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A new convergence theorem for families of asymptotically nonexpansive maps and solution of variational inequality problem

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Abstract A new strong convergence theorem for approximation of common fixed points of family of uniformly asymptotically regular asymptotically nonexpansive mappings, which is also a unique solution of some variational inequality problem is proved in the framework of a real Banach space. The Theorem presented here extend, generalize and unify many recently announced results.

Keywords Asymptotically nonexpansive mappings · Accretive mappings · Uniformly convex Banach spaces

Mathematics Subject Classification 47H09 · 47J25

1 Introduction

Let E be a real Banach space and E^* be the dual space of E. A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a guage function if it is strictly increasing, continuous and $\varphi(0) = 0$. Let φ be a gauge function, a generalized duality mapping with respect to φ, $J_\varphi : E \to 2^{E^*}$ is defined by, $x \in E$,

$$J_\varphi x = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|\varphi(\|x\|), \|x^*\| = \varphi(\|x\|)\},$$

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where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between element of E and that of E^*. If $\varphi(t) = t$, then J_φ is simply called the normalized duality mapping and is denoted by J. For any $x \in E$, an element of $J_\varphi x$ is denoted by $j_\varphi(x)$.
Let $S(E) := \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The space E is said to have *Gâteaux differentiable norm* if for any $x \in S(E)$ the limit

$$\lim_{\lambda \to 0} \frac{\|x + \lambda y\| - \|x\|}{\lambda} \tag{1.1}$$

exists $\forall y \in S(E)$. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S(E)$, the limit (1.1) is attained uniformly for $x \in S(E)$.
If E has a uniformly Gâteaux differentiable, then $J_\varphi : E \to 2^{E^*}$ is uniformly continuous on bounded subsets of E from the strong topology of E to the weak* topology of E^*. All $L_p, \ell_p (1 < p < \infty)$ spaces has uniformly Gâteaux differentiable.

A mapping $T : E \to E$ is said to be L-*Lipschitz* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{forall } x, y \in E. \tag{1.2}$$

If in this case, (1.2) is satisfied with $L \in [0, 1)$, respectively $L = 1$, then the mapping T is called a *contraction,* respectively *nonexpansive.* A mapping $T : E \to E$ is called *asymptotically nonexpansive* if there exists a sequence $\{\rho_n\} \subset [1, \infty)$, $\lim_{n\to\infty} \rho_n = 1$ such that for all $x, y \in E$

$$||T^n x - T^n y|| \leq \rho_n ||x - y|| \quad \text{for all} \quad n \in N. \tag{1.3}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [13] as an important generalization of the class of nonexpansive mappings. A point $x \in E$ is called a *fixed point of* T provided $Tx = x$. We denote by $F(T)$ the set of all fixed point of T (i.e., $F(T) = \{x \in E : Tx = x\}$).
Goebel and Kirk [13] proved that if C is a nonempty, bounded, closed and convex subset of a real uniformly convex Banach space and T is a self asymptotically nonexpansive mapping of C, then T has a fixed point in C.
The mapping T is said to be asymptotically regular if

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all $x \in C$. It is said to be uniformly asymptotically regular if for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K} \|T^{n+1}x - T^n x\| = 0.$$

A mapping $G : E \to E$ is said to be *accretive* if for all $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Gx - Gy, j(x - y) \rangle \geq 0.$$

For some positive real numbers η and μ the mapping G is called η-*strongly accretive* if

$$\langle Gx - Gy, j(x - y) \rangle \geq \eta \|x - y\|^2$$

holds $\forall x, y \in E$ and μ-*strictly pseudocontractive* if

$$\langle Gx - Gy, j(x - y) \rangle \leq \|x - y\|^2 - \mu \|(I - G)x - (I - G)y\|^2$$

holds $\forall x, y \in E$. It is known that if G is μ-strictly pseudocontractive then it is $(1 + \frac{1}{\mu})-$Lipschitzian.

Let C be a nonempty closed convex subset of E, a *variational inequality problem* with respect to C and G, is to find $\bar{x} \in C$ such that

$$\langle G(\bar{x}), j(y-\bar{x})\rangle \geq 0 \quad \forall y \in E. \tag{1.4}$$

The problem of solving variational inequality of the form (1.4) has been intensively studied by numerous authors due to its various applications in several physical problems, such as in operational research, economics, engineering, e.t.c.
A typical problem is to minimize a quadratic function over the set of fixed points of some nonexpansive mapping in a real Hilbert space H:

$$\min_{x\in F} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle. \tag{1.5}$$

Here F is a fixed point set of some nonexpansive mapping T of H, b is a point in H, and A is some bounded, linear and strongly positive operator on H, where a map $A : H \to H$ is said to be strongly positive if there exist a constant $\overline{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2, \quad \forall x \in H.$$

Iterative methods for approximating fixed points of nonexpansive mappings and their generalizations which solves some variational inequalities problems have been studied by a number of authors, see for examples [1–23,26,30,31] and the references contained in them.
In 2000, Moudafi [17] introduced viscosity approximation method for nonexpansive mappings. He proved that if a sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0 \tag{1.6}$$

then $\{x_n\}$ converges strongly to the unique solution $x^* \in F$ of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \geq 0, \quad \forall x \in F \tag{1.7}$$

where $\{\alpha_n\} \subseteq (0, 1)$ is a real sequence satisfying some conditions and $f : H \to H$ is a contraction map.
In 2003, Xu [29] proved that for a strongly positive linear bounded operator A on H a sequence $\{x_n\}$ defined by $x_0 \in H$

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n b, \quad n \geq 0, \tag{1.8}$$

converges strongly to the unique solution of the minimization problem (1.5) provided the sequence $\{\alpha_n\}$ satisfies some control conditions.
In 2006, Marino and Xu [16] combined the iterative methods of Xu [29] and that of Moudafi [17] and studied the following general iterative method:

$$x_{n+1} = (I-\alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.9}$$

They proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A-\gamma f)x^*, x-x^*\rangle \geq 0, \quad \forall x \in F. \tag{1.10}$$

Let $T_k : E \to E$, $k = 1, 2, 3, \dots N$ be a finite family of nonexpansive maps. For $n \in \mathbb{N}$, define a map $W_n : E \to E$ by

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1 - \gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1 - \gamma_{n,2}) I \\ &\vdots \\ W_n = U_{n,N} &= \gamma_{n,N} T_N U_{n,N-1} + (1 - \gamma_{n,N}) I, \end{aligned}$$

where $I = U_{n,0}$ and $\{\gamma_{n,k}\}_k^N \subseteq [0, 1]$. The mapping W_n here is called the W mapping generated by $T_1, T_2, \dots, T_N$ and $\{\gamma_{n,k}\}_{n \geq 1}$, $k \in \{1, 2, \dots, N\}$.
In 2007, Shang et al. [22] introduced a composite iterative scheme as follows: given $x_0 = x \in C$ arbitrarily chosen,

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n, \end{aligned}$$

where f is a contraction, and A is a strongly positive bounded linear operator on H.
In 2009, Kangtunyakarn and Suantai [15] introduced and studied the following scheme for approximation of common fixed point of a finite family of nonexpansive mappings $\{T_k\}_{k=1}^N$, for $n \in \mathbb{N}$;

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1 - \gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1 - \gamma_{n,2}) U_{n,1}, \\ &\vdots \\ K_n = U_{n,N} &= \gamma_{n,N} T_N U_{n,N-1} + (1 - \gamma_{n,N}) U_{n,N-1}. \end{aligned} \tag{1.11}$$

The mapping K_n here is called the K mapping generated by $T_1, T_2, \dots, T_N$ and $\{\gamma_{n,k}\}_{n \geq 1}$, $k \in \{1, 2, \dots, N\}$.
Recently, Singthong and Suantai [24] studied the convergence of the following composite scheme $x_0 \in C$,

$$\begin{aligned} y_n &= \beta_n x_n + (1 - \beta_n) K_n x_n, \\ x_{n+1} &= P_C(\alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n), \end{aligned} \tag{1.12}$$

where C is a nonempty, closed convex subset of Hilbert space H, $f : C \to C$ is a contraction, and A is a strongly positive bounded linear operator on H.
More recently, Ali et al. [2] introduce a modified iterative scheme for approximation of common fixed point of a finite family of nonexpansive mappings $\{T_k\}_{k=1}^N$, for $n \in \mathbb{N}$ and a sequence $\{\gamma_{n,k}\}$, $k \in \{1, 2, \dots, N\}$,

$$\begin{aligned} U_{n,1} &= \gamma_{n,1} T_1 + (1 - \gamma_{n,1}) I, \\ U_{n,2} &= \gamma_{n,2} T_2 U_{n,1} + (1 - \gamma_{n,2}) U_{n,1}, \\ &\vdots \\ U_{n,N-1} &= \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \gamma_{n,N-1}) U_{n,N-2} \\ K_n = U_{n,N} &= \gamma_{n,N} T_N U_{n,N-1} + (1 - \gamma_{n,N}) I. \end{aligned} \tag{1.13}$$

