_{s-1}} + \Delta t\|f(t_k, 0)\|_{\tilde{D}_{s-1}}.$$

Inserting this estimate, as well as (3.31) and (3.32) into (3.30), we deduce owing to Lemma 3.12 and the continuity of f from $[0, T] \times V$ into $\tilde{D}_{s-1}$ that (3.29) is valid. □

Remark 3.14 Under the same hypotheses of Lemma 3.13, if the Euler scheme (3.16) is used to approximate the solution of problem (1.1), then the error estimate (3.29) remains valid.

Corollary 3.15 *Under the hypotheses* $\mathbf{H_1}$ *to* $\mathbf{H_8}$*, we suppose that* $u_0 \in W \cap D_s$*,* $f \in C([0, T] \times V; \tilde{D}_{s-1})$*, that* f *is* $(V, \tilde{D}_{s-1})$*-Lipschitz continuous with respect to the second variable uniformly in* t*. Assume that the solution* u *of problem* (1.1) *exists and belongs to* $C^{p+1}([0, T']; V)$*,* $p = 1$ *or* 2*. Let* $U_{n,h}$ *be its approximated solution given by* (3.16) *for* $p = 1$ *(resp.* (3.17) *for* $p = 2$*). Then we have the global error estimate*

$$\|u(t_n) - U_{n,h}\| \lesssim (\Delta t)^p + h^{q(s)}, \quad \forall n = 1, \ldots, N. \tag{3.33}$$

Proof Direct consequence of Proposition 3.5 and Remark 3.14 for the Euler scheme, and of Proposition 3.7 and Lemma 3.13 for the Runge–Kutta scheme. □

4 Applications to particular semi-linear equations

4.1 Elliptic operators of order two: the regular case

Let Ω be a bounded domain of $\mathbb{R}^d$, $d \geq 1$ with a Lipschitz boundary. For $i = 1, 2$, let L_i be two elliptic operators of order two of the form

$$L_i(x, D_x)u = -\sum_{k,\ell=1}^{d} \partial_k(a_{k,\ell}^{(i)}(x)\partial_\ell u) + \sum_{k=1}^{d} b_k^{(i)}(x)\partial_k u + c^{(i)}(x)u, \tag{4.1}$$

where $a_{k,\ell}^{(i)} = a_{\ell,k}^{(i)} \in C^{0,1}(\bar{\Omega})$, $b_k^{(i)}, c^{(i)} \in L^\infty(\Omega)$. Moreover we suppose that L_1 is strongly elliptic, namely that there exists $\alpha_* > 0$ such that

$$\sum_{k,\ell=1}^{d} a_{k,\ell}^{(1)}(x)\xi_\ell\xi_k \geq \alpha_*|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

In this case, we may introduce the continuous bilinear forms a_i on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$a_i(u, v) = \int_\Omega \left(\sum_{k,\ell=1}^{d} a_{k,\ell}^{(i)}(x)\ \partial_\ell u\ \partial_k v + \sum_{k=1}^{d} b_k^{(i)}(x)\partial_k u\ v + c^{(i)}(x)u\ v \right) dx, \forall u, v \in H_0^1(\Omega).$$

Hence their associated operator A_i

$$\langle A_i u, v\rangle_{V',V} = a_i(u, v), \quad \forall u, v \in H_0^1(\Omega),$$

are continuous from $H_0^1(\Omega)$ into its dual $H^{-1}(\Omega)$. These operators satisfy the hypotheses from Sect. 2 with $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$, if we assume that a_1 is coercive on $H_0^1(\Omega)$ (that is the case if $\operatorname{div} \mathbf{b}^{(1)} = \sum_{k=1}^d \partial_k b_k^{(1)} = 0$ and $c^{(1)} \geq 0$ for example).

Consequently the problem

$$\begin{cases} L_1 u_t + L_2 u = f(t, u), & \text{in } \ \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \ H_0^1(\Omega), \end{cases} \tag{4.2}$$

is well-posedness for an initial datum $u_0 \in H_0^1(\Omega)$ and f continuously differentiable from $[0, T] \times H_0^1(\Omega)$ into $H^{-1}(\Omega)$. This system is a semi-linear Sobolev equation in Ω that has been analyzed in [2,15,28,31,32,39] in some particular situations with Neumann type boundary conditions. Such boundary conditions also enter into our framework by simply replacing $H_0^1(\Omega)$ by $H^1(\Omega)$ (and assuming that a_1 is coercive in $H^1(\Omega)$).

In order to check the assumptions $\mathbf{H_7}$ and $\mathbf{H_8}$, we will characterize the domains of A_1 (as an unbounded operator in H) and of $A_1^{\frac{3}{2}}$ in some particular situations. In the first case we make use of Kadlec's result.

Lemma 4.1 *Under the previous hypotheses on the coefficients of L_i, $i = 1, 2$, and if Ω is convex or has a boundary of class $C^{1,1}$, then*

$$D(A_1) = H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow D(A_2). \tag{4.3}$$

Proof For $i = 1$, or 2, we recall that

$$D(A_i) = \{u \in H_0^1(\Omega) : A_i u \in L^2(\Omega)\}.$$

Hence $u \in H_0^1(\Omega)$ belongs to $D(A_1)$ if and only if there exists $f \in L^2(\Omega)$ such that

$$a_1(u, v) = \int_\Omega f v \, dx, \quad \forall v \in H_0^1(\Omega).$$

This is equivalent to

$$\int_\Omega \sum_{k,\ell=1}^{d} a_{k,\ell}^{(1)}(x) \partial_\ell u \partial_k v \, dx = \int_\Omega h v \, dx, \quad \forall v \in H_0^1(\Omega),$$

where $h = f - \sum_{k=1}^{d} b_k^{(1)} \partial_k u - c^{(1)} u$ belongs to $L^2(\Omega)$. Hence owing to Kadlec's result [22] (see also [18, Thm 3.2.1.3]), we conclude that $u \in H^2(\Omega)$. This proves the embedding

$$D(A_1) \hookrightarrow H^2(\Omega) \cap H_0^1(\Omega).$$

The inverse embedding being trivial, we have shown that

$$\|u\|_{D(A_1)} \sim \|u\|_{H^2(\Omega)}, \quad \forall u \in D(A_1).$$

Clearly we have $D(A_1) = H^2(\Omega) \cap H_0^1(\Omega) \subset D(A_2)$, and therefore for $u \in D(A_1)$, $L_2 u$ belongs to $L^2(\Omega)$ with

$$\|u\|_{D(A_2)} = \|u\|_{H_0^1(\Omega)} + \|L_2 u\|_{L^2(\Omega)} \lesssim \|u\|_{H^2(\Omega)},$$

which proves the continuous embedding of $D(A_1)$ into $D(A_2)$. □

With the help of this result, as a first guess we can take $D_1 = D(A_1)$ and $\tilde{D}_0 = H$, since A_1 is an isomorphism from $D(A_1)$ into H and A_2 is bounded from $D(A_1)$ into H. The characterization of the domain of $D(A_1^{\frac{3}{2}})$ and additional assumptions on the coefficients of A_2 allow to build a second choice of pairs D_s, $\tilde{D}_{s-1}$.

Lemma 4.2 *Suppose that the boundary of Ω is of class $C^{2,1}$, that $a_{k,\ell}^{(i)} = a_{\ell,k}^{(i)} \in C^{1,1}(\bar{\Omega})$, and that $b_k^{(i)}, c^{(i)} \in C^{0,1}(\Omega)$. Then*

$$D(A_1^{3/2}) = A_1^{-1} H_0^1(\Omega) = \{u \in H^3(\Omega) \cap H_0^1(\Omega) : A_1 u = 0 \ \text{ on } \ \partial\Omega\}. \tag{4.4}$$

If furthermore we have

$$a_{k,l}^{(2)} = a_{k,l}^{(1)} \text{ and } d^{(2)} \cdot n = d^{(1)} \cdot n \text{ on } \partial\Omega, \tag{4.5}$$

where n is the unit outward normal vector along the boundary and for $d^{(i)} = (d_l^{(i)})_{l=1}^d$ is the vector given by

$$d_l^{(i)} = -\sum_k^d \partial_k a_{k,l}^{(i)} + b_l^{(i)}.$$

Then A_2 is continuous from $D(A_1^{3/2})$ into $H_0^1(\Omega)$.

Proof Let $u \in D(A_1^{3/2})$. Then there exists $h \in H_0^1(\Omega)$ such that

$$A_1 u = h.$$

By [18, Theorem 2.5.1.1, p. 128], we deduce that $u \in H^3(\Omega)$, hence the embedding

$$D(A_1^{3/2}) \hookrightarrow \{u \in H^3(\Omega) \cap H_0^1(\Omega) : A_1 u = 0 \ \text{ sur } \ \partial\Omega\}.$$

The converse embedding being trivial, the first assertion is proved.

Let us go on with the second assertion. Let us then fix $u \in D(A_1^{3/2})$, then by the regularity of the coefficients of L_2, we directly see that $A_2 u = L_2 u$ belongs to $H^1(\Omega)$. Hence it remains to show that it is zero on the boundary. For that purpose, we notice that

$$L_i u = -\sum_{k,\ell=1}^{d} a_{k,\ell}^{(i)} \partial_k \partial_\ell u + d^{(i)} \cdot \nabla u + c^{(i)} u.$$

Hence on the boundary, splitting the gradient of u into its tangential part and its normal one, and recalling that $u = 0$ on the boundary, we have

$$L_i u = -\sum_{k,\ell=1}^{d} a_{k,\ell}^{(i)} \partial_k \partial_\ell u + (d^{(i)} \cdot n) \partial_n u \text{ on } \partial\Omega.$$

Our assumption (4.5) then implies that

$$L_2 u = L_1 u \text{ on } \partial\Omega,$$

which finishes the proof since $L_1 u$ is zero on the boundary. □

Let us notice that a similar result may remain valid for less regular boundaries. Indeed it holds for instance for a square and for L_1 reduces to the Laplace operator.

