Afr. Mat. (2018) 29:383–398
https://doi.org/10.1007/s13370-018-0548-6

On vector variational-like inequalities involving right upper-Dini-derivative functions

Anurag Jayswal[1] · **Shipra Singh**[1]

Received: 12 August 2016 / Accepted: 23 January 2018 / Published online: 2 February 2018

Abstract In this paper, we introduce (weak) Stampacchia and Minty arcwise connected vector variational-like inequalities in the term of right upper-Dini-derivative and establish not only the relations of introduced inequalities with vector optimization problems but also the existence results, by using KKM-Fan theorem and Brouwer fixed point theorem. Examples are provided to illustrate the derived results.

Keywords Arcwise connected vector variational inequalities · Vector optimization problems · Arcwise connected functions · Existence theorems

Mathematics Subject Classification 90C30 · 90C29 · 49J40

1 Introduction

Nonsmooth analysis has arisen to deal with the problems of minimization or maximization of nondifferentiable functions. These types of problems, called nonsmooth optimization problems are most common in engineering. Here the study of differential properties of non-differentiable functions is stressed however directional derivative does not exist always. Thus, it is necessary to generalize the derivatives into Dini derivatives, Clarke derivatives, Fréchet derivatives, etc. These kinds of generalized derivatives are helpful and practical, when dealing with nonsmooth continuous and discontinuous functions. Also generalized derivatives have been successfully applied in control theory, mechanics, economic, differential equations. For various approaches, we refer to [11,14,20,22].

✉ Anurag Jayswal
anurag_jais123@yahoo.com

Shipra Singh
shiprasingh384@gmail.com

1 Department of Applied Mathematics, Indian Institute of Technology (Indian School of Mines), Dhanbad, Jharkhand 826 004, India

Convexity is an inevitable hypothesis, which serves as an efficient concept to investigate optimization problems. The notion of convexity is based upon the possibility of connecting two points of the space by means of line segment. It is used to obtain sufficient conditions for that conditions, which are only necessary, for example Kuhn–Tucker conditions in nonlinear programming. Due to these extensive attentions, generalization of convexity has grown very rapidly. There has been several extensions and generalizations for classical convexity. Hanson [16] generalized the convex functions to introduce the notion of invexity. Since, Dini derivatives of these type of functions play a significant role, hence recently Yuan and Liu [22] gave some new generalized convexities, using right upper-Dini-derivatives. For more contributions, see [3,4,8,12,15].

Vector variational inequalities and their generalizations have sought more attentions in various real world problems related to physics, mechanics and fluid dynamics because these problems can be transformed into variational inequalities. Initially, the formulation of vector variational inequality was introduced by Giannessi [13]. There are several interesting and important topics, for instance, existence results of vector variational inequalities, which ensure the existence of efficient solutions of vector optimization problems and relationships between both problems. In order to do so, a sizable number of researchers have been attracted towards this directions, see [1,2,6,7,9,10,13,17–19,21].

Motivated by above research works, we present our paper, in which we introduce Stampacchia and Minty arcwise connected vector variational-like inequality with also its weak formulations. We deduce the relations between the solutions of introduced inequalities and vector optimization problems, using (α, ρ)-right upper-Dini derivative locally arcwise connected functions. Further, we also establish the existence results of introduced inequalities, by using KKM-Fan theorem and Brouwer fixed point theorem. The structure of this paper is as follows: Sect. 2 is concerned with some preliminaries, definitions and lemmas, which are applicable in proving our results. Some relationships of arcwise connected vector variational-like inequalities with vector optimization problems are derived in Sect. 3. Furthermore, results for existence of solutions of introduced inequalities, involving monotonicity and without monotonicity are established in Sects. 4 and 5, respectively. Eventually, Sect. 6 concludes our paper.

2 Notations and preliminaries

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, $X \subset \mathbb{R}^n$ and the continuous function $H_{u,x}\colon [0, 1] \mapsto X$ be an arc joining the points $u, x \in X$. For $x, y \in X$, we use the following convention for equalities and inequalities, throughout this paper.

(a) $x \leq y \Leftrightarrow x_i \leq y_i,\ i = 1, \ldots, n$, with strict inequality holding for at least one i;
(b) $x \leqq y \Leftrightarrow x_i \leq y_i,\ i = 1, \ldots, n$;
(c) $x = y \Leftrightarrow x_i = y_i,\ i = 1, \ldots, n$;
(d) $x < y \Leftrightarrow x_i < y_i,\ i = 1, \ldots, n$.

First of all, we recall some known definitions, which will be applicable in sequel of the paper.

Definition 2.1 [12] A set X is said to be arcwise connected, if for any $x, u \in X$, there exists a continuous function $H_{u,x}(t) \in X$ for $t \in (0, 1)$ such that

$$H_{u,x}(0) = u \quad \text{and} \quad H_{u,x}(1) = x.$$

Definition 2.2 [22] A set X is said to be locally arcwise connected at u if for any $x \in X$ and $u \neq x$ there exists a positive number $a(x,u)$, with $0 < a(x,u) \leq 1$ and a continuous function $H_{u,x}$ such that $H_{u,x}(t) \in X$ for any $t \in (0, a(x,u))$ and $H_{u,x}(0) = u$. The set X is locally arcwise connected, if X is locally arcwise connected at any $u \in X$.

Definition 2.3 [22] Let X be arc wise connected set. The right upper-Dini-derivative of the function $h: X \mapsto \mathbb{R}$ with respect to $H_{u,x}(t)$ at $t = 0$ is defined as follows:

$$(dh)^{+}(H_{u,x}(0+)) = \limsup_{t \to 0^{+}} \left(\frac{h(H_{u,x}(t)) - h(u)}{t} \right).$$

Similarly, right lower-Dini-derivative of the function $h: X \mapsto \mathbb{R}$ with respect to $H_{u,x}(t)$ at $t = 0$ is defined as follows:

$$(dh)_{+}(H_{u,x}(0+)) = \liminf_{t \to 0^{+}} \left(\frac{h(H_{u,x}(t)) - h(u)}{t} \right).$$

Remark 2.1 It is easy to see that $(dh)_{+}(H_{u,x}(0+)) \leq (dh)^{+}(H_{u,x}(0+))$.