They proved strong convergence of an iterative scheme to a common fixed point of a finite family of nonexpansive mappings which is also a unique solution of some variational inequality problem in a framework of a Banach space much more general than Hilbert space. They actually proved the following theorems:

Theorem 1.1 (Ali et al. [2]) *Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of E into itself and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f : E \to E$ be a contraction with constant $\alpha \in (0,1)$. Let $G : E \to E$ be an $\eta-$ strongly accretive and $\mu-$ strictly pseudocontractive with $\eta + \mu > 1$ and let $\tau = 1 - \sqrt{\frac{1-\eta}{\mu}}$. Let γ be a real number satisfying $0 < \gamma < \frac{\tau}{\alpha}$ and let $K : E \to E$ be as in (1.13). Given $\beta \in (0,1)$, then for any $t \in (0,1)$. Let $\{z_t\}_{t\in(0,1)}$ be a path defined by*

$$z_t = t\gamma f(z_t) + (I - tG)[\beta z_t + (1-\beta)K z_t]. \tag{1.14}$$

Then $\{z_t\}$ converges strongly to a common fixed point of the family say p which is a unique solution of the variational inequality

$$\langle (G - \gamma f)p, j(q-p)\rangle \geq 0,\ \forall q \in F. \tag{1.15}$$

Theorem 1.2 (Ali et al. [2]) *Let E be a real, reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0,1)$. Let $\{T_k\}_{k=1}^N$ be a finite family of nonexpansive mappings of E into itself and $F = \bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let K_n be as in (1.13). Assume that $0 < \gamma < \frac{\tau}{2\alpha}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$ and let $x_0 \in C$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0,1)$, and suppose that the following conditions are satisfied:*

- (C1) $\alpha_n \to 0$ *as* $n \to \infty$;
- (C2) $\Sigma_{n=0}^\infty \alpha_n = \infty$
- (C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
- (C4) $\Sigma_{n=1}^\infty |\gamma_{n,k} - \gamma_{n-1,k}| < \infty$, *for all* $k = 1, 2, 3, \ldots, N$ *and* $\{\gamma_{n,k}\}_{k=1}^N \subset [a, b]$, *where* $0 < a \leq b < 1$;
- (C5) $\Sigma_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (C6) $\Sigma_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

If $\{x_n\}_{n=1}^\infty$ is a sequence defined by,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)K_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n G)y_n, \qquad n \geq 0, \end{aligned} \tag{1.16}$$

then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which also solves the following variational inequality problem,

$$\langle (G - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F. \tag{1.17}$$

It is our purpose in this paper to continue the study of the above problem and prove a new convergence theorems for approximation of common fixed point of finite family $\{T_k\}_{k=1}^N$ of asymptotically nonexpansive mappings which is also a unique solution of some variational inequality problem. The result presented here generalize and improve those recent ones such as in [2,24]. In particular our Theorem extend the result in [24] to more general Banach space setting than Hilbert and generalizes it to family of asymptotically nonexpansive mappings. On the other hand our result also not only generalizes Theorems 1.1 and 1.2 to the family of asymptotically nonexpansive mappings but also conditions C5 and C6 imposed in both Theorems 1.2 above and Theorem 2.1 of [24] are dispensed with.

2 Preliminaries

The following lemmas will be use for the main result.

Lemma 2.1 *Let E be a real normed linear space. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in E, j(x+y) \in J(x+y).$$

Lemma 2.2 (Suzuki [25]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1-\beta_n)x_n$ for all integer $n \geq 1$ and*

$$\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then, $\lim_{n\to\infty}||y_n - x_n|| = 0$.

Lemma 2.3 (Xu [27]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : ||x|| \leq r\}$ and $\lambda \in [0, 1]$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \ \ g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||).$$

Lemma 2.4 (Xu [28]) *Let $\{a_n\}$ be a sequence of nonegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \ n \geq 0$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

Lemma 2.5 (Chang et al. [9]) *Let E be a uniformily convex Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be an asymptotically nonexpansive mapping, then $I - T$ is demiclosed at zero.*

Lemma 2.6 (Piri and Vaezi [19] see also [1]) *Let E be a real Banach space and $G : E \to E$ be a mapping.*

(i) *If G is η-strongly accretive and μ-strictly pseudo-contractive with $\eta + \mu > 1$, then $I - G$ is contractive with constant $\sqrt{\frac{1-\eta}{\mu}}$.*

(ii) *If G is η-strongly accretive and μ-strictly pseudo-contractive with $\eta + \mu > 1$, then for any fixed number $\kappa \in (0, 1)$, $I - \kappa G$ is contractive with constant $1 - \kappa\left(1 - \sqrt{\frac{1-\eta}{\mu}}\right)$.*

3 Main results

Lemma 3.1 *Let C be a nonempty closed convex subset of a uniformly convex real Banach space E. Let $\{T_k\}_{k=1}^N$ be finite family of uniformly asymtotically regular asymptotically nonexpansive mappings of C into itself with sequences $\{\rho_{n,k}\} \subset [1, \infty)$, let $\{\gamma_{n,k}\}_{k=1}^N$ be a sequence in $(0, 1)$ such that $0 < \liminf_{n\to\infty} \gamma_{n,k} \leq \limsup_{n\to\infty} \gamma_{n,k} < 1$ and*

$\lim_{n\to\infty}|\gamma_{n,k}-\gamma_{n-1,k}| = 0\,\forall k \in \{1,2,3,\ldots,N\}$. *Let* K_n *be a mapping generated by* $T_1, T_2, T_3, \ldots, T_N$ *and* $\gamma_{n,1}, \gamma_{n,2}, \gamma_{n,3}, \ldots, \gamma_{n,N}$ *as follows;*

$$\begin{aligned}
U_{n,1} &= \gamma_{n,1}T_1^n + (1-\gamma_{n,1})I,\\
U_{n,2} &= \gamma_{n,2}T_2^nU_{n,1} + (1-\gamma_{n,2})U_{n,1},\\
&\vdots\\
U_{n,N-1} &= \gamma_{n,N-1}T_{N-1}^nU_{n,N-2} + (1-\gamma_{n,N-1})U_{n,N-2},\\
K_n = U_{n,N} &= \gamma_{n,N}T_N^nU_{n,N-1} + (1-\gamma_{n,N})I.
\end{aligned}\tag{3.1}$$

Then, the following holds:

(i) $\|K_nx - K_ny\| \le (1+\upsilon_n)\|x-y\|$, *where* $\upsilon_n = \rho_{n,N}(1+\lambda_{n,N-1}) - 1$, *and* $\{\lambda_{n,N}\}$ *is some sequence in* $[0,\infty)$, *with* $\lambda_{n,N} \to 0$ *as* $n \to \infty$.

(ii) *If* $\lim_{n\to\infty}\|T_k^{n+1}U_{n,k-1}z_n - T_k^nU_{n,k-1}z_n\| = 0$, *then* $\underset{n\to\infty}{lim}\|K_{n+1}z_n - K_nz_n\| = 0$, *for every bounded sequence* $\{z_n\}$ *in* $E, k = 1, 2, \ldots, N$;

(iii) *For every bounded sequence* $\{z_n\}$ *in* C *such that* $\lim_{n\to\infty}||K_nz_n - z_n|| = 0$, *we have* $\lim_{n\to\infty}||T_kz_n - z_n|| = 0$ *for any* $k \in \{1,2,3,\ldots,N\}$. *Furthermore , we have* $w_w(z_n) \subset \cap_{k=1}^N F(T_k)$ *and* $F(K_n) \overset{N}{=\subset} \cap_{k=1}^N F(T_k)$.

