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A note on starlike functions associated with symmetric points

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Abstract The objective of the present paper is to study results that are defined using the notions of generalization of Janowski classes and k-symmetrical functions. A representation theorem, coefficients inequality, distortion properties and the result on radius of starlikeness are discussed.

Keywords Janowski functions · Subordination · Starlike functions · Convex functions · k-symmetric points

Mathematics Subject Classification 30C45

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all function which are univalent in $\mathcal{U}$.

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For two functions f and g, analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$, if there exists an analytic function w in $\mathcal{U}$ such that $|w(z)| < 1$ with $w(0) = 0$, and $f(z) = g(w(z))$, and we denote this by $f(z) \prec g(z)$. If g is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Using the principle of the subordination we define the class $\mathcal{P}$ of functions with positive real part.

Definition 1.1 [6] Let $\mathcal{P}$ denote the class of analytic functions of the form $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ defined on $\mathcal{U}$ and satisfying $p(0) = 1$, $\Re p(z) > 0$, $z \in \mathcal{U}$.

Any function p in $\mathcal{P}$ has the representation $p(z) = \dfrac{1 + w(z)}{1 - w(z)}$, where $w \in \Omega$ and

$$\Omega = \{w \in \mathcal{A} : w(0) = 0, |w(z)| < 1\}. \tag{1.2}$$

The class of functions with positive real part $\mathcal{P}$ plays a crucial role in geometric function theory. Its significance can be seen from the fact that simple subclasses like class of starlike $\mathcal{S}^*$, class of convex functions $\mathcal{C}$, class of starlike functions with respect to symmetric points $\mathcal{S}_s^*$ have been defined by using the concept of class of functions with positive real part.

Let $\mathcal{P}[A, B]$, with $-1 \leq B < A \leq 1$, denote the class of analytic function p defined on $\mathcal{U}$ with the representation $p(z) = \dfrac{1 + Aw(z)}{1 + Bw(z)}$, $z \in \mathcal{U}$, where $w \in \Omega$.

we note that

$$p \in \mathcal{P}[A, B] \text{ if and only if } p(z) \prec \frac{1 + Az}{1 + Bz} (\text{see}[7]).$$

The class $\mathcal{P}[A, B, \alpha]$ of generalized Janowski functions was introduced in [9]. For arbitrary numbers A, B, α, with $-1 \leq B < A \leq 1$, $0 \leq \alpha < 1$, a function p analytic in $\mathcal{U}$ with $p(0) = 1$ is in the class $\mathcal{P}[A, B, \alpha]$ if and only if

$$p(z) \prec \frac{1 + [(1 - \alpha)A + \alpha B]z}{1 + Bz} \Leftrightarrow p(z) = \frac{1 + [(1 - \alpha)A + \alpha B]w(z)}{1 + Bw(z)}, \quad w \in \Omega.$$

The definition of starlike functions with respect to k-symmetric points is as follows.

Definition 1.2 For a positive integer k, let $\varepsilon = \exp\left(\frac{2\pi i}{k}\right)$ denote the kth root of unity for $f \in \mathcal{A}$, let

$$M_{f,k}(z) = \sum_{v=1}^{k-1} \varepsilon^{-v} f(\varepsilon^v z). \frac{1}{\sum_{v=1}^{k-1} \varepsilon^{-v}}, \tag{1.3}$$

be its k-weighted mean function.

A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}_k^*$ if functions starlike with respect to k-symmetric points if for every r close to 1 , $r < 1$, the angular velocity of f about the point $M_{f_k(z_0)}$ positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction, that is

$$\Re \left\{ \frac{zf'(z)}{f(z) - M_{f,k}(z_0)} \right\} > 0$$

for $z = z_0$, $|z_0| = r$.