From now onwards, we assume that X is nonempty locally arcwise connected set, $h: X \mapsto \mathbb{R}$ and $\alpha,\ \rho: X \times X \mapsto \mathbb{R}$ are real valued functions unless otherwise specified.

Definition 2.4 [22] A function h is said to be (strictly) (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at $u \in X$, if there exist real valued functions α and ρ such that

$$h(x) - h(u)(>) \geq (dh)^{+}(H_{u,x}(0+))\alpha(x,u) + \rho(x,u), \quad \forall\, x \in X,\ (x \neq u).$$

If h is (strictly) (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at for all $u \in X$, then h is called (strictly) (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ on X.

Special cases. For spacial cases, see Remark 2.4 in [22].

Definition 2.5 A function h is said to be pseudo (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at $u \in X$, if there exist real valued functions α and ρ such that

$$(dh)^{+}(H_{u,x}(0+))\alpha(x,u) + \rho(x,u) \geq 0 \Rightarrow h(x) - h(u) \geq 0, \quad \forall\, x \in X.$$

Equivalently,

$$h(x) - h(u) < 0 \Rightarrow (dh)^{+}(H_{u,x}(0+))\alpha(x,u) + \rho(x,u) < 0, \quad \forall\, x \in X.$$

If h is pseudo (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at for all $u \in X$, then h is called pseudo (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ on X.

By keeping the view of definitions of monotonicites given by Al-Homidan et al. [3], we introduce the monotonicity of right upper-Dini-derivative of a real valued function, which will be helpful in proving results.

Definition 2.6 The right upper-Dini-derivative of function h is called monotone on X, if there exist real valued functions α and ρ such that

$$(dh)^{+}(H_{u,x}(0+))\alpha(x,u)+\rho(x,u)$$
$$+(dh)^{+}(H_{x,u}(0+))\alpha(u,x)+\rho(u,x)\leq 0,\quad \forall\, x,u\in X.$$

Consider the following vector optimization problem:

$$\textbf{(VOP)}\qquad \text{Minimize}\quad f(x)=\big(f_1(x), f_2(x),\ldots,f_p(x)\big)$$
$$\text{subject to}\quad x\in X,$$

where $f_i\colon X\ \mapsto \mathbb{R},\ i\in P=\{1,2,\ldots,p\}$.

Definition 2.7 A point $u\in X$ is said to be an efficient solution of (VOP), if there exists no $x\in X$ such that

$$f_i(x)-f_i(u)\leq 0,\quad \forall\, i\in P,$$

with strict inequality for at least one i.

Definition 2.8 A point $u\in X$ is said to be a weak efficient solution of (VOP), if there exists no $x\in X$ such that

$$f_i(x)-f_i(u)<0,\quad \forall\, i\in P.$$

Definition 2.9 [9] Let E be a nonempty subset of a topological vector space Y. A multifunction $\psi\colon E\mapsto 2^Y$ is a KKM mapping, if for any finite subset $\{x_1,x_2\ldots,x_n\}$ of E,

$$co\{x_1,x_2,\ldots,x_n\}\subset\bigcup_{j=1}^{n}\psi(x_j),$$

where $co\{x_1,x_2,\ldots,x_n\}$ denotes the convex hull of $\{x_1,x_2,\ldots,x_n\}$.

From the topological vector space, we recollect the following renowned concept of partition of unity subordinate to an open cover.

Definition 2.10 Let $K\subset X$ be a topological vector space and $N=\{N_{y_1},N_{y_2},\ldots,N_{y_n}\}$ be a finite open cover of K. A partition of unity subordinate to N is a family of smooth functions $\beta_j\colon K\mapsto[0,1],\ \forall\, j\in\{1,2,\ldots,n\}$, satisfying

$$\sum_{j=1}^{n}\beta_j(x)=1\quad\text{and}\quad \text{supp}(\beta_j)\subset N_{y_j},$$

where $\text{supp}(\beta_j)$ stands for support of function β_j and defined as $\text{supp}(\beta_j)=\{x\in K:\beta_j(x)\neq 0\}$.

Remark 2.2 It is well known that there exists a partition of unity subordinate to any open cover.

Lemma 2.1 (KKM-Fan theorem) [9] *Let E be a nonempty convex subset of a Hausdorff topological vector space Y and let $\psi\colon E\mapsto 2^Y$ be a KKM mapping with closed values. If there is a point $x_0\in E$ such that $\psi(x_0)$ is compact, then $\bigcap_{x\in E}\psi(x)\neq\phi$.*

Lemma 2.2 (Brouwer fixed point theorem) [9] *Let K be a nonempty compact convex subset of R^n and let $\phi\colon K\mapsto K$ be a continuous function. Then ϕ has a fixed point, i.e., there exists $x\in K$ such that $\phi(x)=x$.*

Now, we are in the position to introduce the following Stampacchia and Minty arcwise connected vector variational-like inequalities, respectively, with also their weak formulations, which will be used to ensure the existence of efficient solutions of considered vector optimization problem (VOP) with nondifferentiable objective function in sequel of the paper. Let $\rho_i : X \times X \mapsto \mathbb{R}, i \in P = \{1, 2, \ldots, p\}$.