Proof (i) Let $x, y \in C$ then from (3.1), if $N = 1$ the result follows. Assume $N \ne 1$ and $U_{n,0} = I$ (identity map), then for $k \in \{1,2,\ldots,N-1\}$, we have

$$\begin{aligned}
\|U_{n,k}x - U_{n,k}y\| &\le \gamma_{n,k}\|T_k^nU_{n,k-1}x - T_k^nU_{n,k-1}y\|\\
&\quad +(1-\gamma_{n,k})\|U_{n,k-1}x - U_{n,k-1}y\|\\
&\le [\gamma_{n,k}\rho_{n,k} + (1-\gamma_{n,k})]\|U_{n,k-1}x - U_{n,k-1}y\|\\
&= [1+\gamma_{n,k}(\rho_{n,k}-1)]\|U_{n,k-1}x - U_{n,k-1}y\|\\
&\le [1+\gamma_{n,k}(\rho_{n,k}-1)][\gamma_{n,k-1}\|T_{k-1}^nU_{n,k-2}x - T_{k-1}^nU_{n,k-2}y\|\\
&\quad +(1-\gamma_{n,k-1})\|U_{n,k-2}x - U_{n,k-2}y\|]\\
&\le [1+\gamma_{n,k}(\rho_{n,k}-1)][1+\gamma_{n,k-1}(\rho_{n,k-1}-1)]\|U_{n,k-2}x - U_{n,k-2}y\|\\
&\vdots \quad \vdots\\
&\le [1+\gamma_{n,k}(\rho_{n,k}-1)][1+\gamma_{n,k-1}(\rho_{n,k-1}-1)]\ldots[1+\gamma_{n,2}(\rho_{n,2}-1)]\\
&\quad \|U_{n,1}x - U_{n,1}y\|\\
&\le [1+\gamma_{n,k}(\rho_{n,k}-1)][1+\gamma_{n,k-1}(\rho_{n,k-1}-1)]\ldots[1+\gamma_{n,2}(\rho_{n,2}-1)]\\
&\quad [1+\gamma_{n,1}(\rho_{n,1}-1)]\|x-y\|\\
&= \prod_{j=1}^{k}[1+\gamma_{n,j}(\rho_{n,j}-1)]\|x-y\|\\
&= (1+\lambda_{n,k})\|x-y\|,
\end{aligned}$$

where $\prod_{j=1}^{k}\Big(1+\gamma_{n,j}(\rho_{n,j}-1)\Big)=(1+\lambda_{n,k})$, observe that $\lim_{n\to\infty}\lambda_{n,k}=0$. Then,

$$\begin{aligned}\|K_nx-K_ny\| &= \|U_{n,N}x-U_{n,N}y\|\\ &\le \gamma_{n,N}\|T_N^nU_{n,N-1}x-T_N^nU_{n,N-1}y\|+(1-\gamma_{n,N})\|x-y\|\\ &\le \gamma_{n,N}\rho_{n,N}\|U_{n,N-1}x-U_{n,N-1}y\|+(1-\gamma_{n,N})\|x-y\|\\ &\le \gamma_{n,N}\rho_{n,N}(1+\lambda_{n,N-1})\|x-y\|+(1-\gamma_{n,N})\|x-y\|\\ &= [1+\gamma_{n,N}(\rho_{n,N}(1+\lambda_{n,N-1})-1)]\|x-y\|\\ &\le [1+(\rho_{n,N}(1+\lambda_{n,N-1})-1)]\|x-y\|\\ &= (1+v_n)\|x-y\|,\end{aligned}$$

where $v_n=\rho_{n,N}(1+\lambda_{n,N-1})-1$, observe that $\lim_{n\to\infty}v_n=0$.
Next we show (ii). For $k\in\{2,3,\ldots,N-1\}$ and any bounded sequence $\{z_n\}\subset E$, letting $\delta_{n+1,k}:=[1+\gamma_{n+1,k}(\rho_{n+1,k}-1)]$, $M_{n,k}:=[\|T_k^{n+1}U_{n,k}z_n\|+\|U_{n,k}z_n\|]$ and $P_{n,k}:=\|T_{k-1}^{n+1}U_{n,k}z_n-T_{k-1}^nU_{n,k}z_n\|$, we have

$$\begin{aligned}\|U_{n+1,k}z_n-U_{n,k}z_n\| &= \|\gamma_{n+1,k}T_k^{n+1}U_{n+1,k-1}z_n\\ &\quad-\gamma_{n+1,k}T_k^{n+1}U_{n,k-1}z_n\\ &\quad+[\gamma_{n+1,k}-\gamma_{n,k}]T_k^{n+1}U_{n,k-1}z_n\\ &\quad+\gamma_{n,k}[T_k^{n+1}U_{n,k-1}z_n-T_k^nU_{n,k-1}z_n]\\ &\quad+(1-\gamma_{n+1,k})(U_{n+1,k-1}z_n-U_{n,k-1}z_n)\\ &\quad+[(1-\gamma_{n+1,k})-(1-\gamma_{n,k})]U_{n,k-1}z_n\|\\ &\le [1+\gamma_{n+1,k}(\rho_{n+1,k}-1)]\|U_{n+1,k-1}z_n-U_{n,k-1}z_n\|\\ &\quad+|\gamma_{n+1,k}-\gamma_{n,k}|\Big[\|T_k^{n+1}U_{n,k-1}z_n\|+\|U_{n,k-1}z_n\|\Big]\\ &\quad+\gamma_{n,k}\|T_k^{n+1}U_{n,k-1}z_n-T_k^nU_{n,k-1}z_n\|\\ &\le \Big[1+\gamma_{n+1,k}(\rho_{n+1,k}-1)\Big]\Big[\Big[1+\gamma_{n+1,k-1}(\rho_{n+1,k-1}-1)\Big]\|U_{n+1,k-2}z_n-U_{n,k-2}z_n\|\\ &\quad+|\gamma_{n+1,k-1}-\gamma_{n,k-1}|\Big[\|T_{k-1}^{n+1}U_{n,k-2}z_n\|+\|U_{n,k-2}z_n\|\Big]\\ &\quad+\gamma_{n,k-1}\|T_{k-1}^{n+1}U_{n,k-2}z_n-T_{k-1}^nU_{n,k-2}z_n\|\Big]\\ &\quad+|\gamma_{n+1,k}-\gamma_{n,k}|\Big[\|T_k^{n+1}U_{n,k-1}z_n\|+\|U_{n,k-1}z_n\|\Big]\\ &\quad+\gamma_{n,k}\|T_k^{n+1}U_{n,k-1}z_n-T_k^nU_{n,k-1}z_n\|\\ &= \delta_{n+1,k}\delta_{n+1,k-1}\|U_{n+1,k-2}z_n-U_{n,k-2}z_n\|\\ &\quad+\delta_{n+1,k}|\gamma_{n+1,k-1}-\gamma_{n,k-1}|M_{n,k-2}\\ &\quad+\delta_{n+1,k}\gamma_{n,k-1}P_{n,k-2}\\ &\quad+|\gamma_{n+1,k}-\gamma_{n,k}|M_{n,k-1}\\ &\quad+\gamma_{n,k}P_{n,k-1}\\ &\le \delta_{n+1,k}\delta_{n+1,k-1}\Big[\delta_{n+1,k-2}\|U_{n+1,k-3}z_n-U_{n,k-3}z_n\|\\ &\quad+|\gamma_{n+1,k-2}-\gamma_{n,k-2}|[\|T_{k-2}^{n+1}U_{n,k-3}z_n\|+\|U_{n,k-3}z_n\|]\\ &\quad+\gamma_{n,k-1}\|T_{k-2}^{n+1}U_{n,k-3}z_n-T_{k-2}^nU_{n,k-2}z_n\|\Big]\\ &\quad+\delta_{n+1,k}|\gamma_{n+1,k-1}-\gamma_{n,k-1}|M_{n,k-2}\end{aligned}$$

