Afr. Mat. (2018) 29:665–675
https://doi.org/10.1007/s13370-018-0568-2

# Ricci collineations on 3-dimensional paracontact metric manifolds

**I. Küpeli Erken**[1] · **C. Murathan**[2]

Received: 22 February 2017 / Accepted: 6 February 2018 / Published online: 13 February 2018


**Abstract** We classify three-dimensional paracontact metric manifold whose Ricci operator $Q$ is invariant along Reeb vector field, that is, $\mathcal{L}_\xi Q = 0$.

**Keywords** Paracontact metric manifold · Contact metric manifold · Ricci collineation · Reeb vector field

**Mathematics Subject Classification** Primary 53B30 · 53C25; Secondary 53D10

## 1 Introduction

A symmetry in general relativity is a smooth vector field whose local flow diffeomorphisms preserve certain mathematical or physical quantities [8,9]. So, one can regard it as vector fields preserving certain geometric quantities like the metric tensor, the curvature tensor or the Ricci tensor in general relativity. Symmetries of the geometrical or physical relevant quantities of the general relativity theory are known as collineations. The geometrical symmetries of the spacetime are expressible through the vanishing of the Lie derivative of certain tensors with respect to a vector. In literature, these can be represented as $\mathcal{L}_X A = 0$, where $A$ is the geometric or physical object, $X$ is the vector field generating the symmetry and $\mathcal{L}_X$ denotes Lie differentiation with respect to the vector $X$. The collineations of the Ricci tensor are called the Ricci collineations.

✉ I. Küpeli Erken
irem.erken@btu.edu.tr

C. Murathan
cengiz@uludag.edu.tr

1 Department of Mathematics, Faculty of Natural Sciences, Architecture and Engineering, Bursa Technical University, Bursa, Turkey

2 Department of Mathematics, Art and Science Faculty, Uludag University, 16059 Bursa, Turkey

A paracontact metric manifold whose characteristic vector field $\xi$ is a harmonic vector field is called an *H-paracontact* manifold. In [2], Calvaruso and Perrone proved that $\xi$ is *harmonic* if and only if $\xi$ is an eigenvector of the Ricci operator for contact semi-Riemannian manifolds. Calvaruso and Perrone [3] proved that all three-dimensional homogeneous paracontact metric manifolds are $H$-paracontact.

Recently, Küpeli Erken and Murathan [10] have studied the harmonicity of the characteristic vector field of three-dimensional paracontact metric manifolds and proved that $\xi$ is a harmonic vector field if and only if $\xi$ is an eigenvector of the Ricci operator. In the same study, the authors characterized the 3-dimensional $H$-paracontact metric manifolds in terms of $(\kappa, \mu, \nu)$-paracontact metric manifolds.

In [6,7], Cho study contact three manifolds with Ricci collineation respect to Reeb vector field $\xi$. But no effort has been made to investigate the paracontact counterpart. In [4], Calvino-Louzao et.al. determined all left-invariant Ricci collineations on three-dimensional Lie groups. In the present work, we concentrate on three-dimensional paracontact metric manifold whose Ricci operator $Q$ is invariant along the Reeb flow, that is, $\mathcal{L}_\xi Q = 0$.

*Overview* Here is the plan of the paper: Sect. 2 is focused on basic facts for paracontact metric manifolds and harmonicity of the characteristic vector field of three-dimensional paracontact metric manifolds. In Sect. 3, we proved that the Ricci operator $Q$ on a paracontact metric manifold is invariant along the Reeb vector field if and only if the manifold is $\kappa \neq -1$-nullity paracontact manifold and $\eta$-Einstein. Especially we showed that if $M$ is a 3-dimensional $(-1, \mu = const.)$ paracontact metric manifold with $\mathfrak{h}_2$ type then $\mathcal{L}_\xi Q = 0$.

## 2 Preliminaries

An $(2n+1)$-dimensional smooth manifold $M$ is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions:

(i) $\eta(\xi) = 1,\ \varphi^2 = I - \eta \otimes \xi$,

(ii) the tensor field $\varphi$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e. the $\pm 1$-eigendistributions, $\mathcal{D}^{\pm} := \mathcal{D}_\varphi(\pm 1)$ of $\varphi$ have equal dimension $n$.

From the definition it follows that $\varphi\xi = 0$, $\eta \circ \varphi = 0$ and the endomorphism $\varphi$ has rank $2n$. On an almost paracontact manifold, one defines the $(1,2)$-tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi,$$

where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$

$$[\phi, \phi](X, Y) = \phi^2 [X, Y] + [\phi X, \phi Y] - \phi [\phi X, Y] - \phi [X, \phi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal [14]. If an almost paracontact manifold admits a pseudo-Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.1}$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \varphi, \xi, \eta, g)$ is an *almost paracontact* metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $g(X_i, X_j) = \delta_{ij}$, $g(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \varphi X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\varphi$-basis.