Lemma 4.3 *If Ω is the unit square $(0, 1)^2$ of the plane and $L_1 = \Delta$, then (4.4) remains valid.*

Proof We use an argument from the proof of Lemma 2.4 in [21]. Let $v \in D(A_1^{3/2}) \hookrightarrow D(A_1) = H^2(\Omega) \cap H_0^1(\Omega)$, then $g = \Delta v \in H_0^1(\Omega)$. Hence from elliptic regularity, $v \in H^3(\Omega \setminus V)$, where V is any neighborhood of the corners. It then remains to show the H^3 regularity near the corners. By symmetry, it suffices to show such a regularity near 0. Let us then fix a radial cut-off function η such that $\eta = 1$ near 0 with supp $\eta \subset B(0, \frac{1}{2})$. Consequently $u = \eta v$ (extended by zero outside its support) belongs to $H^2((0, \infty)^2) \cap H_0^1((0, \infty)^2)$ and satisfies

$$\Delta u = \eta g + 2\frac{\partial \eta}{\partial r}\frac{\partial v}{\partial r} + v\Delta\eta = \tilde{f} \in H_0^1((0, \infty)^2).$$

We now set

$$U(x, y) = u(|x|, |y|) \text{ sign } xy, \text{ and } F(x, y) = \tilde{f}(|x|, |y|) \text{ sign } xy,$$

and easily check that

$$\Delta U = F,$$

in the distributional sense. But as $F \in H^1(\mathbb{R}^2)$ by [37, p.85], we conclude that $U \in H^3(\mathbb{R}^2)$, and consequently $u \in H^3((0, \infty)^2)$ and finally $v \in H^3(\Omega)$. □

In conclusion if (4.4) and (4.5) are valid, we can take $D_{3/2} = D(A_1^{\frac{3}{2}})$ and $\tilde{D}_{1/2} = D(A_1^{\frac{1}{2}}) = H_0^1(\Omega)$.

Now to build a fully discrete scheme, we shall use a finite element method based on a triangulation of Ω. To this end, we consider a family of meshes $\{\mathcal{T}_h\}_h$ of Ω, where each mesh is made of tetrahedral (or triangular) elements K. To simplify the analysis, we assume that the boundary of Ω is exactly triangulated, and therefore, we consider curved Lagrange finite elements as described in [5]. Also, for each element K, we denote by $\mathcal{F}_K$ the mapping taking the reference element $\hat{K}$ to K.

With the help of this triangulation $\mathcal{T}_h$, we define the approximation space $V_h \subset H_0^1(\Omega)$ by

$$V_h = \left\{ v_h \in H_0^1(\Omega) \ : \ v_h|_K \circ \mathcal{F}_K^{-1} \in \mathbb{P}_p(\hat{K}) \ \forall K \in \mathcal{T}_h \right\}, \tag{4.6}$$

where $\mathbb{P}_p(\hat{K})$ stands for the set of polynomials of total degree less than or equal to p.

In this setting, owing to Corollary 5.2 of [5] (see also Theorem 3.2.2 of [10]), the assumption (3.19) is satisfied for $s = 1$ (under the assumption (4.3)) or $s = \frac{3}{2}$ (under the assumptions (4.4) and (4.5)), with $q(s) = 2s - 1$ and the choice $p \geq q(s)$, i.e., for all $f \in D_s$, one has

$$\|f - Q_h f\| \lesssim h^{q(s)} \|f\|_{D_s}.$$

Finally the fully discrete schemes of problem (4.2) can be formulated as follows: The explicit Euler scheme consists in looking for $U_{n+1,h} \in V_h$ solution of

$$a_1\big(U_{n+1,h}, \chi_h\big) = a_1(U_{n,h}, \chi_h) - \Delta t a_2(U_{n,h}, \chi_h) + \Delta t(f(t_n, U_{n,h}), \chi_h), \quad \forall \chi_h \in V_h, \tag{4.7}$$

that allows to compute $U_{n+1,h}$ by the knowledge of $U_{n,h}$ and of $f(t_n, U_{n,h})$.

Similarly, by the Runge–Kutta method, we look for $U^*_{n+1,h} \in V_h$ solution of

$$a_1(U^*_{n+1,h}, \chi_h) = a_1(U_{n,h}, \chi_h) + \Delta t(f_h(t_n, U_{n,h}), \chi_h) - \Delta t a_2(U_{n,h}, \chi_h), \ \forall \chi_h \in V_h.$$

and then $U_{n+1,h} \in V_h$ solution of

$$\begin{aligned} &a_1(U_{n+1,h}, \chi_h) = a_1(U_{n,h}, \chi_h) \\ &+ \frac{\Delta t}{2}\big[\big(f(t_n, U_{n,h}) + f(t_{n+1}, U^*_{n+1,h}), \chi_h\big) - a_2(U_{n,h} + U^*_{n+1,h}, \chi_h)\big], \ \forall \chi_h \in V_h. \end{aligned} \tag{4.8}$$

4.2 Elliptic operators of order two: the singular case

We now extend the previous results to the case where the domain Ω is a non-smooth two-dimensional domain and the principal part of L_1 and L_2 are piecewise constant. In that case, Lemma 4.1 is no more valid in general (see [24–26,30] for instance), but the use of weighted Sobolev spaces of Kondratiev's type [18,36] will allow to put (4.2) into our abstract framework.

Let us start with the definition of the weighted Sobolev spaces in a polygonal domain D of $\mathbb{R}^2$ (see [36] or [18, Def. 8.4.1.1]).

Definition 4.4 Let $r(x)$ be the distance from a point x of D to the vertices of D. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}^*$, we define

$$\begin{aligned} L^2_\alpha(D) &= \{u \in L^2_{\text{loc}}(D) : r^\alpha u \in L^2(D)\}, \\ V^k_\alpha(D) &= \{u \in L^2_{\alpha-k}(D) : r^{\alpha+|\beta|-k} D^\beta u \in L^2(D), \forall \beta \in \mathbb{N}^2 : |\beta| \leq k\}. \end{aligned}$$

These spaces are Hilbert spaces equipped with their natural norms:

$$\|u\|_{L^2_\alpha(D)} = \|r^\alpha u\|_{L^2(D)}, \|u\|^2_{V^k_\alpha(D)} = \sum_{|\beta| \leq k} \|r^{\alpha+|\beta|-k} D^\beta u\|^2_{L^2(D)}.$$

For any edge e of D, the trace space of $V_\alpha^1(D)$ onto e is denoted by $V_\alpha^{\frac{1}{2}}(e)$ (see [29, Thm 1.31]). Note that $V_\alpha^{\frac{1}{2}}(e)$ has its own definition, see [29, Def. 1.9], in particular we have

$$V_\alpha^{\frac{1}{2}}(e) \hookrightarrow L^2_{\alpha-\frac{1}{2}}(e). \tag{4.9}$$

We now suppose that Ω is a polygonal domain of $\mathbb{R}^2$ that is partitioned into sub-domains Ω_j, $j = 1, \ldots, J$, with a positive integer J so that the Ω_j's are disjoint open polygonal domains and that

$$\bar{\Omega} = \cup_{j=1}^J \bar{\Omega}_j.$$

Let us further denote by e_ℓ, $\ell = 1, \ldots, L$, the set of interior edges, namely the set of straight segments that are the intersection of $\bar{\Omega}_j \cap \bar{\Omega}_{j'}$ with $j \neq j'$ (hence they are not included into the boundary of Ω). Similarly the set $\mathcal{S}$ of vertices of Ω is simply the set of vertices of all Ω_j's.

In the following we need piecewise weighted Sobolev spaces $\mathcal{V}_\alpha^k(\Omega)$, more precisely, we set

$$\mathcal{V}_\alpha^k(\Omega) = \{v \in L^2_{\alpha-k}(\Omega) : v_j \in V_\alpha^k(\Omega_j), \quad \forall j = 1, \ldots, J\},$$

where $v_j := v_{|\Omega_j}$ denotes the restriction of v to Ω_j. Again these spaces are Hilbert spaces equipped with their natural norms.

Now we suppose that the operators L_i are elliptic of order 2 in the previous form (4.1) but with coefficients $a_{k,\ell}^{(i)} = a_{\ell,k}^{(i)}$ piecewise regular, in other words the restriction of $a_{k,\ell}^{(i)}$ to Ω_j are regular ($C^\infty(\bar{\Omega}_j)$). As before we assume that the bilinear form a_1 associated with A_1 is coercive so that A_1 is an isomorphism from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

To facilitate the presentation, for $i = 1$ or 2, let us introduce the symmetric matrix $M_i = (a_{k,\ell}^{(i)})_{k,\ell=1,2}$ and the gradient jumps of u through an edge e_ℓ as follows

$$[\![M_i \nabla u \cdot n]\!]_\ell = \Big((M_i \nabla u)_{|\Omega_j} - (M_i \nabla u)_{|\Omega_{j'}} \Big) \cdot n_\ell,$$

when $e_\ell = \bar{\Omega}_j \cap \bar{\Omega}_{j'}$ and n_ℓ is the unit normal vector along e_ℓ orientated from Ω_j to $\Omega_{j'}$.

We now recall Corollary 4.4 of [30] that is valid in dimension 2 under the assumption $\gamma < 1$ since there exists $r > 1$ such that

$$L^2_\gamma(\Omega) \hookrightarrow L^r(\Omega). \tag{4.10}$$

Theorem 4.5 *Suppose that $\gamma \in (0, 1)$ and the segment $(0, 1-\gamma]$ does not contain singular exponent of A_1 at all corners of Ω. Then for all $f \in L^2_\gamma(\Omega)$ and $h_\ell \in V_\gamma^{\frac{1}{2}}(e_\ell)$, $\ell = 1, \ldots, L$, there exists a unique solution $u \in H_0^1(\Omega) \cap \mathcal{V}_\gamma^2(\Omega)$ to problem*

$$\begin{cases} L_1 u_j = f_j & \text{in } \Omega_j, \; j = 1, \cdots, J, \\ [\![M_1 \nabla u \cdot n]\!]_\ell = h_\ell & \text{on } e_\ell, \; \ell = 1, \cdots, L, \end{cases} \tag{4.11}$$

in the sense that

$$a_1(u, v) = F(v), \quad \forall v \in H_0^1(\Omega),$$

where the linear form F is given by

$$F(v) = \int_\Omega f(x)v(x)\,dx + \sum_{\ell=1}^{L} \int_{e_\ell} h_\ell \gamma v(x)\,d\sigma(x), \quad \forall v \in H_0^1(\Omega).$$

Note that F is well-defined on $H_0^1(\Omega)$ since for all $\gamma \in (0,1)$, there exists $r \in (1,2]$ such that

$$V_\gamma^{\frac{1}{2}}(e_\ell) \hookrightarrow L^r(e_\ell). \tag{4.12}$$

Indeed by (4.9), any $w \in V_\gamma^{\frac{1}{2}}(e_\ell)$ satisfies

$$r^{\gamma-\frac{1}{2}} w \in L^2(e_\ell).$$

If $\gamma \leq \frac{1}{2}$, we directly obtain $w \in L^2(e_\ell)$, on the contrary if $\gamma \in (0, \frac{1}{2})$, as

$$r^{\frac{1}{2}-\gamma} \in L^t(e_\ell),$$

for all $t < (\gamma - \frac{1}{2})^{-1}$, owing to Hölder's inequality we show that $w \in L^r(e_\ell)$ for some $r > 1$.