Definition 1.3 [11] A function f in $\mathcal{A}$ is univalent and starlike with respect to k-symmetric points, or briefly k-starlike if and only if

$$\Re \left\{ \frac{zf'(z)}{f_k(z)} \right\} > 0, \quad z \in \mathcal{U}, \tag{1.4}$$

where

$$f_k(z) = \frac{1}{k}(f(z) - M_{f,k}(z)). \tag{1.5}$$

If $f(z)$ defined by (1.1) then,

$$f_k(z) = z + \sum_{n=2}^{\infty} \chi_n a_n z^n, \quad (k = 2, 3, \dots), \tag{1.6}$$

where

$$\chi_n = \begin{cases} 1, & n = lk + 1, \quad l \in \mathbb{N}_0, \\ 0, & n \neq lk + 1. \end{cases} \tag{1.7}$$

Al-Sarari and Latha in [1–3] (see also, [4]) studied some classes which related to Janowski type functions and symmetric points.

Now using the generalization of Janowski functions and the concept of k-symmetrical functions we define the following:

Definition 1.4 A function f in $\mathcal{A}$ is said to belong to the class $\mathcal{S}^k(A, B, \alpha)$, $(-1 \leq B < A \leq 1), 0 \leq \alpha < 1$ if

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1 + [(1-\alpha)A + \alpha B]z}{1 + Bz}, \quad z \in \mathcal{U},$$

where $f_k(z)$ defined by (1.6).

We note that for special values of k, α, A and B yield the following classes:

(i) $\mathcal{S}^1(A, B, \alpha)$=$\mathcal{S}^*(A, B, \alpha)$ the class introduced by Polatoglu et al. [9];
(ii) $\mathcal{S}^k(A, B, 0)$=$\mathcal{S}^{(k)}(A, B)$ is the class studied by Kwon and Sim [8];
(iii) $\mathcal{S}^k(1, -1, 0)$= $\mathcal{S}_k^*$=$\mathcal{S}_k^*(1, -1)$, the class is studied by Sakaguchi [11] and etc.

We need the following lemmas to prove our main results.

Lemma 1.5 [5] *Let* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}[A, B, \alpha]$, *then for* $n \geq 1$,

$$|p_n| \leq (1-\alpha)(A - B).$$

Lemma 1.6 pol *Any function* $f \in \mathcal{S}^*(A, B, \alpha)$ *can be written in the form*

$$f(z) = \begin{cases} z\,(1 + Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ z\,\exp\left[(1-\alpha)Aw(z)\right], & \text{if } B = 0, \end{cases}$$

where $w \in \Omega$, *and* Ω *was defined by* (1.2).

Lemma 1.7 [10] *Let* ϕ *be convex and* g *starlike Then for* F *analytic in* $\mathcal{U}$ *with* $F(0) = 1$,

$$\frac{\phi * Fg}{\phi * g}(\mathcal{U}) \subset \overline{CO}(F(\mathcal{U}))$$

where $\overline{CO}(F(\mathcal{U}))$ *denotes the closed convex hull of* $F(\mathcal{U})$.

Lemma 1.8 [9] *Let* $p \in \mathcal{P}[A, B, \alpha]$, *then the set of the values of* p *is in the closed disc with center at* $C(r)$ *and having the radius* $\rho(r)$, *where*

$$\begin{cases} C(r) = \left(\frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, 0\right), \quad \rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2} & \text{if } B \neq 0, \\ C(r) = (1, 0), \quad \rho(r) = (1-\alpha)|A|r & \text{if } B = 0. \end{cases}$$

2 Main results

Lemma 2.1 *Let $p \in \mathcal{P}[A, B, \alpha]$. Then*

$$\left.\begin{array}{ll} \dfrac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, & \text{if } B \neq 0, \\ 1-(1-\alpha)Ar, & \text{if } B = 0 \end{array}\right\} \leq |p(z)|$$
$$\leq \begin{cases} \dfrac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, & \text{if } B \neq 0, \\ 1+(1-\alpha)Ar, & \text{if } B = 0. \end{cases}$$