In addition, if $d\eta(X,Y)=g(X,\varphi Y)$ for all vector fields $X, Y$ on $M$ then $(M,\varphi,\xi,\eta,g)$ is said to be a *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $h=\frac{1}{2}\mathcal{L}_{\xi}\varphi$. It is known [14] that $h$ anti-commutes with $\varphi$ and satisfies $\mathrm{tr}h=0=h\xi$ and

$$\nabla\xi=-\varphi+\varphi h, \tag{2.2}$$

where $\nabla$ is the Levi–Civita connection of the pseudo-Riemannian manifold $(M,g)$. Moreover, $h\equiv 0$ if and only if $\xi$ is a Killing vector field and in this case $(M,\varphi,\xi,\eta,g)$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also, in this context, the para-Sasakian condition implies the $K$-paracontact condition and the converse holds only in dimension 3 (see [1]). Moreover, in any para-Sasakian manifold

$$R(X,Y)\xi=-(\eta(Y)X-\eta(X)Y) \tag{2.3}$$

holds, but unlike contact metric geometry the condition (2.3) not necessarily implies that the manifold is para-Sasakian. Using (2.2 ), one can get

$$R(X,Y)\xi=-(\nabla_X\varphi)Y+(\nabla_Y\varphi)X+(\nabla_X\varphi h)Y-(\nabla_Y\varphi h)X. \tag{2.4}$$

In [14], Zamkovoy proved that

$$(\nabla_{\xi}h)X=-\varphi X+h^2\varphi X+\varphi R(\xi,X)\xi. \tag{2.5}$$

Moreover, he showed that Ricci curvature $S$ in the direction of $\xi$ is given by

$$S(\xi,\xi)=-2n+trh^2 \tag{2.6}$$

where $S(X,Y)=g(QX,Y)$. Henceforward, we denote $S_{ij}=S(e_i,e_j)$ for $i,j=1,2$.

An almost paracontact structure $(\varphi,\xi,\eta)$ is said to be *integrable* if $N_{\varphi}(X,Y)\in\Gamma(\mathbb{R}\xi)$ whenever $X,Y\in\Gamma(\mathcal{D})$.

In [13], Welyczko proved that any 3-dimensional paracontact metric manifold is always integrable. So for 3-dimensional paracontact metric manifold, we have

$$(\nabla_X\varphi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX). \tag{2.7}$$

**Definition 2.1** [10] A $2n+1$-dimensional paracontact metric $(\kappa,\mu,\nu)$-manifold is a paracontact metric manifold for which the curvature tensor field satisfies

$$\begin{aligned}R(X,Y)\xi&=\kappa\left(\eta(Y)X-\eta(X)Y\right)+\mu\left(\eta(Y)hX\right.\\&\quad\left.-\eta(X)hY\right)+\nu\left(\eta(Y)\varphi hX-\eta(X)\varphi hY\right),\end{aligned} \tag{2.8}$$

for all $X,Y\in\Gamma(TM)$, where $\kappa,\mu,\nu$ are smooth functions on $M$.

The authors proved following theorems

**Theorem 2.2** [10] *Let $(M,\varphi,\xi,\eta,g)$ be a 3-dimensional paracontact metric manifold. $\xi$ is a harmonic vector field if and only if the characteristic vector field $\xi$ is an eigenvector of the Ricci operator.*

**Theorem 2.3** [10] *Let $(M,\varphi,\xi,\eta,g)$ be a 3-dimensional paracontact metric manifold. If the characteristic vector field $\xi$ is harmonic map then the paracontact metric $(\kappa,\mu,\nu)$-manifold always exists on every open and dense subset of $M$. Conversely, if $M$ is a paracontact metric $(\kappa,\mu,\nu)$-manifold then the characteristic vector field $\xi$ is harmonic map.*

Beside the other results, the different possibilities for the tensor field $h$ are analyzed in [10].

*The tensor h has the canonical form (I)* Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold.

$$U_1 = \{p \in M \mid h(p) \neq 0\} \subset M,$$
$$U_2 = \{p \in M \mid h(p) = 0, \text{ in a neighborhood of p}\} \subset M.$$

Here, $h$ is a smooth function on $M$. Therefore, $U_1 \cup U_2$ is an open and dense subset of $M$. Thus any property satisfied in $U_1 \cup U_2$ is also satisfied in $M$. For any point $p \in U_1 \cup U_2$, there exists a local orthonormal $\varphi$-basis $\{e, \varphi e, \xi\}$ of smooth eigenvectors of $h$ in a neighborhood of $p$, where $-g(e, e) = g(\varphi e, \varphi e) = g(\xi, \xi) = 1$. On $U_1$, we put $he = \lambda e$, where $\lambda$ is a non-vanishing smooth function. Since $trh = 0$, we have $h\varphi e = -\lambda\varphi e$. The eigenvalue function $\lambda$ is continuous on $M$ and smooth on $U_1 \cup U_2$. So, $h$ has following form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

with respect to local orthonormal $\varphi$-basis $\{e, \varphi e, \xi\}$. In this case, we will say the operator $h$ is of $\mathfrak{h}_1$ *type*. Using same method with [11,12], we have