This Theorem allows to check the assumption $\mathbf{H_7}$ with $D_1 = \mathcal{V}_\gamma^2(\Omega) \cap H_0^1(\Omega)$ and $\tilde{D}_0 = L_\gamma^2(\Omega) \times \prod_{\ell=1}^{L} V_\gamma^{\frac{1}{2}}(e_\ell)$, for all $\gamma \in (1-\lambda_a, 1)$, when λ_a is the smallest positive singular exponent associated with A_1. Indeed, the previous result asserts that A_1 is an isomorphism from D_1 into $\tilde{D}_0$, therefore it remains to check the boundedness property of A_2:

Lemma 4.6 *A_2 is bounded from D_1 into $\tilde{D}_0$.*

Proof Fix $v \in \mathcal{V}_\gamma^2(\Omega) \cap H_0^1(\Omega)$, then $A_2 v$ belongs to $H^{-1}(\Omega)$ and is given by

$$\begin{aligned}\langle A_2 v, w\rangle &= a_2(v,w)\\ &= \int_\Omega \left(\sum_{k,\ell=1}^{2} a_{k,\ell}^{(2)}(x)\ \partial_\ell v\ \partial_k w + \sum_{k=1}^{2} b_k^{(2)}(x) \partial_k v\ w + c^{(2)}(x) v\ w \right) dx,\end{aligned}$$

for all $w \in H_0^1(\Omega)$. As $b_k^{(2)} \partial_k v$ and $c^{(2)} v$ are in $L^2(\Omega)$, it remains to transform the first term of this right-hand side. For that purpose, we first fix $w \in \mathcal{D}(\Omega)$. By an application of Hölder's inequality, there exists $r > 1$ such that

$$\partial_\ell v_j \in W^{1,r}(\Omega_j), \forall j = 1, \cdots, J.$$

Therefore Green's formula on each Ω_j yields

$$\begin{aligned}\int_\Omega a_{k,\ell}^{(2)}(x)\ \partial_\ell v\ \partial_k w\,dx = &-\sum_{j=1}^{J} \int_\Omega \partial_k (a_{k,\ell}^{(2)}(x)\ \partial_\ell) v\ w\,dx\\ &+ \sum_{\ell=1}^{L} \int_{e_\ell} [\![M_2 \nabla v \cdot n]\!]_\ell \gamma w(x)\,d\sigma(x), \quad \forall w \in \mathcal{D}(\Omega).\end{aligned}$$

As $\partial_k (a_{k,\ell}^{(2)}\ \partial_\ell) v$ (resp. $[\![M_2 \nabla u \cdot n]\!]_\ell$) belongs to $L_\gamma^2(\Omega)$ (resp. $V_\gamma^{\frac{1}{2}}(e_\ell)$) and since $H_0^1(\Omega)$ (resp. $H^{\frac{1}{2}}(e_\ell)$) is embedded into $L^s(\Omega)$ (resp. $L^s(e_\ell)$) for all $s > 1$ and recalling (4.10), (4.12), the previous identity remains valid for all $w \in H_0^1(\Omega)$, owing to Hölder's inequality. We then deduce that

$$\langle A_2 v, w\rangle = \sum_{j=1}^{J} \int_{\Omega} L_2 v_j \; w_j \, dx + \sum_{\ell=1}^{L} \int_{e_\ell} [\![M_2 \nabla v \cdot n]\!]_\ell \gamma w(x) \, d\sigma(x), \quad \forall w \in H_0^1(\Omega).$$

This ends the proof in view of the regularity of $L_2 v_j$ and of $[\![M_2 \nabla v \cdot n]\!]_\ell$. □

In conclusion problem (4.2) is well-posed if we take an initial datum u_0 in D_1 and if $f(t, u)$ is continuous with value in $\tilde{D}_0$. Nevertheless it is well known that the reduction of regularity diminishes the rate of convergence for a standard FEM based on quasi-uniform meshes, but the use of refined meshes near the singular points allows to restore the optimal order of convergence, namely the estimate (3.19) is valid with $q(1) = 1$ (using [36] or [18, Thm 8.4.1.6] on each subdomain Ω_j), V_h being defined by (4.6) with a triangulation that is conform with the partition of Ω (i.e., each triangle T of $\mathcal{T}_h$ has to be included into one Ω_j).

Remark 4.7 Near an exterior vertex, where Ω is convex or if the coefficients $a_{k,\ell}^{(1)}$ are continuous at an interior vertex, then the shift Theorem is valid in standard Sobolev spaces and therefore it is not necessary to take initial data in weighted Sobolev spaces near such vertices but it suffices to take them in H^2. Consequently near such vertices, quasi-uniform meshes can be used.

Remark 4.8 If L_2 is strongly elliptic, then this operator may have one singularity S_μ near a vertex, with $0 < \mu < 1$, in other words, $S_\mu \in H_0^1(\Omega)$ behaves like r^μ near this vertex, is piecewise regular elsewhere and satisfies

$$L_2 S_\mu = g \in L^2(\Omega).$$

Then the function $u(x, t) = S_\mu(x)$ is clearly a solution of (4.2) with $f = g$ and initial datum S_μ:

$$\begin{cases} L_1 u_t + L_2 u = g, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = S_\mu, & \text{in } H_0^1(\Omega). \end{cases}$$

This solution is indeed furnished by our abstract framework if we fix the parameter γ appropriately, namely if $\gamma > \max\{1 - \mu, 1 - \lambda_a\}$.

To avoid to take initial data in weighted Sobolev spaces, we will extend our previous framework in the following way. For each interior vertex $s \in \mathcal{S}_{\text{int}}$, we fix a smooth cut-off function η_s equal to 1 near s and equal to zero near the other vertices. We then introduce

$$\hat{D}_1 = D_1 \oplus \text{ Span } \{\eta_s : s \in \mathcal{S}_{\text{int}}\},$$

in other words $v \in \hat{D}_1$ if and only if there exists $v_w \in D_1$ and coefficients $c_s \in \mathbb{R}$, $s \in \mathcal{S}_{\text{int}}$ such that

$$v = v_w + \sum_{s \in \mathcal{S}_{\text{int}}} c_s \eta_s. \tag{4.13}$$

This is a Hilbert space with the inner product

$$(v, v') = (v_w, v'_w)_{D_1} + \sum_{s,s'\in\mathcal{S}_{\text{int}}} c_s c'_s,$$

when $v = v_w + \sum_{s\in\mathcal{S}_{\text{int}}} c_s\eta_s$ and $v' = v'_w + \sum_{s\in\mathcal{S}_{\text{int}}} c'_s\eta_s$.

The key point is the next result.

Lemma 4.9 *The operator $A_1^{-1}A_2$ is bounded from $\hat{D}_1$ into itself.*

Proof Take an arbitrary element $v \in \hat{D}_1$, then it admits the splitting (4.13) and hence

$$A_2 v = A_2 v_w + \sum_{s\in\mathcal{S}_{\text{int}}} c_s A_2\eta_s.$$

As we have seen in Lemma 4.6 that $A_2 v_w$ is in $\tilde{D}_0$, it remains to show that $A_2\eta_s$ is in $\tilde{D}_0$ as well. If this is the case, then $A_2 v$ belongs to $\tilde{D}_0$ and we conclude owing to Theorem 4.5.

For $s \in \mathcal{S}_{\text{int}}$, let us characterize $A_2\eta_s$. By definition we have

$$\begin{aligned}\langle A_2\eta_s, w\rangle &= a_2(\eta_s, w)\\ &= \int_\Omega \left(\sum_{k,\ell=1}^{2} a_{k,\ell}^{(2)}(x)\ \partial_\ell\eta_s\ \partial_k w + \sum_{k=1}^{2} b_k^{(2)}(x)\partial_k\eta_s\ w + c^{(2)}(x)\eta_s\ w\right) dx,\end{aligned}$$

for all $w \in H_0^1(\Omega)$. As η_s is regular, we can apply Green's formula on each Ω_j to find

$$\langle A_2\eta_s, w\rangle = \sum_{j=1}^{J}\int_\Omega L_2\eta_{s,j}\ w_j\, dx + \sum_{\ell=1}^{L}\int_{e_\ell} [\![M_2\nabla\eta_s\cdot n]\!]_\ell\gamma w(x)\, d\sigma(x).$$

Since η_s is constant near the vertices of Ω, we deduce that $L_2\eta_{s,j} \in L_\gamma^2(\Omega_j)$ and that $[\![M_2\nabla\eta_s\cdot n]\!]_\ell \in V_\gamma^{\frac{1}{2}}(e_\ell)$, which shows that $A_2\eta_s \in \tilde{D}_0$. □

Corollary 4.10 *If $f \in C([0,T]; L^2(\Omega))$ and $u_0 \in \{v \in H_0^1(\Omega) : v_j \in H^2(\Omega_j), \forall j = 1, \cdots, J\}$, then problem*

$$\begin{cases} L_1 u_t + L_2 u = f, & \text{in } \Omega\times(0,T),\\ u = 0, & \text{on } \partial\Omega\times(0,T),\\ u(0) = u_0, & \text{in } H_0^1(\Omega),\end{cases}$$

has a unique solution $u \in C^1([0,T]; \hat{D}_1)$.