Proof The set of the values of p is in the closed disc with center at $C(r) = \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}$ and having the radius $\rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2}$ using Lemma 1.8, that is

$$\left|p - \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}\right| \leq \frac{(1-\alpha)(A-B)r}{1-B^2r^2}. \tag{2.1}$$

Simplifying (2.1), we get the required result . ∎

Theorem 2.2 *If $f \in \mathcal{S}^k(A, B, \alpha)$, then $f_k \in \mathcal{S}(A, B, \alpha)$, where f_k is defined by* (1.6).

Proof Supposing that $f \in \mathcal{S}^k(A, B, \alpha)$, we can get

$$\frac{zf'(z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}. \tag{2.2}$$

Substituting z by $\varepsilon^\nu z$ in (2.2), it follows

$$\frac{\varepsilon^\nu zf'(\varepsilon^\nu z)}{f_k(\varepsilon^\nu z)} \prec \frac{1+[(1-\alpha)A+\alpha B]\varepsilon^\nu z}{1+B\varepsilon^\nu z} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz},$$

hence

$$\frac{\varepsilon^{\nu-\nu j}zf'(\varepsilon^\nu z)}{f_k'(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}. \tag{2.3}$$

Letting $\nu = 0, 1, 2, \ldots, k-1$ in (2.3) and using the fact that $\mathcal{P}[A, B, \alpha]$ is a convex set, we deduce that

$$\frac{z\frac{1}{k}\sum_{\nu=0}^{k-1}\varepsilon^{\nu-\nu j}f'(\varepsilon^\nu z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz},$$

or equivalently

$$\frac{zf_k'(z)}{f_k(z)} \prec \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz},$$

that is $f_k \in \mathcal{S}(A, B, \alpha)$. ∎

Theorem 2.3 *Let $f \in \mathcal{S}^k(A, B, \alpha)$, with $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$. Then,*

$$f(z) = \begin{cases} \int_0^z \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(\zeta)}{1+B\widetilde{w}(\zeta)}(1+Bw(\zeta))^{\frac{(1-\alpha)(A-B)}{B}}\,d\zeta, & \text{if } B \neq 0, \\ \int_0^z [1+A(1-\alpha)\widetilde{w}(\zeta)]\exp[(1-\alpha)Aw(\zeta)]\,d\zeta, & \text{if } B = 0, \end{cases}$$

for some $w, \widetilde{w} \in \Omega$.

Proof Supposing that $f \in \mathcal{S}^k(A,B,\alpha)$, it follows that there exists a function $\widetilde{w} \in \Omega$ such that

$$\frac{zf'(z)}{f_k(z)} = \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(z)}{1+B\widetilde{w}(z)}, \ z \in \mathcal{U}.$$

Using Theorem 2.2 and Lemma 1.6, we have

$$f'(z) = \begin{cases} \frac{1+[(1-\alpha)A+\alpha B]\widetilde{w}(z)}{1+B\widetilde{w}(z)} (1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & \text{if } B \neq 0, \\ [1+A(1-\alpha)\widetilde{w}(z)]\exp[(1-\alpha)Aw(z)], & \text{if } B = 0, \end{cases}$$

and integrating the above relations along the line connecting the origin with $z \in \mathcal{U}$ we obtain our result. ∎

Theorem 2.4 *Let $f(z) \in \mathcal{S}^k(A,B,\alpha)$ and is of the form (1.1). Then for $n \geq 2$, $-1 \leq B < A \leq 1, 0 \leq \alpha < 1$.*

$$|a_n| \leq \prod_{m=1}^{n-1} \frac{\chi_m[(1-\alpha)(A-B)-1]+m}{m+1-\chi_{m+1}}, \tag{2.4}$$

where χ_n is defined in (1.7).