**Lemma 2.4** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold with h of $\mathfrak{h}_1$ type. Then, the following equations are valid for the covariant derivative on the subset $U_1$.*

$$\begin{aligned}
&\text{(i) } \nabla_e e = \frac{1}{2\lambda}[\sigma(e) - (\varphi e)(\lambda)]\varphi e, \quad \text{(ii) } \nabla_e \varphi e = \frac{1}{2\lambda}[\sigma(e) - (\varphi e)(\lambda)]e + (1-\lambda)\xi, \\
&\text{(iii) } \nabla_e \xi = (\lambda - 1)\varphi e, \\
&\text{(iv) } \nabla_{\varphi e} e = -\frac{1}{2\lambda}[\sigma(\varphi e) + e(\lambda)]\varphi e - (\lambda+1)\xi, \quad \text{(v) } \nabla_{\varphi e}\varphi e = -\frac{1}{2\lambda}[\sigma(\varphi e) + e(\lambda)]e, \\
&\text{(vi) } \nabla_{\varphi e}\xi = -(\lambda+1)e, \\
&\text{(vii) } \nabla_\xi e = b\varphi e, \quad \text{(viii) } \nabla_\xi \varphi e = be, \\
&\text{(ix) } [e, \xi] = (\lambda - 1 - b)\varphi e, \quad \text{(x) } [\varphi e, \xi] = (-\lambda - 1 - b)e, \\
&\text{(xi) } [e, \varphi e] = \frac{1}{2\lambda}[\sigma(e) - (\varphi e)(\lambda)]e + \frac{1}{2\lambda}[\sigma(\varphi e) + e(\lambda)]\varphi e + 2\xi,
\end{aligned} \tag{2.10}$$

*where*

$$b = g(\nabla_\xi e, \varphi e), \sigma = S(\xi, .)_{\ker\eta}.$$

**Proposition 2.5** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold. If h is $\mathfrak{h}_1$ type then on the subset $U_1$ we have*

$$\nabla_\xi h = -2bh\varphi + \xi(\lambda)s, \tag{2.11}$$

*where s is the $(1, 1)$-type tensor defined by $s\xi = 0$, $se = e$, $s\varphi e = -\varphi e$.*

**Lemma 2.6** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold. If h is $\mathfrak{h}_1$ type then the Ricci operator Q is given by*

$$Q = a_1 I + b_1 \eta \otimes \xi - \varphi(\nabla_\xi h) + \sigma(\varphi^2) \otimes \xi - \sigma(e)\eta \otimes e + \sigma(\varphi e)\eta \otimes \varphi e, \tag{2.12}$$

*where $a_1$ and $b_1$ are smooth functions defined by $a_1 = 1 - \lambda^2 + \frac{r}{2}$ and $b_1 = 3(\lambda^2 - 1) - \frac{r}{2}$, and $r$ denotes scalar curvature. Moreover the components of the Ricci operator $Q$ are given by*

$$\begin{aligned} Q\xi &= (a_1 + b_1)\xi - \sigma(e)e + \sigma(\varphi e)\varphi e, \\ Qe &= \sigma(e)\xi + (a_1 - 2b\lambda)e - \xi(\lambda)\varphi e, \\ Q\varphi e &= \sigma(\varphi e)\xi + \xi(\lambda)e + (a_1 + 2b\lambda)\varphi e. \end{aligned} \tag{2.13}$$

*From* (2.13)*, we get*

$$S_{11} = -(a_1 - 2b\lambda),\ S_{12} = -\xi(\lambda),\ S_{22} = a_1 + 2b\lambda,\ S_{11} + S_{22} = 4b\lambda. \tag{2.14}$$

*The tensor $\tilde{h}$ has the canonical form (II)* Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold and $p$ is a point of $M$. Then there exists a local pseudo-orthonormal basis $\{e_1, e_2, \xi\}$ in a neighborhood of $p$ where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$ and $g(e_1, e_2) = 1$.

**Lemma 2.7** [10] *Let $\mathcal{U}$ be the open subset of $M$ where $h \neq 0$. For every $p \in \mathcal{U}$ there exists an open neighborhood of $p$ such that $he_1 = e_2$, $he_2 = 0$, $h\xi = 0$ and $\varphi e_1 = \pm e_1$, $\varphi e_2 = \mp e_2$.*

Hence the tensor $h$ has the form $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ relative a pseudo-orthonormal basis $\{e_1, e_2, \xi\}$. In this case, we call $h$ is of $\mathfrak{h}_2$ *type*.

*Remark 2.8* [10] Without loss of generality, we can assume that $\varphi e_1 = e_1\, \varphi e_2 = -e_2$. Moreover one can easily get $h^2 = 0$ but $h \neq 0$.