Proof Owing to Hardy's inequality [18, p. 28], any function $u_0 \in \{v \in H_0^1(\Omega) : v_j \in H^2(\Omega_j), \forall j = 1, \cdots, J\}$ admits the splitting

$$u_0 = u_w + \sum_{s\in\mathcal{S}_{\text{int}}} u_0(s)\eta_s,$$

with $u_w \in \mathcal{V}_\varepsilon^2(\Omega)$ for all $\varepsilon > 0$. This implies that $u_0 \in \hat{D}_1$. Since $L^2(\Omega) \hookrightarrow L_\gamma^2(\Omega)$, we will have $A_1^{-1}f \in C([0,T], D_1)$ and we conclude owing to the continuity of $A_1^{-1}A_2$ from $\hat{D}_1$ into itself. □

In the framework of this corollary, a solution $u \in C^1([0,T]; \hat{D}_1)$ is found. Therefore our convergence results will be guaranteed if we show (3.19) with $q(1) = 1$ and refined meshes but for any $\varphi \in \hat{D}_1$. For that purpose, write $\varphi \in \hat{D}_1$ into

$$\varphi = \varphi_w + \sum_{s \in \mathcal{S}_{\text{int}}} c_s \eta_s,$$

with $\varphi_w \in D_1$ and real coefficients $c_s \in \mathbb{R}$. For the first term, by [36] or [18, Thm 8.4.1.6] we have

$$\|\varphi_w - I_h \varphi_w\|_{1,\Omega} \lesssim h \|\varphi_w\|_{D_1},$$

where I_h is the Lagrange interpolation operator. For the second term, as η_s belongs to $H^2(\Omega)$, a standard interpolation estimate yields

$$\|\eta_s - I_h \eta_s\|_{1,\Omega} \lesssim h \|\eta_s\|_{H^2(\Omega)} \lesssim h.$$

In conclusion the function $I_h \varphi$ satisfies

$$\|\varphi - I_h \varphi\|_{1,\Omega} \lesssim h \|\varphi\|_{\hat{D}_1},$$

which proves (3.19) with $q(1) = 1$.

4.3 Numerical results

To validate our theoretical results, we propose different test examples. First in (4.2) we take $L_1 = I - \Delta$ and $L_2 = -\Delta$ (Δ being the Laplace operator) in convex and non-convex polygons with an explicit solution and compute the different rates of convergence. Then we will consider a semi-linear equation for which the exact solution is unknown, hence we compute experimental convergence rates. In all cases, we compute two rates of convergence of the error (in the $H_0^1(\Omega)$ norm): one in space and another one in time. Namely, for the first (resp. second) one, we chose Δt (resp. h) small enough with respect to h (resp. Δt) so that the error due to the time (resp. space) discretization is neglectible; and then let vary the parameter h (resp. Δt) from a rough value to finer ones.

In the whole subsection, for a sequence of functions $U_n \in H^1(\Omega)$, $0 \le n \le N$, we set

$$\|U_n\|_\infty = \max_{0 \le n \le N} \|U_n\|_{H^1(\Omega)}.$$

4.3.1 The smooth case

On the unit square $(0,1)^2 \subset \mathbb{R}^2$, we take the exact solution

$$u(t,x,y) = x(1-x)y(1-y)\sin t, \quad \forall t \in [0,T],\ x, y \in (0,1),$$

the right-hand side f being computed accordingly. In such a case, we present the numerical tests for the Euler scheme (4.7), where V_h is based on $\mathbb{P}_1$ elements. The approximated solution obtained by this scheme is illustrated in Fig. 1 for different values of t with the choice $\Delta t = h = 0.1$.

The rate of converge of the error in space (resp. time) is presented in Table 1 (resp. 2) with $\Delta t = 0.001$ (resp. $h = 1/160$). There we can see a rate of convergence of 1, that is in accordance with (3.33).

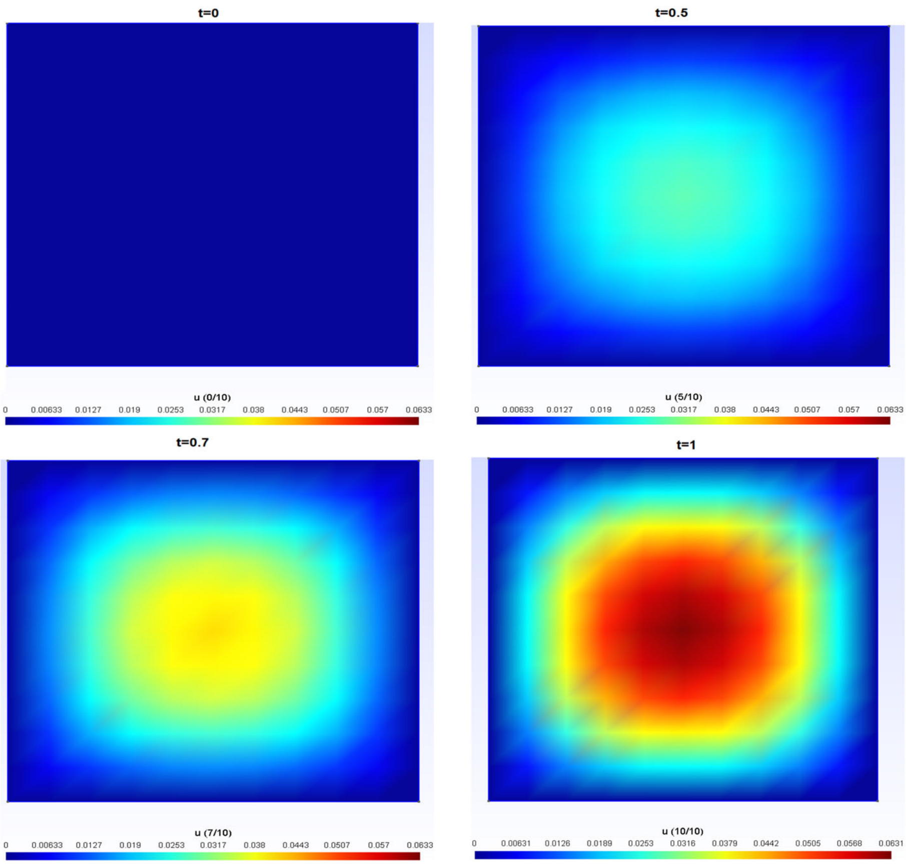


Fig. 1 Approximated solution by Euler's scheme (4.7) of problem (4.2) for $t = 0$, $t = 0.5$, $t = 0.7$ and $t = T$ with $\Delta t = 0.1$ and $h = 0.1$

Table 1 Evolution of the error by Euler's scheme at final time $T = 0.1$ for different h

N	h	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$
10	0.1	1.92e−04
20	0.05	9.72e−05
40	0.025	4.95e−05
80	0.0125	2.65e−05
160	0.00625	1.63e−05

Table 2 Evolution of the error by Euler's scheme at final time $T = 1$ for different Δt

N	Δt	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$
10	0.1	3.98e−02
20	0.05	1.96e−02
40	0.025	9.83e−03
80	0.0125	5.06e−03
160	0.00625	2.85e−03

Table 3 Evolution of the error at final time $T = 1$ for different h for the Runge–Kutta scheme and $\mathbb{P}_2$ el

N	h	$\|u(t_n, x, y) - U_{n,h}\|_\infty$
10	0.1	4.981e−06
20	0.05	1.44e−06
40	0.025	3.78e−07
80	0.0125	9.58e−08
160	0.00625	2.40e−08

Table 4 Evolution of the error at final time $T = 20$ for different Δt for the Runge–Kutta scheme and $\mathbb{P}_2$ el

N	Δt	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$
20	1	0.000145
40	0.5	2.9261e−05
80	0.25	6.4912e−06
160	0.125	1.5455e−06
320	0.0625	4.36447e−07

We now present the numerical results relative to the Runge–Kutta scheme (4.8), where V_h is based on $\mathbb{P}_2$ elements. In this case, as exact solution, we take

$$u(t, x, y) = \left[x(1 - x)y(1 - y)\right]^3 \sin t,$$

that then belongs to $C([0, T]; D(A_1^{\frac{3}{2}}))$.

From Tables 3 and 4, we see that the convergence rate is 2 in space and 2 in time, as expected from (3.33).

4.3.2 The nonsmooth case

In order to illustrate the results from Sect. 4.2, we have decided to take the domain $\Omega = (-1, 1) \times (0, 1) \subset \mathbb{R}^2$, the operator $L_2 = -\Delta$, while the operator $L_1 = -\operatorname{div} a\nabla$, with a piecewise constant, namely

$$a = \begin{cases} \epsilon & \text{in } \Omega_2, \\ 1 & \text{in } \Omega_1 \cup \Omega_3, \end{cases}$$

where ϵ is a positive parameter, we have set

$$\begin{aligned}
\Omega_1 &= \Omega \cap \{(r\cos\theta, r\sin\theta) : r > 0 \text{ and } 0 < \theta < \frac{\pi}{4}\}, \\
\Omega_2 &= \Omega \cap \{(r\cos\theta, r\sin\theta) : r > 0 \text{ and } \frac{\pi}{4} < \theta < \frac{3\pi}{4}\}, \\
\Omega_3 &= \Omega \cap \{(r\cos\theta, r\sin\theta) : r > 0 \text{ and } \frac{3\pi}{4} < \theta < \pi\},
\end{aligned}$$

and, as usual, (r, θ) are the polar coordinates of (x, y) centred at the origin.

Table 5 Evolution of the error by Euler's scheme at final time $T = 0.1$ for different h with uniform/refined meshes for $\Delta t = 0.0001$ and $\epsilon = 3$

N	h	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$ Uniform meshes	Refined meshes
5	0.2	0.0244	0.018
10	0.1	0.0164	0.0087
20	0.05	0.0105	0.0045
40	0.025	0.0064	0.0024
80	0.0125	0.0041	0.0013

In that case if $\epsilon \neq 1$, the operator L_1 with Dirichlet boundary conditions has a singularity at (0, 0) given by (see [30])

$$S_\lambda = \begin{cases} r^\lambda \sin(\lambda\theta) & \text{in } \Omega_1, \\ r^\lambda \left(\frac{2}{\epsilon+1} \sin(\lambda\theta) + \frac{\epsilon-1}{\sqrt{\epsilon}(\epsilon+1)} \cos(\lambda\theta) \right) & \text{in } \Omega_2, \\ r^\lambda \sin(\lambda(\pi - \theta)) & \text{in } \Omega_3, \end{cases}$$

with $\lambda = \frac{4}{\pi} \arcsin \left(\sqrt{\frac{1}{\epsilon+1}} \right)$.