$$
\begin{aligned}
&+\delta_{n+1,k}\gamma_{n,k-1}P_{n,k-2}\\
&+|\gamma_{n+1,k}-\gamma_{n,k}|M_{n,k-1}\\
&+\gamma_{n,k}P_{n,k-1}\\
&=\delta_{n+1,k}\delta_{n+1,k-1}\delta_{n+1,k-2}\|U_{n+1,k-3}z_n-U_{n,k-3}z_n\|\\
&+\delta_{n+1,k}\delta_{n+1,k-1}|\gamma_{n+1,k-2}-\gamma_{n,k-2}|M_{n,k-3}\\
&+\delta_{n+1,k}\delta_{n+1,k-1}\gamma_{n,k-1}P_{n,k-3}\\
&+\delta_{n+1,k}|\gamma_{n+1,k-1}-\gamma_{n,k-1}|M_{n,k-2}\\
&+\delta_{n+1,k}\gamma_{n,k-1}P_{n,k-2}\\
&+|\gamma_{n+1,k}-\gamma_{n,k}|M_{n,k-1}\\
&+\gamma_{n,k}P_{n,k-1}\\
&\leq\\
&\vdots\vdots \qquad (3.2)\\
&\leq \delta_{n+1,k}\delta_{n+1,k-1}\delta_{n+1,k-2}\ldots\delta_{n+1,3}\delta_{n+1,2}\|U_{n+1,1}z_n-U_{n,1}z_n\|\\
&+\Big(\delta_{n+1,k}\delta_{n+1,k-1}\delta_{n+1,k-2}\ldots\delta_{n+1,3}\gamma_{n+1,2}P_{n,1}\\
&+\cdots+\delta_{n+1,k}\delta_{n+1,k-1}\gamma_{n+1,k-2}P_{n,k-3}\\
&+\delta_{n+1,k}\gamma_{n+1,k-1}P_{n,k-2}+\gamma_{n+1,k}P_{n,k-1}\Big)\\
&+\Big(\delta_{n+1,k}\delta_{n+1,k-1}\delta_{n+1,k-2}\ldots\delta_{n+1,3}|\gamma_{n+1,2}-\gamma_{n,2}|M_{n,1}\\
&+\cdots+\delta_{n+1,k}\delta_{n+1,k-1}|\gamma_{n+1,k-2}-\gamma_{n,k-2}|M_{n,k-3}\\
&+\delta_{n+1,k}|\gamma_{n+1,k-1}-\gamma_{n,k-1}|M_{n,k-2}\\
&+|\gamma_{n+1,k}-\gamma_{n,k}|M_{n,k-1}\Big)\\
&=\|U_{n+1,1}z_n-U_{n,1}z_n\|\prod_{j=2}^{k}\delta_{n+1,j}\\
&+\sum_{i=2}^{k}\gamma_{n+1,i}P_{n,i-1}\prod_{j=i+1}^{k}\delta_{n+1,j}+\sum_{i=2}^{k}|\gamma_{n+1,i}-\gamma_{n,i}|M_{n,i-1}\prod_{j=i+1}^{k}\delta_{n+1,j}\\
&\leq\Big[\gamma_{n+1,1}||T_1^{n+1}z_n-T_1^{n}z_n||+|\gamma_{n+1,1}-\gamma_{n,1}|(||z_n||+||T_1^{n}z_n||)\Big]\prod_{j=2}^{k}\delta_{n+1,j}\\
&+\sum_{i=2}^{k-1}\gamma_{n+1,i}P_{n,i-1}\prod_{j=i+1}^{k}\delta_{n+1,j}+\sum_{i=2}^{k}|\gamma_{n+1,i}-\gamma_{n,i}|M_{n,i-1}\prod_{j=i+1}^{k}\delta_{n+1,j}\\
&=\sum_{i=1}^{k}\gamma_{n+1,i}P_{n,i}\prod_{j=i+1}^{k}\delta_{n+1,j}+\sum_{i=1}^{k}|\gamma_{n+1,i}-\gamma_{n,i}|M_{n,i}\prod_{j=i+1}^{k}\delta_{n+1,j} \qquad (3.3)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|K_{n+1}z_n-K_nz_n\| &= \|U_{n+1,N}z_n-U_{n,N}z_n\|\\
&\leq \gamma_{n+1,N}||T_N^{n+1}U_{n+1,N-1}z_n-T_N^{n+1}U_{n,N-1}z_n||\\
&\quad+\gamma_{n+1,N}||T_N^{n+1}U_{n+1,N-1}z_n-T_N^{n}U_{n,N-1}z_n||
\end{aligned}
$$

$$
\begin{aligned}
&+|\gamma_{n+1,N}-\gamma_{n,N}|[||T_N^n U_{n,N-1}z_n||+||z_n||]\\
&\leq \gamma_{n+1,N}\rho_{n+1,N}||U_{n+1,N-1}z_n-U_{n,N-1}z_n||\\
&+\gamma_{n+1,N}||T_N^{n+1}U_{n+1,N-1}z_n-T_N^n U_{n,N-1}z_n||\\
&+|\gamma_{n+1,N}-\gamma_{n,N}|[||T_N^n U_{n,N-1}z_n||+||z_n||]\\
&\leq \rho_{n+1,N}\left[\sum_{i=1}^{N-1}\gamma_{n+1,i}P_{n,i}\prod_{j=i+1}^{N-1}\delta_{n+1,j}\right.\\
&\left.+\sum_{i=1}^{N-1}|\gamma_{n+1,i}-\gamma_{n,i}|M_{n,i}\prod_{j=i+1}^{N-1}\delta_{n+1,j}\right]\\
&+\gamma_{n+1,N}||T_N^{n+1}U_{n+1,N-1}z_n-T_N^n U_{n,N-1}z_n||\\
&+|\gamma_{n+1,N}-\gamma_{n,N}|\big[||T_N^n U_{n,N-1}z_n||+||z_n||\big]. \qquad (3.4)
\end{aligned}
$$

Therefore

$$\lim_{n\to\infty}||K_{n+1}z_n-K_n z_n||=0. \qquad (3.5)$$

Hence (ii) is satisfied.
Next, we show (iii), let $\{z_n\}$ be a bounded sequence in E such that $\lim_{n\to\infty}||K_n z_n-z_n||=0$, then for $x^*\in\cap_{k=1}^N F(T_k)$, we obtain

$$
\begin{aligned}
||K_n z_n-x^*||^2 &\leq \gamma_{n,N}||T_N^n U_{n,N-1}z_n-x^*||^2+(1-\gamma_{n,N})||z_n-x^*||^2\\
&\leq \gamma_{n,N}\rho_{n,N}^2||U_{n,N-1}z_n-x^*||^2+(1-\gamma_{n,N})||z_n-x^*||^2\\
&\leq \gamma_{n,N}\rho_{n,N}^2\big[\gamma_{n,N-1}||T_{N-1}^n U_{n,N-2}z_n-x^*||^2\\
&\quad+(1-\gamma_{n,N-1})||U_{n,N-2}z_n-x^*||^2\big]\\
&\quad+(1-\gamma_{n,N})||z_n-x^*||^2 \qquad (3.6)\\
&\leq \gamma_{n,N}\rho_{n,N}^2\big[\gamma_{n,N-1}\rho_{n,N-1}^2||U_{n,N-2}z_n-x^*||^2\\
&\quad+(1-\gamma_{n,N-1})||U_{n,N-2}z_n-x^*||^2\big]\\
&\quad+(1-\gamma_{n,N})||z_n-x^*||^2\\
&=\gamma_{n,N}\rho_{n,N}^2\Big([1+\gamma_{n,N-1}(\rho_{n,N-1}^2-1)]||U_{n,N-2}z_n-x^*||^2\Big)\\
&\quad+(1-\gamma_{n,N})||z_n-x^*||^2\\
&\leq \gamma_{n,N}\rho_{n,N}^2\big[1+\gamma_{n,N-1}(\rho_{n,N-1}^2-1)\big]\big[1+\gamma_{n,N-2}(\rho_{n,N-2}^2-1)\big]\\
&\quad\ldots\times[1+\gamma_{n,1}(\rho_{n,1}^2-1)]||z_n-x^*||^2+(1-\gamma_{n,N})||z_n-x^*||^2\\
&=\Big(1+\gamma_{n,N}\Big\{\rho_{n,N}^2[1+\gamma_{n,N-1}(\rho_{n,N-1}^2-1)][1+\gamma_{n,N-2}(\rho_{n,N-2}^2-1)]\\
&\quad\ldots\times[1+\gamma_{n,1}(\rho_{n,1}^2-1)]-1\Big\}\Big)||z_n-x^*||^2\\
&=(1+\vartheta_n)||z_n-x^*||^2, \qquad (3.7)
\end{aligned}
$$

where $\vartheta_n := \gamma_{n,N}\Big\{\rho_{n,N}^2[1+\gamma_{n,N-1}(\rho_{n,N-1}^2-1)][1+\gamma_{n,N-2}(\rho_{n,N-2}^2-1)]\ldots[1+\gamma_{n,2}(\rho_{n,2}^2-1)][1+\gamma_{n,1}(\rho_{n,1}^2-1)]-1\Big\}$ and observe that $\lim_{n\to\infty}\vartheta_n=0$.