(SAVVLI) For given functions α and ρ_i, $i \in P$, find $u \in X$ such that there exists no $x \in X$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) \leq 0.$$

(MAVVLI) For given functions α and ρ_i, $i \in P$, find $u \in X$ such that there exists no $x \in X$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)) \geq 0.$$

(WSAVVLI) For given functions α and ρ_i, $i \in P$, find $u \in X$ such that there exists no $x \in X$, satisfying

$$\left((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)\right) < 0.$$

(WMAVVLI) For given functions α and ρ_i, $i \in P$, find $u \in X$ such that there exists no $x \in X$, satisfying

$$\left((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)\right) > 0.$$

Special cases:

(i) If $\alpha = 1$, $\rho_i = 0$, for all $i \in P$ and $H_{u,x}(t) = u + t(x-u)$ in (SAVVLI) and (MAVVLI) (respectively (WSAVVLI) and (WMAVVLI)), then these problems reduce to Stampacchia and Minty (respectively weak) vector variational inequality, treated by Ansari and Lee [5] and if, right upper-Dini-derivative is replaced by right upper derivative of f in (WSAVVLI), then this problem reduces to vector variational-like inequality, defined by Fu and Wang [12].

(ii) If $\alpha \in \mathbb{R}_+\backslash\{0\}$, $\rho_i = 0$, for all $i \in P$, $H_{u,x}(t) = u + t\eta(x,u)$, where $\eta : X \times X \mapsto \mathbb{R}^n$ and right upper-Dini-derivative is replaced by directional derivative of f in (SAVVLI) (respectively (WSAVVLI)), then this problem reduces to vector variational inequality (respectively weak), introduced by Farajzadeh and Lee [10].

Following example shows that, there exists a solution of (SAVVLI) but Stampacchia vector variational inequality (SVVI), introduced by Ansari and Lee [5] is not solvable at that point.

Example 2.1 Consider the functions $f : \mathbb{R} \mapsto \mathbb{R}^2$, $\alpha : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ and $\rho_i : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$, $i = \{1, 2\}$ defined by

$$f(x) = (f_1(x), f_2(x)),\ \alpha(x,u) = (x-u)\sin^2(x-u)$$

and

$$(\rho_1(x,u), \rho_2(x,u)) = \left((x-u)^2\cos^2(x-u), 20(x-u)^2\sin^2(x-u)\right),$$

respectively, where

$$f_1(x) = \begin{cases} 2 - e^x, x \geq 0 \\ 1 + x^2, x < 0 \end{cases}, \quad f_2(x) = -10x.$$

Further, define $H_{u,x}(t) = tx + (1-t)u, \ \forall\, t \in [0,1]$. Then, by definition of right upper-Dini-derivative of f at $u = 0$, we obtain

$$(df_1)^+(H_{u,x}(0+)) = \max\{-x, 0\} \quad \text{and} \quad (df_2)^+(H_{u,x}(0+)) = -10x.$$

Now, for $u = 0$, we have

$$\begin{aligned}&((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), (df_2)^+(H_{u,x}(0+))\alpha(x,u) + \rho_2(x,u))\\ &\quad = (\max\{-x, 0\}x \sin^2 x + x^2 \cos^2 x, 10x^2 \sin^2 x).\end{aligned}$$

From above, it is clear that for $u = 0$, there exists no $x \in \mathbb{R}$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), (df_2)^+(H_{u,x}(0+))\alpha(x,u) + \rho_2(x,u)) \le 0.$$

Therefore, $u = 0$ is a solution of (SAVVLI). Further, for $u = 0$ we attain

$$((df_1)^+(H_{u,x}(0+)), (df_2)^+(H_{u,x}(0+))) = (\max\{-x, 0\}, -10x) \le 0, \quad \forall\, x \in \mathbb{R}_+\backslash\{0\},$$

i.e., for $u = 0$ there exists $x \in \mathbb{R}_+\backslash\{0\}$ such that the following implication holds

$$\big((df_1)^+(H_{u,x}(0+)), (df_2)^+(H_{u,x}(0+))\big) \le 0.$$

Hence, (SVVI) is not solvable at $u = 0$.
Following example enables us to give a solution of (MAVVLI) but does not give the solution of Minty vector variational inequality (MVVI) at the same point, introduced by Ansari and Lee [5].

Example 2.2 Consider the functions $f : \mathbb{R} \mapsto \mathbb{R}^2$, $\alpha : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ and $\rho_i : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$, $i = \{1, 2\}$ defined by

$$f(x) = (f_1(x), f_2(x)), \quad \alpha(u,x) = (x-u)$$

and

$$(\rho_1(u,x), \rho_2(u,x)) = ((x-u)^2 e^x, -10(x-u)^2),$$

respectively, where

$$f_1(x) = \begin{cases} e^x - 2, x > 0 \\ x - e^x, x \le 0 \end{cases}, \quad f_2(x) = 10x.$$

Further, define $H_{x,u}(t) = tu + (1-t)x, \ \forall\, t \in [0,1]$. Then, by definition of right upper-Dini-derivative of f at $u = 0$, we obtain

$$(df_1)^+(H_{x,u}(0+)) = \max\{-xe^x,\ x(e^x - 1)\} \quad \text{and} \quad (df_2)^+(H_{x,u}(0+)) = -10x.$$

Now, for $u = 0$, we have

$$\begin{aligned}&((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), (df_2)^+(H_{x,u}(0+))\alpha(u,x) + \rho_2(u,x))\\ &\quad = (\max\{-xe^x,\ x(e^x - 1)\}x + x^2 e^x, -20x^2).\end{aligned}$$

From above, it is clear that for $u = 0$, there exists no $x \in \mathbb{R}$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), (df_2)^+(H_{x,u}(0+))\alpha(u,x) + \rho_2(u,x)) \ge 0.$$

Therefore, $u = 0$ is a solution of (MAVVLI). Further, for $u = 0$ there exists $x \in \mathbb{R}_-\backslash\{0\}$ such that following implication holds

$$((df_1)^+(H_{x,u}(0+)), (df_2)^+(H_{x,u}(0+))) \ge 0.$$

Hence, (MVVI) is not solvable at $u = 0$.