Consequently we take

$$u(t, x, y) = \sin t S_\lambda(x, y), \forall t \geq 0, x, y \in \Omega,$$

that is seen as the exact solution of

$$A_1 u_t + A_2 u = (\sin t) h,$$

where $h \in \tilde{D}_0$ is the jump of $\frac{\partial u}{\partial n}$ along the edge $e_1 = \bar{\Omega}_1 \cap \bar{\Omega}_2$ and the edge $e_2 = \bar{\Omega}_2 \cap \bar{\Omega}_3$. This means that u is solution of

$$\int_\Omega (a \nabla u_t \cdot \nabla v + \nabla u \cdot \nabla v)\, dx dy = \sum_{j=1,2} \int_{e_j} [\![\tfrac{\partial u}{\partial n}]\!]_\ell \gamma v d\sigma, \forall v \in H_0^1(\Omega).$$

We then have approximated this problem by the Euler scheme (4.7), where V_h is based on $\mathbb{P}_1$ elements on either uniform meshes or refined (near 0) ones with the choice $\epsilon = 3$ that yields $\lambda = 2/3$. The rate of converge of the error in space is presented in Table 5 for uniform and refined meshes (see Fig. 2 for $h = 0.2$) with $\Delta t = 0.0001$ and a final time $T = 0.1$. There we can see a rate of convergence of 2/3 (resp. 1) for uniform (refined) meshes, as expected. Here for shortness, we do not present the rate of converge of the error in time since we are interested in the influence of the space singularities.

4.3.3 A semi-linear equation

Here we consider problem (4.2) on the unit square $\Omega = (0, 1)^2$ and zero initial datum with

$$f(t, u) = \sqrt{1 + t + u^2}, \tag{4.14}$$

that is clearly continously differentiable from $[0, T] \times H_0^1(\Omega)$ into $\mathbb{R}$. In such a case, the exact solution is unknown, hence we shall compute the experimental rates of convergence using succesive solutions: the experimental space convergence rate is computed by

$$\log_2 \left(\frac{\|U_{n,h} - U_{n,2h}\|_\infty}{\|U_{n,h/2} - U_{n,h}\|_\infty} \right),$$

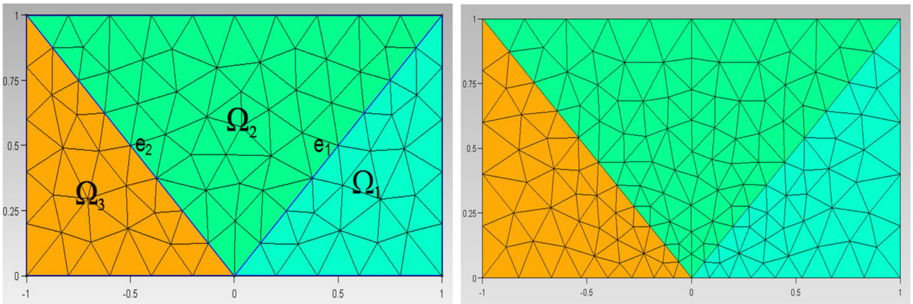


Fig. 2 Uniform (left) and refined (right) meshes

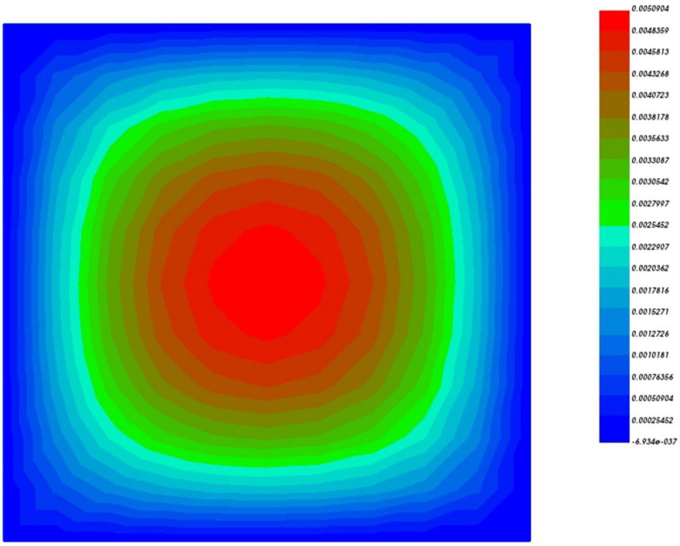


Fig. 3 The fully discrete solution $U_{n,h}$ obtained by Euler's scheme with $\Delta t = 0.1$ and $h = 0.1$ and $\mathbb{P}_1$ el

where $U_{n,2h}$ and $U_{n,h/2}$ are the fully discrete solutions for the meshes $2h$ and $h/2$ respectively and Δt small enough. Similarly, the experimental time convergence rate is computed by

$$\log_2 \left(\frac{\|U_{n,h}^{\Delta t} - U_{n,h}^{2\Delta t}\|_\infty}{\|U_{n,h}^{\Delta t/2} - U_{n,h}^{\Delta t}\|_\infty} \right), \tag{4.15}$$

where $U_{n,h}^{2\Delta t}$ and $U_{n,h}^{\Delta t/2}$ are the fully discrete solutions for the time steps $2\Delta t$ and $\Delta t/2$, respectively and h small enough.

Figure 3 shows the fully discrete solution $U_{n,h}$ obtained by Euler's scheme and $\mathbb{P}_1$ elements at final time $T = 1$ with $\Delta t = 0.1$ and $h = 0.1$. In that case, the experimental time (resp. space) convergence rate is presented in Table 6 (resp. 7), where an order one is detected, as theoretically expected. Additionnally, the experimental time (resp. space) convergence rate is presented in Table 8 (resp. 9) using $\mathbb{P}_2$ elements, where, as theoretically expected, an order one in time and two in space are observed.

Table 6 Experimental time convergence rate for different Δt with $h = \frac{1}{160}$ and $\mathbb{P}_1$ el

Time steps	$\|U_{n,h}^{\Delta t} - U_{n,h}^{2\Delta t}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.0348315	1.01
$\frac{1}{20}$	0.0172607	1.007
$\frac{1}{40}$	0.00858329	1.004
$\frac{1}{80}$	0.0042788	1.002
$\frac{1}{160}$	0.00213605	1.001
$\frac{1}{320}$	0.00106717	

Table 7 Experimental space convergence rate for different h with $\Delta t = \frac{1}{4000}$ and $\mathbb{P}_1$ el

Mesh sizes	$\|U_{n,h} - U_{n,2h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00582633	0.94
$\frac{1}{20}$	0.00302302	0.978
$\frac{1}{40}$	0.00152823	0.99
$\frac{1}{80}$	0.00076712	0.997
$\frac{1}{160}$	0.000383953	0.999
$\frac{1}{320}$	0.000192033	

Table 8 Experimental time convergence rate for different Δt with $h = \frac{1}{320}$ and $\mathbb{P}_2$ el

Time steps	$\|U_{n,h}^{\Delta t} - U_{n,h}^{2\Delta t}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.0324174	0.962
$\frac{1}{20}$	0.0166353	0.9816
$\frac{1}{40}$	0.00842427	0.9908
$\frac{1}{80}$	0.00423879	0.99547
$\frac{1}{160}$	0.00212606	0.997
$\frac{1}{320}$	0.00106468	

Table 9 Experimental space convergence rate for different h with $\Delta t = \frac{1}{4000}$ and $\mathbb{P}_2$ el

Mesh sizes	$\|U_{n,h} - U_{n,2h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.000782864	1.81
$\frac{1}{20}$	0.000223207	1.85
$\frac{1}{40}$	6.18843E−05	1.87
$\frac{1}{80}$	1.68535E−05	1.89
$\frac{1}{160}$	4.53264E−06	

5 Applications to quasi-linear equations

5.1 Non autonomous equations

Here we concentrate on the non autonomous case, namely we suppose that the operators $A_1(t, u)$ and $A_2(t, u)$ depend only on the time variable t, but still corresponds to second order differential operators. More precisely, in a bounded domain Ω of $\mathbb{R}^d$, $d \geq 1$ with a Lipschitz boundary, for $i = 1, 2$, L_i is a differential operator of order two of the form

$$L_i(x, D_x, t)u = -\sum_{k,\ell=1}^{d} \partial_k(a_{k,\ell}^{(i)}(x,t)\partial_\ell u) + \sum_{k=1}^{d} b_k^{(i)}(x,t)\partial_k u + c^{(i)}(x,t)u,$$

where $a_{k,\ell}^{(i)} = a_{\ell,k}^{(i)} \in C([0,T]; C^{0,1}(\bar{\Omega})) \cap C^{0,\gamma}([0,T]; L^\infty(\Omega))$, $b_k^{(i)}, c^{(i)} \in C^{0,\gamma}([0,T]; L^\infty(\Omega))$, for some $\gamma \in (0,1]$. Furthermore, L_1 is supposed to be uniformly elliptic, namely there exists $\alpha_* > 0$ such that

$$\sum_{k,\ell=1}^{d} a_{k,\ell}^{(1)}(x,t)\xi_\ell\xi_k \geq \alpha_*|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, t \in [0,T].$$

In this case, the bilinear form $a_i(t;\cdot,\cdot)$, $i = 1, 2$, is independent of u and is defined by

$$a_i(t; v, w) = \int_\Omega \left(\sum_{k,\ell=1}^{d} a_{k,\ell}^{(i)}(x,t)\partial_\ell v \partial_k w \right. \\ \left. + \sum_{k=1}^{d} b_k^{(i)}(x,t)\partial_k v w + c^{(i)}(x,t) v w \right) dx, \quad \forall v, w \in H_0^1(\Omega),$$

and for all $t \in [0,T]$, the operator $A_i(t)$ defined by

$$\langle A_i(t)u, v\rangle_{V',V} = a_i(t, u, v), \quad \forall u, v \in H_0^1(\Omega),$$

is continous from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Finally if we suppose that a_1 is uniformly coercive in $H_0^1(\Omega)$, then the assumptions $\mathbf{H_1}$–$\mathbf{H_3}$ will be satisfied. As in Sect. 4.1, one can show that $D(A_1(t)) = H^2(\Omega) \cap H_0^1(\Omega)$, for all $t \in [0,T]$ if the boundary of Ω is $C^{1,1}$ or if Ω is convex. Therefore, under this additional hypothesis, the assumptions $\mathbf{H_7}$ and $\mathbf{H_8}$ with $D_1 = H^2(\Omega) \cap H_0^1(\Omega)$ and $\tilde{D}_0 = L^2(\Omega)$ will be satisfied if V_h is defined by (4.6) with $p = 1$.