Then by using Lemma 2.3, (3.6) and (3.7), we have

$$||K_n z_n-x^*||^2=||\gamma_{n,N}(T_N^n U_{n,N-1}z_n-x^*)+(1-\gamma_{n,N})(z_n-x^*)||^2$$

$$\leq \gamma_{n,N}||T_N^n U_{n,N-1}z_n - x^*||^2 + (1-\gamma_{n,N})||z_n - x^*||^2$$
$$-\gamma_{n,N}(1-\gamma_{n,N})g(||T_N^n U_{n,N-1}z_n - z_n||)$$
$$\leq (1+\vartheta_n)||z_n - x^*||^2 - \gamma_{n,N}(1-\gamma_{n,N})g(||T_N^n U_{n,N-1}z_n - z_n||),$$

from this we obtain

$$\begin{aligned}
&\gamma_{n,N}(1-\gamma_{n,N})g(||T_N^n U_{n,N-1}z_n - z_n||) \leq ||z_n - x^*||^2 - ||K_n z_n - x^*||^2 + \vartheta_n||z_n - x^*||^2 \\
&= (||z_n - x^*|| - ||K_n z_n - x^*||)(||z_n - x^*|| + ||K_n z_n - x^*||) + \vartheta_n||z_n - x^*||^2 \\
&\leq ||z_n - K_n z_n||(||z_n - x^*|| + ||K_n z_n - x^*||) + \vartheta_n||z_n - x^*||^2 \\
&\leq (||z_n - K_n z_n|| + \vartheta_n)M_0 \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned}$$

for some $M_0 > 0$. Thus, by the property of g, we obtain that

$$\lim_{n\to\infty}||T_N^n U_{n,N-1}z_n - z_n|| = 0. \tag{3.8}$$

Moreover,

$$\begin{aligned}
&||z_n - x^*||^2 \leq (||z_n - T_N^n U_{n,N-1}z_n|| + ||T_N^n U_{n,N-1}z_n - x^*||)^2 \\
&\quad = ||z_n - T_N^n U_{n,N-1}z_n||(||z_n - T_N^n U_{n,N-1}z_n|| + 2||T_N^n U_{n,N-1}z_n - x^*||) \\
&\qquad + ||T_N^n U_{n,N-1}z_n - x^*||^2 \\
&\quad \leq ||z_n - T_N^n U_{n,N-1}z_n||M_1 + \rho_{n,N}^2||U_{n,N-1}z_n - x^*||^2 \quad (\text{for some } M_1 > 0) \\
&\quad \leq ||z_n - T_N^n U_{n,N-1}z_n||M_1 + \rho_{n,N}^2[\gamma_{n,N-1}||T^{n,N-1}U_{n,N-2}z_n - x^*||^2 \\
&\qquad + (1-\gamma_{n,N-1})||U_{n,N-2}z_n - x^*||^2 \\
&\qquad - \gamma_{n,N-1}(1-\gamma_{n,N-1})g(||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n||)] \\
&\quad \leq ||z_n - T_N^n U_{n,N-1}z_n||M_1 + \rho_{n,N}^2[\gamma_{n,N-1}\rho_{n,N-1}^2||U_{n,N-2}z_n - x^*||^2 \\
&\qquad + (1-\gamma_{n,N-1})||U_{n,N-2}z_n - x^*||^2 \\
&\qquad - \gamma_{n,N-1}(1-\gamma_{n,N-1})g(||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n||)] \\
&\quad \leq ||z_n - T_N^n U_{n,N-1}z_n||M_1 + \rho_{n,N}^2[(1+\gamma_{n,N-1}(\rho_{n,N-1}^2 - 1))||U_{n,N-2}z_n - x^*||^2 \\
&\qquad - \gamma_{n,N-1}(1-\gamma_{n,N-1})g(||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n||)] \\
&\qquad \leq ||z_n - T_N^n U_{n,N-1}z_n||M_1 + (1+\vartheta_n)||z_n - x^*||^2 \\
&\qquad - \rho_{n,N}^2\gamma_{n,N-1}(1-\gamma_{n,N-1})g(||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n||)], \\
&g(||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n||) \\
&\quad \leq \frac{\left(||z_n - T_N^n U_{n,N-1}z_n||M_1 + \vartheta_n||z_n - x^*||^2\right)}{\rho_{n,N}^2\gamma_{n,N-1}(1-\gamma_{n,N-1})} \\
&\quad \leq \frac{\left(||z_n - T_N^n U_{n,N-1}z_n|| + \vartheta_n\right)M}{\rho_{n,N}^2\gamma_{n,N-1}(1-\gamma_{n,N-1})},
\end{aligned}$$

for some $M > 0$. Thus, using property of g,

$$\lim_{n\to\infty}||T_{N-1}^n U_{n,N-2}z_n - U_{n,N-2}z_n|| = 0. \tag{3.9}$$

Continuing in this fashion we observe that for $k \in \{2, 3, 4, \ldots, N-1\}$

$$\lim_{n\to\infty}||T_k^n U_{n,k-1}z_n - U_{n,k-1}z_n|| = 0, \tag{3.10}$$

and

$$\lim_{n\to\infty} ||T_1^n z_n - z_n|| = 0. \tag{3.11}$$

Also

$$\begin{aligned}||U_{n,k}z_n - z_n|| &\leq ||U_{n,k}z_n - T_k^n U_{n,k-1}z_n|| + ||T_k^n U_{n,k-1}z_n - U_{n,k-1}z_n|| \\ &\quad + ||U_{n,k-1}z_n - T_{k-1}^n U_{n,k-2}z_n|| \\ &\quad + ||T_{k-1}^n U_{n,k-2}z_n - U_{n,k-2}z_n|| \\ &\quad + \cdots + ||T_2^n U_{n,1}z_n - U_{n,1}z_n|| + ||U_{n,1}z_n - z_n|| \\ &\leq (1-\gamma_{n,k})||U_{n,k-1}z_n - T_k^n U_{n,k-1}z_n|| \\ &\quad + ||T_k^n U_{n,k-1}z_n - U_{n,k-1}z_n|| \\ &\quad + (1-\gamma_{n,k-1})||U_{n,k-2}z_n - T_{k-1}^n U_{n,k-2}z_n|| \\ &\quad + \cdots + (1-\gamma_{n,2})||U_{n,1}z_n - T_2^n U_{n,1}z_n|| \\ &\quad + \gamma_{n,1}||T_1^n z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Thus

$$\begin{aligned}||T_k^n U_{n,k-1}z_n - z_n|| &\leq ||T_k^n U_{n,k-1}z_n - U_{n,k-1}z_n|| \\ &\quad + ||U_{n,k-1}z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

So for any $k \in \{1, 2, 3, \ldots, N\}$, we obtain

$$\begin{aligned}||z_n - T_k^n z_n|| &\leq ||z_n - T_k^n U_{n,k-1}z_n|| + ||T_k^n U_{n,k-1}z_n - T_k^n z_n|| \\ &\leq ||z_n - T_k^n U_{n,k-1}z_n|| \\ &\quad + \rho_{n,k}||U_{n,k-1}z_n - z_n|| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.12}$$

Hence

$$\begin{aligned}||T_k z_n - z_n|| &\leq ||T_k z_n - T_k(T_k^n)z_n|| + ||T_k(T_k^n)z_n - T_k^n z_n|| + ||T_k^n z_n - z_n|| \\ &\leq (L_k + 1)||z_n - T_k^n z_n|| + ||T_k^{n+1}z_n - T_k^n z_n||.\end{aligned}$$

Therefore, from (3.12), for each $k \in \{1, 2, 3, \ldots, N\}$, we obtain

$$\lim_{n\to\infty} ||T_k z_n - z_n|| = 0. \tag{3.13}$$

Moreover, by Lemma 2.5, we have $w_w(x_n) \subset \cap_{k=1}^N F(T_k)$, also since $\cap_{k=1}^N F(T_k) \subset F(K_n)$ is obvious, we only need to show that $F(K_n) \subset \cap_{k=1}^N F(T_k)$. Let $z^* \in F(K_n)$, and $z_n = z^*$, then, we have that $||z^* - T_k z^*|| = 0$ for each $k \in \{1, 2, 3, \ldots, N\}$ that is $z^* = T_k z^*$, for each $k \in \{1, 2, 3, \ldots, N\}$, so that $z^* \in \cap_{k=1}^N F(T_k)$. Hence (iii) is satisfied. □

Theorem 3.2 *Let E be a real uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0, 1)$. Let $\{T_i\}_{i=1}^N$ be a family of uniformly asymptotically regular asymptotically nonexpansive self mappings of C into itself and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let K_n be as in Lemma 3.1. Assume that $0 < \gamma < \frac{\tau}{\alpha}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$ and let $x_0 \in C$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

(C1) $\alpha_n \to 0$ *and* $\frac{\upsilon_n}{\alpha_n} \to 0$ *as* $n \to \infty$, *where* υ_n *is as in (i) of Lemma 3.1;*
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$

If $\{x_n\}_{n=1}^{\infty}$ *is a sequence defined by,*

$$x_n = \alpha_n \gamma f(x_n) + (I - \alpha_n G)[\beta_n x_n + (1 - \beta_n) K_n x_n], \qquad n \geq 0, \tag{3.14}$$

then $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$, *which also solves the following variational inequality:*

$$\langle \gamma f(p) - Gp, j(q - p) \rangle \leq 0, \quad \forall q \in F. \tag{3.15}$$