3 Relationships between arcwise connected vector variational-like inequalities and vector optimization problems

Al-Homidan and Ansari [1,2] derived relationships among Stampacchia, Minty vector variational-like inequalities and vector optimization problems for nondifferentiable functions, involving Dini upper subdifferential and Clarke's generalized subdifferential, respectively. In continuation of these, we shall study the relationships between Stampacchia and Minty arcwise connected vector variational-like inequalities and considered vector optimization problems, in the setting of (α, ρ)-right upper-Dini-derivative locally arcwise connected functions.

Theorem 3.1 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R}$, $i \in P$ be given real valued functions. Assume that, for each i, function f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at $u \in X$. If u solves (SAVVLI), then u is an efficient solution of (VOP).*

Proof Suppose contrary to the hypothesis that u is not an efficient solution of (VOP), then there exists $x \in X$ such that

$$f_i(x) - f_i(u) \leq 0, \quad \forall\, i \in P, \tag{1}$$

with strict inequality for at least one i. Since, each f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected function with respect to $H_{u,x}$ at $u \in X$, it ensures that

$$f_i(x) - f_i(u) \geq (df_i)^+(H_{u,x}(0+))\alpha(x, u) + \rho_i(x, u), \quad \forall\, x \in X \text{ and } i \in P. \tag{2}$$

On combining inequalities (1) and (2), it follows that, there exists $x \in X$ such that

$$(df_i)^+(H_{u,x}(0+))\alpha(x, u) + \rho_i(x, u) \leq 0, \quad \forall\, i \in P,$$

with strict inequality for at least one i, i.e., there exists $x \in X$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x, u) + \rho_1(x, u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x, u) + \rho_p(x, u)) \leq 0,$$

which leads to a contradiction, that u solves (SAVVLI). Hence the theorem is complete. □

We present the following example to illustrate the result established in the above theorem.

Example 3.1 Consider the functions $f : \mathbb{R} \mapsto \mathbb{R}^2$, $\alpha : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ and $\rho_i : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$, $i = \{1, 2\}$ defined by

$$f(x) = (f_1(x), f_2(x)), \quad \alpha(x, u) = -\frac{1}{2}(x - u) \quad \text{and}$$
$$(\rho_1(x, u), \rho_2(x, u)) = ((x - u)^2, 10(x - u)),$$

respectively, where

$$f_1(x) = \begin{cases} -\frac{1}{2} + e^x, x \geq 0 \\ \frac{1}{2} + x^2, x < 0 \end{cases}, \quad f_2(x) = 10x.$$

Further, define $H_{u,x}(t) = tx + (1 - t)u$, $\forall\, t \in [0, 1]$. Then, by definition of right upper-Dini-derivative of f with respect to $H_{u,x}$ at $u = 0$, we obtain

$$(df_1)^+(H_{u,x}(0+)) = \max\{x, 0\} \quad \text{and} \quad (df_2)^+(H_{u,x}(0+)) = 10x.$$

It can be easily verified that, for $i = \{1, 2\}$, each function f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected function with respect to $H_{u,x}$ at $u = 0$. Further, for $u = 0$, we have

$$\begin{aligned}&((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), (df_2)^+(H_{u,x}(0+))\alpha(x,u) + \rho_2(x,u))\\ &\quad = \left(-\frac{x}{2}\max\{x, 0\} + x^2, 5x(2-x)\right).\end{aligned}$$

From above, it is clear that for $u = 0$, there exists no $x \in \mathbb{R}$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), (df_2)^+(H_{u,x}(0+))\alpha(x,u) + \rho_2(x,u)) \leq 0.$$

Therefore, $u = 0$ is a solution of (SAVVLI). Moreover, for $u = 0$, there exists no $x \in \mathbb{R}$ such that $f(x) - f(u) \leq 0$. Hence, $u = 0$ is an efficient solution of (VOP).

Theorem 3.2 *Let* $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ *be given real valued functions with* $\alpha \in \mathbb{R}_+ \backslash \{0\}$ *and* $\rho_i \in \mathbb{R}_+$. *If* $u \in X$ *is a weak efficient solution of* (VOP), *then* u *solves* (WSAVVLI).

Proof Suppose u is a weak efficient solution of (VOP), then it follows that for all $x \in X$

$$(f_1(x) - f_1(u), \ldots, f_p(x) - f_p(u)) \not< 0. \tag{3}$$

Since X is a locally arcwise connected set, therefore there exists a positive number $a(x, u)$ with $0 < a(x, u) \leq 1$ and a continuous function $H_{u,x}$ such that $H_{u,x}(t) \in X$ for all $t \in (0, a(x, u))$ and $H_{u,x}(0) = u$. Thus, inequality (3) can be rewritten as

$$\left(f_1(H_{u,x}(t)) - f_1(u), \ldots, f_p(H_{u,x}(t)) - f_p(u)\right) \not< 0, \quad \forall\, x \in X. \tag{4}$$

On dividing above inequality by t and taking limit superior as $t \to 0^+$, we attain

$$\left((df_1)^+(H_{u,x}(0+)), \ldots, (df_p)^+(H_{u,x}(0+))\right) \not< 0, \quad \forall\, x \in X,$$

and together with hypothesis, we can write

$$\begin{aligned}&((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) \not< 0,\\ &\quad \forall\, x \in X.\end{aligned}$$

Therefore, u is a solution of (WSAVVLI). Hence the theorem is complete. □

Theorem 3.3 *Let* $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R}, i \in P$ *be given real valued functions. Assume that, for each* i, *function* f_i *is pseudo* (α, ρ_i)*-right upper-Dini-derivative locally arcwise connected with respect to* $H_{u,x}$ *at* u. *If* u *solves* (WSAVVLI), *then* u *is a weak efficient solution of* (VOP).

Proof Suppose contrary to the hypothesis that u is not a weak efficient solution of (VOP), then there exists $x \in X$ such that

$$f_i(x) - f_i(u) < 0, \quad \forall\, i \in P.$$

Since, each f_i is pseudo (α, ρ_i)-right upper-Dini-derivative locally arcwise connected function with respect to $H_{u,x}$ at u, therefore, for $x \in X$ the following implication holds

$$(df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u) < 0, \quad \forall\, i \in P.$$

Above inequality can be rewritten as, there exists $x \in X$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) < 0,$$

which leads to a contradiction, that u solves (WSAVVLI). Hence the theorem is complete. □

Following theorem enables us to give the condition, under which a weak efficient solution will be an efficient solution of (VOP).