**Lemma 2.9** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold with $h$ of $\mathfrak{h}_2$ type. Then, the following equations are valid for the covariant derivative on the subset $\mathcal{U}$*

$$\begin{aligned} &\text{(i) } \nabla_{e_1} e_1 = -b_2 e_1 + \xi, \quad \text{(ii) } \nabla_{e_1} e_2 = b_2 e_2 + \xi, \quad \text{(iii) } \nabla_{e_1}\xi = -e_1 - e_2, \\ &\text{(iv)} \nabla_{e_2} e_1 = -c_2 e_1 - \xi, \quad \text{(v) } \nabla_{e_2} e_2 = c_2 e_2, \quad \text{(vi) } \nabla_{e_2}\xi = e_2, \\ &\text{(vii) } \nabla_{\xi} e_1 = a_2 e_1, \quad \text{(viii) } \nabla_{\xi} e_2 = -a_2 e_2, \\ &\text{(ix) } [e_1, \xi] = -(1 + a_2)e_1 - e_2, \quad \text{(x) } [e_2, \xi] = (1 + a_2)e_2, \\ &\text{(xi) } [e_1, e_2] = c_2 e_1 + b_2 e_2 + 2\xi. \end{aligned} \tag{2.15}$$

*where $a_2 = g(\nabla_\xi e_1, e_2)$, $b_2 = g(\nabla_{e_1} e_2, e_1)$ and $c_2 = -\frac{1}{2}\sigma(e_1) = -\frac{1}{2}S(\xi, e_1)$.*

**Proposition 2.10** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold with $h$ of $\mathfrak{h}_2$ type. Then, we have*

$$\nabla_\xi h = 2a_2\varphi h, \tag{2.16}$$

*on $\mathcal{U}$.*

**Lemma 2.11** [10] *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold with $h$ of $\mathfrak{h}_2$ type. Then, the Ricci operator $Q$ is given by*

$$Q = \left(1 + \frac{r}{2}\right) I - \left(3 + \frac{r}{2}\right) \eta \otimes \xi - \varphi(\nabla_\xi h) + \sigma(\varphi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_2. \tag{2.17}$$

A consequence of Lemma 2.11, we can give the components of the Ricci operator $Q$ by following,

$$\begin{aligned} Q\xi &= -2\xi + \sigma(e_1)e_2, \\ Qe_1 &= \sigma(e_1)\xi + \left(1 + \frac{r}{2}\right) e_1 - 2a_2 e_2, \\ Qe_2 &= \left(1 + \frac{r}{2}\right) e_2. \end{aligned} \tag{2.18}$$

*The tensor* $\tilde{h}$ *has the canonical form (III)* Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold and $p$ is a point of $M$. Then, there exists a local orthonormal $\varphi$-basis $\{e, \varphi e, \xi\}$ in a neighborhood of $p$ where $-g(e, e) = g(\varphi e, \varphi e) = g(\xi, \xi) = 1$. Now, let $U_1$ be the open subset of $M$ where $h \neq 0$ and let $U_2$ be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of $p$. $U_1 \cup U_2$ is an open subset of $M$. For every $p \in U_1$, there exists an open neighborhood of $p$ such that $he = \lambda\varphi e$, $h\varphi e = -\lambda e$ and $h\xi = 0$ where $\lambda$ is a non-vanishing smooth function. Since $trh = 0$, the matrix form of $h$ is given by

$$h = \begin{pmatrix} 0 & -\lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.19}$$

with respect to local orthonormal basis $\{e, \varphi e, \xi\}$. In this case, we say that $h$ is of $\mathfrak{h}_3$ *type*.

**Lemma 2.12** [10] *Let* $(M, \varphi, \xi, \eta, \tilde{g})$ *be a* 3 *-dimensional paracontact metric manifold with* $h$ *of* $\mathfrak{h}_3$ *type. Then, the following equations are valid for the covariant derivative on the subset* $U_1$

$$\begin{aligned} &\text{(i) } \nabla_e e = a_3\varphi e + \lambda\xi, \quad \text{(ii) } \nabla_e \varphi e = a_3 e + \xi, \quad \text{(iii) } \nabla_e \xi = -\varphi e + \lambda e, \\ &\text{(iv) } \nabla_{\varphi e} e = b_3 \varphi e - \xi, \quad \text{(v) } \nabla_{\varphi e}\varphi e = b_3 e + \lambda\xi, \quad \text{(vi) } \nabla_{\varphi e}\xi = -e - \lambda\varphi e, \\ &\text{(vii) } \nabla_\xi e = d_3\varphi e, \quad \text{(viii) } \nabla_\xi \varphi e = d_3 e, \\ &\text{(ix) } [e, \xi] = \lambda e - (1 + d_3)\varphi e, \quad \text{(x) } [\varphi e, \xi] = -(1 + d_3)e - \lambda\varphi e, \\ &\text{(xi) } [e, \varphi e] = a_3 e - b_3\varphi e + 2\xi. \end{aligned} \tag{2.20}$$