Proof By Definition 1.4, we have

$$\frac{zf'(z)}{f_k(z)} = p(z), \quad p \in \mathcal{P}[A,B,\alpha],$$

then we have

$$zf'(z) = [1+\sum_{n=1}^{\infty} p_n z^n]f_k(z)$$

by (1.1) and (1.6), we have

$$(1-\chi_1)z + \sum_{n=2}^{\infty}[n-\chi_n]a_n z^n = \left[\sum_{n=1}^{\infty} p_n z^n\right]\left[\sum_{n=1}^{\infty} \chi_n a_n z^n\right].$$

Equating coefficients of z^n on both sides, we have

$$a_n = \frac{1}{[n-\chi_n]}\sum_{m=1}^{n-1} p_m \chi_{n-m} a_{n-m}, \quad \chi_1 = 1, \tag{2.5}$$

by Lemma 1.5, we have

$$|a_n| \leq \frac{(A-B)(1-\alpha)}{[n-\chi_n]}\sum_{m=1}^{n-1}\chi_m|a_m| \tag{2.6}$$

Now we want to prove that

$$\frac{(A-B)(1-\alpha)}{[n-\chi_n]}\sum_{m=1}^{n-1}\chi_m|a_m| \leq \prod_{m=1}^{n-1}\frac{\chi_m[(1-\alpha)(A-B)-1]+m}{[m+1-\chi_{m+1}]}. \tag{2.7}$$

For this, we use the induction method.

The inequality (2.7) is true for $n=2$ and 3.

Let the hypothesis be true for $n=m$, we have

$$\frac{(A-B)(1-\alpha)}{[m-\chi_m]}\sum_{r=1}^{m-1}\chi_r|a_r| \le \prod_{r=1}^{m-1}\frac{\chi_r[(1-\alpha)(A-B)-1]+r}{[r+1-\chi_{r+1}]},$$

Multiplying both sides by $\frac{\chi_m[(A-B)(1-\alpha)-1]+m}{[m+1-\chi_{m+1}]}$, we get

$$\prod_{r=1}^{m}\frac{\chi_r[(1-\alpha)(A-B)-1]+r}{[r+1-\chi_{r+1}]} \ge \frac{\chi_m[(A-B)(1-\alpha)-1]+m}{[m+1-\chi_{m+1}]}.\frac{(A-B)(1-\alpha)}{[m-\chi_m]}\sum_{r=1}^{m-1}\chi_r|a_r|,$$

since

$$\begin{aligned}
&\frac{\chi_m[(A-B)(1-\alpha)-1]+m}{[m+1-\chi_{m+1}]}.\frac{(A-B)(1-\alpha)}{[m-\chi_m]}\sum_{r=1}^{m-1}\chi_r|a_r|\\
&=\frac{(A-B)(1-\alpha)}{[m+1-\chi_{m+1}]}.\left[1+\frac{\chi_m(A-B)(1-\alpha)}{[m-\chi_m]}\right]\sum_{r=1}^{m-1}\chi_r|a_r|,\\
&\ge\frac{(A-B)(1-\alpha)}{[m+1-\chi_{m+1}]}.\left[\sum_{r=1}^{m-1}\chi_r|a_r|+\chi_m|a_m|\right],\\
&=\frac{(A-B)(1-\alpha)}{[m+1-\chi_{m+1}]}.\left[\sum_{r=1}^{m}\chi_r|a_r|\right].
\end{aligned}$$

That is

$$|a_{m+1}|\le\frac{(A-B)(1-\alpha)}{[m-\chi_m]}\sum_{r=1}^{m}\chi_r|a_r|\le\prod_{r=1}^{m}\frac{\chi_r[(1-\alpha)(A-B)-1]+r}{[r+1-\chi_{r+1}]},$$

which shows that inequality (2.7) is true for $n=m+1$. This completes the proof. ∎

We now prove the distortion theorem for the class $\mathcal{S}^k(A,B,\alpha)$.