Theorem 3.4 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R}$, $i \in P$ be given real valued functions. Assume that, for each i, function f_i is strictly (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{u,x}$ at $u \in X$ and $\alpha \in \mathbb{R}_+ \backslash \{0\}$, $\rho_i \in \mathbb{R}_+$. If u is a weak efficient solution of* (VOP), *then u is an efficient solution of* (VOP).

Proof Suppose u is a weak efficient solution but not an efficient solution of (VOP), then there exists $x \in X$ such that

$$f_i(x) - f_i(u) \leq 0, \quad \forall\, i \in P, \tag{5}$$

with strict inequality for at least one i. Now, by using strict (α, ρ_i)-right upper-Dini-derivative locally arcwise connectivity of each function f_i with respect to $H_{u,x}$ at u, we get

$$f_i(x) - f_i(u) > (df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u), \quad x \neq u,\ \forall\, x \in X \text{ and } i \in P. \tag{6}$$

On combining inequalities (5) and (6), it follows that there exists $x \in X$ such that

$$(df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u) < 0, \quad i \in P.$$

Therefore, u is not a solution of (WSAVVLI). By Theorem 3.2, it follows that u is not a weak efficient solution of (VOP), which leads to a contradiction. Hence the theorem is complete. □

Theorem 3.5 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R}$, $i \in P$ be given real valued functions. Assume that, for each i, function f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X. If $u \in X$ is an efficient solution of* (VOP), *then u solves* (MAVVLI).

Proof Suppose, contrary to the result, that u does not solve (MAVVLI), then there exists $x \in X$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)) \geq 0.$$

Above inequality can be rewritten as, there exists $x \in X$ such that

$$(df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x) \geq 0, \quad \forall\, i \in P, \tag{7}$$

with strict inequality for at least one i. Since, each f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X, therefore we obtain

$$f_i(u) - f_i(x) \geq (df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x), \quad \forall\, x, u \in X \text{ and } i \in P. \tag{8}$$

On combining inequalities (7) and (8), it follows that, there exists $x \in X$, satisfying

$$f_i(u) - f_i(x) \geq 0, \quad \forall\, i \in P,$$

with strict inequality for at least one i, which leads to a contradiction, that u is an efficient solution of (VOP). Hence the theorem is complete. □

Theorem 3.6 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R}$, $i \in P$ be real valued functions. Assume that, for each i, function f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X. If $u \in X$ solves* (WSAVVLI), *then u solves* (WMAVVLI).

Proof Let u be a solution of (WSAVVLI), then there exists no $x \in X$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) < 0.$$

Above inequality can be rewritten as, there exists no $x \in X$, such that

$$(df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u) < 0, \quad \forall\, i \in P. \tag{9}$$

Since, each f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X, therefore we get

$$f_i(u) - f_i(x) \geq (df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x), \quad \forall\, x, u \in X \text{ and } i \in P. \tag{10}$$

On interchanging x and u in inequality (10), we have

$$f_i(x) - f_i(u) \geq (df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u), \quad \forall\, x, u \in X \text{ and } i \in P. \tag{11}$$

Now, by adding inequalities (10) and (11), we ensure that, for all $x, u \in X$ and $i \in P$, the following implication holds

$$(df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x) \leq -\left[(df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u)\right]. \tag{12}$$

On combining inequalities (9) and (12), it follows that, there exists no $x \in X$, such that

$$(df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x) > 0, \quad \forall\, i \in P,$$

i.e., there exists no $x \in X$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)) > 0,$$

which implies that, u solves (WMAVVLI). Hence the theorem is complete. □

Theorem 3.7 *Let α, $\rho_i : X \times X \mapsto \mathbb{R}$, $i \in P$ be real valued functions. Assume that, for each i, function f_i is strictly (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X. If $u \in X$ is a weak efficient solution of* (VOP), *then u solves* (MAVVLI).

Proof Suppose u is a weak efficient solution of (VOP) but does not solve (MAVVLI), then there exists $x \in X$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)) \geq 0,$$

i.e., there exists $x \in X$ such that

$$(df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x) \geq 0, \quad \forall\, i \in P, \tag{13}$$

with strict inequality for at least one i. Since, each f_i is strictly (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X, therefore, we get

$$f_i(u) - f_i(x) > (df_i)^+(H_{x,u}(0+))\alpha(u,x) + \rho_i(u,x), \quad x \neq u,\ \forall\, x, u \in X \text{ and } i \in P. \tag{14}$$

On combining inequalities (13) and (14), it follows that, there exists $x \in X$ such that

$$f_i(u) - f_i(x) > 0, \quad \forall\, i \in P,$$

which leads to a contradiction, that u is a weak efficient solution of (VOP). Hence the theorem is complete. □

Theorem 3.8 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ be given real valued functions. Assume that, for each i, function f_i is (α, ρ_i)-right upper-Dini-derivative locally arcwise connected with respect to $H_{x,u}$ on X. If $u \in X$ is a weak efficient solution of* (VOP), *then u solves* (WMAVVLI).

Proof The proof follows in the similar lines of Theorem 3.5 and hence being omitted.

4 Existence of solutions of arcwise connected vector variational-like inequalities with monotonicity

In this section, we turn our attention towards the existence of solutions of Stampacchia and Minty arcwise connected vector variational-like inequalities, indulging the concept of monotonicity and KKM-Fan theorem. Let K be a nonempty convex Hausdorff vector subset of X and $f_i : K \mapsto \mathbb{R},\ i \in P$.