Finally, under the assumptions $\mathbf{H_4}$ to $\mathbf{H_6}$ on f, the next problem

$$\begin{cases} L_1(t)u_t + L_2(t)u = f(t,u), & \text{in } \Omega \times (0,T), \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, & \text{in } H_0^1(\Omega), \end{cases} \tag{5.1}$$

is well-posed and can be approxiamted by the fully Euler discrete scheme (see (3.16))

$$a_1(t_n; U_{n+1,h}, \chi_h) = a_1(t_n; U_{n,h}, \chi_h) - \Delta t a_2(t_n; U_{n,h}, \chi_h) + \Delta t(f(t_n, U_{n,h}), \chi_h), \ \forall \chi_h \in V_h.$$

We now illustrate this theory by chosing in (5.1), $\Omega = (0,1)^2$, $L_1(t) = I - (1+t)\Delta$ and $L_2(t) = -(1+t)\Delta$. Clearly the bilinear forms $(a_i(t;\cdot,\cdot))_{t\in[0,T]}$ satisfy the previous assumptions, in particular we directly see that

$$|a_i(t; u, v) - a_i(s; u, v)| \lesssim |t-s|\|u\|\|v\|, \quad \forall u, v \in H_0^1(\Omega), \forall\, t, s \in [0,T].$$

We start with a linear problem by taking the exact solution

$$u(t,x,y) = \sin t \sin(\pi x)\sin(\pi y), \quad \forall (x,y) \in \Omega, t > 0.$$

As before, we present the time (resp. space) convergence rate in Table 10 with $h = 0.003125$ (resp. Table 11 with $\Delta t = 0.001$), where again order one is obtained.

Table 10 Evolution of the error at final time $T = 0.1$ for different Δt

N	Δt	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$
5	0.2	0.168
10	0.1	0.0823
20	0.05	0.040
40	0.025	0.0214
80	0.0125	0.0130

Table 11 Evolution of the error at final time $T = 0.1$ for different h

N	h	$\|u(t_n, \cdot) - U_{n,h}\|_\infty$
5	0.2	0.06387
10	0.1	0.032499
20	0.05	0.0163
40	0.025	0.00824
80	0.0125	0.00425201
160	0.00625	0.00236201

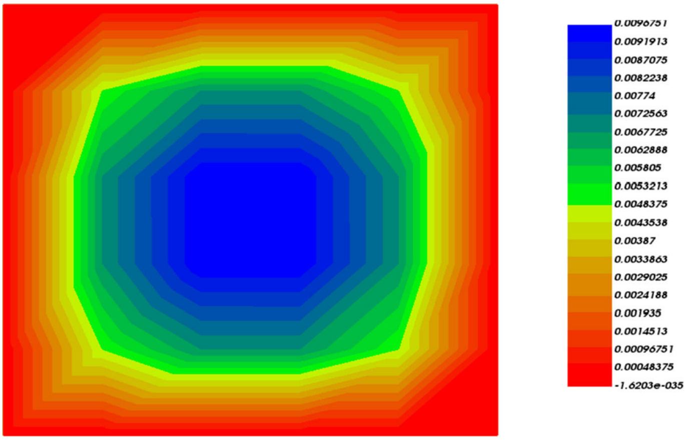


Fig. 4 The fully discrete solution $U_{n,h}$ at final time $T = 1$ obtained by Euler's scheme with $\Delta t = h = 0.1$

We go on with a semi-linear equation by taking the source term $f(t, u)$ defined by (4.14) and a zero initial datum $u(0) = 0$. In Fig. 4, we can see the fully discrete solution (by Euler's scheme) $U_{n,h}$ at final time $T = 1$ with $\Delta t = h = 0.1$. The experimental time (resp. space) convergence rate is presented in Table 12 with $h = \frac{1}{160}$ (resp. 13 with $\Delta t = \frac{1}{4000}$), where again an order one is detected.

5.2 Quasi-linear cases

5.2.1 An example in dimension 1

Here we consider a quasilinear problem

Table 12 Experimental time convergence rate for different Δt with $h = \frac{1}{160}$ and $\mathbb{P}_1$ el

Time steps	$\|U^{\Delta t}_{n,h} - U^{2\Delta t}_{n,h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.01632	0.94
$\frac{1}{20}$	0.00853	0.96
$\frac{1}{40}$	0.00437	0.98
$\frac{1}{80}$	0.00221	0.99
$\frac{1}{160}$	0.00111	1
$\frac{1}{320}$	0.00055	

Table 13 Experimental space convergence rate for different h with $\Delta t = \frac{1}{4000}$ and $\mathbb{P}_1$ el

Mesh sizes	$\|U_{n,h} - U_{n,2h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00559	0.95
$\frac{1}{20}$	0.00290	0.98
$\frac{1}{40}$	0.00147	0.90
$\frac{1}{80}$	0.00074	0.99
$\frac{1}{160}$	0.000337	1
$\frac{1}{320}$	0.00018	

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\rho_1(x,u)\frac{\partial^2 u}{\partial x \partial t} + \rho_2(x,u)\frac{\partial u}{\partial x}\right) = f(t,u), & \text{in } (0,1)\times(0,T),\\ u = 0, & \text{on } \partial(0,1)\times(0,T),\\ u(0) = u_0, & (0,1), \end{cases} \tag{5.2}$$

where $\rho_1, \rho_2 : (0,1)\times\mathbb{R} \longmapsto \mathbb{R}$ are two continuous functions satisfying

- there exist two positive constants β, M such that
$$\beta \le \rho_1(x,u) \le M, \quad \text{and} \quad |\rho_2(x,u)| \le M, \forall (x,u) \in \bar{\Omega}\times\mathbb{R}, \tag{5.3}$$
- the function $\rho_i, i = 1,2$, is globaly Lipschitz, i. e., there exists a constant $L > 0$ such that
$$|\rho_i(x,u) - \rho_i(x,\tilde{u})| \le L|u-\tilde{u}|, \quad \forall (x,u,\tilde{u}) \in \bar{\Omega}\times\mathbb{R}^2. \tag{5.4}$$

With these assumptions, the bilinear forms $a_i(u;\cdot,\cdot), i = 1,2$, defined on $H^1_0(\Omega)\times H^1_0(\Omega)$ as

$$a_i(u; v, w) = \int_0^1 \rho_i(x, u(x))\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}dx,$$

satisfy the assumptions $\mathbf{H_1}$–$\mathbf{H_3}$, this last property following from the Sobolev embedding theorem yielding $H^1(0,1) \hookrightarrow C([0,1])$.

Here we discretize problem (5.2) by explicit Euler's scheme using the finite element space

$$V_h = \{v_h \in H^1_0(\Omega);\ v_h\ |_{[x_i,x_{i+1}]} \in \mathbb{P}_1, 0 \le i \le N, v_h(0) = v_h(1) = 0\},$$

based on a uniform subdivision $x_i = ih$, $0 \le i \le N$, with $h = \frac{1}{N}$ and $N \in \mathbb{N}^*$. For the numerical illustrations, we take the source term $f(t,u)$ defined by (4.14), a zero initial datum $u(0) = 0$ and

$$\rho_1(x,u) = \frac{1}{2} + \frac{u^2}{1+u^2}, \quad \rho_2(x,u) = \frac{u^2}{1+u^2}.$$

Table 14 Experimental time convergence rate for different Δt with $h = 0.00625$

Time steps	$\|U_{n,h}^{\Delta t} - U_{n,h}^{2\Delta t}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00481	0.998
$\frac{1}{20}$	0.002407	1.00
$\frac{1}{40}$	0.001203	1.00
$\frac{1}{80}$	0.000601	0.99
$\frac{1}{160}$	0.0003010	1.00
$\frac{1}{320}$	0.00015	

Table 15 Experimental space convergence rate for different h with $\Delta t = 0.001$

Mesh sizes	$\|U_{n,h} - U_{n,2h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00596	1.00
$\frac{1}{20}$	0.00296	1.01
$\frac{1}{40}$	0.00146	0.9964
$\frac{1}{80}$	0.0007318	1.00
$\frac{1}{160}$	0.0003654	1.001
$\frac{1}{320}$	0.0001825	

As before the experimental time (resp. space) convergence rate is 1, as seen in Table 14 with $h = \frac{1}{320}$ (resp. 15 with $\Delta t = \frac{1}{1000}$).

5.2.2 An example in dimension 2

On the unit square $\Omega = (0, 1)^2$ of $\mathbb{R}^2$, we consider the problem

$$\begin{cases} \Delta\big(\rho_1(x, u)\Delta u_t\big) + \Delta\big(\rho_2(x, u)\Delta u\big) = f(t, u), & \text{in } \ \Omega \times (0, T), \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \times (0, T), \\ u(0) = u_0, & \text{in } \ H_0^2(\Omega), \end{cases} \tag{5.5}$$

where $\frac{\partial u}{\partial n}$ denote the outward normal derivative of u on $\partial\Omega$, ρ_1 and ρ_2 are two functions in $C^2(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ that fulfil the assumptions (5.3)–(5.4) with second order partial derivatives uniformly bounded in x and u.

For all $u \in H_0^2(\Omega)$, and $i = 1$ or 2, we define the bilinear form $a_i(u; \cdot, \cdot)$ by

$$a_i(u; v, w) = \int_\Omega \big(\rho_i(x, u)\Delta v \Delta w\big) dx, \quad \forall v, w \in H_0^2(\Omega), \tag{5.6}$$

that immediately satisfy the assumptions $\mathbf{H_1}$ and $\mathbf{H_3}$, due to the embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ (consequence of the Sobolev embedding theorem). To check that $\mathbf{H_2}$ holds, due to (5.3) we first notice that

$$a_1(u; v, v) \geq \beta \|\Delta v\|_{L^2(\Omega)}^2, \quad \forall v \in H_0^2(\Omega).$$

Secondly as the Laplace operator is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ into $L^2(\Omega)$, we have

$$\|v\|_{H^2(\Omega)} \lesssim \|\Delta v\|_{L^2(\Omega)}, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega).$$

As $H_0^2(\Omega)$ is included into $H^2(\Omega) \cap H_0^1(\Omega)$, these two estimates show that $\mathbf{H_2}$ holds.

In order to check $\mathbf{H_7}$ and $\mathbf{H_8}$, we use the next result.