*where* $a_3$*,* $b_3$ *and* $d_3$ *are defined by*

$$\begin{aligned} a_3 &= -\frac{1}{2\lambda}\left[\sigma(\varphi e) + (\varphi e)(\lambda)\right], \quad \sigma(e) = S(\xi, e), \\ b_3 &= \frac{1}{2\lambda}\left[\sigma(e) - e(\lambda)\right], \quad \sigma(\varphi e) = S(\xi, \varphi e), \\ d_3 &= g(\nabla_\xi e, \varphi e) \end{aligned}$$

*respectively.*

**Proposition 2.13** [10] *Let* $(M, \varphi, \xi, \eta, g)$ *be a* 3*-dimensional paracontact metric manifold with* $h$ *of* $\mathfrak{h}_3$ *type. So, on the subset* $U_1$ *we have*

$$\nabla_\xi h = -2d_3 h\varphi + \xi(\lambda)s, \tag{2.21}$$

*where* $s$ *is the* $(1, 1)$*-type tensor defined by* $s\xi = 0$*,* $se = \varphi e$*,* $s\varphi e = -e$*.*

**Lemma 2.14** [10] *Let* $(M, \varphi, \xi, \eta, g)$ *be a* 3*-dimensional paracontact metric manifold with* $h$ *of* $\mathfrak{h}_3$ *type. Then the Ricci operator* $Q$ *is given by*

$$Q = g_3\, I + f_3\eta \otimes \xi - \varphi(\nabla_\xi h) + \sigma(\tilde{\varphi}^2) \otimes \xi - \sigma(e)\eta \otimes e + \sigma(\varphi e)\eta \otimes \varphi e, \tag{2.22}$$

*where $g_3$ and $f_3$ are smooth functions defined by $g_3 = 1+\lambda^2+\frac{r}{2}$ and $f_3 = -3(\lambda^2+1)-\frac{r}{2}$, respectively. Moreover the components of the Ricci operator $Q$ are given by*

$$\begin{aligned} Q\xi &= (g_3 + f_3)\xi - \sigma(e)e + \sigma(\varphi e)\varphi e,\\ Qe &= \sigma(e)\xi + (g_3 + \xi(\lambda))e - 2d_3\lambda\varphi e,\\ Q\varphi e &= \sigma(\varphi e)\xi + 2d_3\lambda e + (g_3 + \xi(\lambda))\varphi e. \end{aligned} \tag{2.23}$$

*From* (2.23)*, we get*

$$S_{11} = -(g_3+\xi(\lambda)),\ S_{12} = -2d_3\lambda,\ S_{22} = (g_3+\xi(\lambda)),\ S_{11}-S_{22} = -2(g_3+\xi(\lambda)). \tag{2.24}$$

## 3 Ricci collineations on 3-dimensional paracontact metric manifolds

**Lemma 3.1** [13] *If $M$ is a 3-dimensional para-Sasakian manifold, then $h = 0$.*

**Lemma 3.2** [14] *The Ricci curvature $S$ of a $(2n+1)$-dimensional para-Sasakian manifold $M$ satisfies the relation*

$$\begin{aligned} S(\varphi X, \varphi Y) &= -S(X, Y) - 2n\eta(X)\eta(Y),\\ S(X, \xi) &= -2n\eta(X). \end{aligned} \tag{3.1}$$

We recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y\\ & -\frac{r}{2}(g(Y, Z)X - g(X, Z)Y) \end{aligned} \tag{3.2}$$

for all vector fields $X, Y, Z$, where $r$ denotes the scalar curvature.

**Proposition 3.3** *A 3-dimensional para-Sasakian manifold satisfies*

$$S(X, Y) = \left(1 + \frac{r}{2}\right) g(X, Y) - \left(3 + \frac{r}{2}\right) \eta(X)\eta(Y),$$

*which means that it is $\eta$-Einstein.*

**Corollary 3.4** *For a 3-dimensional para-Sasakian manifold $\xi(r) = 0$ if and only if $\mathcal{L}_\xi Q = 0$.*

First of all, we will investigate three possibilities according to canonical form $h$.

*Case 1:* We suppose that $h$ is $\mathfrak{h}_3$ type.

**Lemma 3.5** *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold. If $h$ is $\mathfrak{h}_3$ type then on the subset $U_1$ we have,*

$$\textit{If } \mathcal{L}_\xi Q = 0, \textit{ then } \nabla_\xi Q = 0 \textit{ and } Q\xi = \rho\xi, \textit{ where } \rho \textit{ is a function.}$$

*Proof* Assume that $M$ satisfies $\mathcal{L}_\xi Q = 0$. In this case, we have

$$\begin{aligned} \mathcal{L}_\xi(QX) - Q(\mathcal{L}_\xi X) = 0\\ [\xi, QX] - Q[\xi, X] = 0. \end{aligned}$$

By (2.2), we obtain an equivalent equation to $\mathcal{L}_\xi Q = 0$ as follows

$$(\nabla_\xi Q)X = (Q\varphi - \varphi Q)X + (\varphi h Q - Q\varphi h)X. \tag{3.3}$$