Theorem 4.1 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ be given real valued functions. Assume that*

(i) *for each $i \in P$, $(d(-f_i))^+$ is monotone on K,*
(ii) *$\alpha \in \mathbb{R}_+\backslash\{0\}$ and $\rho_i \in \mathbb{R}_+$,*
(iii) *α and ρ_i are affine functions with respect to its second argument such that $\alpha(x, x) = 0$ and $\rho_i(x, x) = 0,\ \forall\, x \in K$,*
(iv) *for all $x \in K$ the set-valued map $\Gamma : K \mapsto 2^K$ defined by*

$$\Gamma(x) = \big\{u \in K : \big((df_1)^+(H_{u,x}(0+))\alpha(x, u) + \rho_1(x, u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x, u) + \rho_p(x, u)\big) \nleq 0\big\}$$

is closed valued,
(v) *there exists a nonempty compact set $M \subset K$ and a nonempty compact convex set $N \subset K$ such that for each $y \in K\backslash M$, there exists $x \in N$ such that $y \notin \Gamma(x)$.*

Then (SAVVLI) *has a solution in K.*

Proof Define a set-valued map $\widehat{\Gamma}(x) : K \mapsto 2^K$, for all $x \in K$ as

$$\widehat{\Gamma}(x) = \{u \in K : ((df_1)^+(H_{x,u}(0+))\alpha(u, x) + \rho_1(u, x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u, x) + \rho_p(u, x)) \ngeq 0\}.$$

It is clear that $x \in \Gamma(x) \cap \widehat{\Gamma}(x)$, therefore $\Gamma(x)$ and $\widehat{\Gamma}(x)$ are nonempty. Firstly, we shall show that $\widehat{\Gamma}$ is a KKM map on K. For this suppose, contradiction that $\widehat{\Gamma}$ is not a KKM map. Then, it follows that, there exists $\{x_1, x_2 \ldots, x_n\} \subset K$, $t_j \geq 0,\ j = \{1, 2, \ldots, n\}$ with $\sum_{j=1}^{n} t_j = 1$ such that

$$u = \sum_{j=1}^{n} t_j x_j \notin \bigcup_{j=1}^{n} \widehat{\Gamma}(x_j).$$

Hence, for any $j = \{1, 2, \ldots, n\}$

$$\big((df_1)^+(H_{x_j,u}(0+))\alpha(u, x_j) + \rho_1(u, x_j), \ldots, (df_p)^+(H_{x_j,u}(0+))\alpha(u, x_j) + \rho_p(u, x_j)\big) \geq 0.$$

Above inequality can be rewritten as

$$(df_i)^+(H_{x_j,u}(0+))\alpha(u,x_j)+\rho_i(u,x_j)\geq 0,\quad \forall\, i\in P,$$

with strict inequality for at least one i.

For all $i\in P$, $j=\{1,2,\ldots,n\}$, we have

$$\begin{aligned}0&=(df_i)^+(H_{x_j,u}(0+))\alpha(u,u)+\rho_i(u,u)\\&=(df_i)^+(H_{x_j,u}(0+))\alpha\left(u,\sum_{j=1}^{n}t_jx_j\right)+\rho_i\left(u,\sum_{j=1}^{n}t_jx_j\right)\\&=\sum_{j=1}^{n}t_j[(df_i)^+(H_{x_j,u}(0+))\alpha(u,x_j)+\rho_i(u,x_j)]\\&\geq 0,\quad \text{with strict inequality for at least one } i,\end{aligned}$$

which leads to a contradiction. Hence, $\widehat{\Gamma}$ is a KKM map. Further, we have to show that $\widehat{\Gamma}(x)\subset\Gamma(x),\ \forall\, x\in K$. We proceed by letting $u\notin\Gamma(x)$, then there exists $x\in K$ such that

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u)+\rho_1(x,u),\ldots,(df_p)^+(H_{u,x}(0+))\alpha(x,u)+\rho_p(x,u))\leq 0.$$

Above inequality can be rewritten as, there exists $x\in K$ such that

$$(df_i)^+(H_{u,x}(0+))\alpha(x,u)+\rho_i(x,u)\leq 0,\quad \forall\, i\in P, \tag{15}$$

with strict inequality for at least one i. Since, each $(d(-f_i))^+$ is monotone on K, therefore for all $i\in P$ and $x,u\in K$, one has

$$\begin{aligned}&(df_i)_+(H_{u,x}(0+))\alpha(x,u)+\rho_i(x,u)\\&\quad\geq-(df_i)_+(H_{x,u}(0+))\alpha(u,x)+\rho_i(u,x)+2\rho_i(x,u).\end{aligned} \tag{16}$$

By using Remark 2.1 and hypothesis (ii), we conclude that

$$\begin{aligned}&(df_i)_+(H_{u,x}(0+))\alpha(x,u)+\rho_i(x,u)\\&\quad\leq(df_i)^+(H_{u,x}(0+))\alpha(x,u)+\rho_i(x,u),\quad \forall\, x\in K \text{ and } i\in P.\end{aligned} \tag{17}$$

On combining inequalities (15) and (17), it follows that there exists $x\in K$ such that

$$(df_i)_+(H_{u,x}(0+))\alpha(x,u)+\rho_i(x,u)\leq 0,\quad \forall\, i\in P, \tag{18}$$

with strict inequality for at least one i. Now, inequalities (16) and (18) implies that there exists $x\in K$ such that

$$(df_i)_+(H_{x,u}(0+))\alpha(u,x)+\rho_i(u,x)\geq 0,\quad \forall\, i\in P,$$

with strict inequality for at least one i. Together with Remark 2.1 and hypothesis (ii), we can write

$$(df_i)^+(H_{x,u}(0+))\alpha(u,x)+\rho_i(u,x)\geq 0,\quad \forall\, i\in P,$$

with strict inequality for at least one i, i.e., there exists $x\in X$, satisfying

$$((df_1)^+(H_{x,u}(0+))\alpha(u,x)+\rho_1(u,x),\ldots,(df_p)^+(H_{x,u}(0+))\alpha(u,x)+\rho_p(u,x))\geq 0.$$