Lemma 5.1 *For all $u \in H_0^2(\Omega)$, we have*

$$D(A_1(u)) = H^4(\Omega) \cap H_0^2(\Omega) \hookrightarrow D(A_2(u)). \tag{5.7}$$

Proof For a fixed $u \in H_0^2(\Omega)$, we recall that

$$D(A_1(u)) = \{v \in H_0^2(\Omega) : A_1(u)v \in L^2(\Omega)\}.$$

Hence we see that $v \in H_0^2(\Omega)$ belongs to $D(A_1(u))$ if and only if there exists $f \in L^2(\Omega)$ such that

$$\Delta(\rho(x,u)\Delta v) = f,$$

in the distributional sense (here and below, for shortness we write $\rho(x,u)$ for $\rho_1(x,u)$). By Leibniz's product rule, we get equivalently

$$\rho(x,u)\Delta^2 v = \tilde{f} := f - \Delta[\rho(x,u(x))]\Delta v - 2\nabla[\rho(x,u(x))] \cdot \nabla\Delta v. \tag{5.8}$$

The difficulty is that this right-hand side is not automatically in $L^2(\Omega)$, hence we will use a bootstrap argument. First by the chain rule, we notice that

$$\partial_i[\rho(x,u(x))] = \partial_i\rho(x,u(x)) + \partial_u\rho(x,u(x))\partial_i u(x), \tag{5.9}$$

$$\begin{aligned}\partial_{ij}^2[\rho(x,u(x))] = \partial_{ij}^2\rho(x,u(x)) &+ \partial_u^2\rho(x,u(x))\partial_i u(x)\partial_j u(x)\\ &+\partial_u\rho(x,u(x))\partial_{ij}^2 u(x),\end{aligned} \tag{5.10}$$

for any $i, j \in \{1, 2\}$. Note that these identities directly imply that

$$\rho(\cdot,u(\cdot)), \frac{1}{\rho(\cdot,u(\cdot))} \in H^2(\Omega). \tag{5.11}$$

With such identities, for $i = 1$ or 2, we also see that

$$\partial_i[\rho(x,u(x))]\partial_i\Delta v = \partial_i\rho(x,u)\partial_i\Delta v + \partial_u\rho(x,u)\partial_i u\partial_i\Delta v, \tag{5.12}$$

and that

$$\Delta[\rho(x,u(x))]\Delta v = \sum_{j=1}^{3} T_j(x,u)\Delta v, \tag{5.13}$$

where we have set

$$\begin{aligned}T_1(x,u) &= (\Delta\rho)(x,u),\\ T_2(x,u) &= \partial_u^2\rho(x,u)\{(\partial_1 u)^2 + (\partial_2 u)^2\},\\ T_3(x,u) &= \partial_u\rho(x,u)\Delta u.\end{aligned}$$

In a first step, for $i = 1$ or 2, we show that

$$\partial_i[\rho(x,u(x))]\partial_i\Delta v \in H^{-s}(\Omega), \quad \forall s \in (1,2). \tag{5.14}$$

According to the identity (5.12), it suffices to show that each term of its right-hand side belongs to $H^{-s}(\Omega)$, for all $s \in (1, 2)$. For that purpose, we notice that $\partial_i\Delta v \in H^{-1}(\Omega)$, and $\partial_i u \in H^1(\Omega)$. By the regularity assumptions on ρ, we see that

$$\partial_i\rho(x,u), \partial_u\rho(x,u)\partial_i u \in H^1(\Omega). \tag{5.15}$$

As Theorem 1.4.4.2 of [18] implies that the product

$$uv \in H^{-s}(\Omega), \quad \forall s \in (1, 2), \tag{5.16}$$

If $u \in H^1(\Omega)$ and $v \in H^{-1}(\Omega)$, we conclude that (5.14) holds. Similarly for $j = 1, 2$ or 3, we show that

$$T_j(x, u)\Delta v \in H^{-s}(\Omega), \quad \forall s \in (1, 2). \tag{5.17}$$

By the boundedness of $\Delta\rho$ and the regularity $v \in H^2(\Omega)$, we directly get $T_1(x, u)\Delta v \in L^2(\Omega)$, hence (5.17) for $j = 1$. Now by the Sobolev embedding Theorem, $H^1(\Omega) \hookrightarrow L^p(\Omega)$, for all $p \geq 1$, hence by Hölder's inequality, we get

$$(\partial_i u)^2 \Delta v \in L^1(\Omega), \quad \forall i = 1, 2,$$

since the condition $1 = \frac{2}{p} + \frac{1}{2}$ holds if $p = 4$. As $\partial_u^2 \rho(x, u)$ is bounded, we deduce that $T_2(x, u)\Delta v \in L^1(\Omega)$, which implies (5.17) for $j = 2$, because $H_0^s(\Omega) \hookrightarrow C(\bar{\Omega})$, owing to the Sobolev embedding theorem. Finally, the regularities $u, v \in H^2(\Omega)$ simply guarantee that $\Delta u \Delta v \in L^1(\Omega)$ and hence $T_3(x, u)\Delta v \in L^1(\Omega)$, and we conclude as for $j = 2$.

At this stage, by (5.14) and (5.17), we deduce that $\tilde{f}$ (defined in (5.8)) belongs to $H^{-s}(\Omega)$, for all $s \in (1, 2)$. With the regularity property (5.11) and Theorem 1.4.4.2 of [18], we conclude that

$$\Delta^2 v = \frac{\tilde{f}}{\rho(\cdot, u)} \in H^{-s}(\Omega), \quad \forall s \in (1, 2).$$

Owing to Theorem 2 of [6] and Corollary 5.12 of [13], we deduce that

$$v \in H^{4-s}(\Omega), \quad \forall s \in (1, 2),$$

or equivalently

$$v \in H^{3-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, 1).$$

This extra regularity allows to show that

$$\partial_i[\rho(x, u(x))]\partial_i \Delta v \in H^{-\varepsilon'}(\Omega), \quad \forall \varepsilon' \in (0, 1), \tag{5.18}$$

for $i = 1$ or 2 and

$$T_j(x, u)\Delta v \in H^{-\varepsilon'}(\Omega), \quad \forall \varepsilon' \in (0, 1), \tag{5.19}$$

for $j = 1, 2$ or 3. For the first assertion, by the regularity $\partial_i \Delta v \in H^{-\varepsilon}(\Omega)$, the properties (5.15) and Theorem 1.4.4.2 of [18], we conclude that (5.18) holds for $\varepsilon' > \varepsilon$. For the second assertion, by the boundedness of second derivatives of ρ_1, we first notice that

$$T_1(x, u)\Delta v \in L^2(\Omega),$$

hence (5.19) for $j = 1$. For $j = 2$ we remark that $(\partial_1 u)^2 + (\partial_2 u)^2$ belongs to $L^2(\Omega)$ due to the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$, for all $p \geq 1$, and Hölder's inequality. Since we directly get that $\Delta u \in L^2(\Omega)$, we conclude that

$$T_j(x, u) \in L^2(\Omega), \text{ for } j = 2, 3.$$

Again Theorem 1.4.4.2 of [18] leads to (5.19) for $\varepsilon' > \varepsilon$, and $j = 2, 3$.

By its definition, we deduce that $\tilde{f}$ belongs to $H^{-\varepsilon'}(\Omega)$, for all $\varepsilon' \in (0, 1)$, and by (5.11) and Theorem 1.4.4.2 of [18] we get

$$\Delta^2 v \in H^{-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, 1).$$

By Theorem 2 of [6] and Corollary 5.12 of [13], we deduce that

$$v \in H^{4-\varepsilon}(\Omega), \quad \forall \varepsilon \in (0, 1).$$

This regularity implies that $\partial_i \Delta v \in H^{1-\varepsilon}(\Omega)$, for all $\varepsilon \in (0, 1)$ hence $\partial_i \Delta v \in L^p(\Omega)$, for all $p > 1$, which allows to show that

$$\partial_i[\rho(x, u(x))]\partial_i \Delta v \in L^2(\Omega).$$

More simply as Δv belongs to $H^{2-\varepsilon}(\Omega)$, for all $\varepsilon \in (0, 1)$, it is bounded in Ω and consequently

$$T_j(x, u)\Delta v \in L^2(\Omega).$$

This leads to the final property

$$\Delta^2 v \in L^2(\Omega),$$

and by Theorem 2 of [6], we conclude that

$$v \in H^4(\Omega).$$

We have thus shown that $D(A_1(u)) \hookrightarrow H^4(\Omega) \cap H_0^2(\Omega)$. As the converse embedding is direct the proof is complete. □

With the help of this Lemma, the assumption $\mathbf{H_7}$ holds with the choice $D_1 = H^4(\Omega) \cap H_0^2(\Omega)$ and $\tilde{D}_0 = L^2(\Omega)$.

Since the variational space is included into $H^2(\Omega)$, the use of continuous FEM is not appropriate, hence we shall use HCT elements (Hsieh-Cough-Tocher) described in [10] for instance. Such elements are macro-elements (see Fig. 5) since each triangle K is subdivided into three sub-triangles K_i, $i = 1, 2, 3$, namely we define

$$P_K = \{v \in C^1(K) : v\,|_{K_i} \in \mathcal{P}_3(K_i), 1 \le i \le 3\},$$

and then

$$V_h = \left\{v \in C^1(\bar{\Omega}) : v\,|_K \in P_K, \forall K \in \mathcal{T}_h, v = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\right\} \subset H_0^2(\Omega).$$

For more details, we refer to [10, p. 341].

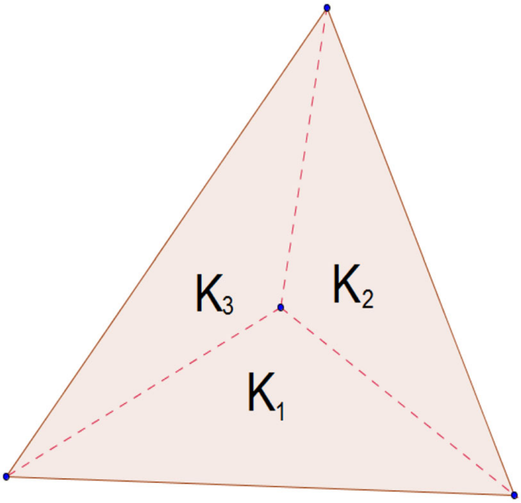
As Theorem 6.1.6 of [10] implies that

$$\|\varphi - P_h\varphi\|_{H^2(\Omega)} \lesssim h^2\|\varphi\|_{H^4(\Omega)}, \quad \forall \varphi \in H^4(\Omega) \cap H_0^2(\Omega),$$

the assumption $\mathbf{H_8}$ holds with $q(1) = 2$.