Proof First, we show that $\{x_n\}$ defined by (3.14) is well defined. For all $n \in \mathbb{N}$, let us define the mapping

$$T_n^f x := \alpha_n \gamma f(x) + (I - \alpha_n G)[\beta_n x + (1 - \beta_n) K_n x].$$

Indeed, for all $x, y \in E$, we have

$$\begin{aligned} ||T_n^f x - T_n^f y|| &= ||\alpha_n \gamma (f(x) - f(y)) + (1 - \alpha_n G)[\beta_n (x - y) + (1 - \beta_n)(K_n x - K_n y)]|| \\ &\leq \alpha_n \gamma \alpha ||x - y|| + (1 - \alpha_n \tau)[\beta_n ||x - y|| + (1 - \beta_n)(1 + \upsilon_n)||x - y||] \\ &\leq [\alpha_n \gamma \alpha + (1 - \alpha_n \tau)(1 + \upsilon_n)]||x - y|| \\ &= \Big(1 - \alpha_n [(\tau - \gamma \alpha) - (1 - \alpha_n \tau)(\upsilon_n / \alpha_n)]\Big) ||x - y||. \end{aligned}$$

Since, $\lim_{n \to \infty} (1 - \alpha_n \tau) \upsilon_n / \alpha_n \to 0$, then there exist $n_0 \in \mathbb{N}$ such that $(1 - \alpha_n \tau) \upsilon_n / \alpha_n < (\tau - \gamma \alpha)/2$ for all $n \geq n_0$. Therefore, for $n \geq n_0$, we have

$$1 - \alpha_n [(\tau - \gamma \alpha) - (1 - \alpha_n \tau)(\upsilon_n / \alpha_n)] < 1 - \alpha_n [(\tau - \gamma \alpha) - (\tau - \gamma \alpha)/2] < 1.$$

Hence,

$$||T_n^f x - T_n^f y|| < ||x - y||.$$

Thus, $\{x_n\}$ defined by (3.14) is well defined. Therefore, by the contraction mapping principle, there exists a unique fixed point $x_n \in C$ of T_n^f which satisfies (3.14).

From the choice of the parameter γ, it is easy to see that the mapping $(G - \gamma f) : E \to E$ is strongly accretive and so the variational inequality (3.15) has unique solution in F. Let $p \in F$ then,

$$\begin{aligned} \|x_n - p\|^2 &= \alpha_n \langle \gamma f(p) - Gp, j(x_n - p) \rangle + \langle (I - \alpha_n G)[\beta_n x_n + (1 - \beta_n) K_n x_n] \\ &\quad -(I - \alpha_n G)p, j(x_n - p) \rangle + \alpha_n \langle \gamma f(x_n) - \gamma f(p), j(x_n - p) \rangle \\ &\leq [1 - \alpha_n (\tau - \gamma \alpha) + (1 - \alpha_n \tau) \upsilon_n] \|x_n - p\|^2 + \alpha_n \langle (\gamma f - G)p, j(x_n - p) \rangle. \end{aligned}$$

Let $d_n = (1 - \alpha_n \tau)(\upsilon_n / \alpha_n)$. Since, $\lim_{n \to \infty} (1 - \alpha_n \tau) \upsilon_n / \alpha_n = 0$, then there exist $n_0 \in \mathbb{N}$ such that $(1 - \alpha_n \tau) \upsilon_n / \alpha_n < (\tau - \gamma \alpha)/2$ for all $n \geq n_0$.

$$\|x_n - p\|^2 \leq \frac{\langle (\gamma f - G)p, j(x_n - p) \rangle}{(\tau - \gamma \alpha) - d_n}, \tag{3.16}$$

that is $||x_n - p|| \leq \frac{2||\gamma f(p) - G(p)||}{\tau - \gamma \alpha}$, for all $n \geq n_0$. Thus $\{x_n\}$ is bounded implies that $\{f(x_n)\}$, $\{G(x_n)\}$ and $\{K_n(x_n)\}$ are also bounded. From (3.14) we also obtain

$$\|x_n - K_n x_n\| \leq \beta_n \|x_n - K_n x_n\| + \alpha_n \|\gamma f(x_n) - G(\beta_n x_n + (1 - \beta_n) K_n(x_n))\|$$

and hence

$$\|x_n - K_n x_n\| \leq \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - G(\beta_n x_n + (1-\beta_n)K_n(x_n))\| \rightarrow 0, \quad (3.17)$$

as $n \rightarrow \infty$. Since $\{x_n\}$ is bounded, using (3.17), it follows from (iii) of Lemma 3.1 that $F = F(K_n)$. We claim that the set $\{x_n\}$ is sequentially compact. Indeed, define a map $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(y) := \mu_n ||x_n - y||^2, \quad \forall y \in E.$$

Then, $\phi(y) \rightarrow \infty$ as $||y|| \rightarrow \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u\in E} \phi(u)$. Hence, the set

$$K^* := \{y \in E : \phi(y) = \min_{u\in E}\phi(u)\} \neq \emptyset.$$

Since $\lim_{n\rightarrow\infty}||x_n - K_n x_n|| = 0$, $\lim_{n\rightarrow\infty}||x_n - K_n^m x_n|| = 0$, for any $m \geq 1$ by induction. Now let $v \in K^*$, we have

$$\begin{aligned}\lim_{n\rightarrow\infty}\phi(K_n v) &= \lim_{n\rightarrow\infty}\mu_n||x_n - K_n v||^2 \\ &= \lim_{n\rightarrow\infty}\mu_n||x_n - K_n x_n + K_n x_n - K_n v||^2 \\ &\leq \lim_{n\rightarrow\infty}\mu_n[(1+v_n)||x_n - v||]^2 = \lim_{n\rightarrow\infty}\phi(v),\end{aligned}$$

and hence $K_n v \in K^*$.
Now let $z \in F$, then $z = K_n z$. Since K^* is a closed convex set, there exists a unique $v^* \in K^*$ such that

$$||z - v^*|| = \min_{u\in K^*}||z - u||.$$

But

$$\lim_{n\rightarrow\infty}||z - K_n v^*|| = \lim_{n\rightarrow\infty}||K_n z - K_n v^*|| \leq \lim_{n\rightarrow\infty}(1+v_n)||z - v^*||,$$

which implies $v^* = K_n v^*$ and so $K^* \cap F \neq \emptyset$.
Let $p \in K^* \cap F$ and $\epsilon \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - \epsilon(G - \gamma f)p)$ and using Lemma 2.1, we obtain that

$$||x_n - p + \epsilon(G - \gamma f)p||^2 \leq ||x_n - p||^2 + 2\epsilon\langle(G - \gamma f)p, j(x_n - p + \epsilon(G - \gamma f)p)\rangle$$

which implies

$$\mu_n\langle(\gamma f - G)p, j(x_n - p + \iota(G - \gamma f)p)\rangle \leq 0.$$

Moreover,

$$\begin{aligned}\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle &= \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G - \gamma f)p)\rangle \\ &\quad +\mu_n\langle(\gamma f - G)p, j(x_n - p + \epsilon(G - \gamma f)p)\rangle \\ &\leq \mu_n\langle(\gamma f - G)p, j(x_n - p) - j(x_n - p + \epsilon(G - \gamma f)p)\rangle.\end{aligned}$$

Since j is *norm*-to-*weak** uniformly continuous on bounded subsets of E, we have that

$$\mu_n\langle(\gamma f - G)p, j(x_n - p)\rangle \leq 0.$$

It follows from (3.16) that

$$\|x_n - p\|^2 \le \frac{\langle(\gamma f - G)p, j(x_n - p)\rangle}{(\tau - \gamma\alpha) - d_n},$$

and so

$$\mu_n \|x_n - p\|^2 \le 0.$$

Thus there exist a subsequence say $\{x_{n_l}\}$ of $\{x_n\}$ such that $\lim_{l\to\infty} x_{n_l} = p$.
Define S_n as $S_n x := \beta_n x + (1-\beta_n)K_n x$, then $\lim_{l\to\infty} S_n x_{n_l} = p$ and $S_n p = p$. Thus for any $z \in F$, using (3.14) we have

$$\begin{aligned}\langle G(x_{n_l}) - \gamma f(x_{n_l}), j(x_{n_l} - z)\rangle &= \frac{-1}{\alpha_{n_l}}\langle (I - S_n)x_{n_l} - (I - S_n)p, j(x_{n_l} - z)\rangle \\ &\quad + \langle Gx_{n_l} - GS_n x_{n_l}, j(x_{n_l} - z)\rangle \\ &\le \langle Gx_{n_l} - GS_n x_{n_l}, j(x_{n_l} - z)\rangle \\ &\le (1 + \frac{1}{\mu})||x_{n_l} - S_n x_{n_l}||||x_{n_l} - z||, \end{aligned} \tag{3.18}$$

since $\langle (I - S_n)x_{n_l} - (I - S_n)p, j(x_{n_l} - z)\rangle \ge 0$ and G is Lipschitzian. Using the fact that $\|x_{n_l} - S_n x_{n_l}\| = (1 - \beta_{n_l})\|x_{n_l} - K_{n_l} x_{n_l}\| \to 0$ as $l \to \infty$, we have $\|x_{n_l} - S_n x_{n_l}\| \to 0$ as $l \to \infty$. From (3.18), taking limit as $l \to \infty$ we obtain

$$\langle (G - \gamma f)p, j(p - z)\rangle \le 0.$$

Hence p is the unique solution of the variational inequality (3.15). Now assume there exists another subsequence of $\{x_n\}$ say $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = p^*$. Then, using (3.17) we have $p^* \in F$. Repeating the above argument with p replaced by p^* we can easily obtain that p^* also solved the variational inequality (3.15). By uniqueness of the solution of the variational inequality, we obtained that $p = p^*$ and this completes the proof. □