We first note that, since $\nabla_\xi Q$ and $Q\varphi - \varphi Q$ are self-adjoint operators, it easily follows that

$$Q\varphi h - \varphi h Q = Qh\varphi - h\varphi Q.$$

Using the anti-commutative property $h$ with $\varphi$ in the last equation, we get

$$Q\varphi h = \varphi h Q. \tag{3.4}$$

Applying $\xi$ to both sides of (3.4), we then get $hQ\xi = 0$. Using this in the first equation of (2.23), we have $Q\xi = \rho\xi$, $\rho = -2(\lambda^2 + 1)$ on the subset $U_1$. Clearly, using relation (3.4) in (3.3), we get

$$\nabla_\xi Q = Q\varphi - \varphi Q. \tag{3.5}$$

In view of the last equation, one can easily calculate $(\nabla_\xi S)(\xi, \xi) = 0$. So, by $\nabla_\xi \xi = 0$, one can get $\xi(\lambda) = 0$. Moreover, from (2.23), we have

$$S_{12} = S_{21}, \quad S_{11} = -S_{22}. \tag{3.6}$$

Applying $e$ to (3.4) and taking inner product with $\varphi e$ with respect to $g$, the last relation returns to

$$S_{12} = S_{21} = 0, \quad S_{11} = -S_{22}. \tag{3.7}$$

on $U_1$. So, using last equation in (2.24), it follows that $d_3 = 0$ and (2.23) can be given by

$$\begin{aligned} Q\xi &= \rho\xi, \\ Qe &= g_3 e, \\ Q\varphi e &= g_3 \varphi e. \end{aligned} \tag{3.8}$$

Obviously, from the last equation, we have $Q\varphi = \varphi Q$ on $U_1$. Hence, from (3.5), we get $\nabla_\xi Q = 0$ on $U_1$. □

We now check whether $\lambda$ is constant or not.

In view of (3.2) and Lemma 3.5, the following formulas hold in $U_1$

$$\begin{aligned} R(e, \varphi e)\varphi e &= Qe + (1 + \lambda^2)e, \\ R(e, \varphi e)e &= Q\varphi e + (1 + \lambda^2)\varphi e, \\ R(\varphi e, \xi)\varphi e &= (1 + \lambda^2)\xi, \\ R(e, \xi)e &= -(1 + \lambda^2)\xi, \\ R(e, \xi)\xi &= -(1 + \lambda^2)e, \\ R(\varphi e, \xi)\xi &= -(1 + \lambda^2)\varphi e, \end{aligned} \tag{3.9}$$

where $R(e_i, e_j)e_k = 0$, for $i \neq j \neq k$.

On the other hand, taking into account, (2.20) and (3.9), direct calculations give

$$\begin{aligned} (\nabla_e R)(\varphi e, \xi)\varphi e &= e(1 + \lambda^2)\xi + \lambda(g_3 + 2(1 + \lambda^2))e, \\ (\nabla_{\varphi e} R)(\xi, e)\varphi e &= -\lambda(g_3 + 2(1 + \lambda^2))e, \\ (\nabla_\xi R)(e, \varphi e)\varphi e &= \xi(g_3)e, \\ (\nabla_{\varphi e} R)(e, \xi)e &= -\tilde{\varphi} e(1 + \lambda^2)\xi + \lambda(g_3 + 2(1 + \lambda^2))\varphi e, \\ (\nabla_e R)(\xi, \varphi e)e &= -\lambda(g_3 + 2(1 + \lambda^2))\varphi e, \\ (\nabla_\xi R)(\varphi e, e)e &= -\xi(g_3)\varphi e. \end{aligned} \tag{3.10}$$

With the help of second bianchi identity and (3.10), we find $e(\lambda) = 0$ and $\varphi e(\lambda) = 0$. Regarding $\xi(\lambda) = 0$, we can conclude that $\lambda$ is constant on $M$.

So we can state following

**Lemma 3.6** *$\lambda$ is constant.*

Using Lemma 3.6, (2.20) returns to

$$\begin{aligned}
&(\text{i})\ \nabla_e e = \lambda\xi, \quad (\text{ii})\ \nabla_e \varphi e = \xi,\\
&(\text{iii})\ \nabla_e \xi = -\varphi e + \lambda e,\\
&(\text{iv})\ \nabla_{\varphi e} e = -\xi, \quad (\text{v})\ \nabla_{\varphi e}\varphi e = \lambda\xi,\\
&(\text{vi})\ \nabla_{\varphi e}\xi = -e - \lambda\varphi e,\\
&(\text{vii})\ \nabla_\xi e = 0, \quad (\text{viii})\ \nabla_\xi \varphi e = 0,\\
&(\text{ix})\ [e,\xi] = -\varphi e + \lambda e, \quad (\text{x})\ [\varphi e,\xi] = -e - \lambda\varphi e,\\
&(\text{xi})\ [e,\varphi e] = 2\xi.
\end{aligned} \tag{3.11}$$

In view of (3.9) and (3.11), we have

$$Qe = 0,\ Q\varphi e = 0, \quad Q\xi = -2(\lambda^2+1)\xi. \tag{3.12}$$

From (3.12) we can easily see that $(\mathcal{L}_\xi Q)e = (\mathcal{L}_\xi Q)\varphi e = 0$.