Theorem 2.5 *If* $f\in\mathcal{S}^k(A,B,\alpha)$, *then*

$$\left.\begin{array}{ll}\dfrac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1-Br)^{\frac{(1-\alpha)(A-B)}{B}}, & \textit{if } B\neq 0,\\ [1-(1-\alpha)Ar]\exp[-(1-\alpha)Ar], & \textit{if } B=0\end{array}\right\}\le|f'(z)|$$

$$\le\left\{\begin{array}{ll}\dfrac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1+Br)^{\frac{(1-\alpha)(A-B)}{B}}, & \textit{if } B\neq 0,\\ [1+(1-\alpha)Ar]\exp[(1-\alpha)Ar], & \textit{if } B=0,\end{array}\right.$$

where $|z|\le r<1$.

Proof For an arbitrary function $f\in\mathcal{S}^k(A,B,\alpha)$, according to Theorem 2.2 and Lemma 1.6 we need to study the following:

(i) If $B\neq 0$, then there exists a function $w\in\Omega$, such that $f_k(z)=z(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}$, by using Lemma 2.1 and therefore

$$\begin{aligned}
&\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}|1+Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}}\le|f'(z)|\\
&\le\frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}|1+Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}},\\
&|z|\le r<1.
\end{aligned}\tag{2.8}$$

Since $w \in \Omega$, we have

$$1-|B|r \leq |1+Bw(z)| \leq 1+|B|r, \quad |z| \leq r < 1.$$

Case 1 If $B > 0$, using the fact that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, we have

$$(1-|B|r)^{\frac{(1-\alpha)(A-B)}{B}} \leq |1+Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}} \leq (1+|B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \quad |z| \leq r < 1,$$

and from (2.8) we obtain

$$\begin{aligned}&\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1-|B|r)^{\frac{(1-\alpha)(A-B)}{B}} \leq |f'(z)| \\ &\leq \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1+|B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \\ &|z| \leq r < 1. \end{aligned} \tag{2.9}$$

Case 2 If $B < 0$, from the fact that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, we have

$$(1-|B|r)^{\frac{(1-\alpha)(A-B)}{B}} \geq |1+Bw(z)|^{\frac{(1-\alpha)(A-B)}{B}} \geq (1+|B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \quad |z| \leq r < 1,$$

and from (2.8) we obtain

$$\begin{aligned}&\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1-|B|r)^{\frac{(1-\alpha)(A-B)}{B}} \geq |f'(z)| \\ &\geq \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1+|B|r)^{\frac{(1-\alpha)(A-B)}{B}}, \\ &|z| \leq r < 1. \end{aligned} \tag{2.10}$$

Now, combining the inequalities (2.9) and (2.10), we finally conclude that

$$\begin{aligned}&\frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1-Br)^{\frac{(1-\alpha)(A-B)}{B}} \leq |f'(z)| \\ &\leq \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}(1+Br)^{\frac{(1-\alpha)(A-B)}{B}}, \\ &|z| \leq r < 1. \end{aligned} \tag{2.11}$$

(ii) If $B = 0$, there exists a function $w \in \Omega$, such that $f_k(z) = z\exp[(1-\alpha)Aw(z)]$, and therefore

$$\begin{aligned}&[1-(1-\alpha)Ar]\,|\exp[(1-\alpha)Aw(z)]| \leq |f'(z)| \leq [1+(1-\alpha)Ar] \\ &\times |\exp[(1-\alpha)Aw(z)]|, \ |z| \leq r < 1. \end{aligned} \tag{2.12}$$

Since $|\exp[(1-\alpha)Aw(z)]| = \exp[(1-\alpha)A\mathrm{Re}w(z)]$, $z \in \mathcal{U}$, using a similar computation as in the previous case, we deduce

$$\exp[-(1-\alpha)Ar] \leq |\exp[(1-\alpha)Aw(z)]| \leq \exp[(1-\alpha)Ar], \ |z| \leq r < 1.$$

Thus, (2.12) yield to

$$\begin{aligned}&[1-(1-\alpha)Ar]\exp[-(1-\alpha)Ar] \leq |f'(z)| \\ &\leq [1+(1-\alpha)Ar]\exp[(1-\alpha)Ar], \ |z| \leq r < 1, \end{aligned} \tag{2.13}$$

which completes the proof of our theorem.