Theorem 3.3 *Let E be a real, uniformly convex Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of E. Let $G : E \to E$ be an η-strongly accretive and μ-strictly pseudocontractive with $\eta + \mu > 1$ and let $f : E \to E$ be a contraction with coefficient $\alpha \in (0, 1)$. Let $\{T_i\}_{i=1}^N$ be family of uniformly asymptotically regular asymptotically nonexpansive self mappings of C into itself and $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let K_n be as in Lemma 3.1. Assume that $0 < \gamma < \frac{\tau}{2\alpha}$, where $\tau := (1 - \sqrt{\frac{1-\eta}{\mu}})$. Let $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ be sequences in $(0, 1)$, and suppose that the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \frac{v_n}{\alpha_n} = 0$, *where v_n is as in* (i) *of Lemma* 3.1;
(C2) $\Sigma_{n=0}^\infty \alpha_n = \infty$
(C3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

Let $\{x_n\}_{n=1}^\infty$ be a sequence defined iteratively by letting $x_0 \in C$ arbitrary and,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)K_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n G)y_n, \quad n \ge 0, \end{aligned} \tag{3.19}$$

then, the following holds

(a) $\{x_n\}_{n=1}^\infty$ *is bounded;*
(b) $\lim_{n\to\infty} ||K_n x_n - x_n|| = 0$;
(c) $F(K_n) = F$;

(d) $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $p \in F$, *where* p *is a solution of the variational inequality:*

$$\langle \gamma f(p) - Gp, j(q-p)\rangle \leq 0, \quad \forall q \in F. \tag{3.20}$$

Proof First, we show that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $u \in F$ then, since $(1 - \alpha_n\tau)(v_n/\alpha_n) \to 0$ as $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $(1-\alpha_n\tau)(v_n/\alpha_n) < (\tau - \gamma\alpha)/2$ for all $n \geq n_0$. Hence, for $n \geq n_0$, we have the following.

$$\begin{aligned}\|y_n - u\| &\leq \beta_n\|x_n - u\| + (1-\beta_n)\|K_n x_n - u\|\\ &\leq \beta_n\|x_n - u\| + (1-\beta_n)(1+v_n)\|x_n - u\|\\ &\leq (1+v_n)\|x_n - u\|,\end{aligned} \tag{3.21}$$

so that,

$$\begin{aligned}\|x_{n+1} - u\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n G)y_n - u\|\\ &= \|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(u) + \alpha_n\gamma f(u) - \alpha_n G(u) + \alpha_n G(u) + (I - \alpha_n G)y_n - u\|\\ &\leq \alpha_n\gamma\|f(x_n) - f(u)\| + \alpha_n\|\gamma f(u) - G(u)\| + \|(I-\alpha_n G)y_n - (I - \alpha_n G)u\|\\ &\leq \alpha_n\gamma\|f(x_n) - f(u)\| + \alpha_n\|\gamma f(u) - G(u)\| + (1-\alpha_n\tau)\|y_n - u\|\\ &\leq \alpha_n\gamma\alpha\|x_n - u\| + \alpha_n\|\gamma f(u) - G(u)\| + (1-\alpha_n\tau)(1+v_n)\|x_n - u\|\\ &= \left[1 - \alpha_n\left((\tau - \alpha\gamma) - (1-\alpha_n\tau)\frac{v_n}{\alpha_n}\right)\right]\|x_n - u\|\\ &\quad + \alpha_n\left((\tau-\alpha\gamma) - (1-\alpha_n\tau)\frac{v_n}{\alpha_n}\right)\frac{2\|\gamma f(u) - G(u)\|}{\tau - \alpha\gamma}\\ &\leq \max\left\{\|x_n - u\|, \frac{2\|\gamma f(u) - G(u)\|}{\tau - \alpha\gamma}\right\}.\end{aligned}$$

Thus by induction, we've

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{2\|\gamma f(u) - G(u)\|}{\tau - \alpha\gamma}\right\}, \quad \forall n \geq n_0. \tag{3.22}$$

Hence, $\{x_n\}$ is bounded. As such $\{y_n\}$, $\{Gy_n\}$ and $\{f(x_n)\}$ are also bounded. Next, we show that $\lim_{n\to\infty}||x_{n+1} - x_n|| = 0$.
Let $z_n := \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, which implies

$$\begin{aligned}z_n &= \frac{\alpha_n\gamma f(x_n) + (I - \alpha_n G)y_n - \beta_n x_n}{1-\beta_n}\\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n) + y_n - \beta_n x_n}{1-\beta_n}\\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n) + (1-\beta_n)K_n x_n}{1-\beta_n}\\ &= \frac{\alpha_n(\gamma f(x_n) - Gy_n)}{1-\beta_n} + K_n x_n\end{aligned}$$

then

$$\begin{aligned}z_{n+1} - z_n = &\frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Gy_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Gy_n)}{1-\beta_n}\\ &+ K_{n+1}x_{n+1} - K_n x_n.\end{aligned}$$

Hence, by letting $M = \sup_n(||\gamma f(x_n)|| + ||Gy_n||)$, we obtain

$$\begin{aligned}||z_{n+1} - z_n|| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(||\gamma f(x_{n+1})|| + ||Gy_{n+1}||) + \frac{\alpha_n}{1-\beta_n}(||\gamma f(x_n)|| + ||Gy_n||)\\ &\quad + ||K_{n+1}x_{n+1} - K_n x_n||\\ &\leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + ||K_{n+1}x_{n+1} - K_{n+1}x_n||\\ &\quad + ||K_{n+1}x_n - K_n x_n||\\ &\leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + (1 + \upsilon_{n+1})||x_{n+1} - x_n||\\ &\quad + ||K_{n+1}x_n - K_n x_n||\end{aligned}$$

Therefore

$$\begin{aligned}||z_{n+1} - z_n|| - ||x_{n+1} - x_n|| &\leq \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right)M + \upsilon_{n+1}||x_{n+1} - x_n||\\ &\quad + ||K_{n+1}x_n - K_n x_n||\end{aligned}$$

which implies

$$\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Hence, by Lemma 2.2, we obtain

$$\lim_{n\to\infty}||z_n - x_n|| = 0$$

thus

$$||x_{n+1} - x_n|| = (1-\beta_n)||z_n - x_n|| \to 0 \quad \text{as} \quad n \to \infty.$$

From (3.19) it follows that,

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n G)y_n - y_n\|\\ &\leq \|\alpha_n\gamma f(x_n)\| + \|(I - \alpha_n G)y_n - y_n\|\\ &= \alpha_n\Big\{\|\gamma f(x_n)\| + \|G(y_n)\|\Big\},\end{aligned}$$

we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. As

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|,$$

we also get

$$\|x_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{3.23}$$

On the other hand, we obtain

$$\begin{aligned}\|K_n x_n - x_n\| &\leq \|x_n - y_n\| + \|y_n - K_n x_n\|\\ &= \|x_n - y_n\| + \|(\beta_n x_n + (1-\beta_n)K_n x_n) - K_n x_n\|\\ &= \|x_n - y_n\| + \beta_n\|x_n - K_n x_n\|,\end{aligned} \tag{3.24}$$

which implies that $(1-\beta_n)\|K_n x_n - x_n\| \leq \|x_n - y_n\|$. From condition $(C3)$ and (3.23) we obtain

$$\|K_n x_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Hence (b) is satisfied.