Therefore, $u \notin \widehat{\Gamma}(x)$, it follows that $\widehat{\Gamma}(x) \subset \Gamma(x),\ \forall\, x \in K$. Hence, Γ is also a KKM map. By hypothesis (iv) and (v), $\Gamma(x)$ is closed subset of compact set. So $\Gamma(x)$ is also compact. Now, by using Lemma 2.1, we get

$$\bigcap_{x\in K} \Gamma(x) \neq \phi,$$

which implies that there exists $u \in K$ such that

$$\big((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, \\ (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)\big) \nleq 0, \quad \forall\, x \in K.$$

Therefore, u is a solution of (SAVVLI). Hence the theorem is complete. □

Theorem 4.2 *Let* $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ *be given real valued functions. Assume that*

(i) *for each* $i \in P$, $(df_i)^+$ *be monotone on* K,
(ii) α *and* ρ_i *are affine functions with respect to its first arguments such that* $\alpha(x,x) = 0$ *and* $\rho_i(x,x) = 0,\ \forall\, x \in K$,
(iii) *for all* $x \in K$ *define the set-valued map* $\Gamma : K \mapsto 2^K$ *as*

$$\Gamma(x) = \{u \in K : ((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, \\ (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)) \ngeq 0\}$$

is closed valued,
(iv) *there exists a nonempty compact set* $M \subset K$ *and a nonempty compact convex set* $N \subset K$ *such that for each* $y \in K\backslash M$, *there exists* $x \in N$ *such that* $y \notin \Gamma(x)$.

Then (MAVVLI) *has a solution in* K.

Proof The proof follows in the similar lines of Theorem 4.1 and hence being omitted.

5 Existence of solutions of arcwise connected vector variational-like inequalities without monotonicity

In this section, we shall derive the existence results of Stampacchia and Minty arcwise connected vector variational-like inequalities, using Brouwer fixed point Theorem. Let K be a nonempty compact convex subset of X and $f_i : K \mapsto \mathbb{R},\ i \in P$.

Theorem 5.1 *Let* $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ *be given real valued functions. Assume that*

(i) α *and* ρ_i *are affine functions with respect to its first arguments such that* $\alpha(x,x) = 0$ *and* $\rho_i(x,x) = 0,\ \forall\, x \in K$,
(ii) *for all* $x \in K$ *the set* N_x *defined by*

$$N_x = \{u \in K : ((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, \\ (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) \leq 0\}$$

is open in K.

Then (SAVVLI) *has a solution in* K.

Proof Suppose contrary to the hypothesis that (SAVVLI) is not solvable, then there exists $x \in K$, satisfying

$$((df_1)^+(H_{u,x}(0+))\alpha(x,u) + \rho_1(x,u), \ldots, (df_p)^+(H_{u,x}(0+))\alpha(x,u) + \rho_p(x,u)) \leq 0.$$

Above inequality can be rewritten as, there exists $x \in K$ such that

$$(df_i)^+(H_{u,x}(0+))\alpha(x,u) + \rho_i(x,u) \leq 0, \quad \forall\, i \in P, \tag{19}$$

with strict inequality for at least one i. Now, we shall show that $K = \bigcup_{x\in K} N_x$. By the assumption (ii), it is obvious that

$$\bigcup_{x\in K} N_x \subset K. \tag{20}$$

Conversely, let $u \in K$, then by inequality (19), $u \in N_x$, for some $x \in K$. Therefore, $u \in \bigcup_{x\in K} N_x$, thus it follows that

$$K \subset \bigcup_{x\in K} N_x. \tag{21}$$

On combining (20) and (21), we have

$$K = \bigcup_{x\in K} N_x. \tag{22}$$

By the hypothesis (ii), the set $\{N_x : x \in K\}$ is open in K, therefore from (22), we can say that the set $\{N_x : x \in K\}$ is an open cover of K. Since, K is compact, it follows that there exists a finite set $\{y_1, y_2, \ldots, y_n\} \subset K$ such that

$$K = \bigcup_{j=1}^{n} N_{y_j}.$$

Obviously, $\{N_{y_1}, N_{y_2}, \ldots, N_{y_n}\}$ is a finite open cover, Definition 2.10 and Remark 2.2 yields that there exists a family of functions $\{\beta_1, \beta_2, \ldots, \beta n\}$ with the following properties:

(a) for each j, $\beta_j : K \mapsto [0, 1]$ is continuous with respect to the weak topology of X,

(b) $\beta_j(x) = \begin{cases} 0, x \notin N_{y_j} \\ > 0, x \in N_{y_j} \end{cases}$

(c) $\sum_{j=1}^{n} \beta_j(x) = 1, \ \forall\, x \in K.$

Let $\phi^\circ : K \mapsto X$ be defined as follows

$$\phi^\circ(x) = \sum_{j=1}^{n} \beta_j(x) y_j, \quad \forall\, x \in K.$$

Since, β_j is continuous with respect to the weak topology on X, then ϕ° is also continuous with respect to the weak topology on X. Let $S = conv\{y_1, y_2, \ldots, y_n\} \subset K$. Then S is compact convex subset of a finite dimensional space and ϕ° maps S into S. By using the Brouwer fixed point theorem, there exists $x_0 \in S$ such that $\phi^\circ(x_0) = x_0$,

$$\text{i.e.,}\ \ x_0 = \phi^\circ(x_0) = \sum_{j=1}^{n} \beta_j(x_0) y_j.$$

Let $x \in K$. Consider the nonempty set of natural number $K(x) = \{j \in N : x \in N_{y_j}\}$.