∎

Theorem 2.6 *Let $f \in \mathcal{S}^k(A, B, \alpha)$ and let ϕ be convex. Then $(f * \phi) \in \mathcal{S}^k(A, B, \alpha)$.*

Proof To prove that $(f * \phi) \in \mathcal{S}^k(A, B, \alpha)$ it is sufficient to show that

$$\frac{z(f * \phi)'(z)}{(f * \phi)_k(z)} \subset \overline{CO}(F(\mathcal{U})),$$

where $F(z) = \frac{zf'(z)}{f_k(z)}$. Now

$$\begin{aligned}\frac{z(f * \phi)'(z)}{(f * \phi)_k(z)} &= \frac{zf'(z) * \phi(z)}{(f_k(z) * \phi(z)} \\ &= \frac{\phi(z) * \frac{zf'(z)}{f_k(z)} . f_k(z)}{\phi(z) * f_k(z)},\end{aligned}$$

by using Lemma 1.7 with $f_k(z) \in \mathcal{S}(A, B, \alpha)$, $F \in \mathcal{P}[A, B, \alpha]$, that complete the proof. ∎

Corollary 2.7 *Let $f \in \mathcal{S}^k(A, B, \alpha)$. Then*

$$F_i(z) \in \mathcal{S}^k(A, B, \alpha), \quad (i = 1, 2, 3, 4),$$

where

$$\begin{aligned}F_1(z) &= \int_0^z \frac{f(t)}{t} dt, \\ F_2(z) &= \int_0^z \frac{f(t) - f(xt)}{t - xt} dt, \quad |x| \le 1, x \ne 1, \\ F_3(z) &= \frac{2}{z} \int_0^z f(t) dt, \\ F_4(z) &= \frac{m+1}{m} \int_0^z t^{m-1} f(t) dt, \quad \Re m > 0.\end{aligned}$$

Proof Since

$$\begin{aligned}&F_1(z) = \phi_1(z) * f(z), \quad \phi_1(z) = \sum_0^\infty \frac{1}{n} z^n = \log(1-z)^{-1}, \\ &F_2(z) = \phi_2(z) * f(z), \\ &\phi_2(z) = \sum_0^\infty \frac{1 - x^n}{n(1-x)} z^n = \frac{1}{1-x} \log\left(\frac{1 - xz}{1 - z}\right), \quad |x| \le 1, x \ne 1, \\ &F_3(z) = \phi_3(z) * f(z), \quad \phi_3(z) = \sum_0^\infty \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z}, \\ &F_4(z) = \phi_4(z) * f(z), \quad \phi_4(z) = \sum_0^\infty \frac{1+m}{n+m} z^n, \quad \Re m > 0.\end{aligned}$$

We note that ϕ_i, $i = 1, 2, 3, 4$ are convex. Now using Theorem 2.6. ∎

Corollary 2.8 *The radius of starlikeness of the class $\mathcal{S}^k(A, B, \alpha)$ is*

$$r_* = \frac{2}{(1-\alpha)(A-B) + \sqrt{[(1-\alpha)(A-B)]^2 + 4B[(1-\alpha)A + \alpha B]}}. \tag{2.14}$$

Proof From Lemma 2.1

$$\Re\left(\frac{zf'(z)}{f_k(z)}\right) \geq \frac{1-(1-\alpha)(A-B)r - B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}.$$

Hence for $r < r_*$ the first hand side of the preceding inequality is positive this implies (2.14). ∎

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