Next, we show that (c) is satisfied, that is $F(K_n) = \cap_{i=1}^{N} F(T_i)$, but from (a), (b) above and (iii) of Lemma 3.1, (c) is satisfied.
Next, we show that

$$\limsup_{n\to\infty}\langle(\gamma f - G)p, j(x_n - p)\rangle \le 0, \tag{3.26}$$

where p is the unique solution of the variational inequality (3.15). Let $z_m = \alpha_m \gamma f(z_m) + (1-\alpha_m G)y_m$, where $y_m = \beta_m z_m + (1-\beta_m)K_m z_m$ and $\{\alpha_m\}$, $\{\beta_m\}$ satisfy the condition of Theorem 3.2. Then it follows from Theorem 3.2 that $p = \lim_{m\to\infty} z_m$, so that

$$z_m - x_n = \alpha_m(\gamma f(z_m) - Gz_m) + \alpha_m(Gz_m - Gy_m) + y_m - x_n$$

Hence

$$\begin{aligned}
||z_m - x_n||^2 &= \alpha_m\langle\gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m\langle Gz_m - Gy_m, j(z_m - x_n)\rangle \\
&\quad + \langle y_m - x_n, j(z_m - x_n)\rangle \\
&\le \alpha_m\langle\gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m||Gz_m - Gy_m||||z_m - x_n|| \\
&\quad + ||y_m - x_n||||z_m - x_n|| \\
&\le \alpha_m\langle\gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m(1 + \frac{1}{\mu})||z_m - y_m||||z_m - x_n|| \\
&\quad + ||y_m - x_n||||z_m - x_n|| \\
&\le \alpha_m\langle\gamma f(z_m) - Gz_m, j(z_m - x_n)\rangle \\
&\quad + \alpha_m(1 + \frac{1}{\mu})(1-\beta_m)||z_m - K_m z_m||||z_m - x_n|| \\
&\quad + ||x_n - z_m||^2 + (1-\beta_m)[v_m||z_m - x_n|| \\
&\quad + ||K_m x_n - x_n||]||z_m - x_n||.
\end{aligned}$$

Therefore

$$\begin{aligned}
\langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle &\le (1 + \frac{1}{\mu})(1-\beta_m)||z_m - K_m z_m||||z_m - x_n|| \\
&\quad + (1-\beta_m)[v_m/\alpha_m]||z_m - x_n||^2 \\
&\quad + \frac{||K_m x_n - x_n||||z_m - x_n||}{\alpha_m}.
\end{aligned}$$

Now, taking limit superior as $n \to \infty$ firstly, and then as $m \to \infty$, we have

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle\gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \le 0 \tag{3.27}$$

Moreover, we note that

$$\begin{aligned}\langle \gamma f(p) - Gp, j(x_n - p)\rangle &= \langle \gamma f(p) - Gp, j(x_n - p)\rangle - \langle \gamma f(p) - Gp, j(x_n - z_m)\rangle \\ &\quad + \langle \gamma f(p) - Gp, j(x_n - z_m)\rangle - \langle \gamma f(p) - Gz_m, j(x_n - z_m)\rangle \\ &\quad + \langle \gamma f(p) - Gz_m, j(x_n - z_m)\rangle - \langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \\ &\quad + \langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \\ &= \langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\ &\quad + \langle Gz_m - Gp, j(x_n - z_m)\rangle \\ &\quad + \langle \gamma f(z_m) - \gamma f(p), j(x_n - z_m)\rangle \\ &\quad + \langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \end{aligned} \tag{3.28}$$

Taking limit superior as $n \to \infty$ in (3.28), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p)\rangle &\le \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\ &\quad + ||Gz_m - Gp|| \limsup_{n\to\infty} ||x_n - z_m|| \\ &\quad + ||\gamma f(z_m) - \gamma f(p)|| \limsup_{n\to\infty} ||x_n - z_m|| \\ &\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \\ &\le \limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle \\ &\quad + \Big((1 + \frac{1}{\mu}) + \alpha\gamma\Big)\|z_m - p\| \limsup_{n\to\infty} ||x_n - z_m|| \\ &\quad + \limsup_{n\to\infty}\langle \gamma f(z_m) - Gz_m, j(x_n - z_m)\rangle \end{aligned}$$

By Theorem 3.2, $z_m \to p \in F$ as $m \to \infty$.
Since j is *norm*-to-*weak** uniformly continuous on bounded subset of E, we obtain

$$\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p) - j(x_n - z_m)\rangle = 0,$$

therefore, from (3.27) we obtain

$$\limsup_{n\to\infty}\langle \gamma f(p) - Gp, j(x_n - p)\rangle \le 0$$

Finally, we show that (d) is satisfied, since $\lim_{n\to\infty}(v_n/\alpha_n) = 0$, if we denote by σ_n the value of $2v_n + v_n^2$ then, clearly $\lim_{n\to\infty}(\sigma_n/\alpha_n) = 0$. Let $N_0 \in \mathbb{N}$ be large enough such that $(1 - \alpha_n\tau)(\sigma_n/\alpha_n) < (\tau - 2\gamma\alpha)/2$, for all $n \ge N_0$. Then, using the recursion formula (3.19) and for all $n \geq N_0$, we obtain.

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||\alpha_n \gamma f(x_n) - \alpha_n G(p) + (1 - \alpha_n G)y_n - (1 - \alpha_n G)p||^2 \\
&\leq ||(1 - \alpha_n G)y_n - (1 - \alpha_n G)p||^2 + 2\alpha_n \langle \gamma f(x_n) - G(p), j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n \tau)^2 ||y_n - p||^2 + 2\alpha_n \langle \gamma f(x_n) - \gamma f(p), j(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n \tau)||y_n - p||^2 + 2\alpha_n \gamma \alpha ||x_n - p|| ||x_{n+1} - p|| \\
&\quad + 2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n \tau)[\beta_n + (1 - \beta_n)(1 + v_n)^2]||x_n - p||^2 \\
&\quad + \alpha_n \gamma \alpha ||x_n - p||^2 + \alpha_n \gamma \alpha ||x_{n+1} - p||^2 \\
&\quad + 2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle \\
&\leq (1 - \alpha_n \tau)[1 + \sigma_n]||x_n - p||^2 \\
&\quad + \alpha_n \gamma \alpha ||x_n - p||^2 + \alpha_n \gamma \alpha ||x_{n+1} - p||^2 \\
&\quad + 2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle \\
&= \Big(1 - \alpha_n [(\tau - \alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)]\Big)||x_n - p||^2 \\
&\quad + \alpha_n \gamma \alpha ||x_{n+1} - p||^2 + 2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
||x_{n+1} - p||^2 &\leq \Big(1 - \alpha_n [\frac{(\tau - 2\alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)}{1 - \alpha_n \alpha\gamma}]\Big)||x_n - p||^2 \\
&\quad + \frac{2\alpha_n [(\tau - 2\alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)]\langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle}{(1 - \alpha_n \gamma\alpha)[(\tau - 2\alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)]}.
\end{aligned}$$

Observe that $\sum \alpha_n [(\tau - 2\alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)] = \infty$ and

$$\limsup \Big(\frac{2\alpha_n \langle \gamma f(p) - G(p), j(x_{n+1} - p)\rangle}{(1 - \alpha_n \gamma\alpha)[(\tau - 2\alpha\gamma) - (1 - \alpha_n \tau)(\sigma_n/\alpha_n)]}\Big) \leq 0$$

Consequently, applying Lemma 2.4, we conclude that $x_n \to p$ as $n \to \infty$.

Corollary 3.4 *Let E be a real uniformly convex Banach space whose duality mapping J is weakly sequentially continuous. Let $G : H \to H$, $f : E \to E$, $\{T_i\}_{i=1}^N$ F, $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem (3.3), then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which is also the unique solution of the variational inequality*

$$\langle \gamma f(p) - Gp, j(q - p)\rangle \leq 0, \quad \forall q \in F$$

Corollary 3.5 *Let H be a real Hilbert space, $\{z_t\}_{t\in(0,1)}$, be as in Theorem 3.2. Then $\{z_t\}$ converges strongly to a common fixed point of the family $\{T_i\}_{i=1}^N$ say p which is a unique solution of the variational inequality*

$$\langle (G - \gamma f)p, q - p\rangle \geq 0, \quad \forall q \in F.$$

Corollary 3.6 *Let H be a real Hilbert space and let C a nonempty closed convex subset of H. Let $G : H \to H$, $f : E \to E$, $\{T_i\}_{i=1}^N$ F, $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem (3.3), then $\{x_n\}_{n=1}^\infty$ converges strongly to $p \in F$, which is also the unique solution of the variational inequality*

$$\langle \gamma f(p) - Gp, q - p\rangle \leq 0, \quad \forall q \in F$$

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