*Case 2:* We suppose that $h$ is $\mathfrak{h}_1$ type.

As the proof of the following lemma is similar to Riemannian case [6], it is not stated here.

**Lemma 3.7** *Let $(M,\varphi,\xi,\eta,g)$ be a 3-dimensional paracontact metric manifold. If $h$ is $\mathfrak{h}_1$ type then on $U_1$ we have,*

$$\mathcal{L}_\xi Q = 0 \text{ if and only if } \nabla_\xi Q = 0 \text{ and } Q\xi = \rho\xi, \text{ where } \rho \text{ is a function.}$$

We now check whether $\lambda$ is constant or not.

In view of (3.2) and Lemma 3.7, the following formulas hold in $U_1$

$$\begin{aligned}
R(e,\varphi e)\varphi e &= Qe + (1-\lambda^2)e,\\
R(e,\varphi e)e &= Q\varphi e + (1-\lambda^2)\varphi e,\\
R(\varphi e,\xi)\varphi e &= (1-\lambda^2)\xi,\\
R(e,\xi)e &= (\lambda^2-1)\xi,\\
R(e,\xi)\xi &= (\lambda^2-1)e,\\
R(\varphi e,\xi)\xi &= (\lambda^2-1)\varphi e,
\end{aligned} \tag{3.13}$$

where $R(e_i,e_j)e_k = 0$, for $i \neq j \neq k$.

On the other hand, taking into account, (2.10) and (3.13), direct calculations give

$$\begin{aligned}
&(\nabla_e R)(\varphi e,\xi)\varphi e = e(1-\lambda^2)\xi,\\
&(\nabla_{\varphi e} R)(\xi,e)\varphi e = 0,\\
&(\nabla_\xi R)(e,\varphi e)\tilde{\varphi} e = \xi(a_1)e,\\
&(\nabla_{\varphi e} R)(e,\xi)e = \varphi e(\lambda^2-1)\xi,\\
&(\nabla_e R)(\xi,\varphi e)e = 0,\\
&(\nabla_\xi R)(\varphi e,e)e = -\xi(a_1)\varphi e.
\end{aligned} \tag{3.14}$$

With the help of second bianchi identity and (3.14), we find $e(\lambda) = 0$ and $\varphi e(\lambda) = 0$. Regarding $\xi(\lambda) = 0$, we can conclude that $\lambda$ is constant on $M$.

So we can state following

**Lemma 3.8** *$\lambda$ is constant.*

Using Lemma 3.8, (2.10) returns to

$$\begin{aligned}
&\text{(i) } \nabla_e e = 0, \quad \text{(ii) } \nabla_e \varphi e = (1-\lambda)\xi,\\
&\text{(iii) } \nabla_e \xi = (\lambda - 1)\varphi e,\\
&\text{(iv) } \nabla_{\varphi e} e = -(\lambda+1)\xi, \quad \text{(v) } \nabla_{\varphi e}\varphi e = 0,\\
&\text{(vi) } \nabla_{\varphi e}\xi = -(\lambda+1)e,\\
&\text{(vii) } \nabla_\xi e = 0, \quad \text{(viii) } \nabla_\xi \varphi e = 0,\\
&\text{(ix) } [e,\xi] = (\tilde{\lambda}-1)\tilde{\varphi} e, \quad \text{(x) } [\varphi e, \xi] = -(\lambda+1)e,\\
&\text{(xi) } [e, \varphi e] = 2\xi.
\end{aligned} \tag{3.15}$$

In view of (3.13) and (3.15), we have

$$Qe = 0,\ Q\varphi e = 0,\ Q\xi = 2(\lambda^2 - 1)\xi. \tag{3.16}$$

From (3.16) we can easily see that $(\mathcal{L}_\xi Q)e = (\mathcal{L}_\xi Q)\varphi e = 0$.

**Theorem 3.9** *Let $M$ be a 3-dimensional paracontact metric manifold. Then $\mathcal{L}_\xi Q = 0$ if and only if $M$ is a $\kappa$-nullity paracontact manifold with $\kappa \neq -1$ and $\eta$-Einstein.*

*Proof* Case 1: Assume that $M$ is a 3-dimensional paracontact metric manifold with $h$ of $\mathfrak{h}_3$ type whose Ricci operator $Q$ satisfies $\mathcal{L}_\xi Q = 0$. Taking into account Theorem 2.2, Theorem 2.3 and Lemma 3.5, together we obtain that $M$ is a $\kappa$ -nullity paracontact manifold with $\kappa = -1 - \lambda^2$. Moreover using [5, Corollary 5.12], we have $M$ is $\eta$-Einstein. Conversely, let $M$ is $\eta$-Einstein. So we have $Q = AI + B\eta \otimes \xi$ where $A$ and $B$ are smooth functions on $M$. If we take the covariant derivative according to $\xi$, we get $\nabla_\xi Q = 0$. We proved that $Q\varphi = \varphi Q$ in the proof of Theorem 2.3. So if we use this we get $\mathcal{L}_\xi Q = 0$. The proof of Case 2 is similar to Case 1. This completes the proof of the theorem. □

*Case 3:* We suppose that $h$ is $\mathfrak{h}_2$ type.