Since, for $j = 1, \ldots, n,\ j \in K(x_\circ)$ and for all $\quad i \in P$

$$\begin{aligned} 0 &= (df_i)^+(H_{x_\circ,y_j}(0+))\alpha(x_\circ,x_\circ) + \rho_i(x_\circ,x_\circ) \\ &= (df_i)^+(H_{x_\circ,y_j}(0+))\alpha\left(\sum_{j=1}^{n}\beta_j(x_0)y_j, x_\circ\right) + \rho_i\left(\sum_{j=1}^{n}\beta_j(x_0)y_j, x_\circ\right) \\ &= \sum_{j=1}^{n}\beta_j(x_\circ)\left[(df_i)^+(H_{x_\circ,y_j}(0+))\alpha(y_j,x_\circ) + \rho_i(y_j,x_\circ)\right] \\ &\le 0, \quad \text{with strict inequality for at least one } i, \end{aligned}$$

which leads to a contradiction. Hence (SAVVLI) is solvable in K. □

Theorem 5.2 *Let $\alpha,\ \rho_i : X \times X \mapsto \mathbb{R},\ i \in P$ be given real valued functions. Assume that*

(i) *α and ρ_i are affine functions with respect to its second arguments such that $\alpha(x,x) = 0$ and $\rho_i(x,x) = 0,\ \forall\, x \in K$,*
(ii) *for all $x \in K$, the set N_x defined by*

$$N_x = \big\{u \in K : \big((df_1)^+(H_{x,u}(0+))\alpha(u,x) + \rho_1(u,x), \ldots, (df_p)^+(H_{x,u}(0+))\alpha(u,x) + \rho_p(u,x)\big) \ge 0\big\}$$

is open in K.

Then (MAVVLI) *is solvable in K.*

Proof The proof follows in the similar lines of Theorem 5.1 and hence being omitted.

6 Conclusion

In the present paper, we have introduced (weak) Stampacchia and Minty arcwise connected vector variational-like inequalities. Moreover, by using the concepts of (α, ρ)-right upper-Dini-derivative locally arcwise connected functions, we have demonstrated the relationships between the solutions of introduced inequalities and vector optimization problems. Ultimately, we have dealt with the existence results of introduced inequalities, under the condition of monotonicity and also without monotonicity. Further, we can extend the results obtained in this paper on reflexive Banach space.

References

1. Al-Homidan, S., Ansari, Q.H.: Relations between generalized vector variational-like inequalities and vector optimization problems. Taiwan. J. Math. **16**, 987–998 (2012)
2. Al-Homidan, S., Ansari, Q.H.: Generalized Minty vector variational-like inequalities and vector optimization problems. J. Optim. Theory Appl. **144**, 1–11 (2010)
3. Al-Homidan, S., Ansari, Q.H., Yao, J.-C.: Nonsmooth invexities, invariant monotonicities and nonsmooth vector variational-like inequalities with applications to vector optimization. In: Ansari, Q.H., Yao, J.-C. (eds.) In Recent Developments in Vector Optimization, pp. 221–274. Springer, Berlin (2012)
4. Ansari, Q.H., Lalitha, C.S., Mehta, M.: Generalized Convexity, Nonsmooth Variational Inequalities and Nonsmooth Optimization. CRC Press, Taylor & Francis Group, Boca Raton (2014)
5. Ansari, Q.H., Lee, G.M.: Nonsmooth vector optimization problems and Minty vector variational inequalities. J. Optim. Theory Appl. **145**, 1–16 (2010)

6. Ansari, Q.H., Rezaie, M., Zafarani, J.: Generalized vector variational-like inequalities and vector optimization. J. Glob. Optim. **53**, 271–284 (2012)
7. Barbagallo, A., Pia, S.: Weighted variational inequalities in non-pivot Hilbert spaces with applications. Comput. Optim. Appl. **48**, 487–514 (2011)
8. Bhatia, D., Gupta, A., Arora, P.: Optimality via generalized approximate convexity and quasiefficiency. Optim. Lett. **7**, 127–135 (2013)
9. Ceng, L.C., Huang, S.: Existence theorems for generalized vector variational inequalities with a variable ordering relation. J. Glob. Optim. **46**, 521–535 (2010)
10. Farajzadeh, A.P., Lee, B.S.: Vector variational-like inequality problem and vector optimization problem. Appl. Math. Lett. **23**, 48–52 (2010)
11. Ferreira, O.P.: Dini derivative and a characterization for Lipschitz and convex functions on Riemannian manifolds. Nonlinear Anal. **68**, 1517–1528 (2008)
12. Fu, J.Y., Wang, Y.H.: Arcwise connected cone-convex functions and mathematical programming. J. Optim. Theory Appl. **118**, 339–352 (2003)
13. Giannessi, F.: Theorems of the alternative quadratic programs and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequalities and Complementarity Problems, pp. 151–186. Wiley, Chichester (1980)
14. Giles, J.R.: Approximate Fréchet subdifferentability of convex functions. N. Z. J. Math. **37**, 21–28 (2008)
15. Gupta, A., Mehra, A., Bhatia, D.: Approximate convexity in vector optimization. Bull. Austral. Math. Soc. **74**, 207–218 (2006)
16. Hanson, M.A.: On sufficiency of the Kuhn–Tucker conditions. J. Math. Anal. Appl. **80**, 545–550 (1981)
17. Lalitha, C.S., Mehta, M.: Vector variational inequalities with cone-pseudomonotone bifunctions. Optimization **54**, 327–338 (2005)
18. Nishimura, H., Ok, E.A.: Solvability of variational inequalities on Hilbert lattices. Math. Oper. Res. **37**, 608–625 (2012)
19. Noor, M.A., Noor, K.I., Yaqoob, H.: On general mixed variational inequalities. Acta Appl. Math. **110**, 227–246 (2010)
20. Roshchina, V.: Mordukhovich subdifferential of pointwise minimum of approximate convex functions. Optim. Methods Softw. **25**, 129–141 (2010)
21. Yang, X.M., Yang, X.Q.: Vector variational-like inequality with pseudoinvexity. Optimization **55**, 157–170 (2006)
22. Yuan, D., Liu, X.: Mathematical programming involving (α, ρ)-right upper-Dini-derivative functions. Filomat **27**, 899–908 (2013)