The fully discrete explicit Euler's scheme of problem (5.5) is therefore given by: find $U_{n+1,h} \in V_h$ solution of

Fig. 5 The HCT element



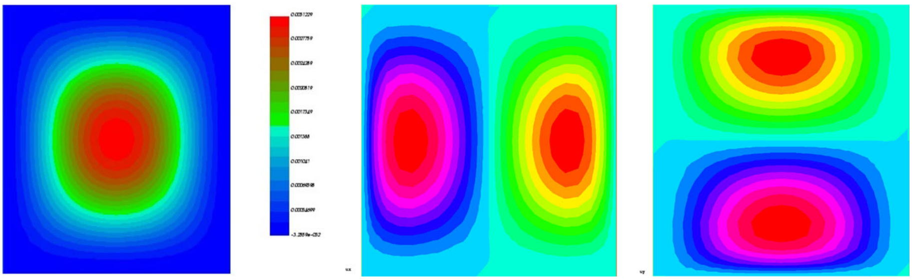


Fig. 6 The fully discrete solution: Left u, middle $\partial_1 u$, right $\partial_2 u$, with $\Delta t = h = 0.1$ at final time $T = 1$

Table 16 Experimental time convergence rate for different Δt with $h = 0.00625$

Time steps	$\|u_{n,h}^{\Delta t} - u_{n,h}^{2\Delta t}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00375	1.027
$\frac{1}{20}$	0.00184	1.015
$\frac{1}{40}$	0.00091	1.04
$\frac{1}{80}$	0.00044	0.97
$\frac{1}{160}$	0.000224	1.01
$\frac{1}{320}$	0.000111	

$$\int_\Omega \{\rho_1(x, U_{n,h})\Delta\Big(\frac{U_{n+1,h} - U_{n,h}}{\Delta t}\Big)\Delta v_h + \rho_2(x, U_{n,h})\Delta U_{n,h}\Delta v_h\}dx$$
$$= \int_\Omega f(t_n, U_{n,h})v_h \, dx, \forall v_h \in V_h.$$

We finally illustrate this case by chosing $\rho_1(x, u) = \frac{1}{2} + \frac{u^2}{1+u^2}$, $\rho_2(x, u) = \frac{u^2}{1+u^2}$, the source term $f(t, u)$ defined by (4.14) and a zero initial datum $u(0) = 0$. In Fig. 6, we can see the fully discrete solution $U_{n,h}$ and its gradient at final time $T = 1$ with $\Delta t = h = 0.1$. The

Table 17 Experimental space convergence rate for different h with $\Delta t = 0.001$

Mesh sizes	$\|u_{n,h} - u_{n,2h}\|_\infty$	Exp. rate of convergence
$\frac{1}{10}$	0.00106	1.82
$\frac{1}{20}$	0.0003	1.72
$\frac{1}{40}$	9.07×10^{-5}	1.94
$\frac{1}{80}$	2.36×10^{-5}	1.97
$\frac{1}{160}$	5.99×10^{-6}	

experimental time (resp. space) convergence rate is presented in Table 16 with $h = 0.00625$ (resp. 17 with $\Delta t = \frac{1}{1000}$), where an order one in time and two in space is obtained, as expected from (3.33).

Note that all our numerical tests are performed with the help of the software freefem++ [1].

References

1. http://www.freefem.org
2. Arnold, D.N., Douglas Jr., J., Thomée, V.: Superconvergence of a finite element approximation to the solution of a Sobolev equation in a single space variable. Math. Comput. **36**(153), 53–63 (1981)
3. Avilez-Valente, P., Seabra-Santos, F.J.: A Petrov-Galerkin finite element scheme for the regularized long wave equation. Comput. Mech. **34**(4), 256–270 (2004)
4. Barenblatt, G.I., Entov, V.M., Ryzhik, V.M.: Theory of Fluid Flow Through Natural Rocks. Kluwer, Dordrecht (1990)
5. Bernardi, C.: Optimal finite-element interpolation on curved domains. SIAM J. Numer. Anal. **26**(5), 1212–1240 (1989)
6. Blum, H., Rannacher, R.: On the boundary value problem of the biharmonic operator on domains with angular corners. Math. Methods Appl. Sci. **2**(4), 556–581 (1980)
7. Carroll, R.W., Showalter, R .E.: Singular and Degenerate Cauchy problems, vol. 127. Academic, New York (1976). Mathematics in Science and Engineering
8. Chatzipantelidis, P.: Explicit multistep methods for nonstiff partial differential equations. Appl. Numer. Math. **27**(1), 13–31 (1998)
9. Chen, P.J., Gurtin, M.E.: On a theory of heat conduction involving two temperatures. Z. Angew. Math. Phys. **19**, 614–627 (1968)
10. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems. North-Holland, Amsterdam (1978)
11. Coleman, B.D., Noll, W.: An approximation theorem for functionals, with applications in continuum mechanics. Arch. Ratl. Mech. Anal. **6**(355–370), 1960 (1960)
12. Cuesta, C., van Duijn, C.J., Hulshof, J.: Infiltration in porous media with dynamic capillary pressure: travelling waves. Eur. J. Appl. Math. **11**(4), 381–397 (2000)
13. Dauge, M.: Elliptic Boundary Value Problems on Corner Domains—Smoothness and Asymptotics of solutions, Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988)
14. Dogan, A.: Numerical solution of regularized long wave equation using Petrov–Galerkin method. Commun. Numer. Methods Eng. **17**(7), 485–494 (2001)
15. Ewing, R.E.: Time-stepping Galerkin methods for nonlinear Sobolev partial differential equations. SIAM J. Numer. Anal. **15**(6), 1125–1150 (1978)
16. Gajewski, H., Zacharias, K.: Über eine weitere Klasse nichtlinearer Differentialgleichungen im Hilbert-Raum. Math. Nachr. **57**, 127–140 (1973)
17. Gao, F., Wang, X.: A modified weak Galerkin finite element method for Sobolev equation. J. Comput. Math. **33**(3), 307–322 (2015)
18. Grisvard, P.: Elliptic Problems in Nonsmooth Domains, Monographs and Studies in Mathematics, vol. 24. Pitman, Boston (1985)
19. Hairer, E., Nø rsett, S.P., Wanner, G.: Solving Ordinary Differential Equations. I, Volume 8 of Springer Series in Computational Mathematics. Springer, Berlin (1993)

20. Hassanizadeh, S.M., Gray, W.G.: Thermodynamic basis of capillary pressure in porous media. Water Resour. Res. **29**, 858–879 (1993)
21. Hell, T., Ostermann, A., Sandbichler, M.: Modification of dimension-splitting methods–overcoming the order reduction due to corner singularities. IMA J. Numer. Anal. **35**(3), 1078–1091 (2015)
22. Kadlec, J.: The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain. Czech. Math. J. **14**(89), 386–393 (1964)
23. Kato, T.: Quasi-linear equations of evolution, with applications to partial differential equations. In: Spectral Theory and Differential Equations. Lecture Notes in Math., vol. 448, pp. 25–70. Springer, Berlin (1975)
24. Kellogg, R.B.: Singularities in interface problems. In: Hubbard, B. (ed.) Numerical Solution of Partial Differential Equations II, pp. 351–400. Academic, New York (1971)
25. Kellogg, R.B.: On the Poisson equation with intersecting interfaces. Appl. Anal. **4**, 101–129 (1975)
26. Lemrabet, K.: Régularité de la solution d'un problème de transmission. J. Math. Pures Appl. **56**, 1–38 (1977)
27. Lions, J.-L.: Équations différentielles opérationnelles et problèmes aux limites. Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer, Berlin (1961)
28. Liu, T., Lin, Y.-P., Rao, M., Cannon, J.R.: Finite element methods for Sobolev equations. J. Comput. Math. **20**(6), 627–642 (2002)
29. Nicaise, S.: Polygonal interface problems, Methoden und Verfahren der Mathematischen Physik, vol. 39. Verlag Peter D. Lang, Frankfurt am Main (1993)
30. Nicaise, S., Sändig, A.-M.: General interface problems. I, II. Math. Methods Appl. Sci. **17**(6):395–429, 431–450 (1994)
31. Ohm, M.R., Lee, H.Y.: L^2-error analysis of fully discrete discontinuous Galerkin approximations for nonlinear Sobolev equations. Bull. Korean Math. Soc. **48**(5), 897–915 (2011)
32. Ohm, M.R., Lee, H.Y., Shin, J.Y.: L^2-error estimates of the extrapolated Crank-Nicolson discontinuous Galerkin approximations for nonlinear Sobolev equations. J. Inequal. Appl., pages Art. ID 895187, 17 (2010)
33. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
34. Ptashnyk, M.: Pseudoparabolic equations with convection. IMA J. Appl. Math. **72**(6), 912–922 (2007)
35. Quarteroni, A.: Fourier spectral methods for pseudoparabolic equations. SIAM J. Numer. Anal. **24**(2), 323–335 (1987)
36. Raugel, G.: Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le laplacien dans un polygone. C. R. Acad. Sci. Paris Sér. A-B **286**(18), A791–A794 (1978)
37. Schwartz, L.: Mathematics for the Physical Sciences. Hermann; Addison-Wesley Publishing Co., Paris, Reading (1966)
38. Showalter, R.E.: The Sobolev equation. I. Appl. Anal. **5**(1), 15–22 (1975)
39. Showalter, R.E.: The Sobolev equation. II. Appl. Anal. **5**(2), 81–99 (1975)
40. Showalter, R.E.: Hilbert Space Methods for Partial Differential Equations, Monographs and Studies in Mathematics, vol. 1. Pitman, London (1977)
41. Ting, T.W.: Certain non-steady flows of second-order fluids. Arch. Ratl. Mech. Anal. **14**, 1–26 (1963)
42. Ting, T.W.: A cooling process according to two-temperature theory of heat conduction. J. Math. Anal. Appl. **45**, 23–31 (1974)
43. Wahlbin, L.: Error estimates for a Galerkin method for a class of model equations for long waves. Numer. Math. **23**, 289–303 (1975)