**Theorem 3.10** *Let $(M, \varphi, \xi, \eta, g)$ be a 3-dimensional paracontact metric manifold. If $h$ is $\mathfrak{h}_2$ type then on $U_1$ we have,*

$\mathcal{L}_\xi Q = 0$ *if and only if* $\xi(\sigma(e_1)) - (1+a_2)\sigma(e_1) = 0,\ \xi(r) = 0$ *and* $\xi(a_2) - 2a_2(1+a_2) = 0$.

*Proof* We suppose $\mathcal{L}_\xi Q = 0$. Using (2.15) and (2.18 ), we have

$$\begin{aligned}
(\mathcal{L}_\xi Q)e_1 &= 0 \Rightarrow [\xi, Qe_1] - Q[\xi, e_1] = 0\\
&\Rightarrow \xi(\sigma(e_1)) - (1+a_2)\sigma(e_1) = 0, \quad \xi(r) = 0, \quad \xi(a_2) - 2a_2(1+a_2) = 0\\
(\mathcal{L}_\xi Q)e_2 &= 0 \Rightarrow [\xi, Qe_2] - Q[\xi, e_2] = 0\\
&\Rightarrow \xi(r) = 0\\
(\mathcal{L}_\xi Q)\xi &= 0 \Rightarrow [\xi, Q\xi] - Q[\xi, \xi] = 0\\
&\Rightarrow \xi(\sigma(e_1)) - (1+a_2)\sigma(e_1) = 0.
\end{aligned}$$

The proof of the converse side is clear. □

**Corollary 3.11** *If $M$ is a 3-dimensional $(-1, \mu = const.)$ paracontact metric manifold with $\mathfrak{h}_2$ type then $\mathcal{L}_\xi Q = 0$.*

*Proof* Since $M$ is a 3-dimensional $(-1, \mu = -2a_2 = const.)$ paracontact metric manifold

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) - 2a_2(\eta(Y)hX - \eta(X)hY)$$

(see [10]). One can easily get $Q\xi = -2\xi$. By (2.18) we obtain $\sigma(e_1) = 0$. Using Theorem 3.10, we get $\mathcal{L}_\xi Q = 0$. This completes proof. □

## References


1. Calvaruso, G.: Homogeneous paracontact metric three-manifolds. Ill. J. Math. **55**, 697–718 (2011)
2. Calvaruso, G., Perrone, D.: H-contact semi-Riemannian manifolds. J. Geom. Phys. **71**, 11–21 (2013)
3. Calvaruso, G., Perrone, D.: Geometry of H-paracontact metric manifolds. Publ. Math. Debrecen **86**(3-4), 325–346 (2015)
4. Calvino-Louzao, E., Seoane-Bascoy, J., Vázquez-Abal, M.E., Vázquez-Lorenzo, R.: Invariant Ricci collineations on three-dimensional Lie groups. J. Geom. Phys. **96**, 59–71 (2015)
5. Cappelletti, Montano B., Küpeli, Erken I., Murathan, C.: Nullity conditions in paracontact geometry. Differ. Geom. Appl. **30**, 665–693 (2012)
6. Cho, J.T.: Contact 3-manifolds with the Reeb flow symmetry. Tohoku Math. J. **66**, 491–500 (2014)
7. Cho, J.T., Kimura, M.: Reeb flow symmetry on almost contact three-manifolds. Differ. Geom. Appl. **35**, 266–273 (2014)
8. Duggal, K.L., Sharma, R.: Symmetries of Spacetimes and Riemannian Manifolds. Kluwer, Dordrecht (1999)
9. Hall, G.S.: Symmetries and geometry in general relativity. Differ. Geom. Appl. **1**, 35–45 (1991)
10. Küpeli Erken, I., Murathan, C.: A study of three-dimensional paracontact $(\kappa, \mu, v)$-spaces. Int. J. Geom. Methods Mod. Phys. **14**(7), 1750106 (2017). https://doi.org/10.1142/S0219887817501067
11. Koufogiorgos, T., Markellos, M., Papantoniou, B.: The harmonicity of the Reeb vector field on contact metric 3-manifolds. Pac. J. Math. **234**, 325–344 (2008)
12. Perrone, D.: Harmonic characteristic vector fields on contact metric three-manifolds. Bull. Austral. Math. Soc. **67**(2), 305–315 (2003)
13. Welyczko, J.: Para-CR structures on almost paracontact metric manifolds. J. Appl. Anal. **20**(2), 105–117 (2014)
14. Zamkovoy, S.: Canonical connections on paracontact manifolds. Ann. Glob. Anal. Geom. **36**, 37–60 (2009)